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# Asymmetric Hermitian and skew-Hermitian splitting methods for positive definite linear systems[☆]

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**Abstract**

In this paper, efficient iterative methods for the large sparse non-Hermitian positive definite systems of linear equations, based on the Hermitian and skew-Hermitian splitting of the coefficient matrix, are studied. These methods include an *asymmetric* Hermitian/skew-Hermitian (AHSS) iteration and its inexact version, the inexact asymmetric Hermitian/skew-Hermitian (IAHSS) iteration, which employs some Krylov subspace methods as its inner process. We theoretically study the convergence properties of the AHSS method and the IAHSS method. Moreover, the contraction factor of the AHSS iteration is derived. Numerical examples illustrating the effectiveness of both AHSS and IAHSS iterations are presented.


*Keywords:* Non-Hermitian positive definite matrix; Hermitian and skew-Hermitian splitting; Iterative method

## 1. Introduction

Many problems in scientific computation give rise to solving the linear system

$$Ax = b, \tag{1}$$

with $A \in C^{n\times n}$ a large sparse non-Hermitian positive definite matrix and $x, b \in C^n$. To solve this problem iteratively, usually efficient splittings of the coefficient matrix $A$ are required. For example, the classic Jacobi and Gauss–Seidel iterations [1–3] split the matrix $A$ into its diagonal and off-diagonal parts. Recently, a Hermitian/skew-Hermitian splitting [4–8] has drawn the authors' attention, which is

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*).$$

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Bai, Golub and Ng [5] presented the *HSS* iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2 \ldots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2}$$

where $\alpha$ is a given positive constant. They have also proved that for any positive $\alpha$ the HSS method converges unconditionally to the unique solution of the system of linear equations. The feasible IHSS iteration is discussed and can be adopted in actual implementations and has quite good performance.

Benzi and Golub [4] and Bai, Golub, and Pan [7] apply this method to the saddle point problem directly or as a preconditioner, which extends the application region of the HSS method to semidefinite linear systems. To make this method more attractive, Bai, Golub, and Ng [9] presented the NSS method, furthermore, Bai, Golub, Lu and Yin [5] proposed the PSS method.

Moreover, based on the HS splitting, in this paper we present a different approach to solve Eq. (1), called the *asymmetric Hermitian/skew-Hermitian splitting iteration*, shortened to the *AHSS* iteration. Let us describe it as follows.

*The AHSS iteration method.* Given an initial guess $x^{(0)}$, for $k = 0, 1, 2 \ldots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{3}$$

where $\alpha$ is a given nonnegative constant and $\beta$ is a given positive constant, and $H$ is the Hermitian part of $A$, $S$ the skew-Hermitian part.

The AHSS iteration alternates between the Hermitian part $H$ and the skew-Hermitian part $S$ of the matrix $A$. Theoretical analysis shows that if the coefficient matrix $A$ is positive definite (Hermitian or non-Hermitian) the AHSS iteration (3) can converge to the unique solution of linear system (1) with any given nonnegative $\alpha$, if $\beta$ is restricted in an appropriate region. And the upper bound of the contraction factor of the AHSS iteration is dependent on the choice of $\alpha$ and $\beta$, the spectrum of the Hermitian part $H$ and the singular-values of the skew-Hermitian part $S$, but is not dependent on the eigenvectors of the matrices $H$, $S$ and $A$.

The two half-steps at each AHSS iterate require exact solutions with the matrices $\alpha I + H$ and $\beta I + S$. However, this is too costly to be practical in actual applications. To overcome this disadvantage, the inexact asymmetric Hermitian/skew-Hermitian splitting (IAHSS) iteration is employed. We solve the system of linear equations with coefficient matrix $\alpha I + H$ by conjugate gradient (CG) method and $\beta I + S$ by Krylov subspace method to some prescribed accuracies at each step of the AHSS iteration. Since the accuracies can be different, therefore, the IAHSS iteration method is essentially a non-stationary method.

Noting that the roles of $H$ and $S$ in the AHSS iteration (3) can be transposed, we can first solve the system of linear equations with coefficient matrix $\alpha I + S$ and then solve the system of linear equations with coefficient matrix $\beta I + H$.

This paper is organized as follows. In Section 2, we analyze the convergence properties of the AHSS iteration for non-Hermitian positive definite linear systems. In Section 3 the IAHSS iteration is presented and its convergence property is studied. Numerical examples are presented in Section 4 to illustrate the effectiveness of our methods.

## 2. Convergence analysis of the AHSS iteration

We study the convergence properties of the AHSS iteration and derive the upper bound of the contraction factor in this section. The AHSS iteration method can be generalized to the two-step splitting iteration framework; we replicate the convergence criterion for a two-step splitting iteration from [5].

**Lemma 2.1.** *Let $A \in C^{n\times n}$, $A = M_i - N_i$ $(i = 1, 2)$ be two splittings of the matrix $A$, and $x^{(0)} \in C^n$ be a given initial vector. If $\{x^{(k)}\}$ is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x^{(k+\frac{1}{2})} = N_1 x^{(k)} + b, \\ M_2 x^{(k+1)} = N_2 x^{(k+\frac{1}{2})} + b, \end{cases}$$

$k = 0, 1, 2, \ldots$ *, then*

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b, \quad k = 0, 1, 2, \ldots.$$

*Moreover, if the spectral radius* $\rho(M_2^{-1} N_2 M_1^{-1} N_1) < 1$*, then the iterative sequence* $\{x^{(k)}\}$ *converges to the unique solution* $x^* \in C^n$ *of the system of linear equations* (1) *for all initial vectors* $x^{(0)} \in C^n$.

Applying this lemma to the AHSS iteration, we obtain the following convergence property.

