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Canonical matrices of isometric operators on indefinite inner product spaces

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Abstract

We give canonical matrices of a pair (A, B) consisting of a nondegenerate form B and a linear operator A satisfying $B(Ax, Ay) = B(x, y)$ on a vector space over $\mathbb{F}$ in the following cases:

- $\mathbb{F}$ is an algebraically closed field of characteristic different from 2 or a real closed field, and B is symmetric or skew-symmetric;
- $\mathbb{F}$ is an algebraically closed field of characteristic 0 or the skew field of quaternions over a real closed field, and B is Hermitian or skew-Hermitian with respect to any nonidentity involution on $\mathbb{F}$.

These classification problems are wild if B may be degenerate.

We use a method that admits to reduce the problem of classifying an arbitrary system of forms and linear mappings to the problem of classifying representations of some quiver. This method was described in [V.V. Sergeichuk, Classification problems for systems of forms and linear mappings, Math. USSR-Izv. 31 (3) (1988) 481–501].
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1. Introduction

Let $\mathbb{F}$ be a field or skew field of characteristic different from 2 with involution (which may be the identity). We consider the problem of classifying pairs

$$(A, B) \tag{1}$$

consisting of a nondegenerate Hermitian or skew-Hermitian form $B : V \times V \to \mathbb{F}$ on a right vector space V over $\mathbb{F}$ and an operator $A : V \to V$ that is *isometric* with respect to B; i.e.,

$$B(Au, Av) = B(u, v) \quad \text{for all } u, v \in V.$$

This problem was solved in [38, Theorem 5] over $\mathbb{F}$ up to classification of Hermitian forms over finite extensions of $\mathbb{F}$, we present the solution in Theorem 2.2. This implies its complete solution over $\mathbb{C}$ and $\mathbb{R}$ since the classification of Hermitian forms over $\mathbb{C}$ and $\mathbb{R}$ is known. But the canonical matrices in [38] are not simple since they are based on the Frobenius canonical form over $\mathbb{F}$ for similarity.

The first purpose of this paper is to give simple canonical matrices of pairs (1) over an algebraically or real closed field of characteristic different from 2 basing on the Jordan canonical form for similarity. We also obtain canonical matrices of (A, B) over the skew field $\mathbb{H}$ of real quaternions; they are given in [40] incorrectly (see the footnote on page 31). This classification problem was studied in [16,17,23,33], other canonical matrices of (A, B) and their applications are given in [2,28] over $\mathbb{C}$ and $\mathbb{R}$, and in [1] over $\mathbb{H}$.

The second purpose of this paper is to present in sufficient detail a technique for classifying systems of forms and linear mappings (we use it to obtain canonical matrices of (1)). It was devised by Roiter [32] and the author [35,36,38]. It is practically unknown although many classification problems solved recently could be easily solved by this method. This linearization technique reduces the "nonlinear" problem of classifying an arbitrary system $\mathscr{S}$ of forms and linear mappings over a field or skew field $\mathbb{F}$ of characteristic different from 2

- to the "linear" problem of classifying some system $\underline{\mathscr{S}}$ of linear mappings over $\mathbb{F}$—i.e., to the problem of classifying representations of a quiver with relations, and
- to the problem of classifying Hermitian forms over fields or skew fields that are finite extensions of the center of $\mathbb{F}$.

The corresponding reduction theorems were extended in [38] to the problem of classifying selfadjoint representations of a linear category with involution and in [41] to the problem of classifying symmetric representations of an algebra with involution. Similar theorems were proved for bilinear and sesquilinear forms by Gabriel, Riehm, and Shrader-Frechette [7,26,27]; for additive categories with quadratic or Hermitian forms on objects by Quebbermann et al. [25,34]; for generalizations of quivers involving linear groups by Derksen et al. [5,44].

Two cases are possible for the system S.

Case 1. $\mathscr{S}$ *is wild.* This means that the problem of classifying the system $\underline{\mathscr{S}}$ contains the problem of classifying pairs of matrices up to simultaneous similarity. The latter problem is hopeless since it contains the problem of classifying an *arbitrary* system of linear mappings [3, Theorems 4.5 and 2.1]. Hence, the problem of classifying the system $\mathscr{S}$ is hopeless too. For example, the wildness of $\underline{\mathscr{S}}$ was proved in [37, Theorems 5.4 and 5.5] for the problems of classifying

– selfadjoint/metric operators on a space with degenerate indefinite scalar product (we replicate this result in Theorem 6.1; this classification problem was considered in [22]) and

– normal operators on a space with degenerate indefinite scalar product (this problem was posed in [9, p. 84]; its wildness was also proved in [11]).

Thus, these problems are hopeless, and so the problem of classifying (1) cannot be solved if B may be degenerate.

Case 2. $\mathscr{S}$ *is not wild.* Then the problem of classifying the system $\mathscr{S}$ can be solved. In each dimension, the set of Belitskiĭ's canonical matrices of the system $\mathscr{S}$ consists of a finite number of matrices and 1-parameter families of matrices and is presented by a finite number of points and straight lines in the affine matrix space (see [43, Theorem 3.1] and also [8]). For example, the system $\mathscr{S}$ is not wild for the problems of classifying

- sesquilinear forms,
- pairs of forms, in which the first form is ε-Hermitian and the second is δ-Hermitian ($\varepsilon, \delta \in \{1, -1\}$), and
- isometric or selfadjoint operators on a space with nondegenerate ε-Hermitian form (an operator A is *selfadjoint* with respect to a form B if $B(Ax, y) = B(x, Ay)$).

Their canonical matrices were obtained by the linearization technique in [36,37] and also in [38, Theorems 3–6] over any field of characteristic different from 2 up to classification of Hermitian forms over its finite extensions.

Theorem 3.2 implies that each system of forms and linear mappings over $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{H}$ decomposes into a direct sum of indecomposable systems uniquely up to isomorphism of summands. Hence, it suffices to classify only indecomposable systems.

A detailed exposition of the theory of operators on spaces with indefinite scalar product is given in the books [9,10].

The paper is organized as follows. In Section 2 we formulate Theorem 2.1 about canonical matrices of pairs (1) over algebraically or real closed fields and skew fields of real quaternions. We also formulate Theorem 2.2, which is a useful generalization of [38, Theorem 5] and gives canonical matrices of (1) over any field of characteristic different from 2 up to classification of Hermitian forms.

Section 3 contains a detailed description of the linearization technique; it can be read independently of Section 2. Theorem 3.2 in this section extents Sylvester's Inertia Theorem to systems of forms and linear mappings.

In Sections 4 and 5 we prove Theorems 2.1 and 2.2.

In Section 6 we present Theorem 5.4 of [37] about the wildness of the problem of classifying pairs (1) in which B may be degenerate.

2. Canonical matrices of isometric operators

We recall some properties of algebraically or real closed fields and skew fields of real quaternions, and formulate Theorems 2.1 and 2.2 about canonical matrices of pairs (1).

2.1. Isometric operators over an algebraically or real closed field and over quaternions

In this paper, $\mathbb{F}$ denotes a field or skew field of characteristic different from 2 with involution $a \mapsto \bar{a}$; that is, a bijection $\mathbb{F} \to \mathbb{F}$ satisfying

$$\overline{a + b} = \bar{a} + \bar{b}, \quad \overline{ab} = \bar{b}\bar{a}, \quad \bar{\bar{a}} = a.$$

Therefore, the involution can be the identity only if $\mathbb{F}$ is a field. All vector spaces are assumed to be finite dimensional right vector spaces.

A mapping $B : U \times V \to \mathbb{F}$ on vector spaces U and V over $\mathbb{F}$ is called a *sesquilinear form* if

$$B(ua + u'a', v) = \bar{a}B(u, v) + \bar{a}'B(u', v),$$
$$B(u, va + v'a') = B(u, v)a + B(u, v')a'$$

for all $u, u' \in U$, $v, v' \in V$, and $a, a' \in \mathbb{F}$. This form is *bilinear* if $\mathbb{F}$ is a field and the involution $a \mapsto \bar{a}$ is the identity (we consider bilinear forms as a special case of sesquilinear forms). If $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ are bases of U and V, then

$$B(u, v) = [u]_e^* B_{ef} [v]_f \quad \text{for all } u \in U \text{ and } v \in V,$$

where $[u]_e$ and $[v]_f$ are the coordinate vectors, $[u]_e^* := \overline{[u]_e}^{\mathrm{T}}$, and $B_{ef} := [B(e_i, f_j)]$ is the matrix of B.

Let ε be an element of the center $C(\mathbb{F})$ of $\mathbb{F}$ such that $\varepsilon\bar{\varepsilon} = 1$. A sesquilinear form $B : V \times V \to \mathbb{F}$ is called ε-*Hermitian* if

$$B(u, v) = \varepsilon \overline{B(v, u)} \quad \text{for all } u, v \in V;$$

it is called *Hermitian* if $\varepsilon = 1$ and *skew-Hermitian* if $\varepsilon = -1$. Clearly, $\varepsilon = \pm 1$ if the involution acts identically on $C(\mathbb{F})$. Without loss of generality, *we will assume that $\varepsilon = 1$ if the involution acts nonidentically on $C(\mathbb{F})$* since then an ε-Hermitian form B can be made Hermitian by multiplying it by $1 + \bar{\varepsilon}$ if $\varepsilon \neq -1$ because

$$(1 + \bar{\varepsilon})B(u, v) = (1 + \bar{\varepsilon})\varepsilon \overline{B(v, u)} = (1 + \varepsilon)\overline{B(v, u)} = \overline{(1 + \bar{\varepsilon})B(v, u)},$$

and by $a - \bar{a}$ for any $a \neq \bar{a}$ from $C(\mathbb{F})$ if $\varepsilon = -1$.

Let (A, B) be a pair consisting of a nondegenerate ε-Hermitian form B and an isometric operator A on a vector space V. Their matrices A_e and B_e in a basis of V satisfy the conditions:

$$B_e = \varepsilon B_e^* = A_e^* B_e A_e, \quad A_e \text{ and } B_e \text{ are nonsingular}, \tag{2}$$

where $A_e^* := \overline{A_e}^{\mathrm{T}}$ (usually the letter H is used instead of B_e, then A_e satisfying (2) is called *H-unitary*, see [2]). Every change of the basis reduces (A_e, B_e) by transformations

$$(A_e, B_e) \mapsto (S^{-1}A_e S, S^* B_e S), \quad S \text{ is nonsingular}. \tag{3}$$

In Theorem 2.1 we give canonical matrices of pairs (A_e, B_e) satisfying (2) with respect to transformations (3) over:

- an *algebraically closed field of characteristic different from* 2,
- a *real closed field*—i.e, a field whose algebraic closure has a finite degree $\neq 1$ (see Lemma 2.1), and
- the *skew field of quaternions*

  $$\mathbb{H} = \{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} | a, b, c, d \in \mathbb{P}\}$$

  over a real closed field $\mathbb{P}$, where $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = \mathrm{k} = -\mathrm{ji}$, $\mathrm{jk} = \mathrm{i} = -\mathrm{kj}$, and $\mathrm{ki} = \mathrm{j} = -\mathrm{ik}$.

Without loss of generality we can consider only two involutions on $\mathbb{H}$: *quaternionic conjugation*

$$a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \longmapsto a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k} \tag{4}$$

and *quaternionic semiconjugation*

$$a + bi + cj + dk \longmapsto a - bi + cj + dk, \quad a, b, c, d \in \mathbb{P}, \tag{5}$$

because by Lemma 2.2 if an involution on $\mathbb{H}$ is not quaternionic conjugation then it has the form (5) in a suitable set of imaginary units i, j, k.

There is a natural one-to-one correspondence

$$\left\{ \begin{array}{l} \text{algebraically closed fields} \\ \text{with nonidentity involution} \end{array} \right\} \quad \longleftrightarrow \quad \{\text{real closed fields}\}$$

sending an algebraically closed field with nonidentity involution to its fixed field. This follows from our next lemma, in which we collect known results about such fields.

Lemma 2.1. (a) *Let* $\mathbb{P}$ *be a real closed field and let* $\mathbb{K}$ *be its algebraic closure. Then* char $\mathbb{P} = 0$ *and*

$$\mathbb{K} = \mathbb{P} + \mathbb{P}i, \quad i^2 = -1. \tag{6}$$

The field $\mathbb{P}$ *has a unique linear ordering* $\leqslant$ *such that*

$$a > 0 \text{ and } b > 0 \implies a + b > 0 \text{ and } ab > 0.$$

The positive elements of $\mathbb{P}$ *with respect to this ordering are the squares of nonzero elements.*

(b) *Let* $\mathbb{K}$ *be an algebraically closed field with nonidentity involution. Then* char $\mathbb{K} = 0$,

$$\mathbb{P} := \{k \in \mathbb{K} | \bar{k} = k\} \tag{7}$$

is a real closed field,

$$\mathbb{K} = \mathbb{P} + \mathbb{P}i, \quad i^2 = -1, \tag{8}$$

and the involution has the form

$$\overline{a + bi} = a - bi, \quad a, b \in \mathbb{P}. \tag{9}$$

(c) *Every algebraically closed field* $\mathbb{F}$ *of characteristic* 0 *contains at least one real closed subfield. Hence,* $\mathbb{F}$ *can be represented in the form* (8) *and possesses the involution* (9).

Proof. (a) Let $\mathbb{K}$ be the algebraic closure of $\mathbb{F}$ and suppose $1 < \dim_{\mathbb{P}} \mathbb{K} < \infty$. By Corollary 2 in [21, Chapter VIII, §9], we have char $\mathbb{P} = 0$ and (6). The other statements of part (a) follow from Proposition 3 and Theorem 1 in [21, Chapter XI, §2].

(b) If $\mathbb{K}$ is an algebraically closed field with nonidentity involution $a \mapsto \bar{a}$, then this involution is an automorphism of order 2. Hence $\mathbb{K}$ has degree 2 over its *fixed field* $\mathbb{P}$ defined in (7). Thus, $\mathbb{P}$ is a real closed field. Let $i \in \mathbb{K}$ be such that $i^2 = -1$. By (a), every element of $\mathbb{K}$ is uniquely represented in the form $k = a + bi$ with $a, b \in \mathbb{P}$. The involution is an automorphism of $\mathbb{K}$, so $\bar{i}^2 = -1$. Thus, $\bar{i} = -i$ and the involution has the form (9).

(c) This statement is proved in [45, §82, Theorem 7c]. $\quad \square$

For each real closed field, we denote by $\leqslant$ the ordering from Lemma 2.1(a). Let $\mathbb{K} = \mathbb{P} + \mathbb{P}i$ be an algebraically closed field with nonidentity involution represented in the form (8). By the *absolute value* of $k = a + bi \in \mathbb{K}$ ($a, b \in \mathbb{P}$) we mean a unique nonnegative real root of $a^2 + b^2$, which we write as

$$|k| := \sqrt{a^2 + b^2} \tag{10}$$

(this definition is unambiguous since $\mathbb{K}$ is represented in the form (8) uniquely up to replacement of i by $-$i). For each $M \in \mathbb{K}^{m \times n}$, its *realification* $M^{\mathbb{P}} \in \mathbb{P}^{2m \times 2n}$ is obtained by replacing every entry $a + b$i of M by the 2×2 block

$$
\begin{matrix} a & -b \\ b & a \end{matrix} \tag{11}
$$

Define the $n \times n$ matrices

$$
J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad \Lambda_n := \begin{bmatrix} 1 & 2 & \cdots & 2 \\ & 1 & \ddots & \vdots \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}, \tag{12}
$$

$$
F_n := \begin{bmatrix} 0 & & & & \cdot^{\cdot^{\cdot}} \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & 0 \end{bmatrix}. \tag{13}
$$

If M is nonsingular it is convenient to write

$$
M^{-*} := (M^{-1})^*, \quad M^{-\mathrm{T}} := (M^{-1})^{\mathrm{T}}.
$$

The *skew sum* of two matrices is defined by

$$
M \backslash N := \begin{bmatrix} 0 & N \\ M & 0 \end{bmatrix}.
$$

The main result of this paper is the following theorem.

Theorem 2.1. *Let $\mathbb{F}$ be one of the following fields or skew fields*:

(a) *an algebraically closed field of characteristic different from* 2 *with the identity involution*;
(b) *an algebraically closed field with nonidentity involution*;
(c) *a real closed field $\mathbb{P}$ (by Lemma* 2.1, *its algebraic closure is represented in the form $\mathbb{P} + \mathbb{P}$i and possesses the involution $a + b$i $\mapsto a - b$i)*;
(d) *the skew field $\mathbb{H} = \mathbb{P} + \mathbb{P}$i $+ \mathbb{P}$j $+ \mathbb{P}$k *of quaternions over a real closed field $\mathbb{P}$, with quaternionic conjugation* (4) *or quaternionic semiconjugation* (5).

