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Local majorities, coalitions and monopolies in graphs: a review [☆]

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Abstract

This paper provides an overview of recent developments concerning the process of local majority voting in graphs, and its basic properties, from graph theoretic and algorithmic standpoints. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

1.1. A puzzle

In each vertex of an n -vertex graph G there lives a citizen. Tomorrow morning, the citizens of G are about to vote “Yes/No” on a critical and highly controversial proposition (whose details are not really of our present concern).

Out of curiosity and boredom, each citizen of G spends the afternoon amusing himself by conducting a private poll among his neighbors (including himself), in order to get a sense of the outcome of this all-important election. The alarming and disappointing result found by each “Yes” voter is an astounding 2 : 1 majority for the “No” voters in his neighborhood.

Should the “Yes” voters despair?! May the “No” voters rejoice?! Or can all these polls lie?? In short: What can be said with certainty about the minimum guaranteed number of “No” voters in the election, given these poll results?

A moment’s reflection reveals that the (perhaps surprising) answer is “very little indeed”. For sure, one can deduce the existence of at least two “No” voters, but in some cases this is all it takes! As a concrete example, consider a complete bipartite

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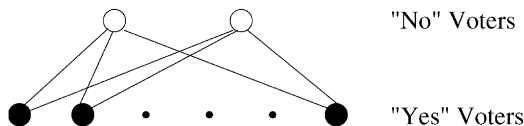


Fig. 1. Two “No” voters controlling the local majority poll of all other vertices. (In all our figures, white circles represent vertices of the coalition, and black circles represent vertices controlled by the coalition. Vertices that fall under neither category, if exist, are represented by dotted circles.)

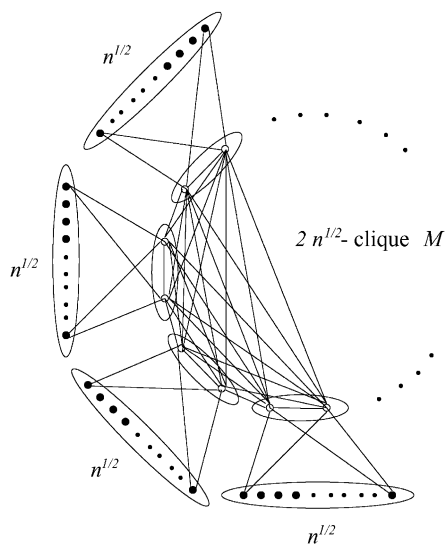


Fig. 2. $2\sqrt{n}$ “No” voters forcing a 2:1 majority in all polls.

graph with the two “No” voters on one side and the $n - 2$ “Yes” voters on the other (see Fig. 1).

The above example clearly points to one potential explanation for these strange poll results: while every “Yes” voter saw a small majority of “No” votes, every “No” voter saw a huge majority of “Yes” votes. So in some sense, a more “balanced” picture would have emerged had we looked at *all* the polls, rather than only at those carried by the “Yes” voters.

This leads us to the following question. Suppose that the result of a 2:1 majority for the “No” voters in the neighborhood was found by *each and every* voter, and not just by the “Yes” voting ones. What could be guaranteed now about the minimum number of “No” voters in the election?

The answer, pointed out in [44], is that it is still possible for the polls to be rather misleading. In particular, it is possible for a negligible minority of $2\sqrt{n}$ “No” voters to cause this outcome in all polls. A graph allowing such behavior is depicted in Fig. 2. The “No” voters form a set M of $2\sqrt{n}$ vertices, u_i, w_i for $1 \leq i \leq \sqrt{n}$, connected by a clique. The remaining vertices (the “Yes” voters) are partitioned into $\sqrt{n} - 2$ groups of \sqrt{n} vertices each, where the vertices of the i th group are attached to u_i and w_i .

A major lesson clearly illuminated by the above two examples is the importance of being well-connected; the influence of a vertex is determined by its degree, and the influence level of a collection of vertices is decided by the distribution of their edges.

1.2. Motivation

Overcoming failures is a central problem in distributed computing. A common theme in a number of approaches to this problem revolves around the notion of *majority ruling*. The idea is to eliminate the damage caused by failed vertices, or at least restrict their influence, by maintaining replicated copies of crucial data, and performing a voting process among the participating processors whenever faults occur, adopting the values stored at the majority of the processors as the correct data.

This method, in one form or another, is used as a component of fault-tolerant algorithms in a wide variety of contexts. Algorithms for agreement and consensus problems, for instance (cf. [43, 11, 19, 48, 6]), are often based on using majority voting among the participating processors in order to eradicate the harmful influence of “Byzantine” (or maliciously behaving) faulty processors. This is, in fact, the inherent reason why it is necessary to limit the number of faults t in an n -processor system to $t \leq n/c$ (typically for some $c \geq 3$) in order to enable agreement algorithms to function correctly. Similarly, majority voting is used as a basic tool in various algorithms for *system-level diagnosis* (cf. [64, 54, 15]). Related studies concerned testing and reconfiguration under *catastrophic fault patterns* [14, 53, 62].

Data redundancy or replication is a commonly used technique in the area of distributed database management algorithms (cf. [13, 37]), and inconsistency resolution protocols are often based on one form or another of majority voting, preferring the version of the data supported by the majority of the available copies.

Voting systems are used also to enforce data consistency by requiring changes to be performed on a *quorum* of the copies (cf. [27, 26, 63, 58, 66]). Informally, a *quorum system* is collection of sets every two of which have a nonempty intersection. Thus, updating whole quorums ensures that each subsequent change will override earlier ones in at least one site. While a variety of quorum system types can be used for that purpose, voting systems are often a favored form due to their ease of implementation and use.

Majority voting systems have found applications also in the context of resource allocation, as a means for ensuring *mutual exclusion* (cf. [61]). For example, once all the processors of a certain quorum permit a user to enter the critical section, no other user will be able to enter the critical section before the first user has released it, since no matter which quorum it asks for permission, it will be refused by at least one processor.

Finally, the applicability of majority voting as a tool for distributed *fault-local mending* was recently investigated in [41, 42]. This approach can be illustrated via the following basic problem. Consider a distributed network, whose nodes collectively store

some function (representing, say, the output of some computation), one bit per node. Suppose that at some point, the memory contents stored at some subset F of the nodes of the network are distorted due to some transient failures. As a result, while the stored values may still look locally legal, the representation of the function stored at the network has changed into some inconsistent function that is no longer valid. The question raised in [41, 42] is whether it is possible to distributedly mend the function in time complexity dependent on the number of *failed nodes*, rather than on the size of the entire network. This operation (if and when possible) is termed *fault-local mending*, and the algorithm developed in [41, 42] for performing it involves majority votings in neighborhoods of different radii.

The majority voting method is usually expected to work well due to the common assumption that given today's reliable technology, at any given moment there can be but a small number of failures in the system. This implies that the required level of replication, and the extent of the voting process, can be limited. In particular, in the distributed context, it is highly desirable to restrict both the replication of data stored originally at a processor v , and the process of majority voting regarding the data of a processor v , to processors in v 's *local vicinity*.

There are two reasons for this focus on locality. First, in many cases, processors in the system are better aware of, and more involved in, whatever happens in their immediate vicinity, than far away. It is thus more natural, and often much cheaper, to store data as locally as possible. Secondly, and perhaps more importantly, the distributed network model allows only for computations which are *local* in nature, namely, in t time units, a processor can only collect data from other processors whose distance from itself in the network does not exceed t . Therefore, voting over large areas might be too expensive in terms of its time consumption.

However, there is an inherent risk in limiting ourselves to local vicinities in this way. Once replication is restricted to local neighborhoods, we run into the danger that a large enough set of faults may manage to gain the majority in some of these neighborhoods. In fact, as vividly demonstrated by our previous puzzle, once the voting is performed over subsets of the vertices, the ability of failed vertices to influence the outcome of the votes becomes not only a function of their *number* but also a function of their *location* in the network: well-situated vertices can acquire greater influence.

This situation has motivated recent interest in the process of local majority voting in graphs, and its basic properties. Among the issues studied were extremal combinatorial aspects such as the possible size of influential coalitions, dynamic aspects such as the behavior of repetitive voting processes, and algorithmic aspects such as the complexity of finding influential sets of vertices. This paper provides an overview of some of the recent developments concerning the local majority voting process.

1.3. Control and polling domains

The central notion we focus on in this paper is that of *control*. Intuitively, a set of vertices M in a graph G is said to control a vertex v if it is always able to determine

the outcome of the local poll carried by v by voting in a coordinated way. We refer to such a set M as a *coalition*.

