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Quasilinear Elliptic Equations with Quadratic Growth in the Gradient*

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1. INTRODUCTION

In this paper we shall be mainly concerned with the existence of bounded weak solutions of the following Dirichlet problem.

Let Ω be a bounded open set of R^n; we seek a function $u\colon \Omega \to R$, such that

$$\begin{aligned} &u \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ &Au(x) = H(x, u, Du). \end{aligned} \tag{1.1}$$

Here A is a linear elliptic second order operator in divergence form,

$$Au(x) = -\sum_{i,j=1}^{n} \partial_i (a_{ij}(x)\, \partial_j u) \tag{1.2}$$

whose coefficients $a_{ij}\colon \Omega \to R$ are measurable functions and the inequalities

$$|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leqslant \Lambda\, |\xi|^2 \tag{1.3}$$

hold a.e. in Ω, $\forall \xi \in R^n$, for some $\Lambda \geqslant 1$.

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Concerning the function $H: \Omega \times R \times R^n \to R$, we assume the hypotheses

$$\forall (z, \xi) \in R \times R^n, \qquad x \to H(x, z, \xi) \text{ is measurable} \tag{1.4}$$

$$\text{for a.e. } x \in \Omega, \qquad (z, \xi) \to H(x, z, \xi) \text{ is continuous} \tag{1.5}$$

((1.4) and (1.5) are the usual Carathéodory assumptions).

Moreover we require the following quadratical behavior of H with respect to ξ: there exist non-decreasing functions k, $\tilde{k}: [0, +\infty) \to [0, +\infty)$ and measurable functions f, $\tilde{f} \in L^p(\Omega)$, $p > n/2$, such that the inequalities

$$\tilde{f}(x) - \tilde{k}(|z|)\, |\xi|^2 \leqslant H(x, z, \xi) \leqslant k(|z|)\, |\xi|^2 + f(x) \tag{1.6}$$

hold for a.e. $x \in \Omega$ and $\forall (z, \xi) \in R \times R^n$.

We write this last hypothesis in the form (1.6) in order to stress the different role of f, $\tilde{f}$, k, $\tilde{k}$ in bounding the positive and the negative part of the solution u. Let us point out that no bound on k, $\tilde{k}$ is required.

Now, problem (1.1) is well defined, in the weak sense, since, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $Au \in H^{-1}(\Omega)$ and $H(\bullet, u, Du) \in L^1(\Omega)$.

Problems of type (1.1) have been largely studied under different assumptions. We recall, for instance, [2, 4, 5, 1, 6]; their hypotheses imply that the functions k, $\tilde{k}$, appearing in (1.6), have to be bounded. The special quadratic growth in the gradient is treated in a series of papers by L. Boccardo, F. Murat, and J. P. Puel, see, e.g., [3], where the existence of a sub- and supersolution is assumed, and [8], where the assumptions on the right hand side are different. We quote also the paper [7] of J. M. Rakotoson, where the so-called "one-sided condition" is requested. For a better understanding of the phenomena related to unboundedness of k, $\tilde{k}$ and to the sign of f, $\tilde{f}$ let us consider the model equation

$$\Delta u + k(|u|)\, |Du|^2 + f(x) = 0. \tag{1.7}$$

By a change of variable

$$v = E(u) = \int_0^u \exp(K(t))\, dt, \tag{1.8}$$

where we put

$$K(t) = \int_0^t k(|s|)\, ds, \tag{1.9}$$

Equation (1.7) takes the form

$$\Delta v + f(x)\, q(v) = 0, \tag{1.10}$$

where

$$q = \exp(K \circ E^{-1}) = E' \circ E^{-1}. \tag{1.11}$$

Note that, when $k > 0$ is a constant, then q is *linear*: $q(v) = 1 + kv$; when k is nondecreasing and unbounded, then $q(v)$ grows at infinity more than v but less than any power $v^{1+\varepsilon}$ with $\varepsilon > 0$. It is apparent that the importance of the sign of f to establish the existence of a solution.

Incidentally, note that u is nonnegative (nonpositive) if f is nonnegative (nonpositive), by maximum principle.

If in particular k and f are constants, and Ω is a ball of radius R, centered at the origin, we obtain explicitly

$$f < 0\colon u(x) = \frac{1}{k} \log \left[\left(\frac{R}{|x|} \right)^{\nu} \frac{I_\nu(|x| \sqrt{k\,|f|})}{I_\nu(R \sqrt{k\,|f|})} \right], \tag{1.12}$$

where $\nu = n/2 - 1$, and $I_\nu(x)$ is the modified Bessel function of order ν.

$$f > 0\colon u(x) = \frac{1}{k} \log \left[\left(\frac{R}{|x|} \right)^{\nu} \frac{J_\nu(|x| \sqrt{kf})}{J_\nu(R \sqrt{kf})} \right] \tag{1.13}$$

provided $R\sqrt{kf} < j_1^{(\nu)}$ (the first zero of the Bessel function J_ν).

Thus, disregarding the sign of f, existence cannot be established without some assumption on the smallness of $|\Omega|$ (the measure of Ω) or of some norm of f.

