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Strongly Gorenstein projective, injective and flat modules ☆

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ABSTRACT

In this paper, we study some properties of strongly Gorenstein projective, injective and flat modules, and we discuss some connections between strongly Gorenstein projective, injective and flat modules, and we consider these properties under change of rings.

1. Introduction

Unless stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let R be a ring. We denote by R-Mod (Mod-R) the category of left (right) R-modules respectively. By $\mathcal{P}(R)$ and $\mathcal{I}(R)$ denote the class of all projective and injective R-modules respectively. For any R-module M, $\mathrm{pd}_R(M)$ denotes the projective dimension of M. The character module $\mathrm{Hom}_Z(M, Q/Z)$ is denoted by M^+.

When R is two-sided noetherian, Auslander and Bridger [2] introduced the G-dimension, $\text{G-dim}_R(M)$ for every finitely generated R-module M. They proved the inequality $\text{G-dim}_R(M) \leqslant \mathrm{pd}_R(M)$, with equality $\text{G-dim}_R(M) = \mathrm{pd}_R(M)$ when $\mathrm{pd}_R(M)$ is finite. Several decades later, Enochs and Jenda [8,9] extended the ideas of Auslander and Bridger and introduced three homological dimensions, called the Gorenstein projective, injective and flat dimensions. These have been studies extensively by their founders and by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others [3,6,9,12,22] over arbitrary associative rings. They proved that these dimensions are similar to the classical homological dimensions; i.e., projective, injective and flat dimensions respectively. D. Bennis and N. Mahdou [5] studied a particular case of Gorenstein projective, injective and flat modules, which they call respectively, strongly Gorenstein projective, injective and flat modules.

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They proved that every Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) module is a direct summand of a strongly Gorenstein projective (resp. strongly Gorenstein injective, strongly Gorenstein flat) module. In this paper, we continue the study of strongly Gorenstein projective, injective and flat modules. In Section 3, we consider these properties under change of rings. Specifically, we consider completions of rings, Morita equivalences, excellent extensions, polynomial extensions and localizations.

We firstly recalled some concepts. Let $\mathcal{X}$ be a class of R-modules. We call $\mathcal{X}$ projectively resolving if $\mathcal{P}(R) \subseteq \mathcal{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. We call $\mathcal{X}$ injectively resolving if $\mathcal{I}(R) \subseteq \mathcal{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X'' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. An R-module M is said to be Gorenstein projective (G-projective for short) if there exists an exact sequence of projective modules

$$\mathbb{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \operatorname{Im}(P_0 \to P^0)$ and such that $\operatorname{Hom}_R(-, Q)$ leaves the sequence $\mathbb{P}$ exact whenever Q is a projective R-module. The exact sequence $\mathbb{P}$ is called a complete projective resolution. The Gorenstein injective (G-injective for short) modules are defined dually. An R-module M is called strongly Gorenstein projective (SG-projective for short) if there exists a complete projective resolution of the form

$$\mathbb{P} = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots$$

such that $M \cong \operatorname{Ker} f$. Every projective module is strongly Gorenstein projective, every strongly Gorenstein projective module is Gorenstein projective. The class of all strongly Gorenstein projective R-modules is denoted by $\mathcal{SGP}(R)$. The strongly Gorenstein injective (SG-injective for short) modules are defined dually. Every injective module is strongly Gorenstein injective, every strongly Gorenstein injective module is Gorenstein injective. The class of all strongly Gorenstein injective R-modules is denoted by $\mathcal{SGI}(R)$. An R-module M is said to be Gorenstein flat (G-flat for short) if there is an exact sequence of flat modules

$$\mathbb{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong \operatorname{Im}(F_0 \to F^0)$ and such that $I \otimes_R -$ leaves the sequence $\mathbb{F}$ exact whenever I is an injective R-module. The exact sequence $\mathbb{F}$ is called a complete flat resolution. An R-module M is called strongly Gorenstein flat (SG-flat for short) if there exists a complete flat resolution of the form

$$\mathbb{F} = \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$$

such that $M \cong \operatorname{Ker} f$. Every flat module is strongly Gorenstein flat, every strongly Gorenstein flat module is Gorenstein flat. The class of all strongly Gorenstein flat R-modules is denoted by $\mathcal{SGF}(R)$.

2. The strongly Gorenstein property

It was shown in [5, Theorem 2.7] that a module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective module. By [5, Example 2.13], {SG-projective modules} $\subsetneq$ {G-projective modules}. Hence direct summands of a strongly Gorenstein projective module need not be strongly Gorenstein projective and the class $\mathcal{SGP}(R)$ of all strongly Gorenstein projective R-modules is not projectively resolving. In fact, assume $\mathcal{SGP}(R)$ is projectively resolving. Let M be a G-projective R-module but not SG-projective. Then there is a G-projective R-module N such that $M \oplus N$ is SG-projective. Set $L = M \oplus N \oplus M \oplus N \oplus \cdots$. Then L is SG-projective by [5, Proposition 2.2]. Consider the exact sequence $0 \to M \to M \oplus N \oplus L \to N \oplus L \to 0$. Since $M \oplus N \oplus L \cong L$ and $N \oplus L \cong L$, we have $0 \to M \to L \to L \to 0$ is exact, and hence M is SG-projective, a contradiction. But we have the following result.

Theorem 2.1. *Let* $0 \to N \to M \to Q \to 0$ *be exact with* Q *projective. Then* N *is SG-projective if and only if* M *is SG-projective.*

Proof. ($\Rightarrow$) If N is SG-projective, then $M \cong N \oplus Q$ is SG-projective by [5, Proposition 2.2]. ($\Leftarrow$) Assume M is SG-projective. There exists an exact sequence $0 \to N \oplus Q \to P \to N \oplus Q \to 0$ with P projective. Consider the pushout of $N \oplus Q \to P$ and $N \oplus Q \to N$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & N \oplus Q & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \longrightarrow & P & \longrightarrow & Q' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N \oplus Q & = & N \oplus Q & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then Q' is G-projective by [12, Theorem 2.5] since N and $N \oplus Q$ are G-projective by [12, Theorem 2.5]. So $\mathrm{Ext}^1_R(Q', Q) = 0$, the sequence $0 \to Q \to P \to Q' \to 0$ splits. Hence Q' is projective. Consider the pullback of $Q' \to N \oplus Q$ and $N \to N \oplus Q$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & \xrightarrow{=} & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Q'' & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & N \oplus Q & \longrightarrow & Q & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $0 \to N \to Q'' \to N \to 0$ is exact and Q'' is projective. Let W be any projective R-module. Then $\mathrm{Ext}^i_R(N, W) = 0$ for all $i \geqslant 1$ since N is G-projective by [12, Theorem 2.5]. It follows that N is SG-projective by [5, Proposition 2.9]. □

By analogy with the proof of Theorem 2.1, we have the following result.

Theorem 2.2. *Let* $0 \to E \to M \to N \to 0$ *be exact with* E *injective. Then* N *is SG-injective if and only if* M *is SG-injective.*

Lemma 2.3. *Let M be a left R-module and P a flat left R-module. Then M is SG-flat if and only if $M \oplus P$ is SG-flat.*

Proof. ($\Rightarrow$) If M is SG-flat, then $M \oplus P$ is SG-flat by [5, Proposition 3.4]. ($\Leftarrow$) Assume $M \oplus P$ is SG-flat. There exists an exact sequence $0 \to M \oplus P \to F \to M \oplus P \to 0$ with F flat. Then $(M \oplus P)^+$ is G-injective by [12, Theorem 3.6], and hence M^+ is G-injective by [12, Theorem 2.6]. Consider the pushout of $M \oplus P \to F$ and $M \oplus P \to M$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & M \oplus P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M \oplus P & = & M \oplus P & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

and consider the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (M \oplus P)^+ & \xrightarrow{=} & (M \oplus P)^+ & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F'^+ & \longrightarrow & F^+ & \longrightarrow & P^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| \downarrow & & \\
0 & \longrightarrow & M^+ & \longrightarrow & (M \oplus P)^+ & \longrightarrow & P^+ & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Then F'^+ is G-injective by [12, Theorem 2.6], and thus $\mathrm{Ext}^1_R(P^+, F'^+) = 0$, the sequence $0 \to F'^+ \to F^+ \to P^+ \to 0$ splits. It follows that F'^+ is injective, and hence F' is flat. Consider the pullback of $F' \to M \oplus P$ and $M \to M \oplus P$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & \xrightarrow{=} & M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F'' & \longrightarrow & F' & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & M \oplus P & \longrightarrow & P & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Then $0 \to M \to F'' \to M \to 0$ is exact and F'' is flat. Let I be any injective right R-module. Then $0 = \mathrm{Tor}_{i+1}^R(I, P) \to \mathrm{Tor}_i^R(I, M) \to \mathrm{Tor}_i^R(I, M \oplus P) = 0$ is exact for all $i \geqslant 1$. Hence $\mathrm{Tor}_i^R(I, M) = 0$ for all $i \geqslant 1$, and therefore M is SG-flat by [5, Proposition 3.6]. □