**Theorem 2.2.** *Let* $A \in C^{n\times n}$ *be a positive definite matrix,* $H = \frac{1}{2}(A + A^*)$ *and* $S = \frac{1}{2}(A - A^*)$ *be its Hermitian and skew-Hermitian parts,* $\alpha$ *be a nonnegative constant and* $\beta$ *be a positive constant. Then the iteration matrix* $M(\alpha, \beta)$ *of the AHSS method is*

$$M(\alpha, \beta) = (\beta I + S)^{-1}(\beta I - H)(\alpha I + H)^{-1}(\alpha I - S), \tag{4}$$

*and its spectral radius* $\rho(M(\alpha, \beta))$ *is bounded by*

$$\delta(\alpha, \beta) \equiv \max_{\sigma_i \in \sigma(S)} \frac{\sqrt{\alpha^2 + \sigma_i^2}}{\sqrt{\beta^2 + \sigma_i^2}} \max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right|, \tag{5}$$

*where* $\lambda(H)$ *is the spectral set of* $H$ *and* $\sigma(S)$ *is the singular-value set of* $S$*. And, for any given parameter* $\alpha$*, if*

$$\frac{\alpha \lambda_{\max}}{2\alpha + \lambda_{\max}} < \beta \le \alpha + 2\lambda_{\min}, \tag{6}$$

*then* $\delta(\alpha, \beta) < 1$*, i.e. the AHSS iteration converges, where* $\lambda_{\max}$ *and* $\lambda_{\min}$ *are the maximum and minimum eigenvalues of* $H$.

**Proof.** Setting

$$M_1 = \alpha I + H, \qquad N_1 = \alpha I - S, \qquad M_2 = \beta I + S \quad \text{and} \quad N_2 = \beta I - H,$$

in Lemma 2.1. Since $\alpha I + H$ and $\beta I + S$ are nonsingular for any nonnegative constant $\alpha$ and positive $\beta$, we get (4).

By similarity transformation, we have

$$\begin{aligned} \rho(M(\alpha, \beta)) &= \rho((\beta I + S)^{-1}(\beta I - H)(\alpha I + H)^{-1}(\alpha I - S)) \\ &\le \|(\beta I + S)^{-1}(\beta I - H)(\alpha I + H)^{-1}(\alpha I - S)\|_2 \\ &\le \|(\beta I + S)^{-1}(\alpha I - S)\|_2 \|(\alpha I + H)^{-1}(\beta I - H)\|_2 \\ &= \max_{\sigma_i \in \sigma(S)} \frac{\sqrt{\alpha^2 + \sigma_i^2}}{\sqrt{\beta^2 + \sigma_i^2}} \max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right|, \end{aligned}$$

then the bound for $\rho(M(\alpha, \beta))$ is given by (5).

Since $\alpha \ge 0$ and $\beta > 0$, the following equality holds:

$$\max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| = \max \left\{ \left| \frac{\beta - \lambda_{\max}}{\alpha + \lambda_{\max}} \right|, \left| \frac{\beta - \lambda_{\min}}{\alpha + \lambda_{\min}} \right| \right\}.$$

Therefore, there exists a $\beta^*$ and $\lambda_{\min} \le \beta^* \le \lambda_{\max}$ such that

$$\max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| = \begin{cases} \dfrac{\lambda_{\max} - \beta}{\lambda_{\max} + \alpha}, & \beta \le \beta^*, \\ \dfrac{\beta - \lambda_{\min}}{\alpha + \lambda_{\min}}, & \beta \ge \beta^*. \end{cases} \tag{7}$$

It should be mentioned that the $\beta^*$ is a function of $\lambda_{\max}$, $\lambda_{\min}$ and $\alpha$.

*Case* 1: If $\beta > \alpha$ then $\max_{\sigma_i \in \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_i^2}}{\sqrt{\beta^2+\sigma_i^2}} < 1$, thus,

$$\delta(\alpha, \beta) < \max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| .$$

When $\beta \leq \beta^*$, if $\frac{\beta - \lambda_{\max}}{\alpha + \lambda_{\max}} \leq 1$ then

$$\delta(\alpha, \beta) < 1,$$

which results in

$$\alpha < \beta \leq \beta^*. \tag{8}$$

And when $\beta \geq \beta^*$, we get

$$\beta^* \leq \beta \leq \alpha + 2\lambda_{\min} \tag{9}$$

by simple computation. Combining (8) and (9), when

$$\alpha < \beta \leq \alpha + 2\lambda_{\min}, \tag{10}$$

we have $\delta(\alpha, \beta) < 1$.

*Case* 2: If $\beta < \alpha$, then $\max_{\sigma_i \in \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_i^2}}{\sqrt{\beta^2+\sigma_i^2}} \leq \frac{\alpha}{\beta}$. Thus,

$$\delta(\alpha, \beta) \leq \frac{\alpha}{\beta} \max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| .$$

In order to make the bound $\delta(\alpha, \beta) < 1$, the following inequality must hold

$$\max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| < \frac{\beta}{\alpha}.$$

Similarly, when $\beta \leq \beta^*$, we have

$$\frac{\alpha \lambda_{\max}}{2\alpha + \lambda_{\max}} < \beta \leq \beta^*, \tag{11}$$

and, when $\beta \geq \beta^*$, we have

$$\beta^* \leq \beta < \alpha. \tag{12}$$

Combining (11) and (12), when

$$\frac{\alpha \lambda_{\max}}{2\alpha + \lambda_{\max}} < \beta < \alpha, \tag{13}$$

we have $\delta(\alpha, \beta) < 1$.

*Case* 3: If $\beta = \alpha$, the AHSS method reduces to the HSS method, then it is unconditionally convergent [5].