A parameter that crucially affects the behavior of local majority polling (and the power of coalitions) is the domain (i.e., the local region) on which the voting is performed. For each vertex v , denote by $Dom(v)$ the set of vertices participating in the local poll carried out by v . Then v is controlled by the coalition M if $|Dom(v) \cap M| > |Dom(v)|/2$, namely, the majority¹ of the vertices of $Dom(v)$ are in M .

There are a number of plausible ways for defining the domain $Dom(v)$. The most straightforward choice is the set of immediate neighbors. Under this choice, a vertex v in a network $G(V, E)$ is controlled by the coalition M if the majority of its neighbors are in M . As this choice severely limits the scope of our majority voting, one may consider the alternative of strengthening the validity of the polls by querying vertices to larger distances. Letting $\Gamma_r(v)$ denote the r -neighborhood of v , i.e., the set of vertices at distance r or less from v (including v itself), it is possible to define $Dom(v) = \Gamma_r(v)$. For clarity, we will henceforth use the term *r-control* for control based on r -neighborhoods as the voting domains.

Two special kinds of coalitions attaining global control are considered in the literature. A coalition M is called an *r-monopoly* if it r -controls every vertex in the graph. A *self-ignoring r-monopoly (r-SIMON)* M is a coalition that r -controls every vertex in $V \setminus M$. Self-ignoring monopolies may be of interest in a context where we think of the control-seeking coalition as a set of faulty (possibly malicious) processors. In such a setting, it may as well be assumed that the coalition M is only interested in gaining control over the neighborhoods of *other* vertices, belonging to $V \setminus M$. This is because, informally speaking, the (faulty) processors of the coalition are not obligated by the rules of the game anyhow, so in order to corrupt the entire system, it suffices for an adversary to attain control over the remaining processors. Such a coalition can therefore be considerably smaller than a true monopoly.

Two notions that turn out to be closely related to monopolies are sphere² packing and covering by spheres (cf. [12]). We will hence discuss also some basic properties of the (weighted or unweighted) packing and covering problems on graphs.

The remainder of this review paper is organized as follows. Section 2 reviews the known bounds on the influence of small coalitions, and the corresponding size bounds on monopolies and self-ignoring monopolies. Section 3 introduces the concept of immune graphs and presents some constructions of such graphs as well as bounds on their immunity. Section 4 considers the problem in special graph families, and Section 5 discusses computational and approximability issues. Section 6 introduces the concepts of graph covering and packing, and their relationships with the voting game. Section 7 introduces control based on a range of polling regions. Finally, Section 8 reviews the literature on dynamic variants of the voting process.

¹ Majority is sometimes defined as a non-strict one; this results in only slight differences asymptotically.

² In the context of graphs, spheres correspond to neighborhoods.

2. Bounds on the maximal influence of coalitions

One of the most fundamental questions concerning local majority voting involves characterizing the potential influence of a small coalition M in a network of processors. Denote the set of vertices r -controlled by M by $Ruled(G, M, r)$. The number of vertices r -controlled by M , denoted $\rho(G, M, r) = |Ruled(G, M, r)|$, can be used as a measure of the *influence* of the coalition M .

For every integer $1 \leq m \leq n$ and n -vertex graph G , let $\Psi_{Ruled}(G, m, r)$ denote the maximum number of vertices that can be r -controlled by an m -vertex coalition in the graph G ,

$$\Psi_{Ruled}(G, m, r) = \max_{M \subseteq V, |M|=m} \{\rho(G, M, r)\}.$$

Dually, let $\Psi_{Mono}(G, r)$ (resp. $\Psi_{Simon}(G, r)$) denote the minimum cardinality of an r -monopoly (resp. r -SIMON) in G ,

$$\begin{aligned} \Psi_{Mono}(G, r) &= \min\{|M| : M \subseteq V, Ruled(G, M, r) = V\}, \\ \Psi_{Simon}(G, r) &= \min\{|M| : M \subseteq V, Ruled(G, M, r) = V \setminus M\}. \end{aligned}$$

The following questions were addressed in [44, 8, 9]:

- (Q1) What is the maximum influence of a set M (as a function of $|M|$), namely, how many vertices can it possibly control?
- (Q2) How small can a monopoly be?
- (Q3) How small can a self-ignoring monopoly be?

For 1-control, the example of Fig. 1 gives us immediate answers to questions (Q1) and (Q3): control of virtually all vertex neighborhoods can be achieved by extremely small coalitions, and in particular, a self-ignoring monopoly exists with as few as two vertices.

As for question (Q2), Fig. 2 illustrates the existence of a 1-monopoly of size $O(\sqrt{n})$. The tightness of this construction was established in [44].

Proposition 2.1 (Linial et al. [44]). (i) For every n -vertex graph, $\Psi_{Mono}(G, 1) = \Omega(\sqrt{n})$.

(ii) There exists a family of n -vertex graphs G_n with $\Psi_{Mono}(G_n, 1) = O(\sqrt{n})$.

Looking at $r \geq 2$, it turns out that an extremal behavior similar to that of the example of Fig. 1 may occur for r -control as well, on certain graphs.

Proposition 2.2 (Bermond and Peleg [8]). For any integer $r \geq 2$ there exists a family of n -vertex graphs G_n such that $\Psi_{Ruled}(G_n, r+1, r) = (n-r-1)/r$.

To illustrate this proposition, consider an integer $r \geq 2$, and fix an integer $p \geq r$ and let $n = rp + (r+1)$. We now construct an n -vertex graph $G_{r,p}$ in which a coalition M of size $r+1$ can r -control as many as $(n-r-1)/r$ vertices. $G_{r,p}$ is *leveled*, namely, the

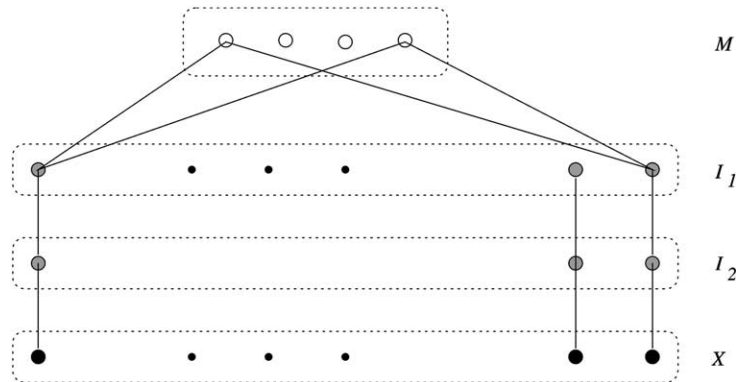


Fig. 3. The graph $G_{r,p}$ for $r=3$ and some p , with a set M of size 4 controlling the majority of 3-neighborhoods of the vertices of a set X of size $p=(n-4)/3$.

vertices are arranged into $r+1$ levels, numbered 1 through $r+1$, with edges connecting only vertices in adjacent levels $\ell, \ell+1$. Each level $2 \leq \ell \leq r+1$ contains p vertices, $v_1^\ell, \dots, v_p^\ell$, and level 1 contains $r+1$ vertices. Let X denote the set of vertices on level $r+1$, and let M denote the set of vertices on level 1. X contains roughly a $1/r$ fraction of the vertices of the graph, yet the edge connections defined next will guarantee that M has the majority in any r -neighborhood around the vertices of X .

The edges connecting two consecutive levels $\ell-1$ and ℓ are defined as follows. The vertices of level 1 (M) are connected by a complete bipartite graph (crossbar) to the vertices of level 2. From level 2 and on, the vertices of the different levels form chains of length r . Namely, for $2 \leq \ell \leq r$, each vertex v_i^ℓ of level ℓ is connected to vertex $v_i^{\ell+1}$ of level $\ell+1$.

It is straightforward to verify that the vertices of M r -control those of X . Fig. 3 depicts an example graph $G_{r,p}$ for $r=3$ and some p .

As for r -monopolies, the following (incomplete) picture is shown in [9].

Proposition 2.3 (Bermond et al. [9]). (i) For every n -vertex graph G and even $r \geq 2$,

$$\Psi_{Mono}(G, r) = \Omega(n^{3/5}).$$

(ii) For any fixed even $r \geq 2$ there exists a family of n -vertex graphs G_n with

$$\Psi_{Mono}(G_n, r) = O(n^{3/5}).$$

(iii) For every n -vertex graph and odd $r \geq 1$, $\Psi_{Mono}(G, 3r) = \Omega(n^{6/11})$.