Theorem 2.4. *Let R be right coherent. Then M is an SG-flat left R-module if and only if M^+ is an SG-injective right R-module.*

Proof. (⇒) There exists an exact sequence $0 \to M \to F \to M \to 0$ in R-Mod with F flat. Then $0 \to M^+ \to F^+ \to M^+ \to 0$ is exact in Mod-R and F^+ is injective. Let I be an injective right R-module. Then $\mathrm{Ext}_R^i(I, M^+) \cong \mathrm{Tor}_i^R(I, M)^+ = 0$ for all $i \geqslant 1$, and hence M^+ is an SG-injective right R-module. (⇐) There exists an exact sequence $0 \to M^+ \to E \to M^+ \to 0$ in Mod-R with E injective. Then there is an injective right R-module E' such that $E \oplus E' = E^{++}$. Let $H = (E' \oplus E)^{\mathbb{N}} \cong (E^{+(\mathbb{N})})^+$. Consider the exact sequence $0 \to M^+ \oplus H \to E \oplus H \oplus H \to M^+ \oplus H \to 0$. Then $0 \to M \oplus E^{+(\mathbb{N})} \to E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})} \to M \oplus E^{+(\mathbb{N})} \to 0$ is exact and $E^{+(\mathbb{N})} \oplus E^{+(\mathbb{N})}$ is flat. Let I be any injective right R-module. Then $\mathrm{Tor}_i^R(I, M \oplus E^{+(\mathbb{N})}) = \mathrm{Tor}_i^R(I, M) \oplus \mathrm{Tor}_i^R(I, E^{+(\mathbb{N})}) = 0$ for all $i \geqslant 1$ since M is G-flat by [12, Theorem 3.6], and thus $M \oplus E^{+(\mathbb{N})}$ is SG-flat. It follows that M is SG-flat by Lemma 2.3. □

Corollary 2.5. *Let R be a commutative coherent ring. Then the following are equivalent*:

(1) *M is SG-flat*;
(2) $\mathrm{Hom}_R(M, E)$ *is SG-injective for all injective R-modules E*;
(3) $\mathrm{Hom}_R(M, E)$ *is SG-injective for any injective cogenerator E for R-*Mod.

Proof. (1) ⇒ (2) By analogy with the proof of Theorem 2.4.

(2) ⇒ (3) is obvious.

(3) ⇒ (1) Since $M^+ \cong \mathrm{Hom}_R(M, R^+)$ is SG-injective, we have M is SG-flat by Theorem 2.4. □

Theorem 2.6. *Let R be right coherent and let $0 \to N \to M \to F \to 0$ be exact with F flat. Then N is SG-flat if and only if M is SG-flat.*

Proof. Use Theorems 2.2 and 2.4. □

Remark 2.7. By analogy with the proof of Theorem 2.1, we can prove that the class of all strongly Gorenstein projective R-modules is closure under direct transfinite extensions.

Let R be a ring and let M, N be left R-modules. Set $T(M) = \{x \in M \mid l_R(x) \neq 0\}$. If $T(M) = 0$, then M is called torsionfree. We denote by τ_N the natural map from $M^* \otimes_R N$ to $\mathrm{Hom}_R(M, N)$ via $\varphi \otimes x \mapsto \tau_N(\varphi \otimes x)(m) = \varphi(m)x$ for any $\varphi \in M^*$, $x \in N$ and $m \in M$, where $M^* = \mathrm{Hom}_R(M, R)$. Recall that an SG-projective module is projective if and only if it has finite projective dimension [12, Proposition 2.27]. It was shown in [5, Proposition 3.7] that an SG-flat module is flat if and only if it has finite flat dimension.

Theorem 2.8. *Let M be a finitely presented torsionfree left R-module. Then the following are equivalent*:

(1) *M is SG-projective*;
(2) *M is SG-flat*;
(3) *The natural map from $M^* \otimes_R M$ to* $\mathrm{Hom}_R(M, M)$ *is an isomorphism*;
(4) *The image of the natural map from $M^* \otimes_R M$ to* $\mathrm{Hom}_R(M, M)$ *contains* Id_M;
(5) *M is projective*;
(6) *M is flat.*

Proof. (1) ⇔ (2) By [5, Proposition 3.9].

(2) ⇒ (3) There exists an exact sequence $0 \to M \to^{f} F \to^{g} M \to 0$ with F flat. Consider the commutative diagram:

$$\begin{array}{ccccccccc} & & M^* \otimes_R M & \xrightarrow{\tau_F M^* \otimes_R f} & M^* \otimes_R F & \xrightarrow{M^* \otimes_R g} & M^* \otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_M} & & \downarrow{\scriptstyle \tau_F} & & \downarrow{\scriptstyle \tau_M} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(M, M) & \xrightarrow{\mathrm{Hom}_R(M,f)} & \mathrm{Hom}_R(M, F) & \xrightarrow{\mathrm{Hom}_R(M,g)} & \mathrm{Hom}_R(M, M). & & \end{array}$$

Let $\varphi \otimes m \in \mathrm{Ker}(M^* \otimes_R f)$. Then for any $m' \in M$, $\tau_F(\varphi \otimes f(m))(m') = f(\varphi(m')m) = 0$. So $\varphi(m')m = 0$, and hence $m = 0$ or $\varphi = 0$ since M is torsionfree. It follows that $\varphi \otimes m = 0$, $M^* \otimes_R f$ is monic, and hence τ_M is an isomorphism since τ_F is an isomorphism by [10, Theorem 3.2.14].

(3) ⇒ (4) and (5) ⇒ (1) are obvious.

(4) ⇔ (5) ⇔ (6) By [15, Theorem 4.19]. □

Proposition 2.9. *Let R be left noetherian. Then every direct limit of finitely generated SG-flat left R-modules is SG-flat.*

Proof. Let $((G_i), (\varphi_{ji}))$ be a direct system over I of finitely generated SG-flat left R-modules. Let $i, j \in I$ with $i \leqslant j$. There are exact sequences $0 \to G_i \to F_i \to G_i \to 0$ and $0 \to G_j \to F_j \to G_j \to 0$ with F_i, F_j flat. Since $\mathrm{Ext}^n_R(G_i, F_j)^+ \cong \mathrm{Tor}^R_n(F_j^+, G_i) = 0$ by [10, Theorem 3.2.13] for all $n \geqslant 1$, then $\mathrm{Ext}^1_R(G_i, F_j) = 0$. Consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_i & \longrightarrow & F_i & \longrightarrow & G_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{ji}} & & \downarrow{\scriptstyle \psi_{ji}} & & \downarrow & & \\ 0 & \longrightarrow & G_j & \longrightarrow & F_j & \longrightarrow & G_j & \longrightarrow & 0. \end{array}$$

Then $((F_i), (\psi_{ji}))$ is a direct system over I. Therefore $0 \to \varinjlim G_i \to \varinjlim F_i \to \varinjlim G_i \to 0$ is exact by [10, Theorem 1.5.6] and $\varinjlim F_i$ is a flat left R-module. Let E be any injective right R-module. Then $\mathrm{Tor}^R_n(E, \varinjlim G_i) \cong \varinjlim \mathrm{Tor}^R_n(E, G_i) = 0$ for all $n \geqslant 1$. Hence $\varinjlim G_i$ is SG-flat by [5, Proposition 3.6]. □

Proposition 2.10. *Let R be a commutative ring and Q a projective R-module. If M is an SG-projective R-module, then $M \otimes_R Q$ is an SG-projective R-module.*

Proof. There is an exact sequence $0 \to M \to P \to M \to 0$ with P projective. Then $0 \to M \otimes_R Q \to P \otimes_R Q \to M \otimes_R Q \to 0$ is exact and $P \otimes_R Q$ is a projective R-module by [21, Ch. 2, $\aleph_1$ Theorem 3]. Let Q' be any projective R-module. Then $\mathrm{Ext}^i_R(M \otimes_R Q, Q') \cong \mathrm{Hom}_R(Q, \mathrm{Ext}^i_R(M, Q')) = 0$ by [18, p. 258, 9.20] for all $i \geqslant 1$. Hence $M \otimes_R Q$ is an SG-projective R-module by [5, Proposition 2.9]. □

Proposition 2.11. *Let K be a field R a commutative K-algebra and suppose that Q is a countably generated free R-module. Then M is an SG-projective R-module if and only if $M \otimes_R Q$ is an SG-projective R-module.*

Proof. (⇒) By Proposition 2.10.