With the combination of Cases 1, 2 and 3, we complete the proof. □

Theorem 2.2 mainly discusses the available $\beta$ for a convergent AHSS iteration for any given nonnegative $\alpha$. It also shows that the choice of $\beta$ is dependent on the spectrum of the Hermitian part $H$ and the choice of $\alpha$, but is not dependent on the spectrum of the skew-Hermitian part $S$ and $A$. Notice that

$$(\alpha + 2\lambda_{\min}) - \frac{\alpha \lambda_{\max}}{2\alpha + \lambda_{\max}} = 2\lambda_{\min} + \frac{2\alpha^2}{2\alpha + \lambda_{\max}},$$

we remark that for any given nonnegative $\alpha$ the available $\beta$ always exists. And if $\lambda_{\min}$ and $\alpha$ are large, $\lambda_{\max}$ is small the restriction put on $\beta$ is loose. The bound $\delta(\alpha, \beta)$ of the convergence rate depends on the spectrum of $H$ and $S$ and the choice of $\alpha$ and $\beta$. Moreover, $\delta(\alpha, \beta)$ is also an upper bound of the contraction factor of the AHSS iteration.

Moreover, from the proof of Theorem 2.2 we can simplify the bound $\delta(\alpha, \beta)$ as

$$\overline{\delta}(\alpha, \beta) = \max_{\lambda_i \in \lambda(H)} \left| \frac{\beta - \lambda_i}{\alpha + \lambda_i} \right| \max \left\{ 1, \frac{\alpha}{\beta} \right\}. \tag{14}$$

If the lower and upper bounds of the eigenvalues of the Hermitian part $H$ are known, then the relationship of the optimal parameters $\alpha$ and $\beta$ for $\delta(\alpha, \beta)$ can be obtained. We conclude this in the following corollary.

**Corollary 2.3.** *Let $A$, $H$ and $S$ be the matrices defined in Theorem 2.1, and $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and the maximum eigenvalues of the matrix $H$. Then for any given nonnegative parameter $\alpha$, the optimal $\beta$ should be*

$$\overline{\beta} = \frac{2\lambda_{\min}\lambda_{\max} + \alpha(\lambda_{\min} + \lambda_{\max})}{2\alpha + \lambda_{\min} + \lambda_{\max}}. \tag{15}$$

*With the simplified bound $\overline{\delta}(\alpha, \beta)$ (14), we obtain the optimal $\alpha^* = 0$, then*

$$\beta^* = \frac{2\lambda_{\max}\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}}. \tag{16}$$

*And the optimal bound is*

$$\delta^*(\alpha^*, \beta^*) = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} < 1. \tag{17}$$

**Proof.** To minimize the bound in (5), the following equality hold:

$$\frac{\beta - \lambda_{\min}}{\alpha + \lambda_{\min}} = \frac{\lambda_{\max} - \beta}{\alpha + \lambda_{\max}}.$$

Then, the relationship between $\alpha$ and $\beta$ of (15) is proved.

$\overline{\beta}$ can be rewritten as

$$\overline{\beta} = \alpha + 2\frac{\lambda_{\max}\lambda_{\min} - \alpha^2}{2\alpha + \lambda_{\max} + \lambda_{\min}} \begin{cases} \le \alpha, & \alpha \le \sqrt{\lambda_{\max}\lambda_{\min}}, \\ > \alpha, & \alpha < \sqrt{\lambda_{\max}\lambda_{\min}}. \end{cases}$$

Let

$$\overline{\delta}(\alpha, \overline{\beta}) = \begin{cases} \dfrac{\lambda_{\max} - \overline{\beta}}{\lambda_{\max} + \alpha}, & \alpha < \sqrt{\lambda_{\max}\lambda_{\min}}, \\ \dfrac{\alpha(\lambda_{\max} - \overline{\beta})}{\overline{\beta}(\lambda_{\max} + \alpha)}, & \alpha > \sqrt{\lambda_{\max}\lambda_{\min}}. \end{cases}$$

If $\overline{\beta}$ and $\overline{\delta}(\alpha, \overline{\beta})$ are considered as functions of $\alpha$, the minimum value of $\overline{\delta}(\alpha, \overline{\beta})$ is obtained when its first derivative is zero, that is

$$\overline{\delta}'(\alpha, \overline{\beta}) = \begin{cases} \left( \dfrac{\lambda_{\max} - \overline{\beta}}{\lambda_{\max} + \alpha} \right)' = 0, & \text{if } \alpha < \sqrt{\lambda_{\max}\lambda_{\min}}, \\ \left( \dfrac{\alpha(\lambda_{\max} - \overline{\beta})}{\overline{\beta}(\lambda_{\max} + \alpha)} \right)' = 0, & \text{if } \alpha > \sqrt{\lambda_{\max}\lambda_{\min}}. \end{cases}$$

Solving these two equations we get $\alpha = -\lambda_{\max} < 0$. However, $\alpha \ge 0$ is required, and when $\alpha > -\lambda_{\max}$ the first derivative $\overline{\delta}'(\alpha, \overline{\beta}) < 0$. Therefore

$$\alpha^* = 0$$

is the minimum point of the simplified bound $\overline{\delta}(\alpha, \overline{\beta})$. The corresponding optimal parameter

$$\beta^* = \frac{2\lambda_{\max}\lambda_{\min}}{\lambda_{\max} + \lambda_{\min}},$$

and formula (17) holds. □

It should be noted that the optimal parameters $\alpha^*$ and $\beta^*$ in Corollary 2.3 minimize the simplified bound $\overline{\delta}(\alpha, \beta)$ of the iteration matrix, not the optimal spectral radius of the iteration matrix. The parameter $\alpha$ is assigned to be nonnegative to guarantee the matrix $\alpha I + H$ is positive definite. However, it is clear that when $\alpha > -\lambda_{\min}$ or $\alpha < -\lambda_{\max}$ the matrix $\alpha I + H$ is definite and the simplified bound $\overline{\delta}(\alpha, \beta)$ may be smaller. To make the situation easier, we don't discuss this issue here.