(iv) For any fixed odd $r \geq 3$ there exists a family of n -vertex graphs G_n with

$$\Psi_{Mono}(G_n, r) = O(n^{4/7}).$$

The second claim of the proposition is illustrated below for $r=2$ (see Fig. 4). Fix a large integer t , and construct the graph G_t as follows. The vertex set of G_t is $V = M_2 \cup M_1 \cup S_1 \cup S_2$. M_2 forms a clique of size t^3 , the vertices being composed of t sets of t^2 vertices.

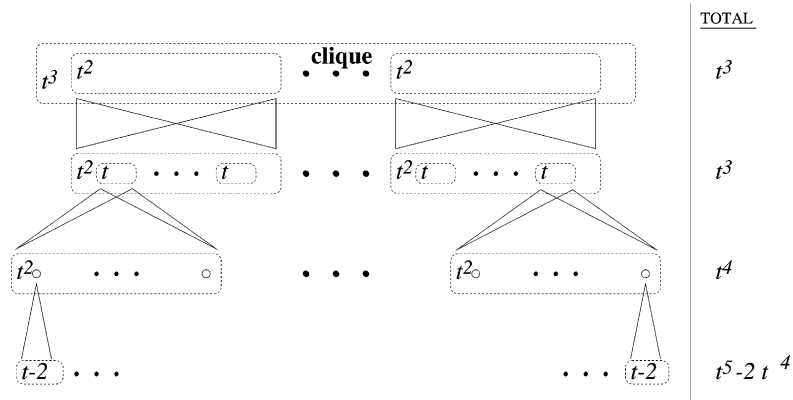


Fig. 4. The graph G_t , with a 2-monopoly $M_1 \cup M_2$ of size $\Theta(n^{3/5})$.

M_1 is an independent set of size t^3 composed of t sets of t^2 elements. The i th set in M_1 is connected to the i th set in M_2 by a complete bipartite graph. Each set of size t^2 is decomposed into subsets of size t .

S_1 is an independent set of size t^4 , composed of t^2 sets of size t^2 . The i th set in S_1 (of size t^2) is connected to the i th subset in M_1 (of size t) by a complete bipartite graph.

S_2 is an independent set of size $t^5 - t^4$, composed of t^4 sets of size $t - 1$. The nodes of the i th set in S_2 are connected to the i th node of S_1 .

Straightforward counting reveals that $M_1 \cup M_2$ is a 2-monopoly.

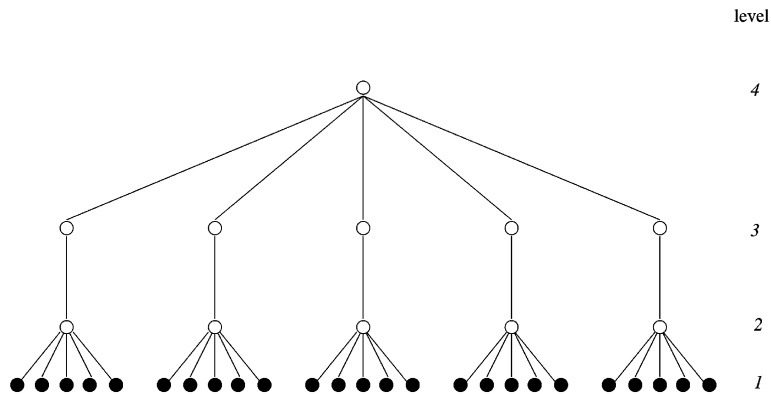
Finally, considering self-ignoring r -monopolies we have:

Proposition 2.4 (Bermond and Peleg [8]). (i) *In every n -vertex graph G and fixed $r \geq 2$, $\Psi_{Simon}(G, r) = \Omega(n^{1/2})$.*

(ii) *For every fixed integer $r \geq 2$ there exists a family of n -vertex graphs G_n such that $\Psi_{Simon}(G_n, r) = \Theta(n^{1/2})$.*

Note that the example of Fig. 1 clearly prevents the possibility of having a similar lower bound for $r = 1$.

As for the upper bound, for $r = 2$, we note that in the graph of Fig. 2, the coalition M (presented there as a 1-monopoly) is also a self-ignoring 2-monopoly. Next, we give an example of such a set for any $r > 2$. For integers r, p , construct the graph $G_{r,p}$ as follows. The graph is again *leveled*, namely, the vertices are arranged into $\lfloor r/2 \rfloor + 2$ levels, numbered 1 through $\lfloor r/2 \rfloor + 2$, with edges connecting only vertices in adjacent levels $\ell, \ell + 1$. Level 1 contains p^2 vertices, each level $2 \leq \ell \leq \lfloor r/2 \rfloor + 1$ contains p vertices, and level $\lfloor r/2 \rfloor + 2$ contains a single vertex. Let X denote the set of vertices on level 1, and let $M = V \setminus X$ be the rest of the vertices. When p is much larger than r , M contains roughly \sqrt{n} of the vertices of the graph, yet our construction will have the property that the vertices of M majorize all r -neighborhoods of X vertices.

Fig. 5. The graph $G_{r,p}$ for $r=5$ and $p=5$.

The edges connecting two consecutive levels $\ell - 1$ and ℓ are now defined as follows. The single vertex of level $\lfloor r/2 \rfloor + 2$ is connected to all the vertices of level $\lfloor r/2 \rfloor + 1$. For level $2 \leq \ell \leq \lfloor r/2 \rfloor + 1$, each vertex v is connected to the corresponding vertex at level $\ell - 1$. For level $\ell = 2$, each vertex of level 2 has p distinct neighbors at level 1 (i.e., each vertex of X has exactly one neighbor on level 2). See Fig. 5 for an example graph $G_{r,p}$ for $r=5$, $p=5$.

A straightforward case analysis reveals that in the graph $G_{r,p}$, for every vertex $v \in X$, the majority of the vertices in $I_r(v)$ are from M .

3. Immune graphs and minimal influence

The problem of controlling coalitions was discussed in the previous section from the angle of looking for extremal constructions allowing small coalitions to control many vertices (namely, graphs with large $\Psi_{Ruled}(G, m, r)$ and small $\Psi_{Mono}(G, r)$ and $\Psi_{Simon}(G, r)$). In contrast, one may consider the dual problem, concerning the existence of graphs *immune* to the influence of small coalitions. These are graphs G for which $Ruled(G, M)$ is small (relative to $|M|$ again) for *every* coalition M , or in other words, $\Psi_{Ruled}(G, m, r)$ is small, and hence also $\Psi_{Mono}(G, r)$ and $\Psi_{Simon}(G, r)$ are large. Upper and lower bounds are derived in [55] on the extent to which such immunity can be achieved.

3.1. Basic properties

To formally discuss the immunity level of a given graph, let us present the following definition.

Definition 3.1. A graph G is (α, β) -immune if $\Psi_{Ruled}(G, m, 1) \leq \alpha m$ for every $1 \leq m \leq \beta$.

Let $\Delta_{\max}(G)$ and $\Delta_{\min}(G)$ denote the maximum and minimum vertex degrees in the graph G , respectively.

Observe that a vertex v of degree $\deg(v)$ can be controlled by a coalition M only if v has more than $\deg(v)/2$ neighbors in M . Hence in a graph G with minimum degree Δ_{\min} , a coalition of cardinality $1 \leq m \leq \Delta_{\min}/2$ can rule nobody, hence $\Psi_{\text{Ruled}}(G, m, 1) = 0$ in that range, and the problem becomes interesting only for $m > \Delta_{\min}/2$.

Claim 1 (Peleg [55]). *For every graph $G(V, E)$,*

- (i) $\Psi_{\text{Ruled}}(G, m, 1) \leq 2\Delta_{\max}(G)m/\Delta_{\min}(G)$ for every $m \geq 1$, and
- (ii) $\Psi_{\text{Mono}}(G, 1) \geq |E|/\Delta_{\max}$.

Indeed, the constructions described in [44, 8, 9] for small coalitions controlling large sets of vertices are all based on large degree gaps, and utilize sets M of high-degree vertices controlling many low-degree vertices. The last claim clearly implies that in a near-regular graph, an m -vertex coalition can rule no more than $O(m)$ vertices, hence we have the following corollary.

Corollary 3.2 (Peleg [55]). *For every near- Δ -regular or bounded-degree graph G ,*

$$\Psi_{\text{Ruled}}(G, m, 1) = O(m) \quad \text{and} \quad \Psi_{\text{Mono}}(G, 1), \Psi_{\text{Simon}}(G, 1) = \Omega(n).$$

This result, however, is not the best one can expect. The existence of near- Δ -regular graphs G for which the power of small coalitions is even more limited than that is established in [55]. Essentially, these graphs guarantee $\Psi_{\text{Ruled}}(G, m, 1) = O(m/\Delta)$. This is the best one can hope for, as indicated by the following lemma, providing a bound on the possible level of immunity one can expect from a graph.