This can be checked, for nondecreasing k, in the one-dimensional case. For, let f be constant, and Ω be the interval $(-R, +R)$. Then, if $f \leqslant 0$, the following representation holds

$$|x| = R - \frac{1}{\sqrt{2\,|f|}} \int_{u(x)}^{0} e^{K(t)} \left(\int_{\underline{u}}^{t} e^{2K(s)}\, ds \right)^{-1/2} dt, \tag{1.14}$$

where $\underline{u} = u(0)$ (the height of the minimum) is implicitly defined by the equation

$$\int_{\underline{u}}^{0} e^{K(t)} \left(\int_{\underline{u}}^{t} e^{2K(s)}\, ds \right)^{-1/2} dt = \sqrt{2\,|f|}\, R. \tag{1.15}$$

Analogously, if $f \geqslant 0$, we have

$$|x| = R - \frac{1}{\sqrt{2f}} \int_{0}^{u(x)} e^{K(t)} \left(\int_{t}^{\bar{u}} e^{2K(s)}\, ds \right)^{-1/2} dt \tag{1.16}$$

and $\bar{u} = u(0)$ (the height of the maximum) is given by

$$\int_0^{\bar{u}} e^{K(t)} \left(\int_t^{\bar{u}} e^{2K(s)}\, ds \right)^{-1/2} dt \equiv \Psi(\bar{u}) = \sqrt{2f}\, R. \tag{1.17}$$

An inspection of the integral appearing in (1.15) shows that it goes monotonically from 0 to $+\infty$ as $\underline{u}$ decreases from 0 to $-\infty$; so Eq. (1.15) gives a unique solution $\underline{u}$ for every fixed right member. The integral in (1.17), i.e., $\Psi(\bar{u})$, starts from 0 and tends to a finite limit when $\bar{u}$ tends to $+\infty$; more precisely, it tends to $\pi/\sqrt{2k(\infty)}$ if k is bounded, to zero if k is unbounded. Thus, a solution of (1.17) exists iff $\sqrt{2f}\, R$ belongs to the range of Ψ; of course this solution is not unique if k is unbounded.

The main purpose of this paper is to prove the existence of a solution to problem (1.1) under the assumptions (1.2)$\cdots$(1.6). The paper has three sections after this introduction. In Section 2 we prove a kind of *maximum principle* for the solutions of (1.1) (Theorem 2.1). The technique used in this section (based on the investigation on the measure of particular level sets of the solution) enables us to obtain estimates of u and Du. As a consequence we obtain in Section 3 an existence result (Theorem 3.2) and a comparison result (Theorem 3.3). With some additional assumptions, in Section 4 special results are proved.

2. A Maximum Principle

In this section we prove our main estimates for a solution of problem (1.1) under the assumption (1.2)$\cdots$(1.6). An important role will be played by the function

$$W(t) = E(t)/E'(t), \tag{2.1}$$

where we put (as we did for the model equation (1.7))

$$E(t) = \int_0^t \exp(K(s))\, ds \tag{2.2}$$

and $K(s)$ is the primitive function of $k(|s|)$ vanishing at the origin (see (1.9)). $\tilde{E}(t)$ and $\tilde{W}(t)$ are defined in analogous way, when k is replaced with $\tilde{k}$.

Note that, when $k(t) = k > 0$ is constant, then $W(t) = (1/k)(1 - e^{-kt})$ so that the range of W is the interval $[0, 1/k)$.

An inspection of W shows that, when k is nondecreasing and bounded two behaviours of W are possible: either it increases monotonically from 0 to $1/k(\infty)$, or it increases from 0 to a maximum value (attained at $t=\lambda$, say) then decreases monotonically to $1/k(\infty)$.

When k is nondecreasing and unbounded, only the second alternative is possible; this is the interesting case, since, when k is bounded, one can follow the methods in [3, 7] to obtain the desired estimates for u and Du.

It is convenient also to introduce the solution V of the following Dirichlet problem:

$$\begin{aligned} -\Delta V &= f_+^\star \quad \text{on} \quad \Omega^\star \\ V &= 0 \quad \text{on} \quad \partial\Omega^\star. \end{aligned} \tag{2.3}$$

Here is f_+ (f_-) the positive (negative) part of f $(f=f_+-f_-)$, $f_+^\star(x)=f_+^*(C_n|x|^n)$ is the spherical rearrangement of f_+ (having denoted by f^* the decreasing rearrangement of f), C_n is the volume of the unit ball in R_n and $\Omega^\star$ is the ball, centered at the origin, with the same measure $|\Omega|$ of Ω. An explicit representation of V is given by

$$V(x)=n^{-2}C_n^{-2/n}\int_{C_n|x|^n}^{|\Omega|} r^{-2+2/n}\,dr\int_0^r f_+^*(s)\,ds. \tag{2.4}$$

It is well known that, when $f_+\in L^p(\Omega)$, $p>n/2$, then V is bounded and $\|V\|_{L^\infty(\Omega^\star)}=V(0)$. In a similar way we can define $\tilde{V}$, when $f_+^\star$ is replaced by $\bar{f}_-^\star$ in (2.3).