## 3. The IAHSS iteration

In the process of AHSS iteration, we need solve two systems of linear equations whose coefficient matrices are $\alpha I + H$ and $\beta I + S$. This is a tough task which is costly and even impractical in actual implementations. To improve computing efficiency of the AHSS iteration, we employ IAHSS iteration, that is to solve the two subproblems iteratively. As assumed, $\alpha I + H$ is Hermitian positive definite, we can solve this system of linear equations by employing conjugate gradient (CG) method, and some Krylov subspace method [1,2,10–12] to solve the system of linear equations with coefficient matrix $\beta I + S$. We write the IAHSS iteration scheme in the following algorithm.

**Algorithm 1.** *IAHSS*: (If $A \in C^{n\times n}$ is a non-Hermitian positive definite matrix, $b \in C^n$ and $x^{(0)} \in C^n$ is the initial guess, then this algorithm leads to the solution of the system of linear equations (1))

$k = 0$;
while (not convergent)
 $r^{(k)} = b - Ax^{(k)}$;
 approximately solve $(\alpha I + H)z^{(k)} = r^{(k)}$ by employing CG method, such that the residual $p^{(k)} = r^{(k)} - (\alpha I + H)z^{(k)}$ of the iteration satisfies $\|p^{(k)}\| \le \eta_k \|r^{(k)}\|$;
 $x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}$;
 $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$;
 approximately solve $(\beta I + S)z^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})}$ by employing some Krylov subspace method, such that the residual $q^{(k)} = r^{(k+\frac{1}{2})} - (\beta I + S)z^{(k+\frac{1}{2})}$ of the iteration satisfies $\|q^{(k)}\| \le \tau_k \|r^{(k+\frac{1}{2})}\|$;
 $x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}$;
 $k = k + 1$;
end

We remark that the convergent criterion is chosen at will. If the inner systems can be solved exactly, the tolerances $\{\eta_k\}$ and $\{\tau_k\}$ are all zeros. Then the IAHSS iteration essentially becomes the AHSS iteration. In fact, to obtain convergent IAHSS iteration, the sequences $\{\eta_k\}$ and $\{\tau_k\}$ are not required to go to zero as $k$ increases. We can deduce that the total work in each step of IAHSS iteration is $o(4n + 2m + \chi_k(H) + \chi_k(S))$, where $m$ is the work of one matrix-vector product $(Ax)$, $\chi_k(H)$ is the number of operations required to solve the inner system with coefficient matrix $\alpha I + H$ inexactly and $\chi_k(S)$ is that to solve the inner system with coefficient matrix $\beta I + S$ inexactly. Bai, Golub and Ng [5] had carefully studied the convergence properties for the two-step iteration, which is represented in the following lemma. We introduce a vector norm

$$|||x|||_{M_2} = \|Mx\|_2 \quad (\forall x \in C^n).$$

**Lemma 3.1.** *Let $A \in C^{n\times n}$ and $A = M_i - N_i (i = 1, 2)$ be two splittings of the matrix $A$. If $\{x^{(k)}\}$ is an iteration sequence defined as follows:*

$$x^{(k+\frac{1}{2})} = x^{(k)} + z^{(k)}, \quad \text{with } M_1 z^{(k)} = r^{(k)} + p^{(k)},$$

*satisfying $\|p^{(k)}\| \le \eta_k \|r^{(k)}\|$, where $r^{(k)} = b - Ax^{(k)}$; and*

$$x^{(k+1)} = x^{(k+\frac{1}{2})} + z^{(k+\frac{1}{2})}, \quad \text{with } M_2 z^{(k+\frac{1}{2})} = r^{(k+\frac{1}{2})} + q^{(k)},$$

*satisfying $\|q^{(k)}\| \le \tau_k \|r^{(k+\frac{1}{2})}\|$, where $r^{(k+\frac{1}{2})} = b - Ax^{(k+\frac{1}{2})}$, then $\{x^{(k)}\}$ is of the form*

$$x^{(k+1)} = M_2^{-1} N_2 M_1^{-1} N_1 x^{(k)} + M_2^{-1}(I + N_2 M_1^{-1})b + M_2^{-1}(N_2 M_1^{-1} p^{(k)} + q^{k+\frac{1}{2}}). \tag{18}$$

*Moreover, if $x^* \in C^n$ is the exact solution of the system (1), then we have*

$$|||x^{(k+1)} - x^*|||_{M_2} \leq (\zeta + \mu\theta\eta_k + \theta(\rho + \theta\nu\eta_k)\tau_k)|||x^{(k)} - x^*|||_{M_2}, \quad k = 0, 1, \ldots, \tag{19}$$

*where*

$$\zeta = \|N_2 M_1^{-1} N_1 M_2^{-1}\|, \qquad \rho = \|M_2 M_1^{-1} N_1 M_2^{-1}\|, \qquad \mu = \|N_2 M_1^{-1}\|,$$
$$\theta = \|A M_2^{-1}\|, \qquad \nu = \|M_2 M_1^{-1}\|.$$

*In particular, if*

$$\zeta + \mu\theta\eta_{\max} + \theta(\rho + \theta\nu\eta_{\max})\tau_{\max} < 1,$$

*then the iteration sequence $\{x^{(k)}\}$ converges to $x^* \in C^n$, where $\eta_{\max} = \max_k\{\eta_k\}$ and $\tau_{\max} = \max_k\{\tau_k\}$.*

According to this lemma, we derive convergence properties for IAHSS iteration.