Lemma 3.3 (Peleg [55]). *For every graph G and every $m \geq 1$,*

$$\Psi_{\text{Ruled}}(G, m, 1) \geq \left\lfloor \frac{2m}{\Delta_{\max}(G) + 1} \right\rfloor.$$

Corollary 3.4 (Peleg [55]). *There is a constant $c_1 > 0$ such that for any $\beta \geq 1$, there does not exist a $(c_1/\Delta_{\max}(G), \beta)$ -immune graph G .*

3.2. Existential proofs and explicit constructions

In view of the above discussion, we next consider regular or near-regular graphs.

Definition 3.5. A graph G is an (a, b) -expander if $|\Gamma(W)| \geq a|W|$ for every subset W of size $|W| \leq bn$.

Strong relationships exist between immune graphs and expanders.

Lemma 3.6 (Peleg [55]). (i) *If a Δ -regular graph G is an (a, b) -expander for $a > 1/2$, then it is (α, β) -immune for $\alpha = 1/(a - \Delta/2)$ and $\beta = (a - \Delta/2)b - \varepsilon$ (for any $\varepsilon > 0$).*

(ii) *If a Δ -regular graph G is (α, β) -immune, then it is an (a, b) -expander for $a = 1/\alpha$ and $b = \alpha\beta$.*

Based on part (i) of the lemma and known results concerning the existence of strong expanders (cf. [39]) we get

Corollary 3.7 (Peleg [55]). *For every sufficiently large constant integer Δ and for every $\alpha > 1/(\Delta/2 - 1)$, there exists some $\beta > 0$ such that a random Δ -regular graph is (α, β) -immune with high probability.*

Definition 3.8. A graph $G(V, E)$ is *near- Δ -regular* if $\Delta/2 \leq \deg(v) \leq 2\Delta$ for every $v \in V$. Let $\mathcal{G}_{n,p}^{REG}$ denote the restriction of the probability distribution $\mathcal{G}_{n,p}$ to near- np -regular graphs.

Proposition 3.9 (Peleg [55]). *There exist constants $c_1, c_2, c_3 > 0$ such that for every $c_1/n \leq p \leq 1$, every graph from $\mathcal{G}_{n,p}^{REG}$ is $(c_2 \log n/np, n/c_3)$ -immune with probability at least $1 - 1/n$.*

Corollary 3.10 (Peleg [55]). (i) *There exists a constant $c_1 > 0$ such that for every $c_1 \log n/n \leq p \leq 1$, every graph from $\mathcal{G}_{n,p}^{REG}$ satisfies $\Psi_{Simon}(G, 1) = \Omega(n)$ (hence also $\Psi_{Mono}(G, 1) = \Omega(n)$) with probability at least $1 - 1/n$.*

(ii) *There exist constants $c_1, c_2, c_3 > 0$ such that for every $c_1/n \leq p \leq c_2 \log n/n$, every graph from $\mathcal{G}_{n,p}^{REG}$ satisfies $\Psi_{Simon}(G, 1) = \Omega(n^2 p / \log n)$ (hence also $\Psi_{Mono}(G, 1) = \Omega(n^2 p / \log n)$) with probability at least $1 - 1/n$.*

The above relationship provides us mainly with existential proofs (for the optimal parameter range, at least). For $\Delta = \Omega(\sqrt{n})$, explicit examples of (asymptotically optimal) immune graphs are established constructively in [55] on the basis of symmetric block designs. A *symmetric block design* $\mathcal{B}(n, \Delta, \lambda)$ is a collection of n blocks, $\mathcal{B} = \{B_1, \dots, B_n\}$, where each block is a size Δ subset of an n -element universe $U = \{u_1, \dots, u_n\}$, each element u_i occurs in exactly Δ blocks, every two blocks B_i and B_j have precisely λ elements in common, and every two elements u_i and u_j occur together in precisely λ blocks. The order q finite projective plane $FPP(q)$ is an example for a symmetric block design with $n = q^2 + q + 1$, $\Delta = q + 1$ and $\lambda = 1$. A comprehensive coverage of block designs, their properties and methods for constructing them can be found in [32].

Given a block design $\mathcal{B} = \mathcal{B}(n, \Delta, \lambda)$, we construct an immune bipartite graph $G(\mathcal{B})$ as follows. Let $G(\mathcal{B}) = (V, W, E)$, where $V = \{v_1, \dots, v_n\}$, $W = \{w_1, \dots, w_n\}$, and $(v_i, w_j) \in E$ iff the element u_i occurs in the block B_j in \mathcal{B} .

Note that a coalition M in this graph can be decomposed into $M_V = M \cap V$ and $M_W = M \cap W$, and their influence is restricted to the other bipartition, namely, $R = Ruled(G, M)$

can be decomposed into $R_V = R \cap V$ and $R_W = R \cap W$ where $R_V = \text{Ruled}(G, M_V)$ and $R_W = \text{Ruled}(G, M_W)$. Therefore, it suffices to concentrate on coalitions restricted to one bipartition. Henceforth, our coalition M will be restricted to V , and the controlled set $\text{Ruled}(G, M)$ will thus be a subset of W .

Lemma 3.11 (Peleg [55]). *For every graph $G(\mathcal{B})$ as above, and for every m -vertex coalition $M \subseteq V$ where $m < n/8$, $\text{Ruled}(G(\mathcal{B}), M, 1) = O(m/\Delta)$.*

Proposition 3.12 (Peleg [55]). *There exist explicitly constructible Δ -regular n -vertex graphs G_n , $\Delta = \Omega(\sqrt{n})$, such that:*

- (i) *For every $m < n/8$, $\Psi_{\text{Ruled}}(G_n, m, 1) = O(m/\Delta)$.*
- (ii) *$\Psi_{\text{Simon}}(G_n, 1), \Psi_{\text{Mono}}(G_n, 1) = \Omega(n)$.*

In particular, using a finite projective plane of order q yields an asymptotically optimally immune regular graph of degree $\Delta = \Theta(\sqrt{n})$. Symmetric block designs with $\lambda > 1$ can be used to derive asymptotically optimally immune regular graphs of degrees higher than \sqrt{n} .

4. Special graph families

We have already noted (in Corollary 3.2) that small coalitions have limited power on regular, near-regular or bounded-degree graphs. In this section we discuss tight bounds for the problem with $r = 1$ on a number of special graph classes, including planar, high girth graphs, diameter-2 graphs, hypercubes and grids.

Given a monopoly M , let $E(M)$ denote the set of edges of G with both endpoints in M .

Lemma 4.1 (Linial et al. [44]). *For every graph $G(V, E)$ and monopoly M , $|E(M)| \geq (|E| - 3|M|)/4$.*

As an immediate consequence we have

Corollary 4.2 (Linial et al. [44]). *For every graph $G(V, E)$, $\Psi_{\text{Mono}}(G, 1) = \Omega(\sqrt{|E|})$.*

These two facts enables us to establish tight bounds on $\Psi_{\text{Mono}}(G, 1)$ for graphs G taken from a number of special classes of graphs. Let us first consider the planar case. Since the subgraph induced by a monopoly M in a planar graph is itself planar, it satisfies $|E(M)| \leq 3|M| - 6$, which by Lemma 4.1 implies that $|M| = \Omega(n)$. Therefore

Proposition 4.3 (Linial et al. [44]). *$\Psi_{\text{Mono}}(G, 1) = \Omega(n)$ for any n -vertex planar graph G .*

A similar argument applies graphs of large girth. A well known result of Erdős states that a graph on m vertices and girth k has at most $m^{1+O(1/k)}$ edges; the estimate is known to be tight (see e.g. [10]). Therefore

Proposition 4.4 (Linial et al. [44]). $\Psi_{Mono}(G, 1) = n^{1-O(1/k)}$ for any n -vertex graph G of girth $\geq k$. The bound is tight.

To see that the bound is tight, consider the following example. Let $K = (M, E')$ be an extremal (maximum possible number of edges) graph on m vertices with girth k . For each vertex $v \in M$ introduce $d(v) - 1$ new vertices (where $d(v)$ is the degree of v), and connect each of these newly created vertices to v . The new graph $G = (V, E)$ has the same girth as K , it has $2E(K) = m^{1+O(1/k)}$ vertices (by the tightness of Erdős' bound), m of which are in the monopoly as required.

Note also that this example is good not only for $r = 1$, but for any $1 \leq r \leq k/2$.

Concerning graphs with diameter 2 we have

Proposition 4.5 (Linial et al. [44]). $\Psi_{Mono}(G, 1) = \Omega(n^{2/3})$ for any n -vertex graph G of diameter 2. The bound is tight.