Let $\varepsilon = \pm 1$ ($\varepsilon = 1$ if $\mathbb{F}$ is (b)) *and let (A, B) be a pair consisting of a nondegenerate ε-Hermitian form B on a right vector space over $\mathbb{F}$ and an operator A on this space that is isometric with respect to B.*

Then there exists a basis in which (A, B) is given by a direct sum, determined uniquely up to permutation of summands, respectively,

(a) *of the following matrix pairs that are given by $0 \neq \lambda \in \mathbb{F}$ determined up to replacement by λ^{-1}*:
    (i) $(J_n(\lambda) \oplus J_n(\lambda)^{-\mathrm{T}}, I_n \backslash \varepsilon I_n)$, *except for $\lambda = \pm 1$ and $\varepsilon = (-1)^{n+1}$*,
    (ii) $(\lambda \Lambda_n, F_n)$ *if $\lambda = \pm 1$ and $\varepsilon = (-1)^{n+1}$*;

(b) *of the following matrix pairs that are given by* $0 \neq \lambda \in \mathbb{F}$ *determined up to replacement by* $\bar{\lambda}^{-1}$:
  (i) $(J_n(\lambda) \oplus J_n(\lambda)^{-*}, I_n \backslash I_n)$ *if* $|\lambda| \neq 1$,
  (ii) $(\lambda \Lambda_n, \pm i^{n-1} F_n)$ *if* $|\lambda| = 1$;
(c) *of the following matrix pairs that are given by* $0 \neq \lambda \in \mathbb{P} + \mathbb{P}i$ *determined up to replacement by* λ^{-1} *(by* λ^{-1}, $\bar{\lambda}$, *and* $\bar{\lambda}^{-1}$ *in* (iii)):
  (i) $(J_n(\lambda) \oplus J_n(\lambda)^{-T}, I_n \backslash \varepsilon I_n)$ *if* $\lambda \in \mathbb{P}$, *except for* $\lambda = \pm 1$ *and* $\varepsilon = (-1)^{n+1}$,
  (ii) $(\lambda \Lambda_n, \pm F_n)$ *if* $\lambda = \pm 1^2$ *and* $\varepsilon = (-1)^{n+1}$,
  (iii) $(J_n(\lambda)^{\mathbb{P}} \oplus (J_n(\lambda)^{\mathbb{P}})^{-T}, I_{2n} \backslash \varepsilon I_{2n})$ *if* $\lambda \notin \mathbb{P}$ *and* $|\lambda| \neq 1$,
  (iv) $((\lambda \Lambda_n)^{\mathbb{P}}, \pm (i^{n-(\varepsilon+1)/2} F_n)^{\mathbb{P}})$ *if* $\lambda \notin \mathbb{P}$ *and* $|\lambda| = 1$;
(d) *of the following matrix pairs that are given by* $0 \neq \lambda \in \mathbb{P} + \mathbb{P}i$ *determined up to replacement by* λ^{-1}, $\bar{\lambda}$, *and* $\bar{\lambda}^{-1}$:
  (i) $(J_n(\lambda) \oplus J_n(\lambda)^{-*}, I_n \backslash \varepsilon I_n)$ *if* $|\lambda| \neq 1$,
  (ii) $(\lambda \Lambda_n, \delta i^{n-(\varepsilon+1)/2} F_n)$ *if* $|\lambda| = 1$, *where*

$$\delta := \begin{cases} 1, & \text{if } \lambda = \pm 1, \text{ the involution is (4)}, \varepsilon = (-1)^n, \\ & \text{and if } \lambda = \pm 1, \text{ the involution is (5)}, \varepsilon = (-1)^{n+1}; \\ \pm 1, & \text{otherwise.} \end{cases}$$

In this theorem "determined up to replacement by" means that a matrix pair reduces by transformations (3) to the matrix pair obtained by making the indicated replacement (i.e., they give the same (A, B) but in different bases).

Remark. The matrix $i^{n-(\varepsilon+1)/2} F_n$ in (c)(iv) and (d)(ii) can be replaced by F_n if $\varepsilon = (-1)^{n+1}$ and by iF_n if $\varepsilon = (-1)^n$. The pairs

$$(\lambda \Lambda_n, \pm i^{n-1} F_n), \quad ((\lambda \Lambda_n)^{\mathbb{P}}, \pm (i^{n-(\varepsilon+1)/2} F_n)^{\mathbb{P}}), \quad (\lambda \Lambda_n, \delta i^{n-(\varepsilon+1)/2} F_n)$$

in (b)(ii), (c)(iv), and (d)(ii) can be replaced by

$$(\lambda \Omega_n, \pm E_n), \quad \left((\lambda \Omega_n)^{\mathbb{P}}, \pm \left(\sqrt{\varepsilon} E_n\right)^{\mathbb{P}}\right), \quad \left(\lambda \Omega_n, \delta \sqrt{\varepsilon} E_n\right),$$

where $\sqrt{-1} = i$ and

$$\Omega_n := \begin{bmatrix} 1 & 2i & 2i^2 & \ddots & 2i^{n-1} \\ & 1 & 2i & \ddots & \ddots \\ & & 1 & \ddots & 2i^2 \\ & & & \ddots & 2i \\ 0 & & & & 1 \end{bmatrix}, \quad E_n := \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} \quad (n\text{-by-}n).$$

This remark follows from the proof of Theorem 2.1 and from the equalities

$$S_n^{-1} \Lambda_n S_n = \Omega_n, \quad S_n^* i^{n-1} F_n S_n = E_n,$$

where $S_n := \operatorname{diag}(1, i, i^2, i^3, \ldots, i^{n-1})$ (i.e., $(\Lambda_n, i^{n-1} F_n)$ and (Ω_n, E_n) gave the same (A, B) but in different bases).

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[2] This gives 4 pairs: (Λ_n, F_n), $(\Lambda_n, -F_n)$, $(-\Lambda_n, F_n)$, and $(-\Lambda_n, -F_n)$.

Due to the following lemma, we have the right to consider only the involutions (4) and (5) on $\mathbb{H}$.

Lemma 2.2. *Let $\mathbb{H}$ be the skew field of quaternions over a real closed field $\mathbb{P}$. If any involution on $\mathbb{H}$ is not quaternionic conjugation* (4), *then it becomes quaternionic semiconjugation* (5) *after a suitable reselection of the imaginary units* i, j, k.[3]

Proof. The *absolute value* of a quaternion $h = a + bi + cj + dk$ is the unique nonnegative real root

$$|h| := \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{h\bar{h}} \in \mathbb{P},$$

where $\bar{h} := a - bi - cj - dk$ is the *conjugate quaternion* ($a^2 + b^2 + c^2 + d^2$ is a square by Lemma 2.1(a)). Then $h^{-1} = |h|^{-2}\bar{h}$ if $h \neq 0$.

The vector space of purely imaginary quaternions

$$\mathbb{E} := \{bi + cj + dk \,|\, b, c, d \in \mathbb{P}\}$$

can be considered as the Euclidean space over $\mathbb{P}$ with scalar product

$$(bi + cj + dk, b'i + c'j + d'k) := bb' + cc' + dd'.$$

Then $\{i, j, k\}$ is an orthonormal basis, $|h|$ is the length of $h \in \mathbb{E}$, and the multiplication of two purely imaginary quaternions can be represented in the form

$$h_1 h_2 = [h_1, h_2] - (h_1, h_2), \quad h_1, h_2 \in \mathbb{E}, \tag{14}$$

where $[h_1, h_2]$ is the vector product (if $\mathbb{P} = \mathbb{R}$ then we may use its geometrical definition, otherwise we use its definition via determinants) and (h_1, h_2) is the scalar product; in particular, $[i, j] = k$ and $(i, j) = 0$.

If $\{i', j'\}$ is a pair of orthonormal quaternions in $\mathbb{E}$ (i.e., $|i'| = |j'| = 1$ and $(i', j') = 0$), then i', j', $k' := i'j'$ can be taken as a new set of imaginary units.

Let $h \mapsto \widehat{h}$ be an involution on $\mathbb{H}$ that is different from quaternionic conjugation (4). Let us prove that it acts identically on $\mathbb{P}$; that is, $\widehat{r} = r$ for all $r \in \mathbb{P}$. Each $r \in \mathbb{P}$ commutes with all $h \in \mathbb{H}$, hence $\widehat{r}$ commutes with all $\widehat{h} \in \mathbb{H}$. Since $\mathbb{P}$ is the center of $\mathbb{H}$, $\widehat{r} \in \mathbb{P}$ and $r \mapsto \widehat{r}$ is an involution on $\mathbb{P}$. If $\mathbb{P}_\circ := \{r \in \mathbb{P} \,|\, \widehat{r} = r\}$ is its fixed field, then the algebraically closed field $\mathbb{P} + \mathbb{P}i$ has a finite degree over $\mathbb{P}_\circ$, and so $\mathbb{P}_\circ$ is a real closed field. By Lemma 2.1(a), this degree is 2, and so $\mathbb{P}_\circ = \mathbb{P}$.

Since $h \mapsto \widehat{h}$ is not quaternionic conjugation, by [4, Chapter 8, §11, Proposition 2] there exists $h = a + bi + cj + dk \notin \mathbb{P}$ such that $\widehat{h} = h$. Put

$$e := (b^2 + c^2 + d^2)^{-1/2}(bi + cj + dk),$$

then $e \in \mathbb{E}$, $|e| = 1$, and $\widehat{e} = e$.

Choose any $f \in \mathbb{E}$ of length 1 being orthogonal to e. Then by (14)

$$e^2 = f^2 = -1, \quad ef = -fe. \tag{15}$$

Write $\mathbb{K} := \mathbb{P} + \mathbb{P}e$. Since $\{e, f, ef\}$ is a basis of $\mathbb{E}$, there are $a, b, c, d \in \mathbb{P}$ such that

$$\widehat{f} = a + be + cf + def = \varepsilon + \delta f, \quad \varepsilon := a + be, \quad \delta := c + de \in \mathbb{K}. \tag{16}$$

Then

$$\widehat{\widehat{f}} = \widehat{\varepsilon} + \widehat{f}\widehat{\delta} = \varepsilon + \widehat{f}\delta = \varepsilon + (\varepsilon + \delta f)\delta = \varepsilon + \varepsilon\delta + \delta f\delta.$$

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[3] Remark at proofreading: a shorter proof of this statement is given in Ref. [46].

By (15), $fe = -ef$, and so $\widehat{\widehat{f}} = \varepsilon(1 + \delta) + \delta\delta' f$ with $\delta' := c - de$. But $\widehat{\widehat{f}} = f$, hence $f = \varepsilon(1 + \delta) + \delta\delta' f$. Since $\varepsilon(1 + \delta), \delta\delta' \in \mathbb{K}$, and $\mathbb{H} = \mathbb{K} + \mathbb{K}f$, we have

$$\varepsilon = 0 \text{ or } \delta = -1, \quad \text{and} \quad \delta\delta' = c^2 + d^2 = 1.$$

Case 1: $\varepsilon = 0$. Then $\widehat{f} = \delta f$. Since $\mathbb{K}$ is the algebraic closure of $\mathbb{P}$, there exist $x, y \in \mathbb{P}$ such that $(x + ye)^2 = c + de = \delta$. In view of $e^2 = -1$,

$$(x^2 + y^2)^2 = ((x + ye)(x - ye))^2 = (c + de)(c - de) = c^2 + d^2 = 1.$$

Thus $x^2 + y^2 = 1$. Let us write $k' := (x + ye)f$ and prove that the quaternions $i' := ek'$, $j' := e$, k' form a desired set of imaginary units.

It suffices to check that they are purely imaginary quaternions satisfying

$$|e| = |k'| = 1, \quad (e, k') = 0, \tag{17}$$

and that the involution $h \mapsto \widehat{h}$ has the form (5) with respect to these imaginary units; i.e.,

$$\widehat{ek'} = -ek', \quad \widehat{e} = e, \quad \widehat{k'} = k'. \tag{18}$$

By (15)

$$\begin{aligned}
k'^2 &= (x + ye)f(x + ye)f = (x + ye)(x - ye)f^2 \\
&= (x^2 + y^2)f^2 = f^2 = -1.
\end{aligned}$$

In view of (14), $(k', k') = 1$, and so $|k'| = 1$. The inclusion

$$ek' = e(x + ye)f = xef - yf \in \mathbb{E}$$

implies $(e, k') = 0$. This proves (17).

Furthermore,

$$\begin{aligned}
\widehat{k'} &= \widehat{f}(x + ye) = (x + ye)^2 f(x + ye) = (x + ye)^2(x - ye)f \\
&= (x + ye)(x^2 + y^2)f = (x + ye)f = k'
\end{aligned}$$

and $\widehat{ek'} = k'e = -ek'$. This proves (18).

Case 2: $\delta = -1$. Let us prove that the quaternions

$$i' := f, \quad j' := e, \quad k' := fe$$

form a desired set of imaginary units. The conditions $|f| = |e| = 1$, $(f, e) = 1$, and $\widehat{e} = e$ hold.

By (16), $\widehat{f} = \varepsilon - f = a + be - f$. In view of (15), $fe = -ef$, $\widehat{f}e = -e\widehat{f}$, $e\widehat{f} = -\widehat{f}e$, and so $e(a + be - f) = -(a + be - f)e$. Since $-ef = fe$, we have $(a + be)e = 0$, hence $\widehat{f} = -f$. Finally, $\widehat{fe} = \widehat{e}\widehat{f} = -ef = fe$. $\quad\square$

2.2. Isometric operators over a field of characteristic different from 2

Canonical matrices of pairs (A, B) consisting of a nondegenerate Hermitian or skew-Hermitian form B and an isometric operator A were obtained in [38, Theorem 5] over any field $\mathbb{F}$ of characteristic different from 2 up to classification of Hermitian forms. They were based on the Frobenius canonical matrices for similarity. We rephrase [38, Theorem 5] in Theorem 2.2 from this section in terms of an *arbitrary* set of canonical matrices for similarity. This flexibility will be used in the proof of Theorem 2.1. An analogous flexibility was used in [13] to simplify over

$\mathbb{C}$ the canonical matrices for congruence and *congruence from [38, Theorems 3] (a direct proof that the matrices from [13] are canonical is given in [14,15]).

For each polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \in \mathbb{F}[x],$$

we define the polynomials

$$\bar{f}(x) := \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_n,$$
$$f^\vee(x) := \bar{a}_n^{-1}(\bar{a}_n x^n + \cdots + \bar{a}_1 x + \bar{a}_0) \quad \text{if } a_n \neq 0.$$

Lemma 2.3 ([38, Lemma 6]). *Let $\mathbb{F}$ be a field with involution $a \mapsto \bar{a}$ (possibly, the identity), let $p(x) = p^\vee(x)$ be an irreducible polynomial over $\mathbb{F}$, and let r be the integer part of $(\deg p(x))/2$. Consider the field*

$$\mathbb{F}(\kappa) = \mathbb{F}[x]/p(x)\mathbb{F}[x], \quad \kappa := x + p(x)\mathbb{F}[x], \tag{19}$$

with involution

$$f(\kappa)^\circ := \bar{f}(\kappa^{-1}). \tag{20}$$

Then each element of $\mathbb{F}(\kappa)$ on which the involution acts identically is uniquely representable in the form $q(\kappa)$, where

$$q(x) = a_r x^r + \cdots + a_1 x + a_0 + \bar{a}_1 x^{-1} + \cdots + \bar{a}_r x^{-r}, \quad a_0 = \bar{a}_0, \tag{21}$$

$a_0, \ldots a_r \in \mathbb{F}$, and if $\deg p(x) = 2r$ then

$$a_r = \begin{cases} 0 & \text{if the involution on } \mathbb{F} \text{ is the identity,} \\ \bar{a}_r & \text{if the involution on } \mathbb{F} \text{ is not the identity and } p(0) \neq 1, \\ -\bar{a}_r & \text{if the involution on } \mathbb{F} \text{ is not the identity and } p(0) = 1. \end{cases} \tag{22}$$

Proof. *Case 1:* $\deg p(x) = 2r + 1$. The elements $\kappa^r, \ldots, 1, \ldots, \kappa^{-r}$ (κ is defined in (19)) form a basis of $\mathbb{F}(\kappa)$ over $\mathbb{F}$. Therefore, each element of $\mathbb{F}(\kappa)$ is uniquely representable in the form

$$a_r \kappa^r + \cdots + a_0 + \cdots + a_{-r} \kappa^{-r}, \quad a_r, \ldots, a_{-r} \in \mathbb{F}. \tag{23}$$

The involution (20) acts identically on (23) if and only if $a_i = \bar{a}_{-i}$ for all $i = 0, 1, \ldots, r$.

Case 2: $\deg p(x) = 2r$ *and the involution on $\mathbb{F}$ is the identity.* Then the involution (20) acts identically on the elements

$$a_{r-1} \kappa^{r-1} + \cdots + a_0 + \cdots + a_{r-1} \kappa^{-r+1}, \quad a_0, \ldots, a_{r-1} \in \mathbb{F};$$

they are distinct and form over $\mathbb{F}$ a subspace of dimension r, which is contained in the fixed field

$$\mathbb{F}(\kappa)_\circ := \{f(\kappa) \in \mathbb{F}(\kappa) \mid \overline{f(\kappa)} = f(\kappa)\} \tag{24}$$

of $\mathbb{F}(\kappa)$ with respect to the involution (20). $\mathbb{F}(\kappa)_\circ$ has the same dimension r over $\mathbb{F}$ because $\dim_\mathbb{F} \mathbb{F}(\kappa) = 2r$, and so the subspace and the fixed field coincide.

Case 3: $\deg p(x) = 2r$ *and the involution on $\mathbb{F}$ is not the identity.* Let

$$p(x) = x^{2r} + p_1 x^{2r-1} + \cdots + p_{2r-1} x + p_{2r},$$

then

$$p^\vee(x) = \bar{p}_{2r}^{-1}(\bar{p}_{2r} x^{2r} + \bar{p}_{2r-1} x^{2r-1} + \cdots + \bar{p}_1 x + 1).$$

The equality $p(0) = p^\vee(0)$ implies $p_{2r} = \bar{p}_{2r}^{-1}$. Taking any $b \in \mathbb{F}$ for which $\bar{b} \neq b$ and putting

$$\delta := \begin{cases} 1 + \bar{p}_{2r} & \text{if } p_{2r} \neq -1, \\ b - \bar{b} & \text{if } p_{2r} = -1, \end{cases}$$

we find that $\delta p_{2r} = \bar{\delta}$. Then $\delta \bar{p}_{2r}^{-1} = \delta p_{2r} = \bar{\delta}$, and so

$$\delta x^{-r} p^\vee(x) = \overline{\delta p}_{2r} x^r + \overline{\delta p}_{2r-1} x^{r-1} + \cdots + \overline{\delta p}_1 x^{1-r} + \bar{\delta} x^{-r}.$$

Since

$$\delta x^{-r} p^\vee(x) = \delta x^{-r} p(x) = \delta x^r + \delta p_1 x^{r-1} + \cdots + \delta p_{2r-1} x^{1-r} + \delta p_{2r} x^{-r},$$

the function $\pi(x) := \delta x^{-r} p(x)$ has the form

$$\pi(x) = c_r x^r + \cdots + c_1 x + c_0 + \bar{c}_1 x^{-1} + \cdots + \bar{c}_r x^{-r}, \quad c_0 = \bar{c}_0, c_r \neq 0.$$

Using the equalities $c_r = \delta$ and $\delta p_{2r} = \bar{\delta}$, we find that $c_r p_{2r} = \bar{c}_r$,

$$c_r \neq \begin{cases} \bar{c}_r & \text{if } p(0) = p_{2r} \neq 1, \\ -\bar{c}_r & \text{if } p(0) = p_{2r} = 1. \end{cases} \tag{25}$$

Let $q(x)$ be of the form (21), and let $q(\kappa) = 0$. Let us prove that $q(x) = 0$. We have

$$\kappa^r q(\kappa) = 0, \quad x^r q(x) \equiv 0 \bmod p(x), \quad x^r q(x) = ap(x)$$

for some $a \in \mathbb{F}$. Thus,

$$q(x) = a\delta^{-1} \delta x^{-r} p(x) = b\pi(x), \quad b := a\delta^{-1};$$

equating the first coefficients and equating the last coefficients, we obtain $a_r = bc_r$ and $\bar{a}_r = b\bar{c}_r$. So $b = \bar{b}$ and in view of (22) and (25) the equality $q(x) = b\pi(x)$ is possible only if $q(x) = 0$.