**Theorem 3.2.** *Let $A \in C^{n\times n}$ be a positive definite matrix, $H$ and $S$ be its Hermitian and skew-Hermitian parts, and $\alpha$ be a nonnegative constant and be a positive constant. If $\{x^{(k)}\}$ is an iterative sequence generated by the IAHSS iteration method (Algorithm 1) and if $x^* \in C^n$ is the exact solution of the system of linear equation (1), then it holds that*

$$|||x^{(k+1)} - x^*||| \leq (\delta(\alpha, \beta) + \eta_k\rho + \tau_k\rho(\omega + \eta_k\theta\rho))|||x^{(k)} - x^*|||, \quad k = 0, 1, \ldots, \tag{20}$$

*where*

$$\rho = \|(\beta I + S)(\alpha I + H)^{-1}\|_2, \qquad \theta = \|A(\beta I + S)^{-1}\|_2,$$
$$\omega = \|(\beta I + S)(\alpha I + H)^{-1}(\alpha I - S)(\beta I + S)^{-1}\|_2.$$

*Particularly, when*

$$(\delta(\alpha, \beta) + \eta_{\max}\rho + \tau_{\max}\rho(\omega + \eta_{\max}\theta\rho)) < 1,$$

*the iterative sequence $\{x^{(k)}\}$ converges to $x^*$, where $\tau_{\max} = \max_k\{\tau_k\}$ and $\eta_{\max} = \max_k\{\eta_k\}$.*

**Proof.** Replacing $M_i, N_i$ $(i = 1, 2)$ in Lemma 3.1 with

$$M_1 = \alpha I + H, \qquad N_1 = \alpha I - S,$$
$$M_2 = \beta I + S, \qquad N_2 = \beta I - H,$$

an the proof is easy. □

Theorem 3.2 tells us the choices of the tolerances $\{\eta_k\}$ and $\{\tau_k\}$ for convergence. Apparently, we find that there is a trade-off between inner and outer iteration with the choices of $\{\eta_k\}$ and $\{\tau_k\}$. However, the optimal tolerances $\{\eta_k\}$ and $\{\tau_k\}$ are hard to analyze. When we solve the system of linear equations with coefficient matrix $\alpha I + H$ with the CG method in the inner iteration, the convergence speed is dependent on the parameter $\alpha$, since the contractor factor of the CG method is $\frac{\lambda_{\max}-\lambda_{\min}}{2\alpha+\lambda_{\max}+\lambda_{\min}}$. And the similar situation is with the solution of the system of linear equations with coefficient matrix $\beta I + S$ by some Krylov subspace method in the inner iteration.

We remark that the convergence rate of the IAHSS iteration method is asymptotically the same as that of the AHSS iteration method.

## 4. Numerical examples

In this section, we give some numerical examples to illustrate the effectiveness of both LHSS and ILHSS iterations. For the convenience of comparison, we consider the three-dimensional convection–diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z) \tag{21}$$

on the unit cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$, with constant coefficient $q$ and subject to Dirichlet-type boundary conditions. Discretizing this equation with seven-point finite difference and assuming the numbers $(n)$ of grid points

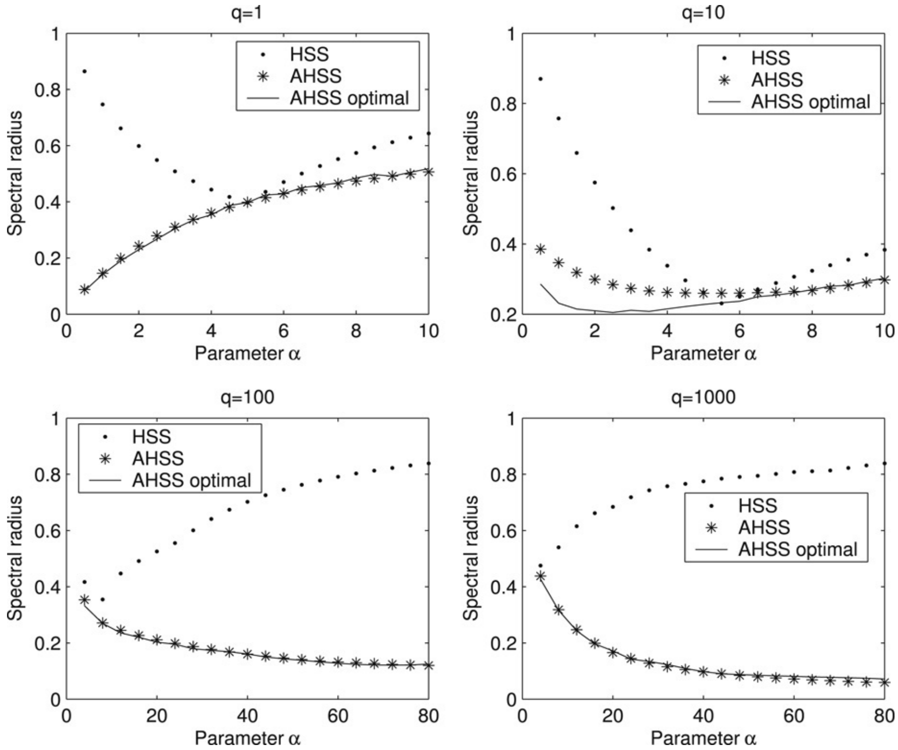


Fig. 1. Centered difference scheme. Spectral radius of iteration matrices of AHSS and HSS methods.

in all three directions are the same, we obtain a positive definite system with linear equations $A$ ($n^3 \times n^3$), and for details, we recommend you turn to [5,13,14]. Different $q$ and $n$ result in different $A$. And our tests are mainly based on this kind of systems.

### 4.1. Spectral radius

In this subsection, we test the spectral radius of the iteration matrix $M(\alpha, \beta)$ (4) with different $q$ and different difference scheme. All the matrices tested are $64 \times 64$ unless otherwise mentioned.

In Figs. 1 and 2, we show the spectral radius of the iteration matrix of the AHSS method and the HSS method with different $\alpha$. *AHSS* represents the spectral radius of the iteration matrix of the AHSS method, where parameter $\beta$ is chosen to be $\overline{\beta}$ in (15), and *HSS* represents that of the HSS method. *AHSS optimal* is the spectral radius of the iteration matrices of the AHSS method, where parameter $\beta$ is tested to be the optimal one.