As an example for the tightness of this bound, consider a projective plane over F_q . There are $q^2 + q + 1$ lines and the same number of points; each line is incident to $q + 1$ points and vice versa. Now, take the incidence graph of these lines and points, add an edge between any two lines, and generate q (disjoint) copies of each point. Take the monopoly M to be the vertices corresponding to lines. There are altogether $(q^2 + q + 1)(q + 1)$ vertices, the graph is of diameter 2 by the properties of the projective plane, and $|M| = q^2 + q + 1$. Also each M -vertex sees $q^2 + q + 1$ M -vertices and $(q + 1)q$ $(V \setminus M)$ -vertices, and each $(V \setminus M)$ -vertex sees $q + 1$ M -vertices and a single $(V \setminus M)$ -vertex, namely itself. The example shows that $|M|$ can indeed be as small as $n^{2/3}$.

Finally, for highly structured graph classes we have tight bounds as well.

Proposition 4.6. (i) $\Psi_{Mono}(G, 1) \geq n/2$ for d -dimensional hypercubes with $d = 2^t - 1$ for integer $t \geq 1$, and $\Psi_{Mono}(G, 1) = \Omega(n)$ for all other values of d .

(ii) $\Psi_{Mono}(G, 1) \geq n/2$ for two-dimensional $m \times k$ grids for $m, k \geq 1$.

5. Computational aspects

Instead of looking for extremal bounds on monopoly size in certain specific graphs, one may ask the following question: given a graph G , find an r -monopoly M of minimum cardinality for G (i.e., a monopoly realizing $\Psi_{Mono}(G, r)$).

This problem can easily be shown NP-hard (say, by a reduction from the minimum dominating set problem). In fact, given recent results [47, 20] on the hardness of approximating the set cover problem and its variants, including the minimum dominating

set problem, it is plausible to make the following conjecture: For any $\varepsilon > 0$, the minimum monopoly problem has no $\ln n - \varepsilon$ approximation unless $\text{NP} \subseteq \text{Dtime}(n^{\log \log n})$.

On the other hand, the following greedy algorithm can be used for obtaining an approximate solution for the problem. Let M denote the intended monopoly, and let W denote the set of uncontrolled vertices. Initially $M = \emptyset$ and $W = V$. While $W \neq \emptyset$, pick a vertex $v \in V \setminus M$ maximizing $|\Gamma(v) \cap W|$, add it to M and remove from W any vertex that is now controlled by M .

Since the problem is submodular, it is easy to see, using an analysis similar to [17, 65], that the resulting monopoly M is at most $\ln |E| + 1$ times greater than the minimum one. Hence we have

Proposition 5.1. *The greedy algorithm yields a ratio $\ln |E| + 1$ approximation for the minimum monopoly problem.*

A similar situation (in terms of hardness and approximability) holds for r -monopolies, as well as for self-ignoring r -monopolies.

6. Sphere covering and packing

Definition 6.1. Given an n -vertex graph G , a collection $\mathcal{C} = \{C_1, \dots, C_t\}$ of neighborhoods in G whose union is V is called a *cover*. The *degree* of a vertex v in the cover (thought of as a hypergraph over V) is denoted by $\deg_{\mathcal{C}}(v)$. The *maximum degree* of \mathcal{C} is defined as $\Delta(\mathcal{C}) = \max_v \{\deg_{\mathcal{C}}(v)\}$.

The low-degree covering problem for a graph G can be formulated as an integer programming problem. Rather than considering the integer program, one may also consider its rational relaxation. In a *weighted cover* of G by a family \mathcal{C} of neighborhoods, one associates a nonnegative weight $\omega_{t,y}$ with all neighborhoods $I_t(y) \in \mathcal{C}$. It is required that for every vertex x , the sum of weights associated with all neighborhoods in \mathcal{C} containing x be at least 1. The *degree* of the weighted cover at a vertex z , denoted $\rho(z)$, is the sum $\sum_{t,y} \omega_{t,y}$ over all neighborhoods $I_t(y) \in \mathcal{C}$ containing z . The maximum degree of the cover is denoted $\Delta(\mathcal{C})$. We ask for weighted covers of least maximal degree.

Proposition 6.2 (Linial et al. [44]). *For every n -vertex graph G and integer $r < n$, G has a weighted cover \mathcal{C} with $\Delta(\mathcal{C}) \leq n^{1/(\lceil \log_3 r \rceil + 1)}$ by neighborhoods with radii in the range $[1, \dots, r]$. All neighborhoods in the cover may be restricted to have a radius which is a power of 3. The degree bound is tight. Moreover, G also has an (integral) cover \mathcal{C}' with the same properties, except*

$$\Delta(\mathcal{C}') \leq n^{1/(\lceil \log_3 r \rceil + 1)} (\ln n + 1).$$

(The claim for integral covers is derived in [44] by a slight variation of [45].)

Next, let us turn to packings.

Definition 6.3. Given an n -vertex graph G , a collection $\mathcal{P} = \{P_1, \dots, P_t\}$ of disjoint neighborhoods in G is called a *packing*. The *volume* of \mathcal{P} is defined as $\mathcal{V}(\mathcal{P}) = \sum_i |P_i|$.

Again, one may consider a *weighted* version of the *packing* problem. Nonnegative weights $\omega_{t,y}$ are associated with neighborhoods $\Gamma_t(y)$ in a family \mathcal{P} of neighborhoods. For every vertex z , the degree $\rho(z)$, namely, the sum of weights associated with neighborhoods containing z should not exceed 1. The sum $\sum \omega_{t,y} |\Gamma_{t,y}|$ over all $\Gamma_t(y) \in \mathcal{P}$ is the *volume* of the fractional packing, denoted $\mathcal{V}(\mathcal{P})$.

Proposition 6.4 (Linial et al. [44]). *For every n -vertex graph G and integer $r < n$, G has a packing \mathcal{P} with volume $\mathcal{V}(\mathcal{P}) \geq n^{1-1/(\lceil \log_2 r \rceil + 1)}$ by neighborhoods with radii in $[1, \dots, r]$. All neighborhoods in the packing may be restricted to have a radius which is a power of 2. Moreover, G also has an (integral) packing \mathcal{P}' with the same properties, except $\mathcal{V}(\mathcal{P}') \geq n^{1-1/(\lceil \log_2 r \rceil + 2)}$. For the integral case, the volume bound is tight.*

There is a strong relationship between packings and voting, allowing us to use bounds on packings in order to derive bounds on monopolies. A basic example for this connection is through the following lemma. Call \mathcal{P} an r -packing if it consists solely of r -neighborhoods.

Lemma 6.5. *For every graph G , integer $r \geq 1$ and r -packing \mathcal{P} in G , $\Psi_{\text{Mono}}(G, r) \geq \frac{1}{2} \mathcal{V}(\mathcal{P})$.*

This lemma follows from the observation that due to the disjointness of the r -neighborhoods in \mathcal{P} , any r -monopoly would have to include at least half of the vertices in each of these neighborhoods.

For instance, the lower bound of Proposition 2.1 on $\Psi_{\text{Mono}}(G, 1)$ readily follows from the simple fact that for every n -vertex graph there exists a 1-packing \mathcal{P} of volume $\mathcal{V}(\mathcal{P}) \geq \sqrt{n}$.

Similarly, Proposition 4.6 follows from Lemma 6.5 and the following observations (cf. [57] for the first).

Lemma 6.6. (i) *The d -dimensional hypercube has a 1-packing \mathcal{P} with $\mathcal{V}(\mathcal{P}) = n$ when $d = 2^t - 1$ for integer $t \geq 1$, and $\mathcal{V}(\mathcal{P}) = \Omega(n)$ in all other cases.*

(ii) *The two-dimensional $m \times k$ grid for $m, k \geq 1$ has a 1-packing \mathcal{P} with $\mathcal{V}(\mathcal{P}) = n$.*

To get similar bounds for self-ignoring monopolies, we need a stronger version of packing, defined next.

Definition 6.7. Given a set of vertices X , a packing \mathcal{P} is said to be X -centered if all the centers of its neighborhoods are from X .

Proposition 6.8 (Bermond and Peleg [8]). *For every n -vertex graph G , set of vertices X and fixed integer r , there exists an X -centered packing \mathcal{P} in G , with neighborhoods of radius at most r , and volume $\mathcal{V}(\mathcal{P}) \geq |X|^{1-1/(\lceil \log_2 r \rceil + 1)}$. All neighborhoods in the packing may be restricted to have a radius which is a power of 2.*

7. A range of polling regions

A somewhat different picture emerges if we strengthen our voting policy, and examine simultaneously *all* i -neighborhoods for a *range* of values of i .