(⇐) There is an exact sequence $0 \to M \otimes_R Q \to P \to M \otimes_R Q \to 0$ with P projective. Consider the pullback of $P \to M \otimes_R Q$ and $M \otimes_R (Q \oplus Q) \to M \otimes_R Q$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M\otimes_R Q & \xrightarrow{=} & M\otimes_R Q & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M\otimes_R Q & \longrightarrow & H & \longrightarrow & M\otimes_R (Q\oplus Q) & \longrightarrow & 0 \\
 & & \| \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M\otimes_R Q & \longrightarrow & P & \longrightarrow & M\otimes_R Q & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then H is SG-projective by Theorem 2.1 and $0 \to M \otimes_R Q \otimes_R Q \to H \otimes_R Q \to P \otimes_R Q \to 0$ is exact. Since Q is countably generated free and $Q \otimes_R R^n \cong (R^n)^{(\mathbb{N})} \cong Q$, we have $Q \otimes_R Q = \varinjlim(Q \otimes_R R^n) \cong Q$. So $0 \to M \otimes_R Q \to H \otimes_R Q \to P \otimes_R Q \to 0$ is exact. Consider the exact sequence $0 \to M \to H \to C \to 0$. Then $C \otimes_R Q \cong P \otimes_R Q$ is projective, and hence C is projective by [21, Ch. 2, $\aleph_1$ Theorem 3]. Thus M is SG-projective by Theorem 2.1. $\square$

Theorem 2.12. *Let R be left artinian and suppose that the injective envelope of every simple left R-module is finitely generated. Then M is an SG-injective left R-module if and only if M^+ is an SG-flat right R-module.*

Proof. ($\Rightarrow$) There exists an exact sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to M^+ \to E^+ \to M^+ \to 0$ is exact and E^+ is a flat right R-module. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [13, Theorem 6.6.4], and hence $\mathrm{Tor}_i^R(M^+, J) \cong \bigoplus_\Lambda \mathrm{Tor}_i^R(M^+, J_\alpha) \cong \bigoplus_\Lambda \mathrm{Ext}_R^i(J_\alpha, M)^+ = 0$ by [10, Theorem 3.2.13] for all $i \geqslant 1$. Therefore M^+ is an SG-flat right R-module.

($\Leftarrow$) There exists an exact sequence $0 \to M^+ \to F \to M^+ \to 0$ in Mod-R with F flat. Then $0 \to M^{++\mathbb{N}} \to F^{+\mathbb{N}} \to M^{++\mathbb{N}} \to 0$ is exact and $F^{+\mathbb{N}}$ is an injective left R-module, and so there is an injective left R-module E such that $F^{+\mathbb{N}} \oplus E = (F^{+\mathbb{N}})^{++}$. Set $L = (F^{+\mathbb{N}} \oplus E)^{\mathbb{N}}$. Then $0 \to M^{++\mathbb{N}} \oplus L \to L \to M^{++\mathbb{N}} \oplus L \to 0$ is exact, and thus $0 \to M \oplus F^{+\mathbb{N}} \to F^{+\mathbb{N}} \to M \oplus F^{+\mathbb{N}} \to 0$ is exact. Let J be any injective left R-module. Then $J = \bigoplus_\Lambda J_\alpha$, where J_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [13, Theorem 6.6.4]. Thus $\mathrm{Ext}_R^i(J_\alpha, M)^+ \cong \mathrm{Tor}_i^R(M^+, J_\alpha) = 0$ by [10, Theorem 3.2.13] for all $i \geqslant 1$ and any $\alpha \in \Lambda$, and hence $\mathrm{Ext}_R^i(J, M) \cong \prod_\Lambda \mathrm{Ext}_R^i(J_\alpha, M) = 0$ for all $i \geqslant 1$. It follows that $M \oplus F^{+\mathbb{N}}$ is an SG-injective left R-module, and so M is an SG-injective left R-module by Theorem 2.2. $\square$

Lemma 2.13. *Let R be left artinian and suppose that the injective envelope of every simple left R-module is finitely generated. Then the class $\mathcal{SGF}(R)$ of all strongly Gorenstein flat right R-modules is closed under arbitrary direct products.*

Proof. Let $M = \prod_{i\in I} M_i$, and $M_i \in \mathcal{SGF}(R)$ for all $i \geqslant 1$. There exists an exact sequence $0 \to M_i \to F_i \to M_i \to 0$ for all $i \geqslant 1$. Then $0 \to \prod_{i\in I} M_i \to \prod_{i\in I} F_i \to \prod_{i\in I} M_i \to 0$ is exact and $\prod_{i\in I} F_i$ is a flat right R-modules. Let E be any injective left R-module. Then $E = \bigoplus_\Lambda E_\alpha$, where E_α is an injective envelope of some simple left R-module for any $\alpha \in \Lambda$ by [13, Theorem 6.6.4]. Thus $\mathrm{Tor}_n^R(\prod_{i\in I} M_i, E) \cong \bigoplus_\Lambda \mathrm{Tor}_n^R(\prod_{i\in I} M_i, E_\alpha) \cong \bigoplus_\Lambda \prod_{i\in I} \mathrm{Tor}_n^R(M_i, E_\alpha) = 0$ by [10, Theorem 3.2.26] for all $n \geqslant 1$. Therefore M is an SG-flat right R-module. $\square$

Corollary 2.14. *Let R be left artinian and suppose that the injective envelope of every simple module is finitely generated. Then the following are equivalent for an (R, S)-bimodule M:*

(1) *M is a G-injective left R-module*;
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E*;
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S;
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F*;
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Proof. (1) ⇒ (2) There is a G-injective left R-module N such that $M \oplus N$ is SG-injective. Let E be any injective right S-module. Then E is isomorphic to a summand of S^{+X} for some set X. So $\mathrm{Hom}_S(M, E)$ is isomorphic to a summand of $\mathrm{Hom}_S(M \oplus N, S^{+X}) \cong (M \oplus N)^{+X}$, and hence $\mathrm{Hom}_S(M, E)$ is a G-flat right R-module by Theorem 2.12, Lemma 2.13 and [5, Theorem 2.7].

(2) ⇒ (3) is obvious.

(3) ⇒ (1) There is a G-injective left R-module N such that $M \oplus N$ is SG-injective. Since $(M \oplus N)^+ \cong \mathrm{Hom}_S(M \oplus N, S^+)$ is an SG-flat right R-module, we have M is a G-injective left R-module by Theorem 2.12 and [5, Theorem 2.7].

(2) ⇒ (4) Let F be any flat left S-module. Then F^+ is an injective right S-module. Hence $(M \otimes_S F)^+ \cong \mathrm{Hom}_S(M, F^+)$ is a G-flat right R-module, and therefore $M \otimes_S F$ is a G-injective left R-module by [12, Theorem 3.6].

(4) ⇒ (5) and (5) ⇒ (1) are obvious. □

A ring R is said to be left V-ring if every simple left R-module is injective. Recall an R-module M is small projective if $\mathrm{Hom}_R(M, -)$ is exact with respect to the exact sequence $0 \to K \to L \to M \to 0$ in R-Mod with $K \ll L$.

Corollary 2.15. *Let R be a left artinian left V-ring. Then the following are equivalent for an (R, S)-bimodule M:*

(1) *M is a G-injective left R-module*;
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E*;
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S;
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F*;
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Corollary 2.16. *Let R be left artinian. If every left R-module is small projective, then the following are equivalent for an (R, S)-bimodule M:*

(1) *M is a G-injective left R-module*;
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E*;
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S;
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F*;
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Corollary 2.17. *Let R be a commutative artinian ring. Then the following are equivalent for an (R, S)-bimodule M:*

(1) *M is a G-injective left R-module*;
(2) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for all injective right S-modules E*;
(3) $\mathrm{Hom}_S(M, E)$ *is a G-flat right R-module for any injective cogenerator E for* Mod-S;
(4) *$M \otimes_S F$ is a G-injective left R-module for all flat left S-modules F*;
(5) *$M \otimes_S F$ is a G-injective left R-module for any faithfully flat left S-module F.*

Proof. If L is a simple R-module, then $E(L)$ is finitely generated by [14, Theorem 3.64]. □

Proposition 2.18. *Let R be a commutative noetherian ring. If M is an SG-flat R-module and Q is a flat R-module, then $M \otimes_R Q$ is an SG-flat R-module.*

Proof. There is an exact sequence $0 \to M \to F \to M \to 0$ with F flat. Then $0 \to M \otimes_R Q \to F \otimes_R Q \to M \otimes_R Q \to 0$ is exact and $F \otimes_R Q$ is a flat R-module by [10, p. 43, Exercise 9]. Let I be any injective R-module and let $\mathbb{F}$ be a flat resolution of I. Then $\mathrm{Tor}_i^R(M \otimes_R Q, I) = \mathrm{H}_i((M \otimes_R Q) \otimes_R \mathbb{F}) \cong \mathrm{H}_i(M \otimes_R (Q \otimes_R \mathbb{F})) = \mathrm{Tor}_i^R(M, Q \otimes_R I) = 0$ for all $i \geqslant 1$ since $Q \otimes_R I$ is an injective R-module by [10, Theorem 3.2.16]. Hence $M \otimes_R Q$ is an SG-flat R-module by [5, Proposition 3.6]. $\square$