Consequently, the elements $q(\kappa)$ with $q(x)$ of the form (21) belong to (24), they are distinct and form a vector space of dimension $2r$ over the fixed field $\mathbb{F}_\circ = \{a \in \mathbb{F} | \bar{a} = a\}$ of $\mathbb{F}$. But this is the dimension over $\mathbb{F}_\circ$ of the whole fixed field (24), so the vector space coincides with (24). $\quad\square$

Two $n \times n$ matrices M and N are said to be *similar* or **congruent* if $S^{-1}MS = N$ or $S^*MS = N$, respectively, for some nonsingular S.

We say that a square matrix is *indecomposable for similarity* if it is not similar to a direct sum of square matrices of smaller sizes. Let $\mathcal{O}_\mathbb{F}$ be any maximal set of nonsingular indecomposable canonical matrices for similarity; this means that each nonsingular indecomposable matrix is similar to exactly one matrix from $\mathcal{O}_\mathbb{F}$.

For example, $\mathcal{O}_\mathbb{F}$ may consist of all nonsingular *Frobenius blocks*—i.e., the matrices

$$\Phi = \begin{bmatrix} 0 & & 0 & -c_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -c_2 \\ 0 & & 1 & -c_1 \end{bmatrix} \tag{26}$$

whose characteristic polynomials $\chi_\Phi(x)$ are powers of irreducible polynomials $p_\Phi(x) \neq x$:

$$\chi_\Phi(x) = p_\Phi(x)^s = x^n + c_1 x^{n-1} + \cdots + c_n. \tag{27}$$

If $\mathbb{F}$ is an algebraically closed field, then $\mathcal{O}_\mathbb{F}$ may consist of all nonsingular Jordan blocks.

For $\varepsilon = \pm 1$ and each nonsingular matrix Φ that is indecomposable for similarity, if there exists a nonsingular M satisfying $M = \varepsilon M^* = \Phi^* M \Phi$ then we fix any and denote it by $\Phi_{(\varepsilon)}$ (we follow the notation in [38]).

It suffices to construct $\Phi_{(\varepsilon)}$ only for the matrices $\Phi \in \mathcal{O}_{\mathbb{F}}$ because if $\Phi_{(\varepsilon)}$ exists and Ψ is similar to Φ then $\Psi_{(\varepsilon)}$ also exists: if

$$\Phi_{(\varepsilon)} = \varepsilon \Phi_{(\varepsilon)}^* = \Phi^* \Phi_{(\varepsilon)} \Phi, \tag{28}$$

then we can take

$$\Psi_{(\varepsilon)} = S^* \Phi_{(\varepsilon)} S, \quad \Psi = S^{-1} \Phi S \tag{29}$$

and obtain

$$\Psi_{(\varepsilon)} = \varepsilon \Psi_{(\varepsilon)}^* = \Psi^* \Psi_{(\varepsilon)} \Psi. \tag{30}$$

Moreover, if $\Psi_{(\varepsilon)}$ is any matrix that is *congruent to $\Phi_{(\varepsilon)}$, then it satisfies (30) with Ψ defined by (29).

Existence conditions and an explicit form of $\Phi_{(\varepsilon)}$ for Frobenius blocks Φ over a field of characteristic not 2 were established in Theorem 9 of [38]; this result is represented in Lemma 2.4 with a detailed proof. Over algebraically or real closed fields, we construct in Lemma 5.1 matrices $\Psi_{(\varepsilon)}$ that are *congruent to $\Phi_{(\varepsilon)}$ from Lemma 2.4 but are much simpler.

Theorem 5 of [38], which was formulated only for the set $\mathcal{O}_{\mathbb{F}}$ of all Frobenius blocks, is extended to any $\mathcal{O}_{\mathbb{F}}$ in the following theorem.

Theorem 2.2. *Let A be an isometric operator on a finite-dimensional vector space with nondegenerate ε-Hermitian form B over a field $\mathbb{F}$ of characteristic different from 2. Let $\mathcal{O}_{\mathbb{F}}$ be a maximal set of nonsingular indecomposable canonical matrices for similarity over $\mathbb{F}$. Then the pair (A, B) can be given in some basis by a direct sum of matrix pairs of the following types*:

(i) $(\Phi \oplus \Phi^{-*}, I \backslash \varepsilon I)$, *where $\Phi \in \mathcal{O}_{\mathbb{F}}$ is such that $\Phi_{(\varepsilon)}$ does not exist (see Lemma 2.4(a)).*

(ii) $\mathscr{A}_{\Phi}^{q(x)} := (\Phi, \Phi_{(\varepsilon)} q(\Phi))$, *where $\Phi \in \mathcal{O}_{\mathbb{F}}$ is such that $\Phi_{(\varepsilon)}$ exists and $q(x) \neq 0$ is of the form (21) in which r is the integer part of $(\deg p_{\Phi}(x))/2$. Here $p_{\Phi}(x)$ is the irreducible divisor of the characteristic polynomial of Φ.*

The summands are determined to the following extent:

Type (i) *up to replacement of Φ by $\Psi \in \mathcal{O}_{\mathbb{F}}$ that is similar to Φ^{-*} (i.e., whose characteristic polynomial is $\chi_{\Psi}(x) = \chi_{\Phi}^{\vee}(x)$).*

Type (ii) *up to replacement of the whole group of summands*

$$\mathscr{A}_{\Phi}^{q_1(x)} \oplus \cdots \oplus \mathscr{A}_{\Phi}^{q_s(x)}$$

with the same Φ by

$$\mathscr{A}_{\Phi}^{q_1'(x)} \oplus \cdots \oplus \mathscr{A}_{\Phi}^{q_s'(x)}$$

such that each $q_i'(x)$ is a nonzero function of the form (21) and the Hermitian forms
$$q_1(\kappa) x_1^{\circ} x_1 + \cdots + q_s(\kappa) x_s^{\circ} x_s,$$
$$q_1'(\kappa) x_1^{\circ} x_1 + \cdots + q_s'(\kappa) x_s^{\circ} x_s$$
are equivalent over the field (19) with involution (20).

The proof of this theorem given in Section 4 is a light modification of the proof of Theorem 5 in [38].

Let

$$f(x) = \gamma_0 x^m + \gamma_1 x^{m-1} + \cdots + \gamma_m \in \mathbb{F}[x], \quad \gamma_0 \neq 0 \neq \gamma_m.$$

A vector $(a_1, a_2, \ldots, a_n)$ over $\mathbb{F}$ is said to be *f*-*recurrent* if $n \leqslant m$, or if

$$\gamma_0 a_l + \gamma_1 a_{l+1} + \cdots + \gamma_m a_{l+m} = 0, \quad l = 1, 2, \ldots, n - m$$

(by definition, it is not *f*-recurrent if $m = 0$). Thus, this vector is completely determined by any fragment of length m.

The following lemma was proved sketchily in [38, Theorem 9].

Lemma 2.4. *Let $\mathbb{F}$ be a field of characteristic different from* 2 *with involution* (*possibly, the identity*). *Let a matrix $\Phi \in \mathbb{F}^{n \times n}$ be nonsingular and indecomposable for similarity; thus, its characteristic polynomial is a power of some irreducible polynomial $p_\Phi(x)$.*

(a) $\Phi_{(\varepsilon)}$ *exists if and only if*

$$p_\Phi(x) = p_\Phi^\vee(x), \quad and \tag{31}$$

if the involution on $\mathbb{F}$ is the identity and $\varepsilon = (-1)^n$, then $\deg p_\Phi(x) > 1$. $\tag{32}$

(b) *If* (31) *and* (32) *are satisfied and if Φ is a nonsingular Frobenius block* (26) *with characteristic polynomial*

$$\chi_\Phi(x) = p_\Phi(x)^s = x^n + c_1 x^{n-1} + \cdots + c_n, \tag{33}$$

then for $\Phi_{(\varepsilon)}$ one can take the Toeplitz matrix

$$\Phi_{(\varepsilon)} := [a_{i-j}] = \begin{bmatrix} a_0 & a_{-1} & \ddots & a_{1-n} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \ddots & a_1 & a_0 \end{bmatrix} \tag{34}$$

whose vector of entries $(a_{1-n}, a_{2-n}, \ldots, a_{n-1})$ is the χ_Φ-recurrent extension of the vector $v = (a_{-m}, \ldots, a_m)$ of length

$$2m + 1 = \begin{cases} n & \text{if } n \text{ is odd,} \\ n + 1 & \text{if } n \text{ is even,} \end{cases}$$

defined as follows:

(i) $v := (c_n - \varepsilon, 0, \ldots, 0, \varepsilon \bar{c}_n - 1)$ *if n is even and $c_n \neq \varepsilon$ (see* (33));

(ii) $v := (c_1, -1, 0, \ldots, 0, -1, c_1)$ ($v := (c_1, -2, c_1)$ *for $n = 2$*) *if n is even, $c_n = \varepsilon$, and the involution on $\mathbb{F}$ is the identity;*

(iii) $v := (a - \bar{a}, 0, \ldots, 0, \bar{a} - a)$ (*with any $a \in \mathbb{F}$ such that $\bar{a} \neq a$*) *if n is even, $c_n = \varepsilon$, the involution is not the identity, and also if n odd, $p_\Phi(x) = x + c$, $c^{n-1} = -1$ (then the involution is not the identity).*

(iv) $v := (1, 0, \ldots, 0, \varepsilon)$ *if n is odd and $p_\Phi(x) \neq x + c$, $c^{n-1} = -1$.*

Proof. (a) Let $\Phi \in \mathbb{F}^{n \times n}$ be nonsingular and indecomposable for similarity. Let us prove that if $\Phi_{(\varepsilon)}$ exists then the conditions (31) and (32) are satisfied; we prove the converse statement in (b).

Let $A := \Phi_{(\varepsilon)}$ exist. By (28), $A = \varepsilon A^* = \Phi^* A \Phi$. Since $A \Phi A^{-1} = \Phi^{-*}$, we have

$$\chi_\Phi(x) = \det(xI - \Phi^{-*}) = \det(xI - \overline{\Phi}^{-1}) = \det((-\overline{\Phi}^{-1})(I - x\overline{\Phi}))$$
$$= \det(-\overline{\Phi}^{-1}) \cdot x^n \cdot \det(x^{-1}I - \overline{\Phi}) = \chi_\Phi^\vee(x),$$

where $n \times n$ is the size of Φ. In the notation (27), $p_\Phi(x)^s = p_\Phi^\vee(x)^s$, which verify (31).

To prove (32), suppose that the involution on $\mathbb{F}$ is the identity.

If $\varepsilon = -1$ then $A = -A^{\mathrm{T}}$. Since A is skew-symmetric and nonsingular, n is even and so $\varepsilon \neq (-1)^n$.

Let $\varepsilon = 1$ and $\deg p_\Phi(x) = 1$. The matrix A is symmetric and by (31) $p_\Phi(x) = x \pm 1$. Due to (28)–(30), we may assume that $\Phi = J_n(\pm 1)$. Then $A = J_n(\pm 1)^{\mathrm{T}} A J_n(\pm 1)$, $J_n(\pm 1)^{-\mathrm{T}} A = A J_n(\pm 1)$, and

$$
\begin{bmatrix} 0 & & & 0 \\ -1 & 0 & & \\ & \ddots & \ddots & \\ * & & -1 & 0 \end{bmatrix} A = A \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}. \tag{35}
$$

This implies that

$$
A = \begin{bmatrix} 0 & & a_n \\ & \iddots & \\ a_1 & & * \end{bmatrix}
$$

for some $a_1, \ldots, a_n$. Then by (35)

$$
\begin{bmatrix} 0 & & & 0 \\ & & \iddots & -a_n \\ & 0 & \iddots & \\ 0 & -a_2 & & * \end{bmatrix} = \begin{bmatrix} 0 & & & 0 \\ & & \iddots & a_{n-1} \\ & 0 & \iddots & \\ 0 & a_1 & & * \end{bmatrix}
$$

and

$$
(a_1, a_2, \ldots, a_n) = (a_1, -a_1, a_1, \ldots, (-1)^{n-1} a_1).
$$

Since A is symmetric, $a_1 = a_n$. If n is even, then $a_1 = a_n = -a_1$, and so $a_1 = 0$, contrary to the nonsingularity of A. Hence, n is odd and $\varepsilon \neq (-1)^n$.

(b) Let Φ be a nonsingular Frobenius block (26) with characteristic polynomial (33) satisfying (31) and (32). Write

$$
\mu_\Phi(x) := p_\Phi(x)^{s-1} = x^t + b_1 x^{t-1} + \cdots + b_t, \quad b_0 := 1. \tag{36}
$$

Let

$$
(a_{1-n}, \ldots, a_{n-1}) \tag{37}
$$

be any vector that is χ_Φ-recurrent but is not μ_Φ-recurrent. Consider the matrix $A := [a_{i-j}]$ of the form (34). By (31),

$$
\begin{aligned}
\chi_\Phi(x) &= x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \\
&= \chi_\Phi^\vee(x) = \bar{c}_n^{-1} (\bar{c}_n x^n + \bar{c}_{n-1} x^{n-1} + \cdots + \bar{c}_1 x + 1),
\end{aligned} \tag{38}
$$

and so the last row of Φ^* is

$$
(-\bar{c}_n, \ldots, -\bar{c}_1) = c_n^{-1}(-1, -c_1, \ldots, -c_{n-1}).
$$

Hence

$$
\Phi^* A \Phi = \Phi^*[a_{i-j-1}] = [a_{i-j}] = A \tag{39}
$$

(a_n is defined by this equality).

Let us show that A is nonsingular. If $w := (a_{n-1}, \ldots, a_0)$ is the last row of A, then

$$w\Phi^{n-1}, w\Phi^{n-2}, \ldots, w \tag{40}$$

are the rows of A. Suppose, on the contrary, that they are linearly dependent. Then $wf(\Phi) = 0$ for some nonzero polynomial $f(x)$ of degree less than n. If $p_\Phi(x)^r$ is the greatest common divisor of $f(x)$ and $\chi_\Phi(x) = p_\Phi(x)^s$, then $r < s$ and

$$p_\Phi(x)^r = f(x)g(x) + \chi_\Phi(x)h(x) \quad \text{for some } g(x), h(x) \in \mathbb{F}[x].$$

Since $wf(\Phi) = 0$ and $w\chi_\Phi(\Phi) = 0$, $wp_\Phi(\Phi)^r = 0$. So $w\mu_\Phi(\Phi) = 0$. Because (40) are the rows of A, for each $i = 0, 1, \ldots, n - t - 1$ we have

$$(0, \ldots, 0, b_0, \ldots, b_t, \underbrace{0, \ldots, 0}_{i})A$$

$$= b_0 w \Phi^{i+t} + b_1 w \Phi^{i+t-1} + \cdots + b_t w \Phi^i = w\mu_\Phi(\Phi)\Phi^i = 0\Phi^i = 0.$$

Hence, $(a_{1-n}, \ldots, a_{n-1})$ is μ_Φ-recurrent, a contradiction.

What is left is to show that the vector $v = (a_{-m}, \ldots, a_m)$ defined in (i)–(iv) is χ_Φ-recurrent but is not μ_Φ-recurrent because this will imply that its χ_Φ-recurrent extension (37) defines the nonsingular matrix $A = [a_{i-j}]$ satisfying (39); since v has the form

$$(\varepsilon\bar{a}_m, \ldots, \varepsilon\bar{a}_1, a_0, a_1 \ldots, a_m), \quad \varepsilon\bar{a}_0 = a_0,$$

we have that $A = \varepsilon A^*$ and so A can be taken for $\Phi_{(\varepsilon)}$.

(i′) The vector (i) of length $n + 1$ is not μ_Φ-recurrent. By (38), $c_n = \bar{c}_n^{-1}$. The vector (i) is χ_Φ-recurrent since $c_n - \varepsilon + c_n(\varepsilon\bar{c}_n - 1) = 0$.

(ii′) Let n be even, $c_n = \varepsilon$, and let the involution on $\mathbb{F}$ be the identity. Then (38) implies $\chi_\Phi(1) = c_n^{-1}\chi_\Phi(1)$.
If $\chi_\Phi(1) = 0$ then $p_\Phi(x) = x - 1$. Since n is even,

$$\varepsilon = c_n = 1 = (-1)^n, \tag{41}$$

contrary to (32).
Hence $\chi_\Phi(1) \neq 0$. This gives $c_n = 1$, and so $c_1 = c_{n-1}$ by (38). The vector (ii) is χ_Φ-recurrent because $c_1 - c_1 - c_{n-1} + c_n c_1 = 0$.
In the same way, $\mu_\Phi(x) = \mu_\Phi^\vee(x)$ implies $\mu_\Phi(1) = b_t^{-1}\mu_\Phi(1)$ and so $b_t = 1$. In view of (41), the condition (32) ensures $\deg p_\Phi(x) > 1$, thus $\deg \mu_\Phi(x) = t \leqslant n - 2$. The vector (ii) is not μ_Φ-recurrent since if $n > 2$ then its fragment $(-1, 0, \ldots, 0, -1)$ of length $n - 1$ is not μ_Φ-recurrent and if $n = 2$ then $\mu_\Phi(x)$ is a scalar.