Definition 7.1. A vertex v in a network $G(V, E)$ is said to be $[1, r]$ -controlled by the set M if for every $1 \leq i \leq r$, the majority of the vertices in $I_i(v)$ are in M .

For every integer $1 \leq m \leq n$ and n -vertex graph G , let $\Psi_{Ruled}(G, m, [1, r])$ denote the maximum number of vertices that can be $[1, r]$ -controlled by an m -vertex coalition in the graph G . Dually, let $\Psi_{Mono}(G, [1, r])$ (resp. $\Psi_{Simon}(G, [1, r])$) denote the minimum cardinality of a $[1, r]$ -monopoly (resp. $[1, r]$ -SIMON) in G .

The number of vertices a set M can $[1, r]$ -control was studied in [44].

Proposition 7.2 (Linial et al. [44]). (i) *For every fixed $r \geq 1$ and n -vertex graph, $\Psi_{Mono}(G, [1, r]) = \Omega(n^{1-1/(\lceil \log_2 r \rceil + 2)})$.*

(ii) *For every fixed $r \geq 1$ there exists a family of n -vertex graphs G_n with $\Psi_{Mono}(G_n, [1, r]) = O(n^{1-1/(\lceil \log_2 r \rceil + 2)})$.*

Proposition 7.3 (Bermond and Peleg [8]). (i) *For every fixed integer $r \geq 1$ there exists a family of n -vertex graphs G_n with $\Psi_{Ruled}(G_n, m, [1, r]) = \Theta(|M_n|^{1+1/\lceil \log_2 r \rceil})$.*

(ii) *For every graph G , fixed integer $r \geq 2$ and integer $m \geq 1$, $\Psi_{Ruled}(G, m, [1, r]) = O(m^{1+1/\lceil \log_2 r \rceil})$.*

Proposition 7.4 (Bermond and Peleg [8]). (i) *In every graph G and for every fixed integer $r \geq 1$, $\Psi_{Simon}(G, [1, r]) = \Omega(n^{1-1/(\lceil \log_2 r \rceil + 1)})$.*

(ii) *For every fixed integer $r \geq 1$ there exists a family of n -vertex graphs G_n with $\Psi_{Simon}(G_n, [1, r]) = \Theta(n^{1-1/(\lceil \log_2 r \rceil + 1)})$.*

Note that the results for monopolies and self-ignoring monopolies have a rather similar structure, except “shifted” downwards (cf. Fig. 6). In fact, the lower bounds are proved using similar techniques (based on Propositions 6.4 and 6.8).

Perhaps even more interesting is the fact that the upper bounds of Propositions 7.2 and 7.4 can be demonstrated on the same example graph $G_{t,p}$ (though the required case analysis is slightly different, given that the bound proved is different too).

For integers t, p , construct the graph $G_{t,p}$ as follows. Let $r = 2^{t+1} - 1$. The graph is *leveled*, namely, the vertices are arranged into $2^t + 1$ levels, numbered 1 through $2^t + 1$, with edges connecting only vertices in adjacent levels $\ell, \ell + 1$. Each level $2 \leq \ell \leq 2^t + 1$

max. radius r	$[1, r]$ -monopoly	self-ignoring $[1, r]$ -monopoly
1	$\Theta(n^{1/2})$	$\Theta(1)$
2,3	$\Theta(n^{2/3})$	$\Theta(n^{1/2})$
2^{t-1} to $2^t - 1$	$\Theta(n^{1-1/(t+1)})$	$\Theta(n^{1-1/t})$

Fig. 6. Size comparison of $[1, r]$ -monopolies vs. self-ignoring $[1, r]$ -monopolies.

contains $m = p^t$ vertices, and level 1 contains p^{t+1} vertices. Let X denote the set of vertices on level 1, and let $M = V \setminus X$ be the rest of the vertices. Note that the graph contains $n = (2^t + p)p^t$ vertices, hence fixing t , and taking $p \gg 2^t$, we have $p = \Theta(n^{1/(t+1)})$, so X contains most of the vertices of the graph, and $|M| = \Theta(2^t p^t) = \Theta(n^{1-1/(t+1)})$.

We break the levels into $t + 2$ classes C_i , $0 \leq i \leq t + 1$, as follows. Class C_0 consists of level 1 alone. For $1 \leq i \leq t$, set $C_i = \{2^{i-1} + 1, \dots, 2^i\}$. Finally, class C_{t+1} consists of level $2^t + 1$ alone. In each level $\ell \in C_i$, for $i \geq 1$, we denote the vertices by $v_1^\ell, \dots, v_m^\ell$, and break them into p^{t-i+1} “blocks” of p^{i-1} consecutive vertices each. Thus the vertices of the top class C_{t+1} form a single block of size m , the vertices in class C_t are broken into p smaller blocks (of size m/p), and so on.

The edges connecting two consecutive levels $\ell - 1$ and ℓ are now defined as follows. For $\ell \geq 3$, all the vertices of a particular block $Q = \{v_x^\ell, \dots, v_y^\ell\}$ are connected by a complete bipartite graph (crossbar) to the corresponding vertices at level $\ell - 1$, $\{v_x^{\ell-1}, \dots, v_y^{\ell-1}\}$. (Note that these vertices may either form a single block or be split into p blocks, according to whether level $\ell - 1$ belongs to the same class as ℓ or one class lower.) For $\ell = 2$, each vertex of level 2 has p distinct neighbors at level 1 (i.e., each vertex of X has exactly one neighbor on level 2). See Fig. 7 for an example graph $G_{t,p}$ for $t = 3$, $p = 2$.

The fact that on this graph, the vertices of M form a self-ignoring $[1, 2^{t+1} - 1]$ -monopoly as well as a $[1, 2^t - 1]$ -monopoly is established by a straightforward case analysis.

8. Repetitive polling processes

Certain *dynamic* variants of majority voting problems were studied in the literature, in the context of discrete time dynamical systems. These variants concentrated on a setting in which the nodes of the graph operate in discrete time steps, and at each step, each node computes the majority in its neighborhood, and adapts the resulting value as its own. The typical problems studied in this setting involve the behavior of the resulting sequence of global states (represented as a vector $X_t = (x_t(v_1), \dots, x_t(v_n))$, where $x_t(v_i)$ represents the value at node v_i after time step t). The resulting process is illustrated in Fig. 8.

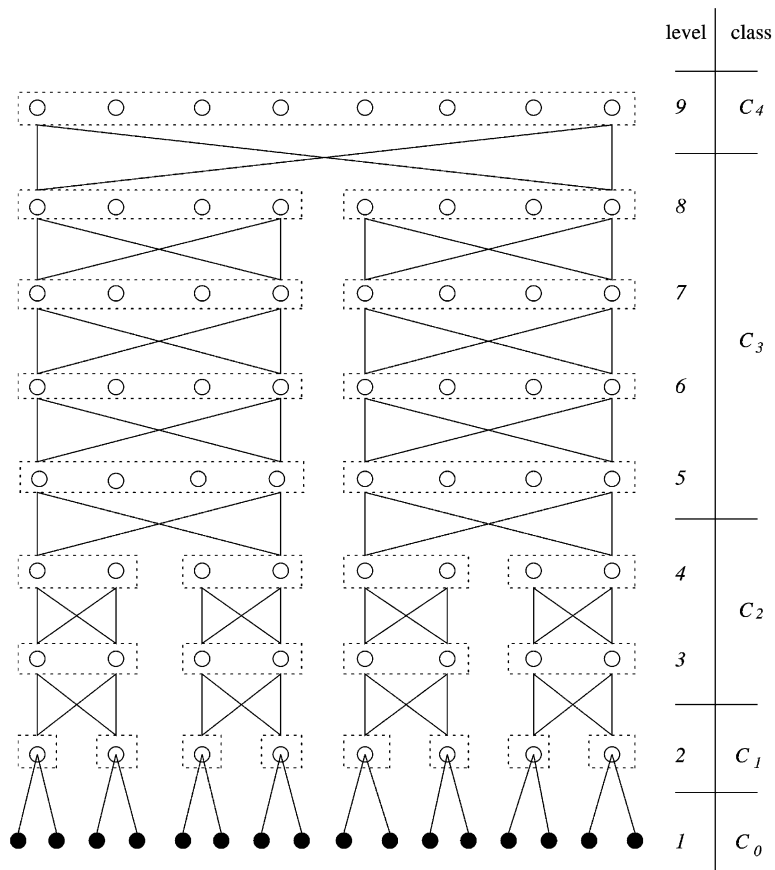


Fig. 7. The graph $G_{t,p}$ for $t=3$ and $p=2$. Dashed lines indicate the partitioning of levels into blocks. The self-ignoring $[1, 2^{t+1} - 1]$ -monopoly (or, $[1, 2^t - 1]$ -monopoly) M consists of all vertices but those of the bottom level.