Proposition 2.19. *If M is a finitely generated SG-projective right R-module, then $M^* = \mathrm{Hom}_R(M, R)$ is a finitely generated SG-projective left R-module.*

Proof. There exists a complete projective resolution of the form $\mathbb{P} = \cdots \to^f P \to^f P \to^f P \to^f \cdots$ such that $M \cong \mathrm{Ker}\, f$ with P finitely generated projective. Then $\mathbb{P}^* = \cdots \to^{f^*} P^* \to^{f^*} P^* \to^{f^*} P^* \to^{f^*} \cdots$ is exact such that $M^* \cong \mathrm{Ker}\, f^*$ since $\mathrm{Ext}_R^i(M, R) = 0$ for all $i \geqslant 1$, and P^* is finitely generated projective by [1, p. 202, Exercise 8]. Let Q be any projective left R-module. Then $\mathrm{Hom}_R(\mathbb{P}^*, Q) \cong \mathbb{P} \otimes_R Q$ is exact by [1, Proposition 20.11]. Hence M^* is a finitely generated SG-projective left R-module. $\square$

3. Change of rings

In this section, let (R, m) be a commutative local noetherian ring with residue field k and let $E(k)$ be the injective envelope of k. $\hat{R}$, $\hat{M}$ will denote the m-adic completion of a ring R and an R-module M, and M^v will denote the Matlis dual $\mathrm{Hom}_R(M, E(k))$. Esmkhani and Tousi in [11] studied Gorenstein projective and flat modules over a noetherian ring R. For an R-module M, they proved that Gorenstein projective dimension of M is finite if and only if Gorenstein flat dimension of M is finite provided the Krull dimension of R is finite.

Proposition 3.1. *Let (R, m) be a commutative local noetherian ring and M a finitely generated R-module. Then*

(1) *$M \in \mathcal{SGP}(R)$ if and only if $\hat{M} \in \mathcal{SGP}(\hat{R})$.*
(2) *If $\hat{R}$ is a projective R-module and $\hat{M} \in \mathcal{SGP}(\hat{R})$, then $\hat{M} \in \mathcal{SGP}(R)$.*

Proof. (1) ($\Rightarrow$) There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P finitely generated projective. Then $0 \to \hat{M} \to \hat{P} \to \hat{M} \to 0$ is exact in $\hat{R}$-Mod by [10, Theorem 2.5.11]. Since $\mathrm{Ext}_{\hat{R}}^i(\hat{P}, -) \cong \mathrm{Ext}_{\hat{R}}^i(\hat{R} \otimes_R P, -) \cong \mathrm{Hom}_R(P, \mathrm{Ext}_{\hat{R}}^i(\hat{R}, -)) = 0$ by [18, p. 258, 9.20] for all $i \geqslant 1$, then $\hat{P}$ is a projective $\hat{R}$-module. Since $\mathrm{Ext}_{\hat{R}}^i(\hat{M}, \hat{R}) \cong \mathrm{Ext}_{\hat{R}}^i(M \otimes_R \hat{R}, R \otimes_R \hat{R}) \cong \mathrm{Ext}_R^i(M, R) \otimes_R \hat{R} = 0$ by [10, Theorem 3.2.5] for all $i \geqslant 1$, we have $\hat{M} \in \mathcal{SGP}(\hat{R})$ by [5, Proposition 2.12].

($\Leftarrow$) There is an exact sequence $0 \to \hat{M} \to \bar{P} \to \hat{M} \to 0$ in $\hat{R}$-Mod with $\bar{P}$ finitely generated projective. Then $\bar{P} = \hat{R}^n$ for some $n \in \mathbb{N}$ by [20, Theorem 2.5.15]. Consider the exact sequence $0 \to M \to R^n \to C \to 0$. Then $0 \to \hat{C} \to \hat{M} \to 0$ is exact. Consider the exact sequence $0 \to C \to M \to L \to 0$. Then $\hat{L} \cong L \otimes_R \hat{R} = 0$, and hence $L = 0$ since $\hat{R}$ is a faithfully flat R-module. Since $0 = \mathrm{Ext}_{\hat{R}}^i(\hat{M}, \hat{R}) \cong \mathrm{Ext}_R^i(M, R) \otimes_R \hat{R}$ by [10, Theorem 3.2.5], we have $\mathrm{Ext}_R^i(M, R) = 0$ for all $i \geqslant 1$. It follows that $M \in \mathcal{SGP}(R)$ by [5, Proposition 2.12].

(2) There is an exact sequence $0 \to \hat{M} \to \bar{P} \to \hat{M} \to 0$ in $\hat{R}$-Mod with $\bar{P}$ finitely generated projective. Then $\bar{P}$ is a projective R-module since $\bar{P}$ is isomorphic to a summand of $\hat{R}^{(X)}$ for some set X and $\hat{R}^{(X)}$ is a projective R-module. Since $0 = \mathrm{Ext}_{\hat{R}}^i(\hat{M}, \hat{R}) \cong \mathrm{Ext}_R^i(M, R) \otimes_R \hat{R}$ by [10, Theorem 3.2.5], we have $\mathrm{Ext}_R^i(M, R) = 0$ for all $i \geqslant 1$, and thus $\mathrm{Ext}_R^i(\hat{M}, R) \cong \mathrm{Ext}_R^i(\hat{R} \otimes_R M, R) \cong \mathrm{Hom}_R(\hat{R}, \mathrm{Ext}_R^i(M, R)) = 0$ by [18, p. 258, 9.20] for all $i \geqslant 1$. Hence $\hat{M} \in \mathcal{SGP}(R)$ by [5, Proposition 2.12]. $\square$

Proposition 3.2. *Let (R, m) be a commutative local noetherian ring and M an R-module. If $\hat{R}$ is a projective R-module, then*:

(1) *If $M \in \mathcal{SGI}(R)$, then $\mathrm{Hom}_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$.*
(2) *If $\mathrm{Hom}_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$, then $\mathrm{Hom}_R(\hat{R}, M) \in \mathcal{SGI}(R)$.*

Proof. (1) There is an exact sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to \mathrm{Hom}_R(\hat{R}, M) \to \mathrm{Hom}_R(\hat{R}, E) \to \mathrm{Hom}_R(\hat{R}, M) \to 0$ is exact in $\hat{R}$-Mod and $\mathrm{Hom}_R(\hat{R}, E)$ is an injective $\hat{R}$-module by [10, Theorem 3.2.9]. Let $\bar{I}$ be any injective $\hat{R}$-module. Then $\mathrm{Ext}^i_R(H, \bar{I}) \otimes_R \hat{R} \cong \mathrm{Ext}^i_{\hat{R}}(H \otimes_R \hat{R}, \bar{I} \otimes_R \hat{R}) = 0$ by [10, Theorem 3.2.15] for any finitely generated R-module H and all $i \geqslant 1$ since $\bar{I} \otimes_R \hat{R}$ is an injective $\hat{R}$-module by [10, Theorem 3.2.16]. So $\mathrm{Ext}^i_R(H, \bar{I}) = 0$, and hence $\bar{I}$ is an injective R-module. Thus $\mathrm{Ext}^i_{\hat{R}}(\bar{I}, \mathrm{Hom}_R(\hat{R}, M)) \cong \mathrm{Ext}^i_R(\bar{I}, M) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$. It follows that $\mathrm{Hom}_R(\hat{R}, M) \in \mathcal{SGI}(\hat{R})$.

(2) There is an exact sequence $0 \to \mathrm{Hom}_R(\hat{R}, M) \to \bar{E} \to \mathrm{Hom}_R(\hat{R}, M) \to 0$ in $\hat{R}$-Mod with $\bar{E}$ injective. Then $\bar{E}$ is an injective R-module by the proof of (1). Let I be any injective R-module. Then I is isomorphic to a summand of $E(k)^X$ for some set X, and hence $I \otimes_R \hat{R}$ is isomorphic to a summand of $E(k)^X \otimes_R \hat{R} \cong E_{\hat{R}}(\hat{R}/\hat{m})^X \otimes_R \hat{R}$ by [10, Theorem 3.4.1]. It follows that $I \otimes_R \hat{R}$ is an injective $\hat{R}$-module by [10, Theorem 3.2.16]. Hence $\mathrm{Ext}^i_R(I, \mathrm{Hom}_R(\hat{R}, M)) \cong \mathrm{Ext}^i_R(I, \mathrm{Hom}_{\hat{R}}(\hat{R}, \mathrm{Hom}_R(\hat{R}, M))) \cong \mathrm{Ext}^i_{\hat{R}}(I \otimes_R \hat{R}, \mathrm{Hom}_R(\hat{R}, M)) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$. So $\mathrm{Hom}_R(\hat{R}, M) \in \mathcal{SGI}(R)$. □

Proposition 3.3. *Let (R, m) be a commutative local noetherian ring and M an R-module. Then*:

(1) *If $M \in \mathcal{SGF}(R)$, then $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$.*
(2) *If $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$, then $\hat{R} \otimes_R M \in \mathcal{SGF}(R)$.*

Proof. (1) There is a complete flat resolution of the form $\mathbb{F} = \cdots \to^f F \to^f F \to^f F \to^f \cdots$ in R-Mod such that $M \cong \mathrm{Ker}\, f$. Then $\hat{R} \otimes_R \mathbb{F} = \cdots \to^{\hat{R} \otimes_R f} \hat{R} \otimes_R F \to^{\hat{R} \otimes_R f} \hat{R} \otimes_R F \to^{\hat{R} \otimes_R f} \hat{R} \otimes_R F \to^{\hat{R} \otimes_R f} \cdots$ is exact in $\hat{R}$-Mod and $\hat{R} \otimes_R M \cong \mathrm{Ker}(\hat{R} \otimes_R f)$, $\hat{R} \otimes_R F$ is a flat $\hat{R}$-module by [10, p. 43, Exercise 9]. Let $\bar{I}$ be any injective $\hat{R}$-module. Then $\bar{I}$ is an injective R-module by the proof of Proposition 3.2(1). Hence $\bar{I} \otimes_{\hat{R}} (\hat{R} \otimes_R \mathbb{F}) \cong \bar{I} \otimes_R \mathbb{F}$ is exact, and therefore $\hat{R} \otimes_R M \in \mathcal{SGF}(\hat{R})$.