We can now prove the following theorem:

THEOREM 2.1. *Let u be a solution of* (1.1) *under the assumptions* (1.2) $\cdots$ (1.6). *Then*

$$W(\sup u_+)\leqslant \|V\|_{L^\infty(\Omega^\star)} \tag{2.5}$$

$$\tilde{W}(\sup u_-)\leqslant \|\tilde{V}\|_{L^\infty(\Omega^\star)}. \tag{2.6}$$

Remark. Clearly the bound in (2.5) becomes effective only when $\|V\|_{L^\infty(\Omega^\star)}<\sup W$. This means, in the case of interest for us, i.e., $k(+\infty)=+\infty$, that $\|V\|_{L^\infty(\Omega^\star)}<W(\lambda)$, where λ is the unique maximum point of W. We are lead to the following alternative: either

$$\sup u_+\leqslant \min\{t: W(t)=\|V\|_{L^\infty(\Omega^\star)}\}$$

or

$$\sup u_+\geqslant \max\{t: W(t)=\|V\|_{L^\infty(\Omega^\star)}\}.$$

An analogous remark holds for the bound (2.6).

Proof of Theorem 2.1. u is a solution of problem (1.1) if the equality

$$\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\, \partial_j u\, \partial_i \varphi\, dx = \int_\Omega H(x, u, Du)\varphi\, dx \tag{2.7}$$

holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

We first prove the estimate for the positive part u_+.

For $t > 0$, let us choose a special test function of the form

$$\varphi_t(x) = [E(u_+(x)) - t]\, E'(u_+(x))\, \chi_{\mathscr{E}_t}(x), \tag{2.8}$$

where $\chi_{\mathscr{E}_t}$ is the characteristic function of the set

$$\mathscr{E}_t = \{x \in \Omega : E(u_+(x)) > t\}. \tag{2.9}$$

It is easy to check that, since $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, φ_t is a genuine test function. If we insert φ_t in (2.7), we obtain (remember that $E'(u) = \exp K(u)$)

$$\int_{\mathscr{E}_t} \sum_{i,j=1}^n a_{ij}(x)\, \partial_i u_+\, \partial_j u_+\, E'(u_+) \cdot \{E'(u_+) + (E(u_+) - t)\, k(u_+)\}\, dx$$
$$= \int_{\mathscr{E}_t} H(x, u_+, Du_+)(E(u_+) - t)\, E'(u_+)\, dx. \tag{2.10}$$

Let us call $\psi(t)$ the right hand side of (2.10).

Claim. $\psi(t)$ is an absolutely continuous function whose derivative is

$$\psi'(t) = -\int_{\mathscr{E}_t} H(x, u_+, Du_+)\, E'(u_+)\, dx \qquad \text{for a.e. } t > 0. \tag{2.11}$$

To prove (2.11) observe that, if $h > 0$,

$$\int_{\mathscr{E}_{t+h}} H(x, u_+, Du_+)(E(u_+) - t - h)\, E'(u_+)\, dx$$
$$- \int_{\mathscr{E}_t} H(x, u_+, Du_+)(E(u_+) - t)\, E'(u_+)\, dx$$
$$= -h \int_{\mathscr{E}_t} H(x, u_+, Du_+)\, E'(u_+)\, dx$$
$$- \int_{\mathscr{E}_t \setminus \mathscr{E}_{t+h}} H(x, u_+, Du_+)(E(u_+) - t - h)\, E'(u_+)\, dx$$
$$\equiv I_1 + I_2$$

On $\mathscr{E}_t \backslash \mathscr{E}_{t+h}$ we have $-h < E(u_+) - t - h \leqslant 0$; therefore

$$\frac{1}{h}|I_2| \leqslant \int_{\mathscr{E}_t \backslash \mathscr{E}_{t+h}} |H(x, u_+, Du_+)| \, E'(u_+) \, dx$$

which goes to 0 when $h \to 0_+$ since

$$\mathscr{E}_t \backslash \mathscr{E}_{t+h} = \{x : E^{-1}(t) < u_+(x) \leqslant E^{-1}(t+h)\}$$

so that

$$|\mathscr{E}_t \backslash \mathscr{E}_{t+h}| \to 0 \qquad \text{as} \quad h \to 0_+.$$

For $h < 0$ one reaches the same conclusion for a.e. $t > 0$. Since either $|\{x : u_+(x) = t\}| = 0$ or on this set $Du_+ = 0$, it follows that ψ is absolutely continuous.

The claim is then completely proved.

Now we find an estimate from below for $-\psi'(t)$.