We find that when $\beta$ is chosen to be the optimal one, the spectral radius of the iteration matrix of the AHSS method is always smaller than that of the HSS method. If centered difference scheme is used $\overline{\beta}$ is a good estimate of the optimal parameter $\beta$ except when $q$ is not very large or small ($q = 10$). If upwind difference scheme is used, when $q$ is small the spectral radius of the iteration matrix of the AHSS method is much smaller than that of the HSS method but when $q$ is very large these two methods seem to perform similarly.

In Fig. 3, we show how the $q$ influence the spectral radius of the iteration matrix of the AHSS method and its upper bound $\delta(\alpha, \beta)$. Here the spectral radius is the optimal one when $\beta$ is chosen to be the $\overline{\beta}$. And it is shown that $\delta(\alpha, \beta)$ is not a very tight bound on the $\rho(M(\alpha, \beta))$.

We depict the eigenvalue distribution of the iteration matrices using the optimal parameters $\alpha$ and $\beta$ in Figs. 4 and 5. Here, the matrix $A$ is $216 \times 216$ ($n = 6$).

### 4.2. Results for AHSS and IAHSS iteration

The AHSS and IAHSS iterations are studied in this subsection. We try to solve the systems of linear equations $Ax = b$, where $A$ is the matrix discretized from (21), and $f(x, y, z)$ is adjusted such that $b = Ae$, ($e$ is

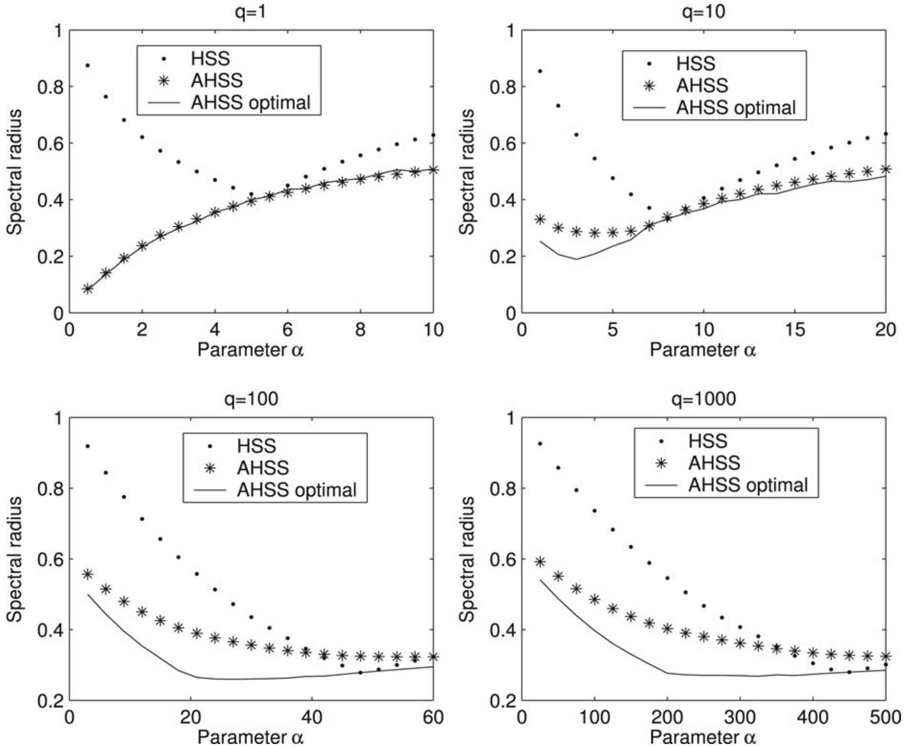


Fig. 2. Upwind difference scheme. Spectral radius of iteration matrices of AHSS and HSS methods.

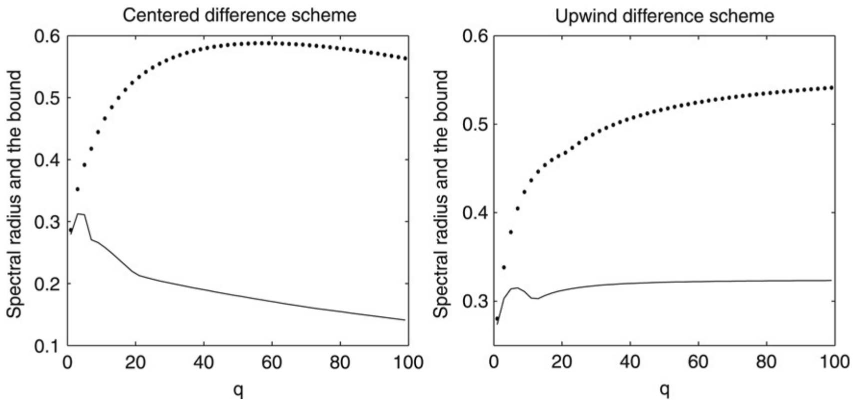


Fig. 3. Spectral radius of iteration matrices and its bound for different $q$.

$(1, 1, \dots, 1)^{\mathrm{T}} \in C^m$). All tests are started from the zero vector, performed in MATLAB with machine precision $10^{-16}$, and terminated when the current iterate satisfies $\|r^{(k)}\|_2 < 10^{-6}$, where $r^{(k)}$ is the residual of the $k$-th AHSS iteration. In Table 1, we show the iteration numbers (it.s) of AHSS method and the associate parameters $\alpha$ and $\beta$ with different differential scheme and different $q$. And for comparison we give the iteration numbers (HSS in the table) of the HSS method and the tested optimal parameter $\alpha$ [5] by $\tilde{\alpha}$. Since when $n$ increases the matrix the scale of $A$ ($n^3 \times n^3$) increases very fast; therefore, it is hard to compute the spectral radius of $A$, and the optimal parameters are hard to obtained, so we only guess them from the previous subsection and test them. We find that when $n$ is large, the optimal parameter $\alpha$ is nearly the same with that when $n$ is small, and the needed iterations is even less than that.