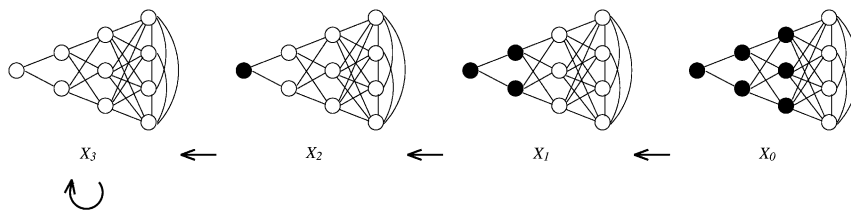


Fig. 8. An example for a multi-round repetitive polling game.

Processes of similar nature can model the influence and flow of information in a variety of different environments, such as societies, genetic processes and distributed multiprocessor systems. In biological and physical systems the prevailing interpretation is that nature operates on the basis of micro rules which cause the macro behavior

observed from outside the system. Hence in all of those cases, it is interesting to understand how the local rules used by the individual participants affect the global dynamic behavior of the system. Indeed, discrete influence systems of this type were studied extensively in areas such as social influence [33, 25, 16] and neural networks [28–30, 59].

8.1. Periodicity behavior

Define the *period* of a repetitive voting process as

$$\text{Period}(X_t) = \min\{\kappa \mid X_{t^*+\kappa} = X_{t^*} \text{ for some } t^*\}.$$

Typical questions dealt with in the literature include the following:

- Is X_t periodic?
- Does it reach a fixed point? (i.e., $\text{Period}(X_t) = 1$).

In finite graphs, the system can have but a finite number of configurations, hence X_t is clearly periodic. In fact, we know the following.

Proposition 8.1 (Goles and Olivos [28]). *For a finite graph, $\text{Period}(X_t)$ is 1 or 2.*

See Fig. 9 for various examples of periodic systems.

This result was extended to more general discrete-time dynamical systems [59, 60], including general threshold functions, and weighted graphs (with a weight function on the edges, $\omega : E \mapsto \mathbb{R}^+$), where

$$x_{t+1}(v) \leftarrow \begin{cases} 1, & \sum_{w \in \Gamma(v)} \omega(w, v) \cdot x_t(w) \geq \theta_0, \\ 0, & \text{otherwise.} \end{cases}$$

Some considered also using multiple colors $\{0, 1, \dots, p\}$ for coloring the vertices, assigning a color for the next time step by the rule

$$C_{t+1}(v) \leftarrow \text{color } j \text{ maximizing } \sum_{w \in \Gamma(v), C_t(w)=j} \omega(w, v).$$

Finally, the question was studied for dynamical systems based on more general gradient mappings. In all of these cases the following was shown.

Proposition 8.2 (Poljak and Sura [59] and Poljak and Turzik [60]). *Assuming symmetric weights ω , the system has period 1 or 2.*

8.2. Stable configurations on the ring

Special attention was given to the problem on the ring, due to its potential applications for modeling biological processes such as the immune system, drug scheduling, gene rearrangement, etc. Consider fixed-point configurations under r -control, namely, colorings that remain invariant after applying majority on r -neighborhoods.

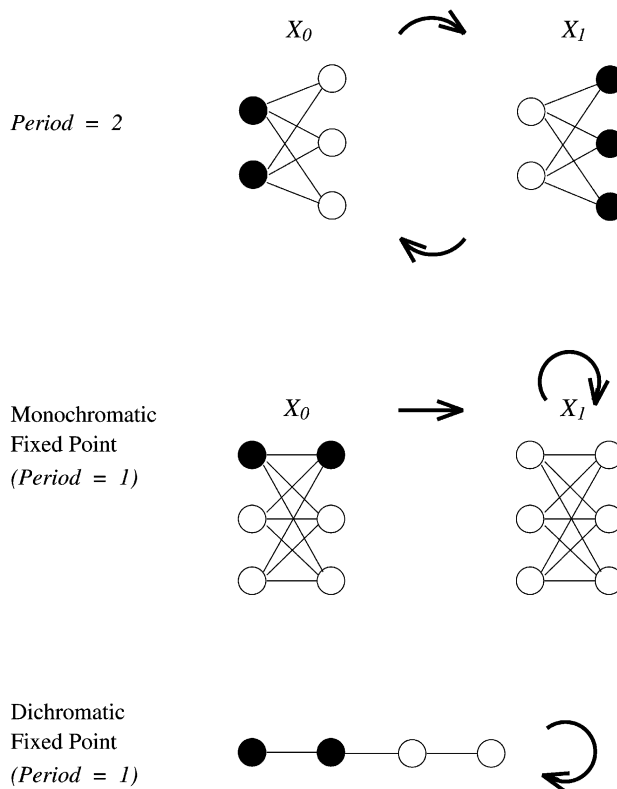


Fig. 9. Examples for repetitive polling games with period 1 or 2.

Definition 8.3. A configuration is *stable* if all its *runs* (maximal segments of identical vote) are longer than r .

Lemma 8.4 (Ager [1]). (i) In an unstable fixed-point configuration, changing a single 0 value to 1 drives the sequence X_t to the all-1 sequence, and vice versa.

(ii) An unstable fixed-point configuration is balanced (i.e., contains an equal number of 0's and 1's).

A number of other problems were studied on the ring. Among these are counting the number of stable fixed-points, done for $r = 1$ [3, 31] and $r > 1$ [2]. The behavior of infinite sequences, and infinite graphs in general, was also studied under the r -majority operator.

For infinite structures the period-2 property is not guaranteed to hold, and conditions sufficient to ensure it were established in [49–51].

8.3. Size bounds

Bounds on the size of monopolies in the dynamic case have also been considered. Here the focus is on cases in which the system reaches a *monochromatic* fixpoint.

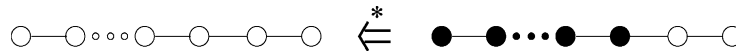


Fig. 10. A dynamo of size 2 for the (PW, SN) model.

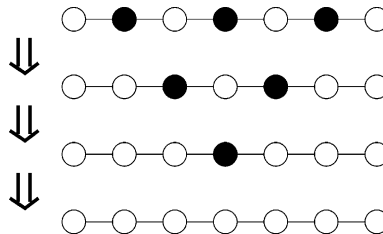


Fig. 11. A nonmonotone dynamo.

A set of vertices M is said to be a *dynamic monopoly*, abbreviated *dynamo*, if starting the game with the vertices of M colored white, the system eventually reaches an all-white global state. Cast in these terms, a monopoly is a dynamo that reaches the final all-white configuration in a single round).

The question of establishing the minimum number of vertices (as a function of n) a dynamic monopoly must contain is raised in [56]. This question seems to be affected by some parameters of the model. In particular, two points were left obscure so far. First, we have not specified how *ties* are to be broken in dynamic majority computations. Two plausible options would be to give priority to one specific color, w.l.o.g. white, or to give priority to the current color of the vertex (i.e., require strict majority in order to *change* the current color). Another (perhaps less well-motivated) option is to give priority to flipping the current color in case of a tie. Let us denote these three options by Prefer-White (PW), Prefer-Current (PC) and Prefer-Flip (PF), respectively. The second parameter concerns the question whether the neighborhood of a vertex v contains v itself or not. Let us denote these two options by Self-Included (SI) and Self-Not-included (SN), respectively. Combinations of these two parameters give rise to a number of possible models, which we will denote by the appropriate pair. (E.g., the (PW, SN) model gives priority to white in case of ties, and computes the majority on the neighborhood including v 's own value.)

Observe that in the (PW, SN) model, there exist graphs with a constant-size dynamo. An example is given in Fig. 10.

Notice that the example of Fig. 10 is *monotone*, in the sense that it never “loses ground”, i.e., once a vertex becomes white, it remains white forever. Formally, letting M_t denote the set of white-colored vertices after round t (with $M_0 = M$), a dynamo is said to be *monotone* if it satisfies $M_t \subseteq M_{t+1}$ for every $t \geq 0$. As another example, the white set in the initial state in Fig. 8 is a monotone dynamo. An example for a nonmonotone dynamo (say, in the (PC, SI) model) is depicted in Fig. 11.

It is shown in [7] that in fact there exist constant-size dynamos in all models.