Take again $h > 0$, small enough; for $t > 0$, from (2.10) we obtain

$$\begin{aligned} -\psi(t+h) + \psi(t) = h \int_{\mathscr{E}_t} \sum_{i,j=1}^{n} a_{ij}(x) \, \partial_i u_+ \, \partial_j u_+ \, E'(u_+) \, k(u_+) \, dx \\ + \int_{\mathscr{E}_t \backslash \mathscr{E}_{t+h}} \sum_{i,j=1}^{n} a_{ij}(x) \, \partial_i u_+ \, \partial_j u_+ \, E'(u_+) \\ \times \{E'(u_+) + (E(u_+) - t - h) \, k(u_+)\} \, dx. \end{aligned}$$

Taking account of the ellipticity condition (1.3) and of the fact that, on $\mathscr{E}_t \backslash \mathscr{E}_{t+h}$, $E^{-1}(t) < u_+ \leqslant E^{-1}(t+h)$, we can write

$$\begin{aligned} -\psi(t+h) + \psi(t) \geqslant h \int_{\mathscr{E}_t} |Du_+|^2 \, E'(u_+) \, k(u_+) \, dx \\ + \int_{\mathscr{E}_t \backslash \mathscr{E}_{t+h}} |Du_+|^2 \, E'(E^{-1}(t)) \{E'(E^{-1}(t)) \\ - hk(E^{-1}(t+h))\} \, dx. \end{aligned}$$

On the other hand, by Schwarz inequality,

$$\int_{\mathscr{E}_t \backslash \mathscr{E}_{t+h}} |Du_+|^2 \, dx \geqslant |\mathscr{E}_t \backslash \mathscr{E}_{t+h}|^{-1} \left(\int_{\mathscr{E}_t \backslash \mathscr{E}_{t+h}} |Du_+| \, dx \right)^2. \tag{2.12}$$

If we set

$$\mu(t) = |\mathscr{E}_t|$$

we can write

$$
\begin{aligned}
-\psi(t+h)+\psi(t) \geqslant h \int_{\mathscr{E}_t} |Du_+|^2 E'(u_+)\, k(u_+)\, dx \\
+ E'(E^{-1}(t))\{E'(E^{-1}(t)) - hk(E^{-1}(t+h))\} \\
\times (\mu(t)-\mu(t+h))^{-1} \left(\int_{\mathscr{E}_t \setminus \mathscr{E}_{t+h}} |Du_+|\, dx \right)^2 .
\end{aligned} \tag{2.13}
$$

The Fleming–Rishel formula and the isoperimetric inequality give

$$
-\frac{d}{dt} \int_{\mathscr{E}_t} |Du_+|\, dx = (E^{-1})'(t)\, P(\mathscr{E}_t) \geqslant n C_n^{1/n} (E^{-1})'(t)\, \mu(t)^{1-1/n}, \tag{2.14}
$$

where $P(\mathscr{E}_t)$ denotes the perimeter of the set $\mathscr{E}_t$ in the sense of De Giorgi. Now divide by h in (2.13) and let h go to zero; making use of (2.14) and recalling that $(E^{-1})'(t) = [E'(E^{-1}(t))]^{-1}$ we obtain, for a.e. $t > 0$

$$
-\psi'(t) \geqslant \int_{\mathscr{E}_t} |Du_+|^2 E'(u_+)\, k(u_+)\, dx + n^2 C_n^{2/n} \mu(t)^{2-2/n} (-\mu'(t))^{-1}. \tag{2.15}
$$

Equation (2.15) is the desired estimate for ψ'. Actually we have shown the calculations for $h > 0$, but, as before, they can be carried out for $h < 0$ in the same way.

Now we combine (2.15) together with (2.11) and the assumption (1.6) (the right hand side) to obtain

$$
\begin{aligned}
& n^2 C_n^{2/n} \mu(t)^{2-2/n} (-\mu'(t))^{-1} \\
& \quad \leqslant \int_{\mathscr{E}_t} E'(u_+)\{H(x, u_+, Du_+) - k(u_+)\, |Du_+|^2\}\, dx \\
& \quad \leqslant \int_{\mathscr{E}_t} E'(u_+)\, f_+(x)\, dx.
\end{aligned} \tag{2.16}
$$

Set now $y(x) = E(u_+(x))$ and $q(y) = E'(E^{-1}(y)) = E'(u_+(x))$. Then

$$
\mu(t) = \int_{y>t} dx.
$$

Furthermore, since q is increasing, the decreasing rearrangement of $q(y)$ is given by $q(y^*(s))$, $0 < s \leqslant |\Omega|$. Then, by the Hardy–Littlewood inequality,

$$
\int_{\mathscr{E}_t} E'(u_+)\, f_+(x)\, dx \leqslant \int_0^{\mu(t)} q(y^*(s))\, f_+^*(s)\, ds.
$$

From (2.16) now we obtain

$$1 \leqslant n^{-2} C_n^{-2/n} \mu(t)^{-2+2/n} (-\mu'(t)) \int_0^{\mu(t)} q(y^*(s))\, f_+^*(s)\, ds. \qquad (2.17)$$

By integration over t we arrive at

$$y^*(0) \leqslant n^{-2} C_n^{-2/n} \int_0^{|\Omega|} dr\, r^{-2+2/n} \int_0^r q(y^*(s))\, f_+^*(s)\, ds$$

$$\leqslant n^{-2} C_n^{-2/n} q(y^*(0)) \int_0^{|\Omega|} dr\, r^{-2+2/n} \int_0^r f_+^*(s)\, ds.$$

Recalling (2.4) (and the definition (2.1) of W) we finally obtain

$$W(\sup u_+) = \frac{y^*(0)}{q(y^*(0))} \leqslant V(0).$$

This ends the proof of (2.5).