In the two half-steps of AHSS iteration, it is required to solve two systems of linear equations with matrices $\alpha I + H$ and $\beta I + S$, which is very costly. We employ the IAHSS method in the actual implementation, that is solving

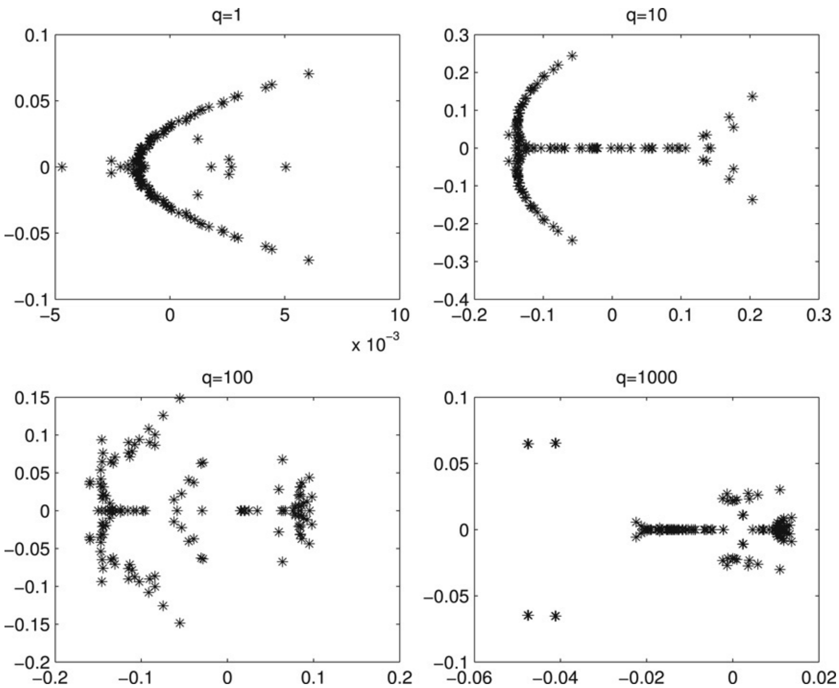


Fig. 4. Eigenvalue distributions of iteration matrices using centered difference scheme.

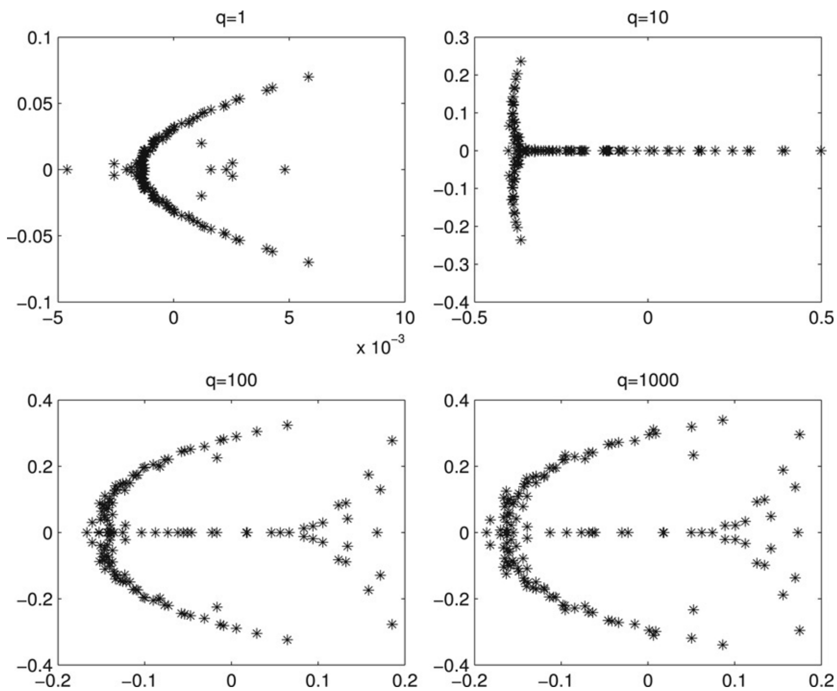


Fig. 5. Eigenvalue distributions of iteration matrices using upwind difference scheme.

the systems with coefficient matrix $\alpha I + H$ iteratively by the conjugate gradient (CG) method and solving the systems with coefficient matrix $\beta I + S$ iteratively by the GMRES method in each outer iteration.

Table 1
Iterations of the AHSS method and HSS method

| $q$ | $n$ | Centered difference | | | | | Upwind difference | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $\beta$ | it.s | $\tilde{\alpha}$ | HSS | $\alpha$ | $\beta$ | it.s | $\tilde{\alpha}$ | HSS |
| $q = 1$ | 8 | 0 | 4 | 5 | 3.2 | 17 | 0 | 4 | 5 | 3.3 | 17 |
| | 16 | 0 | 4 | 5 | 2.8 | 19 | 0 | 3 | 4 | 3 | 19 |
| | 24 | 0 | 5 | 4 | 2.8 | 19 | 0 | 5 | 4 | 2.8 | 19 |
| | 32 | 0 | 5 | 4 | 2.8 | 19 | 0 | 5 | 4 | 2.8 | 19 |
| $q = 10$ | 8 | 0.6 | 4 | 11 | 4.2 | 18 | 0.6 | 4.6 | 10 | 5.4 | 21 |
| | 16 | 0.4 | 3.2 | 10 | 3.2 | 21 | 0.4 | 3.2 | 9 | 3.8 | 25 |
| | 24 | 0 | 5 | 7 | 3 | 22 | 0 | 4 | 7 | 3.4 | 23 |
| | 32 | 0 | 4 | 7 | 3 | 21 | 0 | 4 | 6 | 3.4 | 23 |
| $q = 100$ | 8 | 30 | 6 | 12 | 4 | 17 | 16 | 21 | 16 | 21 | 16 |
| | 16 | 4.5 | 4.5 | 19 | 4.5 | 19 | 8 | 10 | 21 | 10 | 24 |
| | 24 | 3 | 3.5 | 18 | 4 | 19 | 5.5 | 7 | 24 | 7 | 29 |
| | 32 | 2.5 | 3.5 | 19 | 3 | 22 | 3.5 | 5.5 | 24 | 6 | 33 |