(2) There is a complete flat resolution of the form $\bar{\mathbb{F}} = \cdots \to^{\bar{f}} \bar{F} \to^{\bar{f}} \bar{F} \to^{\bar{f}} \bar{F} \to^{\bar{f}} \cdots$ in $\hat{R}$-Mod such that $\hat{R} \otimes_R M \cong \mathrm{Ker}\, \bar{f}$. Then $\bar{F}$ is a flat R-module. Let I be any injective R-module, Then $I \otimes_R \hat{R}$ is an injective $\hat{R}$-module by the proof of Proposition 3.2. Hence $I \otimes_R \bar{\mathbb{F}} \cong (I \otimes_R \hat{R}) \otimes_{\hat{R}} \bar{\mathbb{F}}$ is exact, and therefore $\hat{R} \otimes_R M \in \mathcal{SGF}(R)$. □

Proposition 3.4. *Let (R, m) be a complete local ring and M a nonzero artinian R-module. Then the following are equivalent*:

(1) *M is an SG-injective R-module*;
(2) *M^v is an SG-projective R-module*;
(3) *$\mathrm{Hom}_R(E(k), M)$ is a nonzero SG-projective R-module.*

Proof. (1) ⇒ (2) There is an exact sequence $0 \to M \to E \to M \to 0$ with E injective. Then $E \oplus E' = E(k)^n$ for some injective R-module E' and some $n \in \mathbb{N}$ by [10, Theorem 3.4.3], and thus $E^v \oplus E'^v = R^n$ by [8, Lemma 4.5] and E'^v is a projective R-module. Consider the exact sequence $0 \to M \oplus E' \to E(k)^n \oplus E' \to M \oplus E' \to 0$. Then $0 \to M^v \oplus E'^v \to R^n \oplus E'^v \to M^v \oplus E'^v \to 0$ is exact with $R^n \oplus E'^v$ projective by [8, Lemma 4.5]. Let Q be any projective R-module. Then $\mathrm{Ext}^i_R(M^v \oplus E'^v, Q) \cong \mathrm{Ext}^i_R(M^v, Q) \oplus \mathrm{Ext}^i_R(E'^v, Q) = 0$ by [8, Theorem 4.8]. Thus $M^v \oplus E'^v$ is SG-projective, and hence M^v is SG-projective by Theorem 2.1.

(2) ⇒ (1) There is an exact sequence $0 \to M^v \to P \to M^v \to 0$ with P finitely generated projective by [10, Theorem 3.4.7]. Then $P = R^m$ for some $m \in \mathbb{N}$ by [20, Theorem 2.5.15], and hence $0 \to M \to E(k)^m \to M \to 0$ is exact by [10, Lemma 3.4.6]. Thus M is SG-injective by [8, Theorem 4.8].

(2) ⇔ (3) We first note that if M^v is SG-projective, then $\mathrm{Hom}_R(E(k), M) \cong (M^v)^* \neq 0$ by [8, Lemma 4.1] since $M^v \neq 0$. Let N be a finitely generated R-module. If N^* is SG-projective, then N is G-projective by the proof of [8, Theorem 4.8] and there exists an exact sequence $0 \to N^* \to P \to N^* \to 0$ with P projective, and hence $0 \to N \to P^* \to N \to 0$ is exact by [7, Theorem 4.2.6] and P^* is projective by [1, p. 202, Exercise 8]. It follows that N is SG-projective iff N^* is SG-projective by Proposition 2.19. Therefore M^v is SG-projective iff $(M^v)^*$ is SG-projective iff $\mathrm{Hom}_R(E(k), M)$ is SG-projective by [8, Lemma 4.1].

(⇐) There is an exact sequence $0 \to M^v \to P \to M^v \to 0$ with P finitely generated projective by [10, Theorem 3.4.7]. Then $P = R^m$ for some $m \in \mathbb{N}$ by [20, Theorem 2.5.15]. Thus $0 \to M \to E(k)^m \to M \to 0$ is exact by [10, Lemma 3.4.6], and so M is SG-injective by [8, Theorem 4.8]. □

Corollary 3.5. *Let (R, m) be a complete local ring and M a nonzero R-module. Then the following are equivalent:*

(1) *M is a finitely generated SG-injective R-module;*
(2) *M is of finite length and M^v is SG-projective;*
(3) *M is of finite length and $\mathrm{Hom}_R(E(k), M)$ is a nonzero SG-projective R-module.*

Proof. By [8, Lemma 4.10] and Proposition 3.4. □

Proposition 3.6. *Let R and S be equivalent rings via equivalences $F : R\text{-Mod} \to S\text{-Mod}$ and $G : S\text{-Mod} \to R\text{-Mod}$. Then*

(1) *$M \in \mathcal{SGP}(R)$ if and only if $F(M) \in \mathcal{SGP}(S)$ for all $M \in R$-Mod;*
(2) *$M \in \mathcal{SGI}(R)$ if and only if $F(M) \in \mathcal{SGI}(S)$ for all $M \in R$-Mod;*
(3) *$M \in \mathcal{SGF}(R)$ if and only if $F(M) \in \mathcal{SGF}(S)$ for all $M \in R$-Mod.*

Proof. (1) (⇒) There is a complete projective resolution of the form $\mathbb{P} = \cdots \to^f P \to^f P \to^f P \to^f \cdots$ in R-Mod such that $M \cong \mathrm{Ker}\, f$. Then $F(\mathbb{P}) = \cdots \to^{F(f)} F(P) \to^{F(f)} F(P) \to^{F(f)} F(P) \to^{F(f)} \cdots$ is exact in S-Mod such that $F(M) \cong \mathrm{Ker}(F(f))$ and $F(P)$ is a projective S-module. Let Q be any projective S-module. Then $\mathrm{Hom}_S(F(\mathbb{P}), Q) \cong \mathrm{Hom}_R(\mathbb{P}, G(Q))$ is exact. Hence $F(M) \in \mathcal{SGP}(S)$.

(⇐) By $GF(M) \cong M$.

(2) and (3) By analogy with the proof of (1). □

Corollary 3.7. *Let R and S be equivalent rings via equivalences $F : R\text{-Mod} \to S\text{-Mod}$ and $G : S\text{-Mod} \to R\text{-Mod}$. Then*

(1) *For all $M \in R$-Mod, ${}_RM$ is G-projective if and only if ${}_SF(M)$ is G-projective;*
(1) *For all $M \in R$-Mod, ${}_RM$ is G-injective if and only if ${}_SF(M)$ is G-injective;*
(3) *For all $M \in R$-Mod, ${}_RM$ is G-flat if and only if ${}_SF(M)$ is G-flat.*

Proof. Easy. □

Corollary 3.8. *Let R be a ring and let $e \in R$ be a nonzero idempotent. If $ReR = R$, then*

(1) *$M \in \mathcal{SGP}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGP}(eRe)$ for all $M \in R$-Mod;*
(2) *$M \in \mathcal{SGP}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGP}(R)$ for all $M \in eRe$-Mod;*
(3) *$M \in \mathcal{SGI}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGI}(eRe)$ for all $M \in R$-Mod;*
(4) *$M \in \mathcal{SGI}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGI}(R)$ for all $M \in eRe$-Mod;*
(5) *$M \in \mathcal{SGF}(R)$ if and only if $eR \otimes_R M \in \mathcal{SGF}(eRe)$ for all $M \in R$-Mod;*
(6) *$M \in \mathcal{SGF}(eRe)$ if and only if $Re \otimes_{eRe} M \in \mathcal{SGF}(R)$ for all $M \in eRe$-Mod.*

Corollary 3.9. *Let R be a ring and let $n \geqslant 1$ be a natural number. Then*

(1) *$M \in \mathcal{SGP}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGP}(M_n(R))$ for all $M \in R$-*Mod;
(2) *$M \in \mathcal{SGP}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGP}(R)$ for all $M \in M_n(R)$-*Mod;
(3) *$M \in \mathcal{SGI}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGI}(M_n(R))$ for all $M \in R$-*Mod;
(4) *$M \in \mathcal{SGI}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGI}(R)$ for all $M \in M_n(R)$-*Mod;
(5) *$M \in \mathcal{SGF}(R)$ if and only if $M_n(R)e_{ii} \otimes_R M \in \mathcal{SGF}(M_n(R))$ for all $M \in R$-*Mod;
(6) *$M \in \mathcal{SGF}(M_n(R))$ if and only if $e_{ii}M_n(R) \otimes_{M_n(R)} M \in \mathcal{SGF}(R)$ for all $M \in M_n(R)$-*Mod,

where e_{ii} is matrix unit for all $i = 1, \ldots, n$.