The proof of (2.6) follows exactly the same lines, if we choose as a test function

$$\tilde{\varphi}_t(x) = (\tilde{E}(u_-(x)) - t)\, \tilde{E}'(u_-(x))\, \chi_{\tilde{\mathscr{E}}_t}(x),$$

where

$$\tilde{\mathscr{E}}_t = \{x \in \Omega : \tilde{E}(u_-(x)) > t\} \qquad (t > 0).$$

The proof of Theorem 1 is complete.

3. Existence and Comparison

The estimate in Theorem 2.1 has, as an easy consequence, an a priori estimate for solutions of a family of Dirichlet problems, depending on a positive parameter λ. Set

$$H_\lambda(x, z, \xi) = \begin{cases} H(x, z, \xi) & \text{when} \quad \max(z, 0) \leqslant \lambda \\ H(x, \lambda, \xi) & \text{when} \quad \max(z, 0) > \lambda. \end{cases} \qquad (3.1)$$

Then, from (1.6), when $z \geqslant 0$ we have

$$H_\lambda(x, z, \xi) \leqslant k(\lambda)\, |\xi|^2 + f_+(x). \qquad (3.2)$$

Consider now, for $\lambda > 0$ fixed, the problem

$$\begin{aligned} &u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ &Au_\lambda(x) = H_\lambda(x, u_\lambda, Du_\lambda), \end{aligned} \tag{3.3}$$

where A is the same operator as defined in (1.2), (1.3).

THEOREM 3.1. *Let u_λ be a solution of* (3.3). *If*

$$k(\lambda)\, V(0) < 1 \tag{3.4}$$

then

$$\sup u_{\lambda_+} \leqslant \frac{1}{k(\lambda)} \log[1 - k(\lambda)\, V(0)]^{-1} \tag{3.5}$$

$$\int_\Omega |Du_{\lambda_+}|^2\, dx \leqslant n^{-2} C_n^{-2/n} [1 - k(\lambda)\, V(0)]^{-2} \int_0^{|\Omega|} r^{-2+2/n} \times \left(\int_0^r f_+^*(s)\, ds \right)^2 dr. \tag{3.6}$$

Proof. Estimate (3.5) is a particular case of (2.5) if we choose $k(t)$ constant $= k(\lambda)$ and take (3.4) into consideration.

To prove (3.6) observe that, from (2.14) and (2.12), we have

$$1 \leqslant n^{-2} C_n^{-2/n} (-\mu'(t))\, \mu(t)^{-2+2/n}\, [E'(E^{-1}(t))]^2 \left(-\frac{d}{dt} \int_{\mathscr{E}_t} |Du_{\lambda_+}|^2\, dx \right). \tag{3.7}$$

Since $\int_{\mathscr{E}_t} |Du_{\lambda_+}|^2\, dx$ is an absolutely continuous function for $t > 0$, we can write

$$\begin{aligned} \int_\Omega |Du_{\lambda_+}|^2\, dx &= \int_0^{\sup u_{\lambda_+}} dt \left(-\frac{d}{dt} \int_{\mathscr{E}_t} |Du_{\lambda_+}|^2\, dx \right) \qquad \text{(from 3.7)} \\ &\leqslant \int_0^{\sup u_{\lambda_+}} n^{-2} C_n^{-2/n} (-\mu'(t))\, \mu(t)^{-2+2/n}\, [E'(E^{-1}(t))]^2 \\ &\quad \times \left(-\frac{d}{dt} \int_{\mathscr{E}_t} |Du_{\lambda_+}|^2\, dx \right)^2 dt. \end{aligned}$$

On the other hand, as in the proof of Theorem 2.1, we have

$$\left(-\frac{d}{dt} \int_{\mathscr{E}_t} |Du_{\lambda_+}|^2\, dx \right) [E'(E^{-1}(t))]^2 \leqslant \int_{\mathscr{E}_t} E'(u_{\lambda_+})\, f_+(x)\, dx.$$

Recall now that, being $k = k(\lambda)$ constant, we have $E(s) = (1/k(\lambda))(e^{sk(\lambda)} - 1)$, $E'(s) = e^{sk(\lambda)}$, and $E'(E^{-1}(t)) = 1 + k(\lambda)t > 1$. Hence we obtain

$$\int_\Omega |Du_{\lambda+}|^2\,dx \leqslant n^{-2}C_n^{-2/n} \int_0^{\sup u_{\lambda+}} (-\mu'(t))\,\mu(t)^{-2+2/n} \times \left(\int_{\mathscr{E}_t} E'(u_{\lambda+})\,f_+(x)\,dx\right)^2 dt$$

$$\leqslant n^{-2}C_n^{-2/n} \int_0^{\sup u_{\lambda+}} (-\mu'(t))\,\mu(t)^{-2+2/n}\,e^{2k(\lambda)\sup u_{\lambda+}} \times \left(\int_0^{\mu(t)} f_+^*(s)\,ds\right)^2 dt$$

using again the Hardy–Littlewood estimate for $\int_{\mathscr{E}_t} f_+$.

Now, by using (3.5), we finally obtain (3.6).