Table 2
Centered difference scheme and $q = 1$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.9$ | | | $\tau = 0.8$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 6 | 3.8 | 1 | 6 | 4.3 | 1.5 | 6 | 4.7 | 1.7 |
| 16 | 7 | 4.4 | 1 | 6 | 4.7 | 1 | 6 | 5.7 | 1.3 |
| 24 | 6 | 5 | 1 | 6 | 5.3 | 1 | 6 | 6.2 | 1 |
| 32 | 6 | 5 | 1 | 6 | 5.3 | 1 | 6 | 6.2 | 1 |

Table 3
Centered difference scheme and $q = 10$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.9$ | | | $\tau = 0.8$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 12 | 4.2 | 2.8 | 12 | 4.9 | 3.3 | 12 | 5.8 | 3.9 |
| 16 | 11 | 3.7 | 2.5 | 10 | 5.1 | 2.9 | 10 | 6.1 | 3.4 |
| 24 | 9 | 4.9 | 1.8 | 8 | 6.0 | 1.9 | 8 | 7 | 1.9 |
| 32 | 8 | 5 | 1.5 | 8 | 6.1 | 1.8 | 8 | 7 | 1.9 |

In our computations, the inner CG and GMRES iterates are terminated if the current residual of the inner iterations satisfy

$$\frac{\|p^{(j)}\|_2}{\|r^{(k)}\|_2} \le 0.1\tau^k \quad \text{and} \quad \frac{\|q^{(j)}\|_2}{\|r^{(k)}\|_2} \le 0.1\tau^k$$

(cf. Algorithm 1) where $p^{(j)}$ and $q^{(j)}$ are respectively the residuals of the $j$-th inner CG and GMRES, $r^{(k)}$ is the $k$-th outer IAHSS iteration, $\tau$ is a tolerance. In Tables 2–7, we list numerical results for the centered difference and upwind difference schemes when $q = 1$, 10 and 100. And the parameters $\alpha$ and $\beta$ are the same parameters displayed in Table 1 respectively.

We report the numbers of outer IAHSS iterations (it.s) and the average numbers of inner CG and GMRES iterations. It should be remarked that we adopt restarted GMRES(20) in the inner IAHSS step, the GMRES iteration number is calculated by $20 \times o + i$, where $o$ is the outer iteration and $i$ is the last inner iteration in the process of GMRES(20). According to these tables, the number of IAHSS iterations generally increases when $\tau$ increases and the inner iterations generally decrease. In our numerical tests we don't employ preconditioning techniques in the inner iterations, even though we find that our IAHSS method is very effective.

Table 4
Centered difference scheme and $q = 100$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.98$ | | | $\tau = 0.9$ | | | $\tau = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 16 | 1.8 | 15.8 | 15 | 2 | 20.5 | 17 | 1.4 | 12.9 |
| 16 | 18 | 2.1 | 8.9 | 19 | 2.8 | 12.1 | 18 | 3.6 | 15.8 |
| 24 | 22 | 1.7 | 7.2 | 22 | 2.5 | 10.4 | 20 | 3.9 | 14.9 |
| 32 | 20 | 2.5 | 6.0 | 19 | 3.2 | 7.9 | 19 | 4.5 | 10.8 |

Table 5
Upwind difference scheme and $q = 1$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.9$ | | | $\tau = 0.8$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 6 | 4.2 | 1 | 6 | 4.3 | 1.5 | 6 | 5 | 1.7 |
| 16 | 7 | 4 | 1 | 7 | 4.7 | 1.1 | 6 | 5.2 | 1.3 |
| 24 | 6 | 5 | 1 | 6 | 5.3 | 1 | 6 | 6.2 | 1 |
| 32 | 6 | 5 | 1 | 6 | 5.3 | 1 | 6 | 6.2 | 1 |

Table 6
Upwind difference scheme and $q = 10$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.9$ | | | $\tau = 0.8$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 13 | 3.2 | 3.2 | 12 | 4 | 3.7 | 12 | 4.8 | 4.5 |
| 16 | 11 | 3.6 | 2.6 | 11 | 4.5 | 3 | 11 | 6.1 | 3.4 |
| 24 | 9 | 4.7 | 1.9 | 8 | 5.9 | 2.1 | 8 | 6.8 | 2.5 |
| 32 | 8 | 4.8 | 1.8 | 8 | 5.9 | 1.9 | 7 | 6.7 | 2.1 |

Table 7
Upwind difference scheme and $q = 100$, the number of IAHSS iterations and inner iterations

| $n$ | $\tau = 0.98$ | | | $\tau = 0.9$ | | | $\tau = 0.8$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | it.s | CG | GMRES | it.s | CG | GMRES | it.s | CG | GMRES |
| 8 | 18 | 2.5 | 3.9 | 16 | 3.6 | 5.1 | 16 | 4.6 | 6.6 |
| 16 | 24 | 2.6 | 4.0 | 23 | 3.7 | 6 | 24 | 5.3 | 8.4 |
| 24 | 26 | 2.8 | 3.9 | 24 | 4.2 | 5.5 | 23 | 5.7 | 7.5 |
| 32 | 26 | 2.8 | 3.9 | 24 | 4.8 | 5.1 | 24 | 6.5 | 7.1 |

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