(iii′) Let first n be even, $c_n = \varepsilon$, and the involution be not the identity. Then $c_n = \varepsilon = 1$, so the vector (iii) of length $n + 1$ is χ_Φ-recurrent and is not μ_Φ-recurrent.
Let now n be odd, $p_\Phi(x) = x + c$, and $c^{n-1} = -1$. Then the involution is not the identity: otherwise $p_\Phi(x) = p_\Phi^\vee(x) = x \pm 1$ contradicts $c^{n-1} = -1$. The vector (iii) is χ_Φ-recurrent because of its length $n < n + 1$. It is not μ_Φ-recurrent since $\mu_\Phi(x) = (x + c)^{n-1}$, and so $b_t = c^{n-1} = -1$ in (36).

(iv′) Let n be odd, and if $p_\Phi(x) = x + c$ then $c^{n-1} \neq -1$. The vector (iv) is χ_Φ-recurrent since its length $n < n + 1$.
If $\deg p_\Phi(x) > 1$ then the length of the vector (iv) is greater than $\deg \mu_\Phi(x) = t + 1$, thus (iv) is not μ_Φ-recurrent.
If $p_\Phi(x) = x + c$ then $b_t = c^{n-1} \neq -1$. By (32), $\varepsilon = 1$, hence (iv) is not μ_Φ-recurrent. $\quad\square$

3. Systems of forms and linear mappings

In this section we present in detail the method of articles [32,35,38] for reducing the problem of classifying systems of forms and linear mappings to the problem of classifying systems of linear mappings.

Let V be a vector space over $\mathbb{F}$. A mapping $\varphi : V \to \mathbb{F}$ is called *semilinear* if

$$\varphi(ua + vb) = \bar{a}\varphi(u) + \bar{b}\varphi(v) \quad \text{for all } u, v \in V, a, b \in \mathbb{F}.$$

The set of all semilinear mappings on V is a vector space, we call it the **dual space* to V and denote by V^*.

We identify V with V^{**} by identifying $v \in V$ with $\varphi \mapsto \overline{\varphi v}, \varphi \in V^*$.

For every linear mapping $A : U \to V$, we define the **adjoint mapping* $A^* : V^* \to U^*$, in which $A^*\varphi := \varphi A$ for all $\varphi \in V^*$.

3.1. Representations of dographs

Classification problems for systems of linear mappings can be formulated in terms of quivers and their representations introduced by Gabriel [6]. A *quiver* is an oriented graph. Its *representation* is given by assigning to every vertex a vector space and to every arrow a linear mapping of the corresponding vector spaces. To include into consideration systems of forms and linear mappings, I extended in [35] the notion of quiver representations as follows. A *dograph* (a doubly oriented graph, or an *oriented schema* in terms of [38]) is, by definition, a graph with nonoriented, oriented, and doubly oriented edges; for example,

$$(42)$$

We suppose that the vertices of each dograph are $1, 2, \ldots, n$, and that there can be any number of edges between two vertices.

A *representation* $\mathscr{A}$ of a dograph D over $\mathbb{F}$ is given by assigning

- a vector space V_i over $\mathbb{F}$ to each vertex i,
- a linear mapping $A_\alpha : V_i \to V_j$ to each arrow $\alpha : i \to j$,
- a sesquilinear form $B_\beta : V_i \times V_j \to \mathbb{F}$ to each nonoriented edge $\beta : i \,—\, j$ $(i \leqslant j)$,[4] and
- a sesquilinear form $C_\gamma : V_i^* \times V_j^* \to \mathbb{F}$ on the **dual vector spaces V_i^* and V_j^* to each doubly oriented edge $\gamma : i \longleftrightarrow j$ $(i \leqslant j)$.

Instead of V_i, A_α, B_β, C_γ we sometimes write $\mathscr{A}_i$, $\mathscr{A}_\alpha$, $\mathscr{A}_\beta$, $\mathscr{A}_\gamma$. The *dimension* of a representation $\mathscr{A}$ is the vector

$$\dim \mathscr{A} := (\dim V_1, \ldots, \dim V_n).\tag{43}$$

---

[4] Thus, B_β is semilinear on V_i and linear on V_j if $i \leqslant j$. This condition is imposed for definiteness and it is unessential because each sesquilinear form $B : U \times V \to \mathbb{F}$ defines in one-to-one manner the sesquilinear form $B^* : V \times U \to \mathbb{F}$ as follows: $B^*(v, u) := \overline{B(u, v)}$.

For example, each representation of the dograph (42) is a system
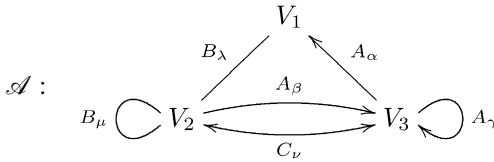
of vector spaces V_1, V_2, V_3 over $\mathbb{F}$, linear mappings A_α, A_β, A_γ, and forms

$$B_\lambda : V_1 \times V_2 \to \mathbb{F}, \quad B_\mu : V_2 \times V_2 \to \mathbb{F}, \quad C_\nu : V_2^* \times V_3^* \to \mathbb{F}.$$

A *morphism*

$$f = (f_1, \ldots, f_n) : \mathscr{A} \to \mathscr{A}' \tag{44}$$

of representations $\mathscr{A}$ and $\mathscr{A}'$ of D is a set of linear mappings $f_i : V_i \to V_i'$ that transform $\mathscr{A}$ to $\mathscr{A}'$; this means that

$$f_j A_\alpha = A_\alpha' f_i, \quad B_\beta(x, y) = B_\beta'(f_i x, f_j y), \quad C_\gamma(x f_i, y f_j) = C_\gamma'(x, y)$$

for all oriented edges $\alpha : i \longrightarrow j$, nonoriented edges $\beta : i \text{---} j$ $(i \leqslant j)$, and doubly oriented edges $\gamma : i \longleftrightarrow j$ $(i \leqslant j)$. The composition of two morphisms is a morphism. A morphism $f : \mathscr{A} \to \mathscr{A}'$ is called an *isomorphism* and is denoted by $f : \mathscr{A} \xrightarrow{\sim} \mathscr{A}'$ if all $f_i : V_i \to V_i'$ are bijections. In this case we say that $\mathscr{A}$ is *isomorphic* to $\mathscr{A}'$ and write $\mathscr{A} \simeq \mathscr{A}'$. If $\mathscr{A} = \mathscr{A}'$, then morphisms are called *endomorphisms* and isomorphisms are called *automorphisms*.

The *direct sum* $\mathscr{A} \oplus \mathscr{A}'$ of representations $\mathscr{A}$ and $\mathscr{A}'$ of D is the representation consisting of the vector spaces $V_i \oplus V_i'$ $(i = 1, \ldots, n)$, the linear mappings

$$A_\alpha \oplus A_\alpha' : V_i \oplus V_i' \to V_j \oplus V_j', \quad \alpha : i \longrightarrow j,$$

and the forms

$$B_\beta \oplus B_\beta' : (V_i \oplus V_i') \times (V_j \oplus V_j') \to \mathbb{F}, \quad \beta : i \text{---} j \ (i \leqslant j),$$
$$C_\gamma \oplus C_\gamma' : (V_i \oplus V_i')^* \times (V_j \oplus V_j')^* \to \mathbb{F}, \quad \gamma : i \longleftrightarrow j \ (i \leqslant j).$$

A representation $\mathscr{A}$ is *indecomposable* if

$$\mathscr{A} \simeq \mathscr{B} \oplus \mathscr{C} \implies \mathscr{B} = 0 \quad \text{or} \quad \mathscr{C} = 0,$$

where 0 is the representation in which all vector spaces are 0.

The set $\mathrm{Rep}(D, \mathbb{F})$ of representations of a dograph D over $\mathbb{F}$ is a category with morphisms (44). But this category is not additive since the sum of two morphisms usually is not a morphism. So the properties of dograph representations are more complicated than the properties of quiver representations, whose morphisms form vector spaces.

Let us denote by $\mathrm{Is}(D, \mathbb{F})$ the subcategory of $\mathrm{Rep}(D, \mathbb{F})$ consisting of the same objects and whose morphisms are the isomorphisms of $\mathrm{Rep}(D, \mathbb{F})$. Roiter [32] proposed to study representations of a dograph D embedding $\mathrm{Is}(D, \mathbb{F})$ into the additive category $\mathrm{Rep}(\underline{D}, \mathbb{F})$ of representations of some quiver $\underline{D}$ with involution. In Section 3.2 we introduce the notion of a quiver with involution and define an involution on the category of its representations. In Section 3.3 we construct the

embedding of Is$(D, \mathbb{F})$ to the category Rep$(\underline{D}, \mathbb{F})$. In Section 3.4 we deal with dographs with relations, they admit to consider systems of forms and linear mappings satisfying relations. In Section 3.5 we reduce the problem of classifying representations of a dograph D with relations to the problems of classifying representations of the quiver $\underline{D}$ with relations and Hermitian forms over finite extensions of the center of $\mathbb{F}$.

3.2. Representations of quivers with involution

By a *quiver with involution*, we mean a quiver Q, in which to every vertex i we associate some vertex i^* and to each arrow $\alpha : i \to j$ some arrow $\alpha^* : j^* \to i^*$ such that $i^* \neq i = i^{**}$ and $\alpha^* \neq \alpha = \alpha^{**}$.
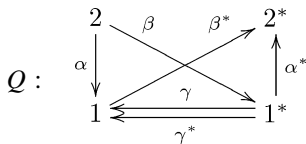
The involution on Q induces the following involution on the category of its representations Rep$(Q, \mathbb{F})$:

- *Involution on representations.* To each representation $\mathscr{M}$ of Q we associate the *adjoint representation* $\mathscr{M}^\circ$ of Q that assigns the vector spaces $\mathscr{M}_i^\circ := \mathscr{M}_{i^*}^*$ and the linear mappings $\mathscr{M}_\alpha^\circ := \mathscr{M}_{\alpha^*}^*$ to all vertices i and arrows α of Q.
- *Involution on morphisms.* To each morphism $f : \mathscr{M} \to \mathscr{N}$ of representations of Q we associate the *adjoint morphism*

$$f^\circ : \mathscr{N}^\circ \to \mathscr{M}^\circ, \quad \text{in which } f_i^\circ := f_{i^*}^* \tag{45}$$
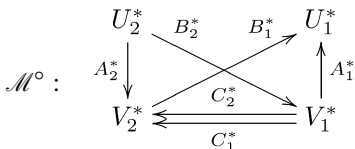
for all vertices i of Q.

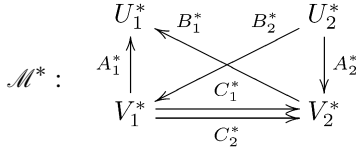For example, consider the quiver with involution

$$Q : \quad \begin{array}{c} 2 \xrightarrow{\beta} \quad \xrightarrow{\beta^*} 2^* \\ \alpha \downarrow \qquad\qquad \uparrow \alpha^* \\ 1 \underset{\gamma^*}{\overset{\gamma}{\rightleftarrows}} 1^* \end{array}$$

- For its representation

$$\mathscr{M} : \quad \begin{array}{c} U_1 \xrightarrow{B_1} \quad \xrightarrow{B_2} U_2 \\ A_1 \downarrow \qquad\qquad \uparrow A_2 \\ V_1 \underset{C_2}{\overset{C_1}{\rightleftarrows}} V_2 \end{array} \tag{46}$$

the adjoint representation

$$\mathscr{M}^\circ : \quad \begin{array}{c} U_2^* \xrightarrow{B_2^*} \quad \xrightarrow{B_1^*} U_1^* \\ A_2^* \downarrow \qquad\qquad \uparrow A_1^* \\ V_2^* \underset{C_1^*}{\overset{C_2^*}{\rightleftarrows}} V_1^* \end{array}$$
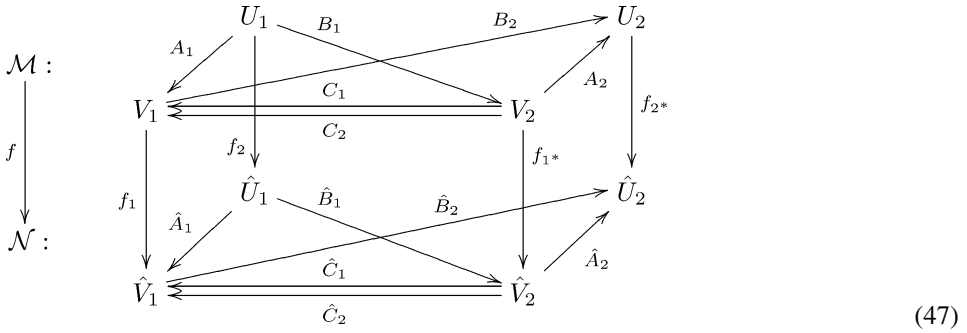
is constructed as follows: we replace all vector spaces of $\mathscr{M}$ by the *dual spaces, all linear mappings by the *adjoint mappings, which reverses the direction of each arrow:
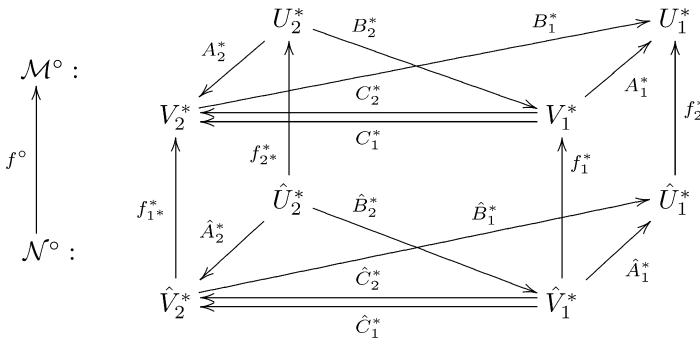
$$\mathscr{M}^*:\quad \begin{array}{c} U_1^* \xleftarrow{\;B_1^*\;} \quad \xleftarrow{\;B_2^*\;} U_2^* \\ A_1^* \uparrow \qquad \qquad \downarrow A_2^* \\ V_1^* \underset{C_2^*}{\overset{C_1^*}{\rightrightarrows}} V_2^* \end{array}$$

rotate the obtained representation around the vertical axis, and interchange C_1^* and C_2^*.

- For a morphism

(47)

of its representations $\mathscr{M}$ and $\mathscr{N}$, the adjoint morphism

is obtained as follows: we replace all vector spaces in (47) by the *dual spaces, all linear mappings by the *adjoint mappings, rotate around the vertical axis, and interchange C_1^* with C_2^* and $\widehat{C}_1^*$ with $\widehat{C}_2^*$.

An isomorphism $f : \mathscr{M} \xrightarrow{\sim} \mathscr{N}$ of selfadjoint representations $\mathscr{M} = \mathscr{M}^\circ$ and $\mathscr{N} = \mathscr{N}^\circ$ is called a *congruence* if $f^\circ = f^{-1}$.

3.3. Representations of dographs as selfadjoint representations of quivers with involution

For every dograph D, we denote by $\underline{D}$ the quiver with involution obtained from D by replacing

- each vertex i of D by the vertices i and i^*,
- each arrow $\alpha : i \to j$ by the arrows $\alpha : i \to j$ and $\alpha^* : j^* \to i^*$,
- each nonoriented edge $\beta : i - j$ $(i \leqslant j)$ by the arrows $\beta : j \to i^*$ and $\beta^* : i \to j^*$,
- each doubly oriented edge $\gamma : i \longleftrightarrow j$ $(i \leqslant j)$ by the arrows $\gamma : j^* \to i$ and $\gamma^* : i^* \to j$.

We define $i^{**} := i$ and $\alpha^{**} := \alpha$ for all vertices i and arrows α of the quiver $\underline{D}$. For example,

$$\tag{48}$$

The embedding of $\mathrm{Is}(D, \mathbb{F})$ into $\mathrm{Rep}(\underline{D}, \mathbb{F})$ (see page 171) is constructed as follows:

- *Embedding of representations.* To each representation $\mathscr{A}$ of D over $\mathbb{F}$, we associate the *selfadjoint representation* $\underline{\mathscr{A}}$ of $\underline{D}$ obtained from $\mathscr{A}$ by replacing

  – each vector space V of $\mathscr{A}$ by the spaces V and V^* (= the *dual space of all semilinear forms $V \to \mathbb{F}$),
  – each linear mapping $A : U \to V$ by the mutually *adjoint mappings $A : U \to V$ and $A^* : V^* \to U^*$,
  – each sesquilinear form $B : U \times V \to \mathbb{F}$ by the mutually *adjoint mappings

  $$B : v \in V \mapsto B(?, v) \in U^*, \quad B^* : u \in U \mapsto \overline{B(u, ?)} \in V^*,$$

  – each sesquilinear form $C : U^* \times V^* \to \mathbb{F}$ by the mutually *adjoint mappings

  $$C : v^* \in V^* \mapsto C(?, v^*) \in U^{**} = U, \quad C^* : u^* \in U^* \mapsto \overline{C(u^*, ?)} \in V.$$
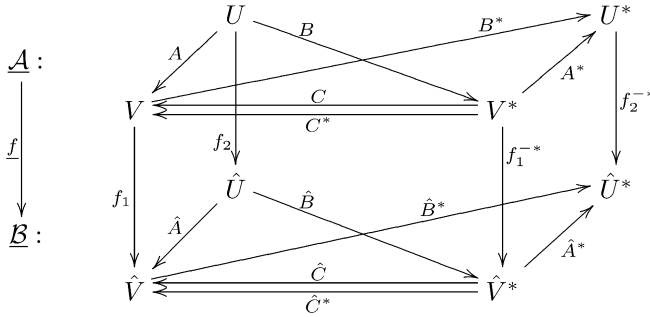
(We use the same letter for a sesquilinear form $B : U \times V \to \mathbb{F}$ and for the corresponding mapping $B : V \to U^*$. They have the same matrices in any bases $\{u_i\}$ of U, $\{v_i\}$ of V, and in the *dual basis $\{u_i^*\}$ of U^* defined by $u_i^*(u_j) = 0$ if $i \neq j$ and $u_i^*(u_i) = 1$.)