Remark 1. If we set

$$\tilde{H}_\lambda(x, z, \xi) = \begin{cases} H(x, z, \xi) & \text{when} \quad \max(-z, 0) \leqslant \lambda \\ H(x, \lambda, \xi) & \text{when} \quad \max(-z, 0) > \lambda \end{cases} \tag{3.1$'$}$$

we have, when $z \leqslant 0$, from (1.6),

$$\tilde{H}_\lambda(x, z, \xi) \geqslant \tilde{k}(\lambda)\,|\xi|^2 - \tilde{f}_-(x). \tag{3.2$'$}$$

Let now $\tilde{u}_\lambda$ be a solution of the problem

$$\tilde{u}_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{3.3$'$}$$

$$A\tilde{u}_\lambda(x) = \tilde{H}_\lambda(x, \tilde{u}_\lambda, D\tilde{u}_\lambda).$$

Then, if

$$\tilde{k}(\lambda)\,\tilde{V}(0) < 1 \tag{3.4$'$}$$

the following estimates hold

$$\sup \tilde{u}_{\lambda-} \leqslant \frac{1}{\tilde{k}(\lambda)} \log[1 - \tilde{k}(\lambda)\,\tilde{V}(0)]^{-1} \tag{3.5$'$}$$

$$\int_\Omega |D\tilde{u}_{\lambda-}|^2\,dx \leqslant n^{-2}C_n^{-2/n}[1 - \tilde{k}(\lambda)\,\tilde{V}(0)]^{-2} \times \int_0^{|\Omega|} r^{-2+2/n} \left(\int_0^r \tilde{f}_-^*(s)\,ds\right) dr. \tag{3.6$'$}$$

Remark 2. Let us set now

$$\bar{H}(x, z, \xi) = \begin{cases} H_\lambda(x, z, \xi) & \text{when } z \geqslant 0 \\ \tilde{H}_\lambda(x, z, \xi) & \text{when } z \leqslant 0 \end{cases}$$

and consider the problem, for $\lambda > 0$ fixed,

$$\begin{aligned} &w_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ &Aw_\lambda(x) = \bar{H}(x, w_\lambda, Dw_\lambda). \end{aligned} \tag{3.8}$$

If

$$\max(k(\lambda)\, V(0), \tilde{k}(\lambda)\, \tilde{V}(0)) < 1 \tag{3.9}$$

then estimates (3.5), (3.6), (3.5′), and (3.6′) hold; from these estimates, by a well-known technique (see, e.g., [3, 7]) it is possible to prove the existence of a solution of problem (3.8).

As a consequence we have the following existence theorem.

THEOREM 3.2. *Assume that, together with the usual hypotheses* (1.2) ... (1.6), *the inequalities hold,*

$$\begin{aligned} V(0) &< \sup_{\lambda > 0} \frac{1}{k(\lambda)} (1 - e^{-\lambda k(\lambda)}) \\ \tilde{V}(0) &< \sup_{\lambda > 0} \frac{1}{\tilde{k}(\lambda)} (1 - e^{-\lambda \tilde{k}(\lambda)}), \end{aligned} \tag{3.10}$$

where the strict inequality sign is not necessary if k (or $\tilde{k}$) is unbounded. Then there exists a solution u of problem (1.1).

Proof. Let λ_1 be such that $V(0) \leqslant (1/k(\lambda_1))(1 - e^{-\lambda_1 k(\lambda_1)})$ and $\tilde{V}(0) \leqslant (1/\tilde{k}(\lambda_1))(1 - e^{-\lambda_1 \tilde{k}(\lambda_1)})$. Then (3.9) holds with $\lambda = \lambda_1$ and problem (3.8) admits a solution u such that $\sup u_\pm \leqslant \lambda_1$; one easily realizes that such a solution also solves problem (1.1).

Let us now go back to formula (2.17). If we integrate it over $(0, t)$ and read the result in terms of decreasing rearrangement we obtain for $0 \leqslant s \leqslant |\Omega|$,

$$y^*(s) \leqslant n^{-2} C_n^{-2/n} \int_s^{|\Omega|} dr\, r^{-2+2/n} \int_0^r q(y^*(t))\, f_+^*(t)\, dt \equiv T_1(y^*)(s). \tag{3.11}$$

Observe that if $z(s)$ is a continuous solution of the integral equation

$$z(s) = T_1(z)(s), \qquad 0 \leqslant s \leqslant |\Omega| \tag{3.12}$$

then the function $Z(x)=z(C_n |x|^n)$ is a solution in $H_0^1(\Omega^\star)\cap L^\infty(\Omega^\star)$ of the (radial) equation

$$-\Delta v = f_+^\star(x)\, q(v). \tag{3.13}$$

It turns out that, under the assumptions of Theorem 3.2 and if $V(0)<\sup_{t>0} W(t)$, Eq. (3.13) has exactly one solution in $H_0^1(\Omega^\star)\cap L^\infty(\Omega^\star)$, the function Z defined above.

Our purpose is to compare Z with a solution of (1.1). This is the content of the following theorem.