(1) The ring S is called right R-projective in case for any right S-module M_S with an S-submodule N_S, $N_R|M_R$ implies $N_S|M_S$. For example, every $n \times n$ matrix ring $M_n(R)$ is right R-projective.
(2) The ring extension $S \geqslant R$ is called a finite normalizing extension in case there is a finite subset $\{s_1, \ldots, s_n\}$ of S such that $S = \sum_{i=1}^{n} s_i R$ and $s_i R = R s_i$ for $i = 1, \ldots, n$.
(3) A finite normalizing extension $S \geqslant R$ is called an excellent extension in case condition (1) is satisfied and ${}_RS$, S_R are free modules with a common basis $\{s_1, \ldots, s_n\}$. Excellent extensions were introduced by Passman [16]. Examples include $n \times n$ matrix rings [16], and crossed products $R * G$ where G is a finite group with $|G|^{-1} \in R$ [17].

Proposition 3.10. *Assume that $S \geqslant R$ is an excellent extension. Then*

(a) *${}_RM \in \mathcal{SGP}(R)$ if and only if $S \otimes_R M \in \mathcal{SGP}(S)$ for all $M \in R$-*Mod;
(b) *${}_RM \in \mathcal{SGI}(R)$ if and only if* $\mathrm{Hom}_R(S, M) \in \mathcal{SGI}(S)$ *for all $M \in R$-*Mod;
(c) *$M_R \in \mathcal{SGF}(R)$ if and only if $M \otimes_R S \in \mathcal{SGF}(S)$ for all $M \in$* Mod-R.

Proof. (a) ($\Rightarrow$) There exists an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P projective. Then $0 \to S \otimes_R M \to S \otimes_R P \to S \otimes_R M \to 0$ is exact in S-Mod with $S \otimes_R P$ projective. Let $\bar{Q}$ be any projective left S-module. Then $\bar{Q}$ is a projective left R-module, and so $\mathrm{Ext}_S^i(S \otimes_R M, \bar{Q}) \cong \mathrm{Ext}_R^i(M, \bar{Q}) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$. It follows that $S \otimes_R M \in \mathcal{SGP}(S)$.

($\Leftarrow$) There exists an exact sequence $0 \to S \otimes_R M \to \bar{P} \to S \otimes_R M \to 0$ in S-Mod with $\bar{P}$ projective. Then there is a projective left S-module $\bar{P}'$ such that $\bar{P} \oplus \bar{P}' = S \otimes_R \bar{P}$. Set $L = (\bar{P} \oplus \bar{P}')^{(\mathbb{N})}$. Consider the exact sequence $0 \to (S \otimes_R M) \oplus L \to \bar{P} \oplus L \oplus L \to (S \otimes_R M) \oplus L \to 0$. Then $0 \to S \otimes_R (M \oplus \bar{P}^{(\mathbb{N})}) \to S \otimes_R \bar{P}^{(\mathbb{N})} \to S \otimes_R (M \oplus \bar{P}^{(\mathbb{N})}) \to 0$ is exact, and so $0 \to M \oplus \bar{P}^{(\mathbb{N})} \to \bar{P}^{(\mathbb{N})} \to M \oplus \bar{P}^{(\mathbb{N})} \to 0$ is exact in R-Mod with $\bar{P}^{(\mathbb{N})}$ projective since S is a faithfully flat R-module. Let Q be any projective left R-module. Then $S \otimes_R Q$ is a projective left S-module. Thus $0 = \mathrm{Ext}_S^i(S \otimes_R M, S \otimes_R Q) \cong \mathrm{Ext}_R^i(M, S \otimes_R Q)$ by [18, p. 258, 9.21], and so $\mathrm{Ext}_R^i(M, Q) = 0$ for all $i \geqslant 1$ since Q is isomorphic to a summand of $S \otimes_R Q$. It follows that $M \in \mathcal{SGP}(R)$.

(b) ($\Rightarrow$) There exists an exact sequence $0 \to M \to E \to M \to 0$ in R-Mod with E injective. Then $0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, E) \to \mathrm{Hom}_R(S, M) \to 0$ is exact in S-Mod with $\mathrm{Hom}_R(S, E)$ injective. Let $\bar{I}$ be any injective left S-module. Then $\bar{I}$ is an injective left R-module, and thus $\mathrm{Ext}_S^i(\bar{I}, \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}_R^i(\bar{I}, M) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$. Hence $\mathrm{Hom}_R(S, M) \in \mathcal{SGP}(S)$.

($\Leftarrow$) There exists an exact sequence $0 \to \mathrm{Hom}_R(S, M) \to \bar{E} \to \mathrm{Hom}_R(S, M) \to 0$ in S-Mod with $\bar{E}$ injective. Then there is an injective left S-module $\bar{E}'$ such that $\bar{E} \oplus \bar{E}' = \mathrm{Hom}_R(S, \bar{E})$. Set $H = (\bar{E} \oplus \bar{E}')^{\mathbb{N}}$. Consider the exact sequence $0 \to \mathrm{Hom}_R(S, M) \oplus H \to \bar{E} \oplus H \oplus H \to \mathrm{Hom}_R(S, M) \oplus H \to 0$. Then $0 \to \mathrm{Hom}_R(S, M \oplus \bar{E}^{\mathbb{N}}) \to \mathrm{Hom}_R(S, \bar{E}^{\mathbb{N}}) \to \mathrm{Hom}_R(S, M \oplus \bar{E}^{\mathbb{N}}) \to 0$ is exact, and so $0 \to M \oplus \bar{E}^{\mathbb{N}} \to \bar{E}^{\mathbb{N}} \to M \oplus \bar{E}^{\mathbb{N}} \to 0$ is exact in R-Mod with $\bar{E}^{\mathbb{N}}$ injective. Let I be any injective left R-module. Then $\mathrm{Hom}_R(S, I)$ is an injective left S-module. Thus $0 = \mathrm{Ext}_S^i(\mathrm{Hom}_R(S, I), \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}_R^i(\mathrm{Hom}_R(S, I), M)$ by [18, p. 258, 9.21], and so $\mathrm{Ext}_R^i(I, M) = 0$ for all $i \geqslant 1$ since I is isomorphic to a summand of $\mathrm{Hom}_R(S, I)$. Hence $M \in \mathcal{SGI}(R)$.

(c) ($\Rightarrow$) There exists an exact sequence $0 \to M \to F \to M \to 0$ in Mod-R with F flat. Then $0 \to M \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0$ is exact in Mod-S with $F \otimes_R S$ flat. Let $\bar{I}$ be any injective left S-

module and let $\mathbb{F}$ be a flat resolution of $\bar{I}$. Then $\mathrm{Tor}_i^S(M \otimes_R S, \bar{I}) = \mathrm{H}_i((M \otimes_R S) \otimes_S \mathbb{F}) \cong \mathrm{H}_i(M \otimes_R \mathbb{F}) = \mathrm{Tor}_i^R(M, \bar{I}) = 0$ for all $i \geqslant 1$, and so $M \otimes_R S \in \mathcal{SGF}(S)$.