For example, for the dograph and the quiver (48):

$$\tag{49}$$

- *Embedding of isomorphisms.* To each isomorphism $f : \mathscr{A} \xrightarrow{\sim} \mathscr{B}$ of representations of a dograph D, we associate the *congruence* $\underline{f} : \underline{\mathscr{A}} \xrightarrow{\sim} \underline{\mathscr{B}}$ of the corresponding selfadjoint representations of $\underline{D}$ by defining $\underline{f}_i := f_i$ and $\underline{f}_{i^*} := f_i^{-*}$ for each vertex i of D. For example, an isomorphism

defines the congruence

Clearly, each selfadjoint representation of $\underline{D}$ has the form $\underline{\mathscr{A}}$ and each congruence of selfadjoint representations has the form $\underline{f} : \underline{\mathscr{A}} \to \underline{\mathscr{B}}$. Two representations $\mathscr{A}$ and $\mathscr{B}$ of D are isomorphic if and only if the corresponding selfadjoint representations $\underline{\mathscr{A}}$ and $\underline{\mathscr{B}}$ of $\underline{D}$ are congruent. Therefore, *the problem of classifying representations of a dograph D up to isomorphism reduces to the problem of classifying selfadjoint representations of the quiver $\underline{D}$ up to congruence.*

3.4. Dographs with relations

A *relation* on a quiver Q over a field or skew field $\mathbb{F}$ is a formal expression of the form

$$\sum_{i=1}^{m} c_i \alpha_{ip_i} \cdots \alpha_{i2}\alpha_{i1} = 0, \tag{50}$$

in which all c_i are nonzero elements of the center of $\mathbb{F}$ and

$$u \xrightarrow{\alpha_{i1}} u_{i2} \xrightarrow{\alpha_{i2}} \cdots \xrightarrow{\alpha_{i,p_i-1}} u_{ip_i} \xrightarrow{\alpha_{ip_i}} v$$

are oriented paths on Q with the same initial vertex u and the same final vertex v (u_{ij} and α_{ij} are vertices and arrows). A path may have length 0 if $u = v$. This "lazy" path (without arrows) is replaced by 1 in (50) and gives a summand of the form $c_i 1$. Therefore, if $u = v$ then (50) may have '1' instead of '0' in its right-hand side.

A representation $\mathscr{A}$ of Q *satisfies the relation* (50) if

$$\sum_{i=1}^{m} c_i \mathscr{A}_{\alpha_{ip_i}} \cdots \mathscr{A}_{\alpha_{i2}} \mathscr{A}_{\alpha_{i1}} = 0.$$

For example, the problem of classifying representations of the quiver with relations

$$\alpha \, \circlearrowright \, 1 \, \circlearrowleft \, \beta \qquad \alpha\beta = \beta\alpha = 0$$

is the problem of classifying pairs of mutually annihilating linear operators, which was solved over a field in [24]. The notion of a quiver with relations arose in the theory of representations of finite dimensional algebras over a field: every algebra can be given by a quiver with relations and there is a natural one-to-one correspondence between representations of the algebra and representations of the quiver with relations.

By a *dograph with relations*, we mean a dograph D with a finite set of relations on its quiver with involution $\underline{D}$, and consider only those representations $\mathscr{A}$ of D, for which the corresponding selfadjoint representations $\underline{\mathscr{A}}$ of $\underline{D}$ satisfy these relations. Clearly, if $\underline{\mathscr{A}}$ satisfies the relation (50), then it satisfies also the *adjoint relation*

$$\sum_{i=1}^{m} \bar{c}_i \alpha_{i1}^* \alpha_{i2}^* \cdots \alpha_{ip_i}^* = 0. \tag{51}$$

For example, the problems of classifying representations of the dographs

$$\tag{52}$$

in which $\varepsilon, \delta \in \{-1, 1\}$ (due to the edges γ and the relations $\gamma\beta = 1$, $\beta\gamma = 1$, the form assigned to β in each representation is nondegenerate) are the problems of classifying, respectively:

- sesquilinear forms,
- pairs of forms, in which the first is ε-Hermitian and the second is δ-Hermitian,
- isometric operators on a space with nondegenerate ε-Hermitian form, and
- selfadjoint operators on a space with nondegenerate ε-Hermitian form (an operator A is *selfadjoint* with respect to B if $B(Au, v) = B(u, Av)$ for all u and v).

These problems were solved in [37] and in [38, Theorems 3–6] over any field of characteristic different from 2 up to classification of Hermitian forms over its finite extensions. An analogous description of pairs of subspaces in a space with an indefinite scalar product was given in [39] by reducing it to the problem of classifying representations of the dograph

3.5. Reduction theorems

If D is a dograph with relations, then we consider $\underline{D}$ as the quiver with relations, whose *set of relations consists of the relations of D and the adjoint relations* (defined in (51)). Suppose we know any maximal set ind($\underline{D}$) of nonisomorphic indecomposable representations of the quiver $\underline{D}$ (this means that every indecomposable representation of $\underline{D}$ satisfying the relations is isomorphic to exactly one representation from ind($\underline{D}$)). Transform ind($\underline{D}$) as follows:

- First replace each representation in ind($\underline{D}$) that is *isomorphic* to a selfadjoint representation by one that is *actually* selfadjoint—i.e., has the form $\underline{\mathscr{A}}$, and denote the set of these $\underline{\mathscr{A}}$ by $\mathrm{ind}_0(\underline{D})$.

- Then in each of the one- or two-element subsets

$$\{\mathscr{M}, \mathscr{L}\} \subset \operatorname{ind}(\underline{D}) \smallsetminus \operatorname{ind}_0(\underline{D}) \quad \text{such that } \mathscr{M}^\circ \simeq \mathscr{L},$$

select one representation and denote the set of selected representations by $\operatorname{ind}_1(\underline{D})$. (If $\mathscr{M} \sim \mathscr{M}^\circ$ then $\{\mathscr{M}, \mathscr{L}\}$ consists of one representation and we take it.)

We obtain a new set $\operatorname{ind}(\underline{D})$ that we partition into 3 subsets:

$$\operatorname{ind}(\underline{D}) = \boxed{\begin{array}{c} \mathscr{M} \mid \mathscr{M}^\circ \text{ (if } \mathscr{M}^\circ \not\simeq \mathscr{M}) \\ \hline \mathscr{A} \end{array}}, \quad \begin{array}{l} \mathscr{M} \in \operatorname{ind}_1(\underline{D}), \\ \mathscr{A} \in \operatorname{ind}_0(\underline{D}). \end{array} \tag{53}$$

For each representation $\mathscr{M}$ of $\underline{D}$, we define a representation $\mathscr{M}^+$ of D by setting $\mathscr{M}_i^+ := \mathscr{M}_i \oplus \mathscr{M}_{i^*}^*$ for all vertices i of D and

$$\mathscr{M}_\alpha^+ := \begin{bmatrix} \mathscr{M}_\alpha & 0 \\ 0 & \mathscr{M}_{\alpha^*}^* \end{bmatrix}, \quad \mathscr{M}_\beta^+ := \begin{bmatrix} 0 & \mathscr{M}_{\beta^*}^* \\ \mathscr{M}_\beta & 0 \end{bmatrix}, \quad \mathscr{M}_\gamma^+ := \begin{bmatrix} 0 & \mathscr{M}_\gamma \\ \mathscr{M}_{\gamma^*}^* & 0 \end{bmatrix} \tag{54}$$

for all edges $\alpha : i \longrightarrow j$, $\beta : i \!-\! j$ $(i \leqslant j)$, and $\gamma : i \longleftrightarrow j$ $(i \leqslant j)$.

The representations $\mathscr{M}^+$ arise as follows: each representation $\mathscr{M}$ of $\underline{D}$ defines the selfadjoint representation $\mathscr{M} \oplus \mathscr{M}^\circ$; the corresponding representation of D is $\mathscr{M}^+$ (and so $\underline{\mathscr{M}}^+ = \mathscr{M} \oplus \mathscr{M}^\circ$).

For example, if $\mathscr{M}$ is the representation (46), then the selfadjointness of

$$\mathscr{M} \oplus \mathscr{M}^\circ :$$
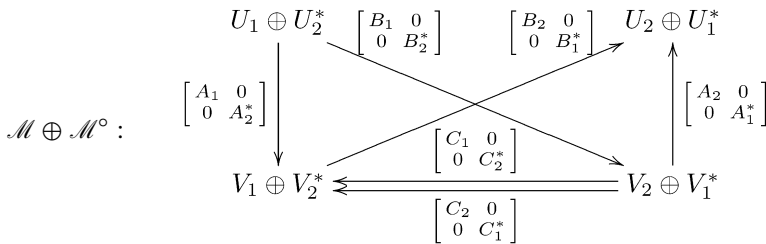
becomes clear if we interchange the summands in each vector space on the right, interchanging respectively the corresponding strips in the matrices of linear mappings:

The corresponding representation of D is

$$\mathcal{M}^+ :$$
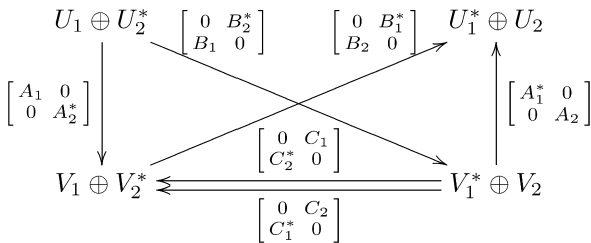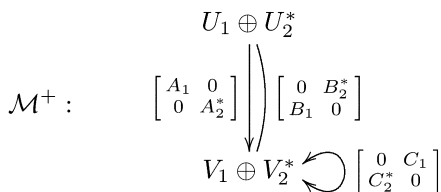
For every representation $\mathscr{A}$ of D and for every selfadjoint automorphism $f = f^\circ : \underline{\mathscr{A}} \xrightarrow{\sim} \underline{\mathscr{A}}$, we denote by $\mathscr{A}^f$ the representation of D obtained from $\mathscr{A}$ by replacing

- each form $\mathscr{A}_\beta$ ($\beta : i\!\!-\!\!j, i \leqslant j$) by $\mathscr{A}_\beta^f := \mathscr{A}_\beta f_j$,
- each form $\mathscr{A}_\gamma$ ($\gamma : i \longleftrightarrow j, i \leqslant j$) by $\mathscr{A}_\gamma^f := f_i^{-1} \mathscr{A}_\gamma$.

The corresponding selfadjoint representation $\underline{\mathscr{A}^f}$ of $\underline{D}$ can be visualized as the diagonal of the rectangle

$$(55)$$
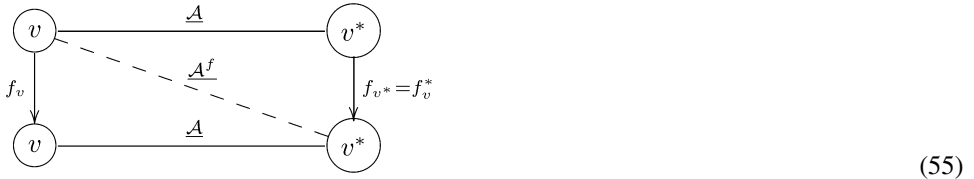
in which v represents the vertices of D (thus, $\underline{\mathscr{A}} \simeq \underline{\mathscr{A}^f}$).

For example, if $\mathscr{A}$ is the first representation in (49), then a selfadjoint automorphism
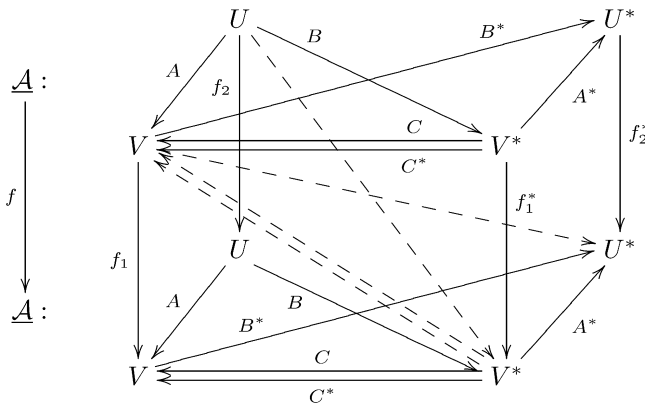
defines the representation

$$\mathscr{A}^f : \quad \begin{array}{c} U \\ A \Big\downarrow \, | \, Bf_2 \\ V \leftarrow \!\!\!\!| \, f_1^{-1}C \end{array}$$

Let $\mathrm{ind}(\underline{D})$ be partitioned as in (53), and let $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{D})$. By [38, Lemma 1], the set R of noninvertible elements of the endomorphism ring $\mathrm{End}(\underline{\mathscr{A}})$ is the radical. Therefore, $\mathbb{T}(\mathscr{A}) := \mathrm{End}(\underline{\mathscr{A}})/R$ is a field or skew field, on which we define the involution

$$(f + R)^\circ := f^\circ + R. \tag{56}$$

For each nonzero $a = a^\circ \in \mathbb{T}(\mathscr{A})$, we fix a selfadjoint automorphism

$$f_a = f_a^\circ \in a, \quad \text{and define } \mathscr{A}^a := \mathscr{A}^{f_a} \tag{57}$$

(we can take $f_a := (f + f^\circ)/2$ for any $f \in a$). The set of representations $\mathscr{A}^a$ is called the *orbit* of $\mathscr{A}$. Note that the corresponding representations $\underline{\mathscr{A}^a}$ of $\underline{D}$ are isomorphic to $\underline{\mathscr{A}}$. Conversely, if $\underline{\mathscr{B}} \simeq \underline{\mathscr{A}}$ then $\mathscr{B} \simeq \mathscr{A}^a$ for some nonzero $a = a^\circ \in \mathbb{T}(\mathscr{A})$; this follows from the next theorem.

For each Hermitian form

$$\varphi(x) = x_1^\circ a_1 x_1 + \cdots + x_r^\circ a_r x_r, \quad 0 \neq a_i = a_i^\circ \in \mathbb{T}(\mathscr{A}),$$

we write

$$\mathscr{A}^{\varphi(x)} := \mathscr{A}^{a_1} \oplus \cdots \oplus \mathscr{A}^{a_r}.$$

Theorem 3.1. *Over a field or skew field* $\mathbb{F}$ *of characteristic different from* 2 *with involution* $a \mapsto \bar{a}$ (*possibly, the identity*), *every representation of a dograph D with relations is isomorphic to a direct sum*

$$\mathscr{M}_1^+ \oplus \cdots \oplus \mathscr{M}_p^+ \oplus \mathscr{A}_1^{\varphi_1(x)} \oplus \cdots \oplus \mathscr{A}_q^{\varphi_q(x)}, \tag{58}$$

where

$$\mathscr{M}_i \in \mathrm{ind}_1(\underline{D}), \quad \underline{\mathscr{A}}_j \in \mathrm{ind}_0(\underline{D}),$$

$\mathscr{A}_j \neq \mathscr{A}_{j'}$ *if* $j \neq j'$, *and each* $\varphi_j(x)$ *is a Hermitian form over* $\mathbb{T}(\mathscr{A}_j)$ *with involution* (56). *This sum is determined by the original representation uniquely up to permutation of summands and replacement of* $\mathscr{A}_j^{\varphi_j(x)}$ *by* $\mathscr{A}_j^{\psi_j(x)}$, *where* $\psi_j(x)$ *is a Hermitian form over* $\mathbb{T}(\mathscr{A}_j)$ *that is equivalent to* $\varphi_j(x)$.

Proof. An analogous statement was proved in [38, Theorem 1] for selfadjoint representations of a linear category with involution. This ensures Theorem 3.1 since every dograph D with relations defines the following category $\mathscr{C}$ (see [38, §2]): its objects are the vertices of $\underline{D}$; if u and v are two vertices of $\underline{D}$ then the set of morphisms from u to v is the vector space over the center of $\mathbb{F}$ spanned by all oriented paths from u to v on $\underline{D}$ and factorized by the relations on $\underline{D}$. An involution on $\mathscr{C}$ is defined in the same way as the involution on relations (see (50) and (51)):

$$\sum_{i=1}^m c_i \alpha_{ip_i} \cdots \alpha_{i2} \alpha_{i1} \longmapsto \sum_{i=1}^m \bar{c}_i \alpha_{i1}^* \alpha_{i2}^* \cdots \alpha_{ip_i}^*. \quad \square$$

Theorem 3.1 was extended in [41] to symmetric representations of algebras with involution.

For each representation $\mathscr{A}$ of D, we write $\mathscr{A}^- := \mathscr{A}^{-1}$, where $-1 \in \mathrm{Aut}\,\underline{\mathscr{A}}$; this means that the representation $\mathscr{A}^-$ is obtained from $\mathscr{A}$ by multiplying all the forms by -1:

$$\mathcal{A}: \qquad \begin{matrix} U \\ A\Big\downarrow\!\!\!\Big) B \\ V \circlearrowleft C \end{matrix} \qquad\qquad \mathcal{A}^-: \qquad \begin{matrix} U \\ A\Big\downarrow\!\!\!\Big) -B \\ V \circlearrowleft -C \end{matrix}$$

Theorem 3.1 implies the following generalization of Sylvester's Inertia Theorem.

Theorem 3.2. *Let* $\mathbb{F}$ *be either*

(i) *an algebraically closed field of characteristic different from* 2 *with the identity involution, or*

(ii) *an algebraically closed field with nonidentity involution, or*

(iii) *a real closed field, or the skew field of quaternions over a real closed field.*

Then every representation of a dograph D with relations over $\mathbb{F}$ *is isomorphic to a direct sum, determined uniquely up to permutation of summands, of representations of the following types:*

$$\mathscr{M}^+, \begin{cases} \mathscr{A} & \text{if } \mathscr{A}^- \simeq \mathscr{A}, \\ \mathscr{A}, \mathscr{A}^- & \text{if } \mathscr{A}^- \not\simeq \mathscr{A}, \end{cases} \quad (\text{where } \mathscr{M} \in \text{ind}_1(\underline{P}), \underline{\mathscr{A}} \in \text{ind}_0(\underline{P})),$$

or, *respectively to the cases* (i)–(iii),

(i) $\mathscr{M}^+, \mathscr{A}$,

(ii) $\mathscr{M}^+, \mathscr{A}, \mathscr{A}^-$,

(iii) $\mathscr{M}^+, \begin{cases} \mathscr{A}, & \textit{if } \mathbb{T}(\mathscr{A}) \textit{ is an algebraically closed field with the identity} \\ & \textit{involution or a skew field of quaternions with} \\ & \textit{involution different from quaternionic conjugation, and} \\ \mathscr{A}, \mathscr{A}^-, & \textit{otherwise.} \end{cases}$

Proof. Theorem 3.1 reduces the classification of representations of any dograph D to the classification of Hermitian forms over the fields or skew fields $\mathbb{T}(\mathscr{A})$, $\underline{\mathscr{A}} \in \text{ind}_0(\underline{D})$, assuming known $\text{ind}_1(\underline{D})$, $\text{ind}_0(\underline{D})$, and the orbit of the representations $\mathscr{A}$ for each $\underline{\mathscr{A}} \in \text{ind}_0(\underline{D})$.