THEOREM 3.3. *Let the assumptions of Theorem* 3.2 *hold and* $V(0)<\sup_{t>0} W(t)$. *Then*

(i) *Eq.* (3.13) *has exactly one solution* $Z\in H_0^1(\Omega^\star)\cap L^\infty(\Omega^\star)$

(ii) *if u is a solution of problem* (1.1) *the following comparison holds*:

$$u_+^\star(x)\leqslant E^{-1}(Z(x)) \qquad \textit{for a.e.} \quad x\in\Omega^\star. \tag{3.14}$$

An analogous result clearly holds for u_- *if* $\tilde{Z}$ *is the unique solution of* $-\Delta v=\tilde{f}_-^\star(x)\, q(v)$ *in* $\Omega^\star$ *and* $\tilde{V}(0)<\sup_{t>0}\tilde{W}(t)$.

Proof. Let $\lambda>0$ such that $V(0)\leqslant W(\lambda)$ and $V(0)\leqslant (1/k(\lambda))(1-e^{-\lambda k(\lambda)})$. Then by Theorem 3.2 there exists a solution $U\in H_0^1(\Omega^\star)\cap L^\infty(\Omega^\star)$ of the equation

$$-\Delta U = k(|U|)\,|DU|^2+f_+^\star.$$

Moreover for every solution we have $\|U\|_{L^\infty(\Omega^\star)}\leqslant\lambda$.

Recalling the discussion in Section 1 on Eqs. (1.7), (1.10), we conclude that for every solution v of Eq. (3.13) we have $\|v\|_{L^\infty(\Omega^\star)}\leqslant E(\lambda)$.

On the other hand, such solutions are fixed points of the integral equation

$$v(x)=\int_{\Omega^\star} G(x,\xi)\, f_+^\star(\xi)\, q((v(\xi))\, d\xi\equiv T(v)(x), \qquad x\in\Omega^\star, \tag{3.15}$$

where $G(x,\xi)$ is the Green's function for $\Omega^\star$.

It turns out that the operator $T(v)$ is a contraction on the subset of $L^\infty(\Omega^\star)$ given by $Y\equiv\{v\in L^\infty(\Omega^\star): 0\leqslant v(x)\leqslant E(\lambda)\}$.

Taking this for granted we deduce the first part of the theorem. Denote now by Z the unique solution of Eq. (3.13).

It follows that Z must coincide with the function $z(C_n|x|^n)$, where $z(s)$, $s\in[0,|\Omega|]$, is the unique solution of (3.12). Note that also the operator

$T_1(y)$ defined in (3.11) is a contraction in the subset of $L^\infty([0, |\Omega|])$ given by $Y_1 = \{z \in L^\infty([0, |\Omega|]) : 0 \leqslant z(s) \leqslant E(\lambda)\}$.

Furthermore T_1 is monotonically increasing with y, since q is increasing. Starting the iteration process at $z_0(s) = u_+^*(s)$, $z_n = T_1 z_{n-1}$, we easily infer $u_+^*(s) \leqslant z(s)$ for any $s \in [0, |\Omega|]$. This gives (3.14).

We have only to prove that the operator T defined in (3.15) is a contraction in Y. In the same way it follows that T_1 is a contraction in Y_1.

We have, for $v \in Y$, since $V(0) \leqslant W(\lambda) = E(\lambda)/E'(\lambda)$

$$T(v)(x) \leqslant q(E(\lambda)) \int_{\Omega^\star} f_+^\star(\xi)\, G(x, \xi)\, d\xi \leqslant E'(\lambda) \cdot V(0) \leqslant E(\lambda).$$

Therefore T carries Y into itself.

Observe now that the function $E'(t) = \exp K(t)$ is convex, hence, for $0 \leqslant y_2 \leqslant y_1 \leqslant E(\lambda)$

$$\begin{aligned} q(y_1) - q(y_2) &= E'(E^{-1}(y_1)) - E'(E^{-1}(y_2)) \\ &\leqslant E''_-(\lambda)\{E^{-1}(y_1) - E^{-1}(y_2)\}, \end{aligned} \tag{3.16}$$

where E''_- denotes the left second derivative of E.

On the other hand, we have

$$E''_-(\lambda) = E'(\lambda)\, k(\lambda_-)$$

and

$$E^{-1}(y_1) - E^{-1}(y_2) = \frac{1}{E'(E^{-1}(\bar{y}))}(y_1 - y_2) \leqslant y_1 - y_2$$

by the mean value theorem ($\bar{y} \in (y_2, y_1)$).

So, from (3.16) we obtain

$$q(y_1) - q(y_2) \leqslant E'(\lambda)\, k(\lambda_-)(y_1 - y_2). \tag{3.17}$$

It follows that the function q is Lipschitz continuous on $[0, E(\lambda)]$. Therefore we can write

$$q(t) = 1 + \int_0^t q'(s)\, ds = 1 + \int_0^t k(E_{-1}(s))\, ds.$$

Since k and E^{-1} are increasing functions, we obtain the following inequality, stronger than (3.17)

$$q(y_1) - q(y_2) \leqslant k(\lambda)(y_1 - y_2). \tag{3.18}$$

Now it is easy to see from (3.18) that for $v_1, v_2 \in Y$

$$\|T(v_1) - T(v_2)\|_{L^\infty(\Omega^\star)} \leqslant k(\lambda)\, V(0)\, \|v_1 - v_2\|_{L^\infty(\Omega^\star)}.$$

Since $k(\lambda)\, V(0) < 1$ it follows that T is a contraction in Y and the proof of the theorem is complete.