($\Leftarrow$) There exists an exact sequence $0 \to M \otimes_R S \to \bar{F} \to M \otimes_R S \to 0$ in Mod-S with $\bar{F}$ flat. Then there is a flat right S-module $\bar{F}'$ such that $\bar{F} \oplus \bar{F}' = \bar{F} \otimes_R S$. Set $L = (\bar{F} \oplus \bar{F}')^{(\mathbb{N})}$. Then $0 \to M \oplus \bar{F}^{(\mathbb{N})} \to \bar{F}^{(\mathbb{N})} \to M \oplus \bar{F}^{(\mathbb{N})} \to 0$ is exact in Mod-R with $\bar{F}^{(\mathbb{N})}$ flat by analogy with the proof of (a). Let I be any injective left R-module. Then $\mathrm{Hom}_R(S, I)$ is an injective left S-module. Let $\mathbb{F}$ be a flat resolution of M over R. Then $0 = \mathrm{Tor}_i^S(M \otimes_R S, \mathrm{Hom}_R(S, I)) = \mathrm{H}_i((\mathbb{F} \otimes_R S) \otimes_S \mathrm{Hom}_R(S, I)) \cong \mathrm{H}_i(\mathbb{F} \otimes_R \mathrm{Hom}_R(S, I)) = \mathrm{Tor}_i^R(M, \mathrm{Hom}_R(S, I))$ for all $i \geqslant 1$, and so $\mathrm{Tor}_i^R(M, I) = 0$. Hence $M \in \mathcal{SGF}(R)$. $\square$

Corollary 3.11. *Let $R * G$ be a crossed product, where G is a finite group with $|G|^{-1} \in R$. Then*:

(a) *For any $M \in (R * G)$-Mod, ${}_RM$ is SG-projective if and only if $(R * G) \otimes_R M$ is SG-projective*;
(b) *For any $M \in (R * G)$-*Mod*, ${}_RM$ is SG-injective if and only if* $\mathrm{Hom}_R(R * G, M)$ *is SG-injective*;
(c) *For any $M \in$* Mod-$(R * G)$, *M_R is SG-flat if and only if $M \otimes_R (R * G)$ is SG-flat.*

Corollary 3.12. *Let R be a ring n any positive integer. Then*:

(a) *For any $M \in M_n(R)$*-Mod, *${}_RM$ is SG-projective if and only if $M_n(R) \otimes_R M$ is SG-projective*;
(b) *For any $M \in M_n(R)$*-Mod, *${}_RM$ is SG-injective if and only if* $\mathrm{Hon}_R(M_n(R), M)$ *is SG-injective*;
(c) *For any $M \in$* Mod-$M_n(R)$, *M_R is SG-flat if and only if $M \otimes_R M_n(R)$ is SG-flat.*

Proposition 3.13. *Let R be a ring and a a central nonzero divisor. Let M be a finitely generated R-module on which a acts simply, that is, such that $ax = 0$, $x \in M$ implies $x = 0$. Set $\bar{R} = R/Ra$ and $\bar{M} = M/aM$. If M is an SG-projective left R-module, then $\bar{M}$ is an SG-projective left $\bar{R}$-module.*

Proof. There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P finitely generated projective. Then $0 \to \bar{M} \to \bar{P} \to \bar{M} \to 0$ is exact in $\bar{R}$-Mod since $\mathrm{pd}_R(\bar{R}) \leqslant 1$, and $\bar{P}$ is a projective $\bar{R}$-module by [15, Exercise 2]. Let $-^\natural = \mathrm{Hom}_{\bar{R}}(-, \bar{R})$. Consider the exact sequence $0 \to Ra \to R \to \bar{R} \to 0$. Then $0 \to \bar{R} \to R^\natural \to Ra^\natural \to 0$ is exact and $0 \to Ra \otimes_R M \to M \to \bar{R} \otimes_R M \to 0$ is exact. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
M^\natural & \longrightarrow & (Ra \otimes_R M)^\natural & \longrightarrow & \mathrm{Ext}^1_{\bar{R}}(\bar{R} \otimes_R M, \bar{R}) & \longrightarrow & \mathrm{Ext}^1_{\bar{R}}(M, \bar{R}) & \longrightarrow & \mathrm{Ext}^1_{\bar{R}}(Ra \otimes_R M, \bar{R}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathrm{Hom}_R(M, R^\natural) & \longrightarrow & \mathrm{Hom}_R(M, Ra^\natural) & \longrightarrow & \mathrm{Ext}^1_R(M, \bar{R}) & \longrightarrow & \mathrm{Ext}^1_R(M, R^\natural) & \longrightarrow & \mathrm{Ext}^1_R(M, Ra^\natural).
\end{array}$$

Then $\mathrm{Ext}^1_{\bar{R}}(\bar{M}, \bar{R}) \cong \mathrm{Ext}^1_{\bar{R}}(\bar{R} \otimes_R M, \bar{R}) \cong \mathrm{Ext}^1_R(M, \bar{R}) = 0$, and hence $\bar{M}$ is an SG-projective left $\bar{R}$-module by [5, Proposition 2.12]. $\square$

If R is a ring, then $R[x]$ is the polynomial ring. If M is a left R-module, write $M[x] = R[x] \otimes_R M$. Since $R[x]$ is a free R-module and since tensor product commutes with sums, we may regard the elements of $M[x]$ as 'Vectors' $(x^i \otimes_R m_i)$, $i \geqslant 0$, $M_i \in M$ with almost all $m_i = 0$.

Proposition 3.14. *Let R be a commutative ring. If M is an SG-projective R-module, then $M[x]$ is an SG-projective $R[x]$-module.*

Proof. There is an exact sequence $0 \to M \to P \to M \to 0$ in R-Mod with P projective. So $0 \to M[x] \to P[x] \to M[x] \to 0$ is exact in $R[x]$-Mod and $P[x]$ is a projective $R[x]$-module. Let Q be any projective $R[x]$-module. Then $Q[x] \cong R[x] \otimes_R Q \cong R^{(\mathbb{N})} \otimes_R Q \cong Q^{(\mathbb{N})}$. Hence $Q[x]$ is a projective

$R[x]$-module, and so Q is a projective R-module by [15, Proposition 5.11]. Thus $\mathrm{Ext}^i_{R[x]}(M[x], Q) \cong \mathrm{Ext}^i_R(M, Q) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$, and hence $M[x]$ is an SG-projective $R[x]$-module. □

Corollary 3.15. *Let K be a field, R a commutative noetherian K-algebra and M a finitely generated R-module. Then M is an SG-projective R-module if and only if $M[x]$ is an SG-projective $R[x]$-module.*

Proof. (⇒) By Proposition 3.14.

(⇐) There is an exact sequence $0 \to M[x] \to \bar{P} \to M[x] \to 0$ in $R[x]$-Mod with $\bar{P}$ projective. Then $\bar{P}$ is a projective R-module by the proof of Proposition 3.14. Since $\mathrm{Ext}^i_R(M[x], R) \otimes_R R[x] \cong \mathrm{Ext}^i_R(R[x] \otimes_R M, R) \otimes_R R[x] \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_R(R[x], R)) \otimes_R R[x] \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_R(R[x], R) \otimes_R R[x]) \cong \mathrm{Ext}^i_R(M, R[x])^{\mathbb{N}} \cong \mathrm{Ext}^i_R(M, \mathrm{Hom}_{R[x]}(R[x], R[x]))^{\mathbb{N}} \cong \mathrm{Ext}^i_{R[x]}(M[x], R[x])^{\mathbb{N}} = 0$ by [18, p. 258, 9.21] and [10, Theorem 3.2.15] and $R[x]$ is a countably generated free R-module for all $i \geqslant 1$, we have $M[x] \cong M \otimes_R R[x]$ is an SG-projective R-module by [5, Proposition 2.12], and hence M is SG-projective by Proposition 2.11. □

Let R be a commutative ring and S a multiplicatively closed set of R. Then $S^{-1}R = (R \times S)/\!\sim\, = \{a/s \mid a \in R,\ s \in S\}$ is a ring and $S^{-1}M = (M \times S)/\!\sim\, = \{x/s \mid x \in M,\ s \in S\}$ is an $S^{-1}R$-module. If P is a prime ideal of R and $S = R - P$, then we will denote $S^{-1}M$, $S^{-1}R$ by M_P, R_P respectively. The spectrum of R is denoted by $\mathrm{Spec}(R)$ and the maximal spectrum of R is denoted by $\mathrm{Max}(R)$.

Lemma 3.16. *Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then $\bar{A}$ is a projective R-module if and only if $\bar{A}$ is a projective $S^{-1}R$-module for any $\bar{A} \in S^{-1}R$-*Mod.

Proof. (⇒) Since $\bar{A} \cong S^{-1}\bar{A}$ by [15, Proposition 5.17], so $\bar{A}$ is a projective $S^{-1}R$-module by [20, Proposition 2.5.10].

(⇐) Since $\bar{A}$ is isomorphic to a summand of $S^{-1}R^{(X)}$ for some set X, we have $\bar{A}$ is a projective R-module. □

Proposition 3.17. *Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then*:

(1) *If A is an SG-projective R-module, then $S^{-1}A$ is an SG-projective $S^{-1}R$-module*;
(2) *If $S^{-1}R$ is a finitely generated R-module, then $\bar{B}$ is an SG-projective R-module if and only if $\bar{B}$ is an SG-projective $S^{-1}R$-module for any $\bar{B} \in S^{-1}R$-*Mod.

Proof. (1) There is an exact sequence $0 \to A \to P \to A \to 0$ in R-Mod with P projective. Then $0 \to S^{-1}A \to S^{-1}P \to S^{-1}A \to 0$ is exact in $S^{-1}R$-Mod and $S^{-1}P$ is a projective $S^{-1}R$-module. Let $\bar{Q}$ be any projective $S^{-1}R$-module. Then $\bar{Q}$ is a projective R-module by Lemma 3.16. So $\mathrm{Ext}^i_{S^{-1}R}(S^{-1}A, \bar{Q}) \cong \mathrm{Ext}^i_{S^{-1}R}(S^{-1}R \otimes_R A, \bar{Q}) \cong \mathrm{Ext}^i_R(A, \bar{Q}) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$. Hence $S^{-1}A$ is an SG-projective $S^{-1}R$-module.