If $\mathbb{F}$ is finite dimensional over its center $C(\mathbb{F})$, then $\mathbb{T}(\mathscr{A})$ is also finite dimensional over $C(\mathbb{F})$ under the natural imbedding of $C(\mathbb{F})$ into the center of $\mathbb{T}(\mathscr{A})$, and the involution on $\mathbb{T}(\mathscr{A})$ extends the involution on $C(\mathbb{F})$.

(i) If $\mathbb{F}$ is an algebraically closed field of characteristic different from 2 with the identity involution, then it has no finite extensions. Hence, $\mathbb{T}(\mathscr{A}) = \mathbb{F}$ for each $\underline{\mathscr{A}} \in \text{ind}_0(\underline{D})$, and so each Hermitian form

$$a_1 x_1^2 + \cdots + a_r x_r^2, \quad 0 \neq a_i \in \mathbb{F}, \tag{59}$$

is equivalent to $x_1^2 + \cdots + x_r^2$. We can replace all $\mathscr{A}_j^{\varphi_j(x)}$ in (58) by $\mathscr{A}_j \oplus \cdots \oplus \mathscr{A}_j$. In view of Theorem 3.1, the obtained direct sum is determined by the original representation uniquely up to permutation of summands.

(ii) Let $\mathbb{F}$ be an algebraically closed field with nonidentity involution. Its characteristic is 0 by Lemma 2.1(b), $\mathbb{T}(\mathscr{A}) = \mathbb{F}$ for each $\underline{\mathscr{A}} \in \text{ind}_0(\underline{D})$, and the involution $a \mapsto a^\circ$ on $\mathbb{T}(\mathscr{A})$ coincides with the involution $a \mapsto \bar{a}$ on $\mathbb{F}$. Due to the law of inertia [4], each Hermitian form

$$a_1 \bar{x}_1 x_1 + \cdots + a_r \bar{x}_r x_r, \quad 0 \neq a_i = \bar{a}_i \in \mathbb{F},$$

is equivalent to exactly one form

$$\bar{x}_1 x_1 + \cdots + \bar{x}_l x_l - \bar{x}_{l+1} x_{l+1} - \cdots - \bar{x}_r x_r.$$

Therefore, we can replace each $\mathscr{A}_j^{\varphi_j(x)}$ in (58) by exactly one direct sum of the form

$$\mathscr{A}_j \oplus \cdots \oplus \mathscr{A}_j \oplus \mathscr{A}_j^- \oplus \cdots \oplus \mathscr{A}_j^-. \tag{60}$$

(iii) Let $\mathbb{F}$ be a real closed field $\mathbb{P}$ or the skew field $\mathbb{H}$ of quaternions over a real closed field $\mathbb{P}$. By Lemma 2.1(a), char($\mathbb{F}$) = 0. The center of $\mathbb{F}$ is $\mathbb{P}$. Hence, $\mathbb{T}(\mathscr{A})$ for each $\underline{\mathscr{A}} \in \text{ind}_0(\underline{D})$ is a finite extension[5] of $\mathbb{P}$. By the Frobenius theorem [4], $\mathbb{T}(\mathscr{A})$ is either $\mathbb{P}$, or its algebraic closure $\mathbb{P} + \mathbb{P}i$, or $\mathbb{H}$.

---

[5] Formulating Theorem 2 in [38], I erroneously thought that all $\mathbb{T}(\mathscr{A}) = \mathbb{H}$ if $\mathbb{F} = \mathbb{H}$. To correct it, remove "or the algebra of quaternions …" in (a) and (b) and add "or the algebra of quaternions over a maximal ordered field" in (c). The paper [40] is based on the incorrect Theorem 2 in [38] and so the signs $\pm$ of the sesquilinear forms in the indecomposable direct summands in [40, Theorems 1–4] are incorrect. Correct canonical forms are given for pairs of symmetric/skew-symmetric matrices in [29–31], for selfadjoint operators in [20], and for isometries in Theorem 2.1.

If $\mathbb{T}(\mathscr{A})$ is either $\mathbb{P}$, or $\mathbb{P} + \mathbb{P}i$ with nonidentity involution, or $\mathbb{H}$ with quaternionic conjugation (4), then by the law of inertia [4] each Hermitian form

$$\varphi(x) = x_1^\circ a_1 x_1 + \cdots + x_r^\circ a_r x_r, \quad 0 \neq a_i = a_i^\circ \in \mathbb{T}(\mathscr{A}), \tag{61}$$

is equivalent to exactly one form

$$x_1^\circ x_1 + \cdots + x_l^\circ x_l - x_{l+1}^\circ x_{l+1} - \ldots - x_r^\circ x_r.$$

Therefore, we can replace each $\mathscr{A}_j^{\varphi_j(x)}$ in (58) by (60).

If $\mathbb{T}(\mathscr{A}) = \mathbb{P} + \mathbb{P}i$ with the identity involution, then every Hermitian form over it is equivalent to $x_1^2 + \cdots + x_r^2$. We can replace each $\mathscr{A}_j^{\varphi_j(x)}$ in (58) by $\mathscr{A}_j \oplus \cdots \oplus \mathscr{A}_j$.

If $\mathbb{T}(\mathscr{A}) = \mathbb{H}$ with quaternionic semiconjugation (5), then every Hermitian form (61) is equivalent to $x_1^\circ x_1 + \cdots + x_r^\circ x_r$ since each a_i is represented in the form $a_i = b_i^\circ b_i$, where $b_i := \sqrt{a_i}$ is taken in the field $\mathbb{P} + \mathbb{P}j$ if $a_i \in \mathbb{P}$, or in the field $\mathbb{P}(a_i)$ if $a_i \notin \mathbb{P}$; these fields are algebraically closed and the involution (5) acts identically on them. Therefore, we can replace each $\mathscr{A}_j^{\varphi_j(x)}$ in (58) by $\mathscr{A}_j \oplus \cdots \oplus \mathscr{A}_j$. $\quad\square$

Example. The problem of classifying Hermitian forms over $\mathbb{F}$ is given by the dograph

$$D: \qquad 1 \,\bigcirc\, \alpha \qquad \alpha^* = \alpha.$$

Its quiver is

$$\underline{D}: \qquad 1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 1^* \qquad \alpha^* = \alpha.$$

Each representation

$$\mathscr{M}: \qquad U \underset{B}{\overset{A}{\rightleftarrows}} V \qquad A = B$$

of $\underline{D}$ is given by a linear mapping $A: U \to V$, which is a direct sum of mappings of the types $\mathbb{F} \xrightarrow{1} \mathbb{F}$, $0 \to \mathbb{F}$, and $\mathbb{F} \to 0$ (because each matrix reduces by equivalence transformations to $I_r \oplus 0$, which is a direct sum of matrices of the types I_1, 0_{10}, and 0_{01}). Thus, the set (53) is

$$\mathrm{ind}(\underline{D}) = \boxed{\begin{array}{c} 0 \rightrightarrows \mathbb{F} \mid \mathbb{F} \rightrightarrows 0 \\ \hline \mathbb{F} \rightrightarrows \mathbb{F} \end{array}}.$$

Theorem (3.2) ensures that every representation of D is isomorphic to a direct sum of representations of the types $\mathbb{F} \,\bigcirc\, 0$ (which is $[0 \rightrightarrows \mathbb{F}]^+$) and $\mathbb{F} \,\bigcirc\, 1$ if $\mathbb{F}$ is an algebraically closed field of characteristic not 2 with the identity involution, and of representations of the types $\mathbb{F} \,\bigcirc\, 0$, $\mathbb{F} \,\bigcirc\, 1$, and $\mathbb{F} \,\bigcirc\, -1$ if $\mathbb{F}$ is an algebraically closed field with nonidentity involution.

Corollary. *Each system of linear mappings and bilinear/sesquilinear forms on vector spaces over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ decomposes into a direct sum of indecomposable systems uniquely up to isomorphisms of summands.*

By [36], the set of dimensions (43) of indecomposable representations of a dograph does not depend on the orientation of its edges, and so by Kac's theorem [19] it coincides with the set of

positive roots of the dograph. An analogous description of the set of dimensions of indecomposable Euclidean or unitary representations of a quiver (i.e., each vertex is assigned by a Euclidean or unitary space) is given in [42].

4. Proof of Theorem 2.2

Each pair (A, B) consisting of a nondegenerate ε-Hermitian form B and an isometric operator A on a vector space V over a field or skew field of characteristic different from 2 determines the representation

$$\mathscr{A}: \qquad A \circlearrowright V \underset{B^{-1}}{\overset{B}{\rightleftharpoons}}$$
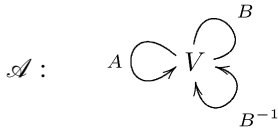
of the dograph D defined in (52); if B is given by a matrix B_e in some basis of V then B^{-1} is given by B_e^{-1} in the *dual basis of V^*.

The quiver with involution of the dograph D is

$$\underline{D}: \qquad \alpha \circlearrowright 1 \underset{\gamma}{\overset{\beta}{\underset{\beta^*}{\rightleftarrows}}} 1^* \circlearrowright \alpha^* \qquad \begin{array}{l} \beta = \varepsilon\beta^* = \alpha^*\beta\alpha, \\ \gamma\beta = 1, \quad \beta\gamma = 1, \\ \gamma^*\beta^* = 1, \quad \beta^*\gamma^* = 1. \end{array}$$
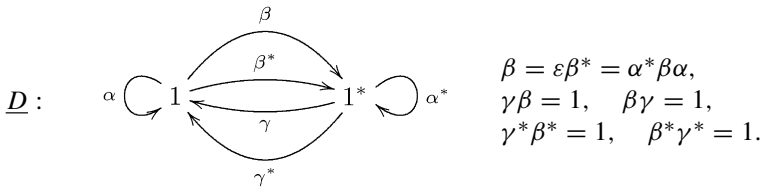
We will prove Theorem 2.2 using Theorem 3.1; to do this, we first identify in Lemma 4.1 the sets $\mathrm{ind}_1(\underline{D})$ and $\mathrm{ind}_0(\underline{D})$, and the orbit of $\mathscr{A}$ for each $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{D})$.

The arrow γ was appended in D with the only purpose: each form assigned to β must be nonsingular. So we will omit γ and γ^* and represent D and $\underline{D}$ as follows:

$$D: \qquad \alpha \circlearrowright 1 \circlearrowleft \beta \qquad \begin{array}{l} \beta = \varepsilon\beta^* = \alpha^*\beta\alpha, \\ \beta \text{ is nonsingular,} \end{array} \tag{62}$$

$$\underline{D}: \qquad \alpha \circlearrowright 1 \overset{\beta}{\underset{\beta^*}{\rightleftarrows}} 1^* \circlearrowright \alpha^* \qquad \begin{array}{l} \beta = \varepsilon\beta^* = \alpha^*\beta\alpha, \\ \beta \text{ is bijective.} \end{array} \tag{63}$$

Every representation of D or $\underline{D}$ over $\mathbb{F}$ is isomorphic to a representation in which all vector spaces are $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$. We will consider only such representations of D and $\underline{D}$, they can be given by matrix pairs (A, B):

$$\mathscr{A}: \qquad A \circlearrowright \bullet \circlearrowleft B \qquad \begin{array}{l} B = \varepsilon B^* = A^*BA, \\ B \text{ is nonsingular,} \end{array}$$

and, respectively, by matrix triples $\mathscr{M} = (A, B, C)$:

$$\mathscr{M}: \qquad A \circlearrowright \bullet \overset{B}{\underset{\varepsilon B}{\rightrightarrows}} \bullet \circlearrowright C \qquad \begin{array}{l} B = CBA, \\ B \text{ is nonsingular} \end{array} \tag{64}$$

(we omit the spaces $\mathbb{F} \oplus \cdots \oplus \mathbb{F}$ since they are completely determined by the sizes of the matrices).

The adjoint representation

$$\mathscr{M}^{\circ} : \qquad C^* \;\overset{\varepsilon B^*}{\underset{B^*}{\rightrightarrows}}\; A^*$$
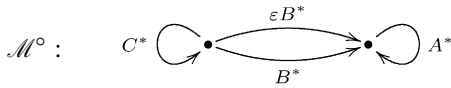
is given by the matrix triple

$$(A, B, C)^{\circ} = (C^*, \varepsilon B^*, A^*). \tag{65}$$

A morphism of representations

$$
\begin{array}{c}
\mathscr{M} : \qquad A \;\overset{B}{\underset{\varepsilon B}{\rightrightarrows}}\; C \\[2pt]
\;\;g \downarrow \qquad\qquad G_1 \qquad\qquad G_2 \\[2pt]
\mathscr{M}' : \qquad A' \;\overset{B'}{\underset{\varepsilon B'}{\rightrightarrows}}\; C'
\end{array}
$$
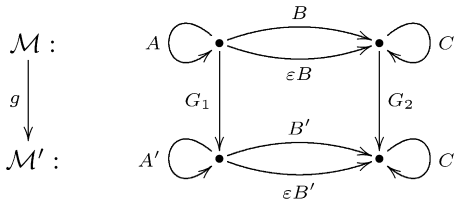
is given by the matrix pair

$$g = [G_1, G_2] : \mathscr{M} \to \mathscr{M}'$$

(for morphisms we use square brackets) satisfying

$$G_1 A = A' G_1, \quad G_2 B = B' G_1, \quad G_2 C = C' G_2, \tag{66}$$

and the adjoint morphism is given by the matrix pair

$$g^{\circ} = [G_2^*, G_1^*] : \mathscr{M}'^{\circ} \to \mathscr{M}^{\circ}. \tag{67}$$

Lemma 4.1. *Let $\mathbb{F}$ be a field or skew field of characteristic different from* 2. *Let $\mathcal{O}_{\mathbb{F}}$ be a maximal set of nonsingular indecomposable canonical matrices over $\mathbb{F}$ for similarity* (*see page 165*). *Let D be the dograph* (62). *Then the following statements hold*:

(a) *The set $\mathrm{ind}(\underline{D})$ can be taken to be all representations (Φ, I, Φ^{-1}) with $\Phi \in \mathcal{O}_{\mathbb{F}}$.*

(b) *The set $\mathrm{ind}_1(\underline{D})$ can be taken to be all representations*

$$\mathscr{M}_{\Phi} := (\Phi, I, \Phi^{-1}),$$

in which $\Phi \in \mathcal{O}_{\mathbb{F}}$ is such that $\Phi_{(\varepsilon)}$ (defined in (28)) *does not exist, and*

$$\begin{array}{l}
\Phi \text{ is determined up to replacement} \\
\text{by } \Psi \in \mathcal{O}_{\mathbb{F}} \text{ that is similar to } \Phi^{-*}.
\end{array} \tag{68}$$

The corresponding representation (54) *of D has the form*

$$\mathscr{M}_{\Phi}^{+} : \quad \begin{bmatrix} \Phi & 0 \\ 0 & \Phi^{-*} \end{bmatrix} \;\circlearrowright\!\bullet\!\circlearrowleft\; \begin{bmatrix} 0 & \varepsilon I \\ I & 0 \end{bmatrix}. \tag{69}$$

(c) *The set $\mathrm{ind}_0(\underline{D})$ can be taken to be all representations*

$$\underline{\mathscr{A}}_{\Phi} := (\Phi, \Phi_{(\varepsilon)}, \Phi^*), \tag{70}$$

in which $\Phi \in \mathcal{O}_{\mathbb{F}}$ is such that $\Phi_{(\varepsilon)}$ exists. The corresponding representations of D have the form

$$\mathscr{A}_\Phi: \quad \Phi \,\overset{\curvearrowright}{\bullet}\, \Phi_{(\varepsilon)}, \qquad \mathscr{A}_\Phi^-: \quad \Phi \,\overset{\curvearrowright}{\bullet}\, - \Phi_{(\varepsilon)}.$$

(d) *Let* $\mathbb{F}$ *be a field and let* $\underline{\mathscr{A}}_\Phi := (\Phi, \Phi_{(\varepsilon)}, \Phi^*) \in \mathrm{ind}_0(\underline{D})$.

(i) *The ring* $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ *of endomorphisms of* $\underline{\mathscr{A}}_\Phi$ *consists of the matrix pairs*

$$[f(\Phi), f(\Phi^{-*})], \quad f(x) \in \mathbb{F}[x], \tag{71}$$

and the involution on $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ *is*

$$[f(\Phi), f(\Phi^{-*})]^\circ = [\bar{f}(\Phi^{-1}), \bar{f}(\Phi^*)].$$

(ii) $\mathbb{T}(\underline{\mathscr{A}}_\Phi)$ *can be identified with the field*

$$\mathbb{F}(\kappa) = \mathbb{F}[x]/p_\Phi(x)\mathbb{F}[x], \quad \kappa := x + p_\Phi(x)\mathbb{F}[x], \tag{72}$$

$(p_\Phi(x)$ *is defined in* (27)*) with involution*

$$f(\kappa)^\circ = \bar{f}(\kappa^{-1}). \tag{73}$$

Each element of $\mathbb{T}(\underline{\mathscr{A}}_\Phi)$ *on which this involution acts identically is uniquely represented in the form* $q(\kappa)$ *for some nonzero function* (21). *The representations*

$$\mathscr{A}_\Phi^{q(\kappa)}: \quad \Phi \,\overset{\curvearrowright}{\bullet}\, \Phi_{(\varepsilon)}q(\Phi) \tag{74}$$

(see (57)*) constitute the orbit of* $\mathscr{A}_\Phi$.