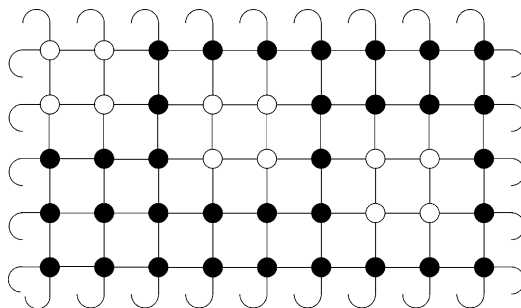


Fig. 12. A 12-node monotone dynamo on the 5×9 toroidal mesh in the (PW,SN) model.

Proposition 8.5 (Berger [7]). *In all models, for every $n \geq 1$, there exists a graph G of n or more vertices with a dynamo of size $O(1)$.*

In contrast, the following is proven in [56] for the limited class of monotone dynamos, in a number of the above models.

Proposition 8.6 (Peleg [56]). *In the (PW,SI), (PC,SI), (PF,SI) and (PC,SN) models, for every n -vertex graph G , every monotone dynamo is of size $\Omega(\sqrt{n})$.*

Near-tight upper and lower bounds on the size of monotone dynamos for various families of tori, including the classical toroidal mesh, the torus cordalis and the torus serpentinus, are derived in [24]. These results reveal marked differences between the (PW,SN) and (PC,SN) models. For instance, in $n \times n$ tori of all three types, the former model allows monotone dynamos of size $\Theta(n)$, whereas the latter one admits only monotone dynamos of size $\Theta(n^2)$. Fig. 12 depicts a 12-node monotone dynamo on the 5×9 toroidal mesh in the (PW,SN) model.

The problem was studied also in the so-called *irreversible* model, in which white nodes are *not allowed* to reverse their color back to black. In the context of fault-tolerant computing, the irreversible model captures situations in which the initial faults are *permanent*. Under such assumptions, situations in which the system converges to a monochromatic state correspond to initial fault distributions that cause the *entire system* to fail.

Characterizations of irreversible dynamos have recently been given for various graph classes, including chordal rings, tori and butterflies [21–24,46]. Fig. 13 depicts a 4-node irreversible dynamo on the 5×5 toroidal mesh in the (PW,SN) model. While most of those bounds are rather tight, there is a wide gap between the upper and lower bounds obtained for the butterfly in [46], and tightening the lower bound remains an interesting open problem.

8.4. Mixing variants

Some natural variants of the problem one may consider are based on mixing the influences, through applying *averaging* instead of majority in each time step. The rule

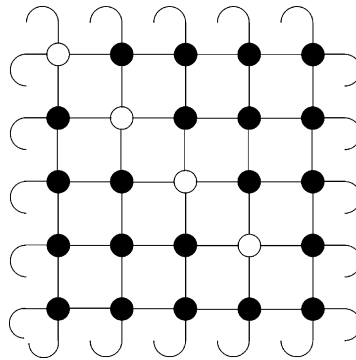


Fig. 13. A 5-node irreversible dynamo on the 5×5 toroidal mesh in the (PW, SN) model.

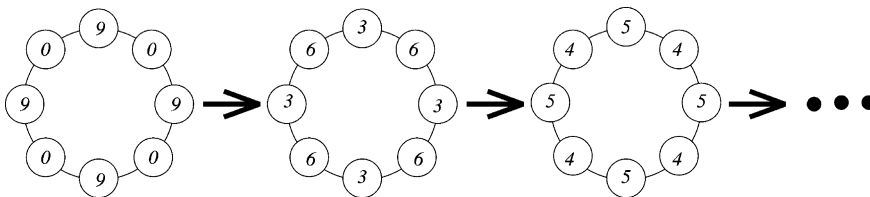


Fig. 14. An example for the application of a local averaging distribution as a tool for load balancing.

to be applied is

$$\begin{aligned}
 x_{t+1} &\leftarrow [\textit{weighted}] \textit{AVERAGE}_{w \in \Gamma(v)} \{x_t(w)\} \\
 &= \frac{\sum_{w \in \Gamma(v)} \omega(w, v) \cdot x_t(w)}{\sum_{w \in \Gamma(v)} \omega(w, v)}.
 \end{aligned}$$

Such a rule may find applications in modeling social influences of opinions on quantitative questions (e.g., deciding on the sum of money to be spent on a certain project). Nonlocal variants of this problem (namely, on a complete graph), with different weights assigned to different opinions, were studied by several authors (cf. [16]). In an entirely different context, such a rule can be used for balancing the workload in a distributed network of servers (see Fig. 14).

8.5. Randomized variants

The repetitive polling process was studied also in the probabilistic model, in which a vertex colors itself according to some probabilistic rule in each round, based on the colors of its neighbors. One natural probabilistic rule requires each vertex to choose at random one of its neighbors and adopt its color. Equivalently, if the vertex has w white neighbors and b black neighbors, it recolors itself white with probability $w/(w + b)$ and black with probability $b/(w + b)$.

This natural process is studied in [34, 35], where the main question addressed is determining the probability of ending up in the all-white state, for any given graph and initial coloring. In fact, the problem is analyzed in the slightly more general *weighted* setting, where real nonnegative weights are assigned to the edges, and the probability to pick a particular neighbor depends on the weight of the edge leading to it. The paper analyzes the underlying Markov chain and proves that the probability of the process ending in the all-white state is proportional to the sum of the edge weights of the initially colored white nodes. In the special unweighted case, this probability is $\sum_{i \in W} d_i / 2m$, where m is the number of edges, W is the set of nodes initially colored white and d_i is the degree of node i . The theorem is generalized also to processes on graphs with multiple colors (instead of two). A similar result, although only for the unweighted case, was obtained independently in [52], using a different proof. Some extremal questions related to such processes are studied in [36].

In [35] it is also observed that this process has applications in the area of consensus protocols (cf. [43, 11, 19, 48, 6]), especially under benign fault models. While the probabilistic polling process is much slower to converge than other (structured) consensus algorithms, and it does not provide termination detection, it is in fact suitable for solving a “repetitive” variant of the consensus problem, in which the variables are occasionally changed by failures or other external forces, and the process must repeatedly bring the system back to a monochromatic view. Put another way, this process is “self-stabilizing” in a weak sense (cf. [5, 40]), i.e., assuming an adversary is allowed to arbitrarily change the colors of some of the nodes, the process will eventually converge back to a monochromatic state, albeit not necessarily to the original one. This type of stability is achieved even in the rather powerful *dynamic network* model of [4].

Another advantage of the probabilistic polling process is that it yields *proportionate* agreement, i.e., the final consensus value is zero or one with probability based on their proportion in the initial inputs.

In addition, some lower and upper bounds are proven in [35] for the size of small monopolies which can bring the process to the all-white state with high probability. It is shown that for reasonably high probabilities ($1 - \varepsilon$ for $\frac{1}{2} \geq \varepsilon \geq 1/n$) the smallest monopolies are of size $\Theta(\sqrt{n/\varepsilon})$. This extremal result indicates that the probabilistic model is not very different from the deterministic model, when considering the size of the smallest monopolies.

Some related results concern a probabilistic asynchronous model called the *voter model*, introduced by Holley and Liggett [38]. This is a continuous time Markov process with a state space consisting of all the 2-colorings of V . The process evolves according to the following mechanism. Attached to each vertex is a clock which rings at an exponentially distributed (unit) rate independently of all other clocks. When its clock rings, the vertex chooses a neighbor at random and adopts its color. This process was extensively studied in infinite grid graphs [38], and the model for arbitrary finite connected graphs is studied in [18].

Among the parameters studied on such models are convergence time (i.e., the time it takes to reach a fixed point) and convergence probability (i.e., the probability to end

up with all nodes colored white, given some initial coloring). Although the underlying Markov chain is quite different from the synchronous probabilistic model of [35], it turns out that the probabilities to end in the all-white state in the two models are the same.

8.6. Other variants

Finally, one may also consider other computational modes. For instance, little is known about the basic properties of the two-party game in which one of the two players attempts to paint the entire graph white and the other black. Each round is divided into two sub-rounds, and the two players alternate in picking one vertex and ordering it to recolor itself (obeying the majority rule).

Another possible model to consider is the asynchronous one, in which a scheduler determines the order in which vertices perform their polls. It is easy to see that in this model, there are graphs and initial small sets that the scheduler can help color the entire graph. For example, the two white vertices of Fig. 1 can color the entire graph white if the scheduler starts the schedule by asking each of the black vertices to perform a poll. On the other hand, it is also easy to see that certain initial sets are “resistant”, or *self-protective*. For example, the two initial sets in the third example in Fig. 9 are stable (say, in the (PC, SI) -model) under any strategy of the scheduler. Again, not much is known about the problem in this model either.

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