(2) (⇒) By (1), since $\bar{B} \cong S^{-1}\bar{B}$ by [15, Proposition 5.17].

(⇐) There is an exact sequence $0 \to \bar{B} \to \bar{P} \to \bar{B} \to 0$ in $S^{-1}R$-Mod with $\bar{P}$ projective. Then $\bar{P}$ is a projective R-module by Lemma 3.16. Let Q be any projective R-module. Then $\mathrm{Hom}_R(S^{-1}R, Q)$ is a projective $S^{-1}R$-module since $S^{-1}R$ is a finitely generated projective R-module by Lemma 3.16. So $\mathrm{Ext}^i_R(\bar{B}, Q) \cong \mathrm{Ext}^i_R(S^{-1}R \otimes_{S^{-1}R} \bar{B}, Q) \cong \mathrm{Ext}^i_{S^{-1}R}(\bar{B}, \mathrm{Hom}_R(S^{-1}R, Q)) = 0$ by [15, Proposition 5.17] and [18, p. 258, 9.21] for all $i \geqslant 1$, and hence $\bar{B}$ is an SG-projective R-module. □

Proposition 3.18. *Let R be a commutative noetherian ring and S a multiplicatively closed set of R. If $\bar{B}$ is a finitely generated SG-projective $S^{-1}R$-module, then $\bar{B}$ is an SG-flat R-module.*

Proof. There is an exact sequence $0 \to \bar{B} \to \bar{P} \to \bar{B} \to 0$ in $S^{-1}R$-Mod with $\bar{P}$ finitely generated projective. Then $\bar{P}$ is a flat R-module by [15, Theorem 5.18]. Let I be any injective R-module. Then $0 =$

$\mathrm{Hom}_{S^{-1}R}(\mathrm{Ext}^i_{S^{-1}R}(\bar{B}, S^{-1}R), S^{-1}I) \cong \mathrm{Tor}_i^{S^{-1}R}(S^{-1}I, \bar{B}) \cong \mathrm{Tor}_i^R(I, \bar{B}) \otimes_R S^{-1}R$ by [10, Theorem 3.2.13], and hence $\mathrm{Tor}_i^R(I, \bar{B}) = 0$ by [19, Condition O_r] for all $i \geqslant 1$. So $\bar{B}$ is an SG-flat R-module. □

Proposition 3.19. *Let R be a commutative ring and S a multiplicatively closed set of R. If $S^{-1}R$ is a projective R-module, then*:

(1) *If A is an SG-injective R-module, then $\mathrm{Hom}_R(S^{-1}R, A)$ is an SG-injective $S^{-1}R$-module*;
(2) *For any $B \in R$-Mod, $\mathrm{Hom}_R(S^{-1}R, B)$ is an SG-injective R-module if and only if $\mathrm{Hom}_R(S^{-1}R, B)$ is an SG-injective $S^{-1}R$-module.*

Proof. (1) There is an exact sequence $0 \to A \to E \to A \to 0$ in R-Mod with E injective. Then $0 \to \mathrm{Hom}_R(S^{-1}R, A) \to \mathrm{Hom}_R(S^{-1}R, E) \to \mathrm{Hom}_R(S^{-1}R, A) \to 0$ is exact in $S^{-1}R$-Mod and $\mathrm{Hom}_R(S^{-1}R, E)$ is an injective $S^{-1}R$-module by [10, Theorem 3.2.9]. Let $\bar{I}$ be any injective $S^{-1}R$-module. Then $\bar{I}$ is an injective R-module by [4, Lemma 1.2]. So $\mathrm{Ext}^i_{S^{-1}R}(\bar{I}, \mathrm{Hom}_R(S^{-1}R, A)) \cong \mathrm{Ext}^i_R(\bar{I}, A) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$, and hence $\mathrm{Hom}_R(S^{-1}R, A)$ is an SG-injective $S^{-1}R$-module.

(2) ($\Rightarrow$) is obvious.

($\Leftarrow$) There is an exact sequence $0 \to \mathrm{Hom}_R(S^{-1}R, B) \to \bar{E} \to \mathrm{Hom}_R(S^{-1}R, B) \to 0$ in $S^{-1}R$-Mod with $\bar{E}$ injective. Then $\bar{E}$ is an injective R-module. Let I be any injective R-module. Then $S^{-1}I$ is an injective $S^{-1}R$-module. So $\mathrm{Ext}^i_R(I, \mathrm{Hom}_R(S^{-1}R, B)) \cong \mathrm{Ext}^i_R(I, \mathrm{Hom}_{S^{-1}R}(S^{-1}R, \mathrm{Hom}_R(S^{-1}R, B))) \cong \mathrm{Ext}^i_{S^{-1}R}(S^{-1}I, \mathrm{Hom}_R(S^{-1}R, B)) = 0$ by [18, p. 258, 9.21] for all $i \geqslant 1$, and hence $\mathrm{Hom}_R(S^{-1}R, B)$ is an SG-injective R-module. □

Proposition 3.20. *Let R be a commutative ring and S a multiplicatively closed set of R. Then*:

(a) *If A is an SG-flat R-module, then $S^{-1}A$ is an SG-flat R-module for any $A \in R$-Mod*;
(b) *If A is an SG-flat R-module, then $S^{-1}A$ is an SG-flat $S^{-1}R$-module for any $A \in R$-Mod*;
(c) *For any $\bar{B} \in S^{-1}R$-Mod, $\bar{B}$ is an SG-flat R-module if and only if $\bar{B}$ is an SG-flat $S^{-1}R$-module.*

Proof. (a) There is a complete flat resolution of the form $\mathbb{F} = \cdots \to^f F \to^f F \to^f F \to^f \cdots$ in R-Mod such that $A \cong \mathrm{Ker}\, f$. Then $S^{-1}\mathbb{F} = \cdots \to^{S^{-1}f} S^{-1}F \to^{S^{-1}f} S^{-1}F \to^{S^{-1}f} S^{-1}F \to^{S^{-1}f} \cdots$ is exact such that $S^{-1}A \cong \mathrm{Ker}(S^{-1}f)$ and $S^{-1}F$ is a flat $S^{-1}R$-module. Hence $S^{-1}F$ is a flat R-module. Let I be any injective R-module. Then $I \otimes_R S^{-1}\mathbb{F} \cong S^{-1}I \otimes_R \mathbb{F}$ is exact by [15, Proposition 5.17] since $S^{-1}I$ is an injective R-module by [4, Lemma 1.2]. Hence $S^{-1}A$ is an SG-flat R-module.

(b) There is an exact sequence $0 \to A \to F \to A \to 0$ in R-Mod with F flat. Then $0 \to S^{-1}A \to S^{-1}F \to S^{-1}A \to 0$ is exact in $S^{-1}R$-Mod and $S^{-1}F$ is a flat $S^{-1}R$-module. Let $\bar{I}$ be any injective $S^{-1}R$-module. Then $\bar{I}$ is an injective R-module by [4, Lemma 1.2]. So $\mathrm{Tor}_i^{S^{-1}R}(\bar{I}, S^{-1}A) \cong \mathrm{Tor}_i^R(\bar{I}, A) \otimes_R S^{-1}R = 0$ for all $i \geqslant 1$, and hence $S^{-1}A$ is an SG-flat $S^{-1}R$-module.

(c) ($\Rightarrow$) By (b).

($\Leftarrow$) There is a complete flat resolution of the form $\bar{\mathbb{F}} = \cdots \to^{\bar{f}} \bar{F} \to^{\bar{f}} \bar{F} \to^{\bar{f}} \bar{F} \to^{\bar{f}} \cdots$ in $S^{-1}R$-Mod such that $\bar{B} \cong \mathrm{Ker}\, \bar{f}$. Then $\bar{F}$ is a flat R-module. Let I be any injective R-module. Then $I \otimes_R \bar{\mathbb{F}} \cong S^{-1}I \otimes_{S^{-1}R} \bar{\mathbb{F}}$ is exact by [15, Proposition 5.17]. So $\bar{B}$ is an SG-flat R-module. □

Corollary 3.21. *Let R be a commutative ring and S a multiplicatively closed set of R. Then*:

(a) *If A is a G-flat R-module, then $S^{-1}A$ is a G-flat R-module for any $A \in R$-Mod.*
(b) *If A is a G-flat R-module, then $S^{-1}A$ is a G-flat $S^{-1}R$-module for any $A \in R$-Mod.*
(c) *For any $\bar{B} \in S^{-1}R$-Mod, $\bar{B}$ is a G-flat R-module if and only if $\bar{B}$ is a G-flat $S^{-1}R$-module.*

Proof. Easy. □

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