Proof. (a) Every representation of the quiver $\underline{D}$ is isomorphic to one of the form (A, I, A^{-1}). By (66), $(A, I, A^{-1}) \simeq (B, I, B^{-1})$ if and only if the matrices A and B are similar.

(b) and (c) Let $\Phi, \Psi \in \mathcal{O}_\mathbb{F}$. In view of (65),

$$(\Psi, I, \Psi^{-1}) \simeq (\Phi, I, \Phi^{-1})^\circ = (\Phi^{-*}, \varepsilon I, \Phi^*) \tag{75}$$

if and only if Ψ is similar to Φ^{-*}.

Suppose (Φ, I, Φ^{-1}) is isomorphic to a selfadjoint representation:

$$[G_1, G_2]: (\Phi, I, \Phi^{-1}) \overset{\sim}{\to} (C, D, C^*), \quad D = \varepsilon D^*. \tag{76}$$

Define a selfadjoint representation (A, B, A^*) by the congruence

$$[G_1^{-1}, G_1^*]: (C, D, C^*) \overset{\sim}{\to} (A, B, A^*), \quad B = \varepsilon B^*. \tag{77}$$

The composition of (76) and (77) is the isomorphism
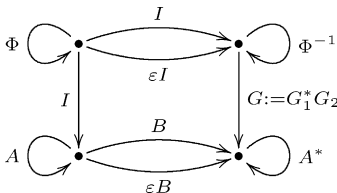
By (66), $A = \Phi$, $B = G$, and $A^*G = G\Phi^{-1}$; hence $B = \varepsilon B^* = \Phi^* B \Phi$. We can replace B by $\Phi_{(\varepsilon)}$ and obtain

$$[I, \Phi_{(\varepsilon)}]: (\Phi, I, \Phi^{-1}) \overset{\sim}{\to} (\Phi, \Phi_{(\varepsilon)}, \Phi^*).$$

This means that if $(\Phi, I, \Phi^{-1}) \in \mathrm{ind}(\underline{D})$ is isomorphic to a selfadjoint representation then it is isomorphic to (70). Hence the representations (70) form $\mathrm{ind}_0(\underline{D})$. Due to (75), we can

identify isomorphic representations in the set of remaining representations $(\Phi, I, \Phi^{-1}) \in \mathrm{ind}(\underline{D})$ by imposing the condition (68), and obtain $\mathrm{ind}_1(\underline{D})$.

(d) Let $\mathbb{F}$ be a field. It is known that if Φ is a square matrix over $\mathbb{F}$ being indecomposable for similarity, then each matrix over $\mathbb{F}$ commuting with Φ is a polynomial in Φ. Let us recall the proof. We can assume that Φ is an $n \times n$ Frobenius block (26). Then the vectors

$$e := (1, 0, \ldots, 0)^{\mathrm{T}}, \ \Phi e, \ldots, \Phi^{n-1} e \tag{78}$$

form a basis of $\mathbb{F}^n$. Let $S \in \mathbb{F}^{n \times n}$ commute with Φ and let

$$Se = a_0 e + a_1 \Phi e + \cdots + a_{n-1} \Phi^{n-1} e, \quad a_0, \ldots, a_{n-1} \in \mathbb{F}.$$

Define

$$f(x) := a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathbb{F}[x].$$

Then $Se = f(\Phi)e$,

$$S\Phi e = \Phi Se = \Phi f(\Phi)e = f(\Phi)\Phi e, \ldots, S\Phi^{n-1}e = f(\Phi)\Phi^{n-1}e.$$

Since (78) is a basis, $S = f(\Phi)$.

(i) Let $g = [G_1, G_2] \in \mathrm{End}(\underline{\mathscr{A}}_\Phi)$, where $\underline{\mathscr{A}}_\Phi = (\Phi, \Phi_{(\varepsilon)}, \Phi^*) \in \mathrm{ind}_0(\underline{D})$. Then by (66)

$$G_1 \Phi = \Phi G_1, \quad G_2 \Phi_{(\varepsilon)} = \Phi_{(\varepsilon)} G_1, \quad G_2 \Phi^* = \Phi^* G_2. \tag{79}$$

By (28),

$$\Phi^{-*} = \Phi_{(\varepsilon)} \Phi \Phi_{(\varepsilon)}^{-1}. \tag{80}$$

Since G_1 commutes with Φ, we have

$$G_1 = f(\Phi) \ (f(x) \in \mathbb{F}[x]), \quad G_2 = \Phi_{(\varepsilon)} f(\Phi) \Phi_{(\varepsilon)}^{-1} = f(\Phi^{-*}).$$

Consequently, the ring $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ of endomorphisms of $\underline{\mathscr{A}}_\Phi$ consists of the matrix pairs (71), and the involution (45) has the form:

$$[f(\Phi), f(\Phi^{-*})]^\circ = [f(\Phi^{-*})^*, f(\Phi)^*] = [\bar{f}(\Phi^{-1}), \bar{f}(\Phi^*)].$$

(ii) Since $\Phi_{(\varepsilon)}$ is fixed and $G_2 = \Phi_{(\varepsilon)} f(\Phi) \Phi_{(\varepsilon)}^{-1}$, each endomorphism $[f(\Phi), f(\Phi^{-*})]$ is completely determined by $f(\Phi)$, and so $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ can be identified with the ring

$$\mathbb{F}[\Phi] = \{f(\Phi) \,|\, f \in \mathbb{F}[x]\} \quad \text{with involution } f(\Phi) \mapsto \bar{f}(\Phi^{-1}),$$

which is isomorphic to $\mathbb{F}[x]/p_\Phi(x)^s \mathbb{F}[x]$, where $p_\Phi(x)^s$ is the characteristic polynomial (27) of Φ. The radical of $\mathrm{End}(\underline{\mathscr{A}}_\Phi)$ is generated by $p_\Phi(\Phi)$, hence $\mathbb{T}(\underline{\mathscr{A}}_\Phi)$ is naturally isomorphic to the field (72) with involution $f(\kappa)^\circ = \bar{f}(\kappa^{-1})$.

According to Lemma 2.3, each element of this field on which the involution acts identically is uniquely representable in the form $q(\kappa)$ for some nonzero function (21).

The pair $[q(\Phi), \Phi_{(\varepsilon)} q(\Phi) \Phi_{(\varepsilon)}^{-1}]$ is an endomorphism of $\underline{\mathscr{A}}_\Phi$ due to (79). This endomorphism is selfadjoint since the function (21) fulfils $q(x^{-1}) = \bar{q}(x)$, and so by (80)

$$\Phi_{(\varepsilon)} q(\Phi) \Phi_{(\varepsilon)}^{-1} = q(\Phi^{-*}) = \bar{q}(\Phi^*) = q(\Phi)^*.$$

Since distinct functions $q(x)$ give distinct $q(\kappa)$ and

$$q(\Phi) \in q(\kappa) = q(\Phi) + p_\Phi(\Phi)\mathbb{F}[\Phi],$$

we can take in (57)

$$f_{q(\kappa)} := [q(\Phi), q(\Phi)^*] \in \mathrm{End}(\underline{\mathscr{A}}_\Phi).$$

By (55), the corresponding representations $\mathscr{A}_\Phi^{q(\kappa)} = \mathscr{A}_\Phi^{f_{q(\kappa)}}$ have the form (74) and constitute the orbit of $\mathscr{A}_\Phi$. $\square$

Proof of Theorem 2.2. Each pair (A, B) consisting of a nondegenerate ε-Hermitian form B and an isometric operator A gives a representation of the dograph (62). By Theorem 3.1, each representation of (62) over a field $\mathbb{F}$ of characteristic different from 2 is isomorphic to a direct sum of representations of the form $\mathscr{M}^+$ and $\mathscr{A}^a$, where $\mathscr{M} \in \mathrm{ind}_1(\underline{D})$, $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{D})$, and $0 \neq a = a^\circ \in \mathbb{T}(\mathscr{A})$. This direct sum is determined uniquely up to permutation of summands and replacement of the whole group of summands $\mathscr{A}^{a_1} \oplus \cdots \oplus \mathscr{A}^{a_s}$ with the same $\mathscr{A}$ by $\mathscr{A}^{b_1} \oplus \cdots \oplus \mathscr{A}^{b_s}$ such that the Hermitian forms $a_1 x_1^\circ x_1 + \cdots + a_s x_s^\circ x_s$ and $b_1 x_1^\circ x_1 + \cdots + b_s x_s^\circ x_s$ are equivalent over the field $\mathbb{T}(\mathscr{A})$ (see (72)).

This proves Theorem 2.2 since we can use the sets $\mathrm{ind}_1(\underline{D})$ and $\mathrm{ind}_0(\underline{D})$ from Lemma 4.1, the field $\mathbb{T}(\mathscr{A})$ is determined in (72), and the representations $\mathscr{M}^+$ and $\mathscr{A}^a$ have the form (69) and (74). $\square$

5. Proof of Theorem 2.1

Theorem 2.1 gives canonical matrices of representations of the dograph (62) over algebraically or real closed fields and over skew fields of quaternions. We will prove it basing on the next lemma, in which we concretize Lemma 4.1: we give the sets $\mathscr{O}_\mathbb{F}$, establish when Ψ is similar to Φ^{-*} (see (68)) and when $\Phi_{(\varepsilon)}$ exists for $\Phi \in \mathscr{O}_\mathbb{F}$, construct the matrices $\Phi_{(\varepsilon)}$ simpler than in Lemma 2.4, and find the field $\mathbb{T}(\mathscr{A})$ for each $\underline{\mathscr{A}} \in \mathrm{ind}_0(\underline{D})$.

Recall the n-by-n matrices defined in (12) and (13):

$$\Lambda_n = \begin{bmatrix} 1 & 2 & \cdots & 2 \\ & 1 & \ddots & \vdots \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}, \quad F_n = \begin{bmatrix} 0 & & & \iddots \\ & & 1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix}.$$

Lemma 5.1. (a) *Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2 with the identity involution, and let $\varepsilon = \pm 1$. One can take $\mathscr{O}_\mathbb{F}$ to be all nonsingular Jordan blocks. For nonzero $\lambda, \mu \in \mathbb{F}$,*

$$J_n(\lambda) \text{ is similar to } J_n(\mu)^{-\mathrm{T}} \iff \lambda = \mu^{-1},$$
$$J_n(\lambda)_{(\varepsilon)} \text{ exists } \iff \lambda = \pm 1 \text{ and } \varepsilon = (-1)^{n+1}.$$

If it exists then $J_n(\lambda)$ is similar to

$$\Psi = \lambda \Lambda_n, \quad \text{with } \Psi_{(\varepsilon)} = F_n. \tag{81}$$

(b) *Let $\mathbb{F}$ be an algebraically closed field with nonidentity involution. One can take $\mathscr{O}_\mathbb{F}$ to be all nonsingular Jordan blocks. For nonzero $\lambda, \mu \in \mathbb{F}$,*

$$J_n(\lambda) \text{ is similar to } J_n(\mu)^{-*} \iff \lambda = \bar{\mu}^{-1},$$
$$J_n(\lambda)_{(1)} \text{ exists } \iff |\lambda| = 1 \text{ (see (10))}.$$

If it exists then $J_n(\lambda)$ is similar to

$$\Psi = \lambda \Lambda_n, \quad with \ \Psi_{(1)} = i^{n-1} F_n. \tag{82}$$

(c) *Let $\mathbb{F}$ be a real closed field $\mathbb{P}$, let $\mathbb{P} + \mathbb{P}i$ (see (6)) be its algebraic closure with involution $a + bi \mapsto a - bi$, and let $\varepsilon = \pm 1$. One can take $\mathcal{O}_\mathbb{F}$ to be all $J_n(\lambda)$ with nonzero $\lambda \in \mathbb{P}$ and all realifications $J_n(\lambda)^\mathbb{P}$ with $\lambda \in (\mathbb{P} + \mathbb{P}i) \smallsetminus \mathbb{P}$ determined up to replacement by $\bar{\lambda}$.*

(i) *For $\lambda \in \mathbb{P}$,*

   $J_n(\lambda)_{(\varepsilon)}$ *exists* $\iff \lambda = \pm 1$ *and* $\varepsilon = (-1)^{n+1}$.

   If it exists then $J_n(\lambda)$ is similar to

   $\Psi = \lambda \Lambda_n, \quad with \ \Psi_{(\varepsilon)} = F_n$.

   The field $\mathbb{T}(\mathscr{A}_\Psi)$, which is constructed basing on the endomorphisms of the corresponding selfadjoint representation

   $\underline{\mathscr{A}}_\Psi = (\lambda \Lambda_n, F_n, (\lambda \Lambda_n)^*) \quad$ (*see (70)*),

   is naturally isomorphic to $\mathbb{P}$.

(ii) *For $\lambda, \mu \in (\mathbb{P} + \mathbb{P}i) \smallsetminus \mathbb{P}$,*

   $$\begin{aligned} J_n(\lambda)^\mathbb{P} \text{ is similar to } (J_n(\mu)^\mathbb{P})^{-T} &\iff \lambda \in \{\mu^{-1}, \bar{\mu}^{-1}\}, \\ J_n(\lambda)^\mathbb{P}_{(\varepsilon)} \text{ exists } &\iff |\lambda| = 1. \end{aligned}$$

   If it exists then $J_n(\lambda)^\mathbb{P}$ is similar to

   $$\Psi = (\lambda \Lambda_n)^\mathbb{P}, \quad with \ \Psi_{(\varepsilon)} = (i^{n-(\varepsilon+1)/2} F_n)^\mathbb{P}. \tag{83}$$

   The field $\mathbb{T}(\mathscr{A}_\Psi)$ is naturally isomorphic to $\mathbb{P} + \mathbb{P}i$ with involution $a + bi \mapsto a - bi$.

(d) *Let $\mathbb{F}$ be the skew field $\mathbb{H}$ of quaternions with quaternionic conjugation (4) or quaternionic semiconjugation (5) over a real closed field $\mathbb{P}$, and let $\varepsilon = \pm 1$. One can take $\mathcal{O}_\mathbb{F}$ to be all $J_n(\lambda)$ with nonzero $\lambda = a + bi \in \mathbb{P} + \mathbb{P}i$ determined up to replacement by $a - bi$. For nonzero $\lambda, \mu \in \mathbb{P} + \mathbb{P}i$,*

   $$J_n(\lambda) \text{ is similar to } J_n(\mu)^{-*} \iff \lambda \in \{\mu^{-1}, \bar{\mu}^{-1}\}, \tag{84}$$

   $J_n(\lambda)_{(\varepsilon)}$ *exists* $\iff |\lambda| = 1$.

If it exists then $J_n(\lambda)$ is similar to

$$\Psi = \lambda \Lambda_n, \quad with \ \Psi_{(\varepsilon)} = i^{n-(\varepsilon+1)/2} F_n, \tag{85}$$

and

(i) *if $\lambda \neq \pm 1$, then the field $\mathbb{T}(\mathscr{A}_\Psi)$ is naturally isomorphic to $\mathbb{P} + \mathbb{P}i$ with involution $a + bi \mapsto a - bi$,*

(ii) *if $\lambda = \pm 1$ and $\varepsilon = (-1)^{n+1}$, then $\mathbb{T}(\mathscr{A}_\Psi)$ is naturally isomorphic to $\mathbb{F}$ and this isomorphism preserves the involution,*

(iii) *if $\lambda = \pm 1$ and $\varepsilon = (-1)^n$, then $\mathbb{T}(\mathscr{A}_\Psi)$ is naturally isomorphic to $\mathbb{F}$ and if the involution on $\mathbb{F}$ is quaternionic conjugation (4) or quaternionic semiconjugation (5) then the involution on $\mathbb{T}(\mathscr{A}_\Psi)$ is (5) or (4), respectively.*

Proof. (a) Let $\mathbb{F}$ be an algebraically closed field with the identity involution and $\varepsilon = \pm 1$. By (31) and (32), if $J_n(\lambda)_{(\varepsilon)}$ exists then $\lambda = \pm 1$ and $\varepsilon = (-1)^{n+1}$. Let these conditions be satisfied. Since $\lambda \Lambda_n$ is similar to $\Psi = J_n(\lambda)$, it remains to check that $\Psi_{(\varepsilon)} = F_n$ fulfils (30), that is,

$$\Psi_{(\varepsilon)} = \varepsilon \Psi_{(\varepsilon)}^*, \qquad \Psi_{(\varepsilon)} = \Psi^* \Psi_{(\varepsilon)} \Psi. \tag{86}$$

The first equality is obvious. The second is satisfied because

$$\Psi_{(\varepsilon)}^{-1} \cdot \Psi^* \Psi_{(\varepsilon)} \Psi = F_n^{-1} \Lambda_n^{\mathrm{T}} F_n \Lambda_n$$

$$= \begin{bmatrix} 0 & & & 1 \\ & & -1 & \\ & 1 & & \\ \ddots & & & 0 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ 2 & 1 & & \\ \vdots & \ddots & \ddots & \\ 2 & \cdots & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & & & \ddots \\ & & 1 & \\ & -1 & & \\ 1 & & & 0 \end{bmatrix} \Lambda_n$$

$$= \begin{bmatrix} 1 & -2 & 2 & \ddots & (-1)^{n-1}2 \\ & 1 & -2 & \ddots & \ddots \\ & & 1 & \ddots & 2 \\ & & & \ddots & -2 \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & \cdots & 2 \\ & 1 & \ddots & \vdots \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix} \qquad \text{(write } J := J_n(0))$$

$$= (I_n - 2J + 2J^2 - 2J^3 + 2J^4 - \cdots)(I_n + 2J + 2J^2 + 2J^3 + \cdots) = I_n.$$

(b) Let $\mathbb{F} = \mathbb{P} + \mathbb{P}\mathrm{i}$ be an algebraically closed field with nonidentity involution represented in the form (8). By (31), if $J_n(\lambda)_{(1)}$ exists for $\lambda = a + b\mathrm{i}$ then $x - \lambda = x - \bar{\lambda}^{-1}$. Thus, $\lambda = \bar{\lambda}^{-1}$ and by (9) $1 = \lambda \bar{\lambda} = a^2 + b^2 = |\lambda|^2$.