4. Special Cases

As we have seen, problem (1.1) has a solution if some restrictions are prescribed on the size of f, $\tilde{f}$ or of $|\Omega|$; but we have also seen, in the model equation (1.7), that a solution does exist without any such restriction, provided the known term f has the "right sign."

For instance, let us focus our attention on estimates of the negative part u_- of the solution; obvious changes will give analogous results for the positive part u_+.

Then we replace assumption (1.6) on H with the more restrictive hypothesis: there exists a nondecreasing function $\tilde{k}$: $[0, +\infty) \to [0, +\infty)$ and a measurable function $\tilde{g} \in L^p(\Omega)$, $p > n/2$, $\tilde{g} \geqslant 0$ a.e., such that the inequality

$$H(x, z, \xi) \geqslant \Lambda \tilde{k}(|z|)\, |\xi|^2 - \tilde{g}(x) \tag{4.1}$$

holds for a.e. $x \in \Omega$ and every $(z, \xi) \in R_- \times R^n$.

(Note that, since $\tilde{k}$ has only to be nondecreasing, in (4.1) we could omit the ellipticity constant Λ simply by considering $k = \Lambda \tilde{k}$ instead of $\tilde{k}$).

To state the result set

$$G(t) = \int_0^t \exp(-\tilde{K}(s))\, ds, \tag{4.2}$$

where $\tilde{K}$ is the primitive function of $\tilde{k}$ vanishing at the origin.

Furthermore denote by U the solution of the Dirichlet problem

$$\begin{aligned} \Delta U &= \tilde{g}^\star \quad &&\text{in} \quad \Omega^\star \\ U &= 0 \quad &&\text{on} \quad \partial\Omega^\star. \end{aligned} \tag{4.3}$$

THEOREM 4.1. *Let u be a solution of (1.1) under the assumptions (1.2), ..., (1.5), (4.1). Then*

$$(u_-)^\star (x) \leqslant G(U(x)) \qquad \textit{for a.e.} \quad x \in \Omega^\star. \tag{4.4}$$

Proof of Theorem 4.1. Let $\alpha = \lim_{t \to +\infty} G(t)$, $0 < \alpha < +\infty$; choose, as a test function, for $t \in (0, \alpha)$

$$\varphi_t(x) = [G[U_-(x)) - t]\, G'(u_-(x))\, \chi_{\mathscr{E}_t}(x), \tag{4.5}$$

where now

$$\mathscr{E}_t = \{x \in \Omega : G(u_-(x)) > t\}. \tag{4.6}$$

Inserting φ_t into (2.7) we obtain

$$\int_{\mathscr{E}_t} \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_i u_-\, \partial_j u_-\, G'(u_-)\{-G'(u_-) + (G(u_-) - t)\, \tilde{k}(u_-)\}\, dx$$

$$= \int_{\mathscr{E}_t} H(x, -u_-, -Du_-)(G(u_-) - t)\, G'(u_-)\, dx.$$

Performing calculations analogous to those in the proof of Theorem 2.1, we reach the following inequality, which holds for a.e. $t \in (0, \alpha)$,

$$1 \leqslant n^{-2} C_n^{-2/n} \mu(t)^{-2+2/n} (-\mu'(t)) \int_{\mathscr{E}_t} \tilde{g}(x)\, G'(u_-(x))\, dx,$$

where again $\mu(t) = |\mathscr{E}_t|$.

Since on $\mathscr{E}_t$, $G'(u_-(x)) \leqslant G'(G^{-1}(t))$, we have, by Hardy–Littlewood inequality,

$$1 \leqslant n^{-2} C_n^{-2/n} \mu(t)^{-2+2/n} (-\mu'(t))\, G'(G^{-1}(t)) \int_0^{\mu(t)} \tilde{g}^*(r)\, dr.$$

By integration of both members of the previous inequality over $(0, s)$, one obtains

$$G^{-1}(s) = \int_0^s \frac{d}{dt}(G^{-1}(t))\, dt$$

$$\leqslant n^{-2} C_n^{-2/n} \int_0^s dt \mu(t)^{-2+2/n} (-\mu'(t)) \int_0^{\mu(t)} \tilde{g}^*(r)\, dr.$$

The conclusion now follows by routine arguments.

Remark. Let hypothesis (1.6) be replaced by

$$-\tilde{g}(x) + \tilde{k}(|z|)\, |\xi|^2 \leqslant H(x, z, \xi) \leqslant k(|z|)\, |\xi|^2 - g(x) \tag{4.7}$$

for a.e. $x \in \Omega$ and every $(z, \xi) \in R \times R^n$, g, $\tilde{g}$ being nonnegative functions.

Then, using the right inequality in (4.7), from the results of Section 3, we deduce that a solution u of problem (1.1) is nonpositive, and, using the left inequality in (4.7), we obtain from Theorem 4.1, an estimate for $\sup u_-$.

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