Let $|\lambda| = 1$. The matrix $\Psi = \lambda \Lambda_n$ in (82) is similar to $J_n(\lambda)$, the first equality in (86) is obvious for $\Psi_{(1)} = \mathrm{i}^{n-1} F_n$ and the second holds since it holds for (81).

(c) Let $\mathbb{F} = \mathbb{P}$ be a real closed field and $\varepsilon = \pm 1$. Let $\mathbb{K} := \mathbb{P} + \mathbb{P}\mathrm{i}$ be the algebraic closure of $\mathbb{P}$ represented in the form (6) with involution $a + b\mathrm{i} \mapsto a - b\mathrm{i}$. By [12, Theorem 3.4.5], we can take $\mathcal{O}_{\mathbb{P}}$ to be all $J_n(\lambda)$ with $0 \neq \lambda \in \mathbb{P}$ and all $J_n(\lambda)^{\mathbb{P}}$ with $\lambda \in \mathbb{K} \smallsetminus \mathbb{P}$ determined up to replacement by $\bar{\lambda}$.

Let us consider $J_n(\lambda)$ with $\lambda \in \mathbb{P}$. By (31) and (32), if $J_n(\lambda)_{(\varepsilon)}$ exists then $\lambda = \pm 1$ and $\varepsilon = (-1)^{n+1}$. Hence we can use Ψ and $\Psi_{(\varepsilon)}$ from (81). In view of (72) and since $p_\Psi(x) = x - \lambda$,

$$\mathbb{T}(\mathscr{A}_\Psi) \simeq \mathbb{P}(\kappa) = \mathbb{P}[x]/p_\Psi(x)\mathbb{P}[x] \simeq \mathbb{P}.$$

Let now $\Phi := J_n(\lambda)^{\mathbb{P}}$ with $\lambda \in \mathbb{K} \smallsetminus \mathbb{P}$. Then

$$p_\Phi(x) = (x - \lambda)(x - \bar{\lambda}) = x^2 - (\lambda + \bar{\lambda}) + |\lambda|^2. \tag{87}$$

If $J_n(\lambda)_{(\varepsilon)}^{\mathbb{P}}$ exists, then $|\lambda| = 1$ by (31) and (87).

If $\lambda \in \mathbb{K} \smallsetminus \mathbb{P}$ and $|\lambda| = 1$, then we can take Ψ and $\Psi_{(\varepsilon)}$ as in (83) due to the following observation. The equalities (86) hold for (85) since for $\varepsilon = 1$ they were checked in (b) and so they are fulfilled for $\varepsilon = -1$ too. Therefore, (86) hold true for (83) due to the following property of realification: *if matrices $M_1, \ldots, M_l, M_1^*, \ldots, M_l^*$ over $\mathbb{P} + \mathbb{P}\mathrm{i}$ satisfy an equation with coefficients in $\mathbb{P}$, then their realifications also satisfy the same equation.* This property is valid since for each matrix $M = A + B\mathrm{i}$ with A and B over $\mathbb{P}$, its realification $M^{\mathbb{P}}$ (see (11)), up to simultaneous permutations of rows and columns, has the form

$$M_{\mathbb{P}} := \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = S^{-1}(M \oplus \overline{M})S = S^*(M \oplus \overline{M})S \qquad (88)$$

with

$$S := \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} = S^{-*};$$

the middle equality in (88) follows from

$$\begin{bmatrix} A + Bi & 0 \\ 0 & A - Bi \end{bmatrix} \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} = \begin{bmatrix} I & iI \\ I & -iI \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

By (72) and since $\deg p_\Psi(x) = 2$, $\mathbb{T}(\mathscr{A}_\Psi) \simeq \mathbb{P}(\kappa) \simeq \mathbb{P} + \mathbb{P}i$. The involution (73) on $\mathbb{T}(\mathscr{A}_\Psi)$ is not the identity (otherwise $\kappa = \kappa^{-1}$, $\kappa^2 - 1 = 0$, i.e. $p_\Psi(x) = x^2 - 1$, which contradicts the irreducibility of $p_\Psi(x)$).

(d) Let $\mathbb{F}$ be the skew field $\mathbb{H}$ of quaternions over a real closed field $\mathbb{P}$, and let $\varepsilon = \pm 1$. By [18, Section 3, §12], we can take $\mathscr{O}_{\mathbb{F}}$ to be all $J_n(\lambda)$ with $\lambda = a + bi \in \mathbb{P} + \mathbb{P}i$ determined up to replacement by $a - bi$. For any nonzero $\mu \in \mathbb{P} + \mathbb{P}i$, the matrix $J_n(\mu)^{-*}$ is similar to $J_n(\bar{\mu}^{-1})$, by (b) it is similar to $J_n(\lambda)$ with $\lambda \in \mathbb{P} + \mathbb{P}i$ if and only if $\lambda \in \{\mu^{-1}, \bar{\mu}^{-1}\}$. This proves (84).

Using (31) and reasoning as in (b), we make sure that if $J_n(a + bi)_{(\varepsilon)}$ exists then $a^2 + b^2 = 1$. We can take Ψ and $\Psi_{(\varepsilon)}$ as in (85) since the equalities (86) for them were checked in (c).

Due to (79), $[G_1, G_2] \in \mathrm{End}(\mathscr{A}_\Psi)$ if and only if

$$G_1\Psi = \Psi G_1, \quad G_2\Psi_{(\varepsilon)} = \Psi_{(\varepsilon)}G_1, \quad G_2\Psi^* = \Psi^* G_2. \qquad (89)$$

The last equality follows from the others:

$$\begin{aligned} \Psi^* G_2 &= \Psi^* \Psi_{(\varepsilon)} G_1 \Psi_{(\varepsilon)}^{-1} = \Psi_{(\varepsilon)} \Psi^{-1} G_1 \Psi_{(\varepsilon)}^{-1} \\ &= \Psi_{(\varepsilon)} G_1 \Psi^{-1} \Psi_{(\varepsilon)}^{-1} = \Psi_{(\varepsilon)} G_1 \Psi_{(\varepsilon)}^{-1} \Psi^* = G_2 \Psi^*. \end{aligned}$$

Let $\lambda \in \mathbb{K} := \mathbb{P} + \mathbb{P}i$ and $|\lambda| = 1$.

(i) First we consider the case $\lambda \neq \pm 1$. Represent G_1 in the form $U + Vj$, where $U, V \in \mathbb{K}^{n \times n}$. Then the first equality in (89) becomes $(U + Vj)\lambda\Lambda = \lambda\Lambda(U + Vj)$ and falls into two equalities

$$U\lambda\Lambda = \lambda\Lambda U, \quad V\bar{\lambda}\Lambda j = \lambda\Lambda Vj$$

(quaternionic conjugation (4) and quaternionic semiconjugation (5) coincide on $\mathbb{K}$). By the second equality,

$$(\bar{\lambda} - \lambda)V = \lambda(\Lambda - I)V - \bar{\lambda}V(\Lambda - I)$$

and so $V = 0$ since $\lambda \neq \bar{\lambda}$ and because $\Lambda - I$ is nilpotent upper triangular. By the first equality (which is over the field $\mathbb{K}$), $G_1 = U = f(\lambda\Lambda) = f(\Psi)$ for some $f \in \mathbb{K}[x]$; see the beginning of the proof of Lemma 4.1(d). Since $\Psi_{(\varepsilon)}$ is over $\mathbb{K}$ and in view of (89),

$$G_2 = \Psi_{(\varepsilon)} G_1 \Psi_{(\varepsilon)}^{-1} = f(\Psi_{(\varepsilon)} \Psi \Psi_{(\varepsilon)}^{-1}) = f(\Psi^{-*}).$$

Due to $G_2 = \Psi_{(\varepsilon)} G_1 \Psi_{(\varepsilon)}^{-1}$, the homomorphism $[G_1, G_2] \in \mathrm{End}(\mathscr{A}_\Psi)$ is completely determined by $G_1 = f(\Psi)$. The matrix $\Psi = \lambda\Lambda$ is upper triangular, so the mapping $f(\Psi) \mapsto f(\lambda)$, $f \in \mathbb{K}[x]$, defines an endomorphism of rings $\mathrm{End}(\mathscr{A}_\Psi) \to \mathbb{K}$, its kernel is the radical of $\mathrm{End}(\mathscr{A}_\Psi)$. Hence $\mathbb{T}(\mathscr{A}_\Psi)$ can be identified with $\mathbb{K}$. In view of (67), the involution on $\mathbb{T}(\mathscr{A}_\Psi)$ is induced by the mapping $G_1 \mapsto G_2^*$ of the form

$$f(\lambda\Lambda) \longmapsto f((\lambda\Lambda)^{-*})^* = \bar{f}((\lambda\Lambda)^{-1}).$$

Therefore, the involution is

$$f(\lambda) \longmapsto \bar{f}(\lambda^{-1}) = \bar{f}(\bar{\lambda}) = \overline{f(\lambda)}$$

and coincides with the involution $a + bi \mapsto a - bi$.

(ii) and (iii) Let $\lambda = \pm 1$. Define

$$\check{h} := a + bi - cj - dk \quad \text{for each } h = a + bi + cj + dk \in \mathbb{H},$$
$$\check{f}(x) := \sum_l \check{h}_l x^l \quad \text{for each } f(x) = \sum_l h_l x^l \in \mathbb{H}[x].$$

Because $\lambda = \pm 1$ and by the first equality in (89), G_1 has the form

$$G_1 = \begin{bmatrix} a_1 & a_2 & \ddots & a_n \\ & a_1 & \ddots & \ddots \\ & & \ddots & a_2 \\ 0 & & & a_1 \end{bmatrix}, \quad a_1, \ldots, a_n \in \mathbb{H}.$$

Thus, $G_1 = f(\Psi)$ for some $f(x) \in \mathbb{H}[x]$.

Using the second equalities in (89) and (85) and the identity $i f(x) = \check{f}(ix)$, we obtain

$$\begin{aligned} G_2 &= \Psi_{(\varepsilon)} G_1 \Psi_{(\varepsilon)}^{-1} = \Psi_{(\varepsilon)} f(\Psi) \Psi_{(\varepsilon)}^{-1} \\ &= \begin{cases} f(\Psi_{(\varepsilon)} \Psi \Psi_{(\varepsilon)}^{-1}) = f(\Psi^{-*}) & \text{if } \varepsilon = (-1)^{n+1}, \\ \check{f}(\Psi_{(\varepsilon)} \Psi \Psi_{(\varepsilon)}^{-1}) = \check{f}(\Psi^{-*}) & \text{if } \varepsilon = (-1)^n. \end{cases} \end{aligned}$$

Since the homomorphism $[G_1, G_2]$ is completely determined by $G_1 = f(\Psi)$, the matrix $\Psi = \lambda \Lambda_n$ is upper triangular, its main diagonal is $(\lambda, \ldots, \lambda)$, and $\lambda = \pm 1$, we conclude that the mapping $f(\Psi) \mapsto f(\lambda)$ defines an endomorphism of rings $\mathrm{End}(\underline{\mathscr{A}}_\psi) \to \mathbb{H}$ and its kernel is the radical of $\mathrm{End}(\underline{\mathscr{A}}_\psi)$. Hence $\mathbb{T}(\mathscr{A}_\psi)$ can be identified with $\mathbb{H}$. The involution on $\mathbb{T}(\mathscr{A}_\psi)$ is induced by the mapping $G_1 \mapsto G_2^*$, that is, by

$$f(\Psi) \mapsto \begin{cases} \bar{f}(\Psi^{-1}) & \text{if } \varepsilon = (-1)^{n+1}, \\ \widehat{f}(\Psi^{-1}) & \text{if } \varepsilon = (-1)^n. \end{cases}$$

Here $h \mapsto \bar{h}$ is the involution on $\mathbb{F}$ that is quaternionic conjugation (4) or quaternionic semiconjugation (5), and $h \mapsto \widehat{h}$ denotes the remaining involution (5) or (4). Thus, the involution on $\mathbb{T}(\mathscr{A}_\psi)$ is $h \mapsto \bar{h}$ if $\varepsilon = (-1)^{n+1}$ and $h \mapsto \widehat{h}$ if $\varepsilon = (-1)^n$. $\quad\square$

Proof of Theorem 2.1. Let $\mathbb{F}$ be one of the fields and skew fields considered in Theorem 2.1. By Theorem 3.2, each representation of a dograph D over $\mathbb{F}$ is uniquely, up to isomorphism of summands, decomposes into a direct sum of indecomposable representations. Hence the problem of classifying its representations reduces to the problem of classifying indecomposable representations.

Let $\mathscr{O}_\mathbb{F}$ be a maximal set of nonsingular indecomposable canonical matrices over $\mathbb{F}$ for similarity. Due to Theorem 3.2 and Lemma 4.1, the following representations form a maximal set of nonisomorphic indecomposable representations of the dograph D defined in (62):

(i) $\mathscr{M}_\Phi^+ = (\Phi \oplus \Phi^{-*}, I \backslash \varepsilon I)$, in which $\Phi \in \mathscr{O}_\mathbb{F}$ is such that $\Phi_{(\varepsilon)}$ does not exist; Φ is determined up to replacement by the matrix $\Psi \in \mathscr{O}_\mathbb{F}$ that is similar to Φ^{-*}.

(ii) $\mathscr{A}_\Phi = (\Phi, \Phi_{(\varepsilon)})$ and $\mathscr{A}_\Phi^- = (\Phi, -\Phi_{(\varepsilon)})$, in which $\Phi \in \mathbb{O}_\mathbb{F}$ is such that $\Phi_{(\varepsilon)}$ exists. The representation $\mathscr{A}_\Phi^-$ is withdrawn if $\mathscr{A}_\Phi \simeq \mathscr{A}_\Phi^-$, this occurs if and only if

  – $\mathbb{F}$ is an algebraically closed field with the identity involution, or
  – $\mathbb{F}$ is not an algebraically closed field and either $\mathbb{T}(\mathscr{A}_\Phi)$ is an algebraically closed field with the identity involution, or $\mathbb{T}(\mathscr{A}_\Phi)$ is a skew field of quaternions with involution different from quaternionic conjugation (4).

Thus, the statements (a)–(d) of Theorem 2.1 follow from the statements (a)–(d) of Lemma 5.1. $\square$

6. Metric and selfadjoint operators with respect to degenerate forms

Recall that a classification problem is called *wild* if it contains the problem of classifying pairs of matrices up to simultaneous similarity and hence (see [3]) it contains the problem of classifying representations of each quiver. A linear operator is called *metric* or *selfadjoint* with respect to a form B (possibly, degenerate) if $B(Au, Av) = B(u, v)$ or $B(Au, v) = B(u, Av)$, respectively, for all u and v. The following theorem was proved in [37, Theorem 5.4].

Theorem 6.1. *The problem of classifying pairs (A, B) consisting of a form B on a vector space V over a field of characteristic different from 2 and an operator A that is metric with respect to B is wild in each of the following three cases: B is symmetric, B is skew-symmetric, or B is Hermitian. This statement also holds if 'metric' is replaced by "selfadjoint".*

Proof. (a) Suppose first that A is metric. The problem of classifying pairs (A, B) is given by the dograph (62) without the condition "β is nonsingular" and reduces to the problem of classifying representations of the corresponding quiver (63) without the condition "β is bijective". Each representation of this quiver has the form (64) without the condition "B is nonsingular", i.e., it is given by matrices A, B, C of sizes $m \times m, n \times m, n \times n$ satisfying the relation:

$$B = CBA. \tag{90}$$

By (66), two matrix triples (A, B, C) and (A', B', C') give isomorphic representations if and only if there exist nonsingular matrices R and S such that

$$RA = A'R, \quad SB = B'R, \quad SC = C'S. \tag{91}$$

Since $B' = SBR^{-1} = I_r \oplus 0$ for some nonsingular R and S, it suffices to consider only the triples (A, B, C) with $B = I_r \oplus 0$. Such a triple satisfies (90) if and only if it has the form

$$(A, B, C) = \left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} \right), \tag{92}$$

in which A_{11} and C_{11} are $r \times r$ matrices and $C_{11} A_{11} = I_r$. Triples (92) and

$$(A', B', C') = \left(\begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix}, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} C'_{11} & C'_{12} \\ 0 & C'_{22} \end{bmatrix} \right) \tag{93}$$

give isomorphic representations if and only if there exist nonsingular R and S satisfying (91). The equality $SB = B'R$ with $B' = B = I_r \oplus 0$ implies

$$R = \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \quad R_{11} = S_{11}. \tag{94}$$

The remaining equalities in (91) take the form

$$
\begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{bmatrix},
\tag{95}
$$

$$
\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} \\ 0 & C'_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}.
\tag{96}
$$

Therefore, the problem of classifying pairs (A, B) contains the problem of classifying upper block-triangular matrices

$$
\begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix},
\tag{97}
$$

in which C_{11} is nonsingular, with respect to upper block-triangular similarity. The wildness of this problem and many analogous problems was proved, for example, in [43, Section 3.3.1].

(b) Suppose now that A is selfadjoint. The problem of classifying pairs (A, B) is given by the dograph (62) in which all the relations are replaced by

$$
\beta = \varepsilon \beta^*, \quad \beta\alpha = \alpha^*\beta.
\tag{98}
$$

It reduces to the problem of classifying representations of the corresponding quiver (63) with relations (98). Each representation of this quiver is given by matrices A, B, C of sizes $m \times m$, $n \times m$, $n \times n$ such that

$$
BA = CB.
\tag{99}
$$

Let us consider triples (A, B, C) with $B = I_r \oplus 0$. Such a triple satisfies (99) if and only if it has the form (92) with $C_{11} = A_{11}$. Triples (92) and (93) give isomorphic representations if and only if the equalities (95) and (96) are valid for some nonsingular R and S of the form (94).

Therefore, the problem of classifying pairs (A, B) contains the wild problem of classifying upper block-triangular matrices (97) with respect to upper block-triangular similarity. $\quad\square$

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