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# Asymptotic Stability of the Bounded or Almost Periodic Solution of the Wave Equation with Nonlinear Dissipative Term

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## 1. INTRODUCTION

Let us consider the wave equation with a nonlinear dissipative term:

$$\frac{\partial^2}{\partial t^2} u - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} u\right) + a_0(x)\, u + \beta\left(x, \frac{\partial u}{\partial t}\right) = f(x, t), \tag{1}$$

together with boundary condition

$$u|_{\partial\Omega} = 0. \tag{2}$$

where $\Omega$ is an open bounded domain in the $n$-dimensional Euclidean space $R^n$ and $\partial\Omega$ is its boundary.

Throughout the paper we shall assume:

$H_1$. The coefficients $a_{ij}(x)$, $a_0(x)$ are measurable and bounded functions on $\Omega$, and $\forall \xi \in R^n$, $\forall x \in \Omega$,

$$a_{ij}(x) = a_{ji}(x), \qquad a_0(x) \geqslant 0,$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geqslant \nu^{-1} |\xi|^2 \qquad (\nu > 0).$$

$H_2$. $\beta(x, z)$ is measurable in $(x, z) \in \Omega \times R$ and satisfies the following conditions

$$\beta(x, 0) \equiv 0, \qquad |\beta(x, z_1) - \beta(x, z_2)| \leqslant k_0(1 + |z_1|^\gamma + |z_2|^\gamma)\, |z_1 - z_2|$$

and

$$(\beta(x, z_1) - \beta(x, z_2))\,(z_1 - z_2) \geqslant k_1 |z_1 - z_2|^{\gamma+2}$$





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with some constants $k_0$, $k_1 > 0$, and $\gamma$ such that

$$0 \leqslant \gamma \leqslant \frac{4}{n-2} \quad \text{if} \quad n \geqslant 3 \qquad \text{and} \qquad 0 \leqslant \gamma < \infty \quad \text{if} \quad n = 1, 2.$$

H$_3$.

$$f(t) \in L_{\text{loc}}^{\gamma+2/\gamma+1}(R; L^{\gamma+2/\gamma+1}(\Omega)) \qquad \text{and} \qquad \sup_{t\in R} \| f(t)\|_S < +\infty,$$

where

$$\| f(t)\|_S = \left(\int_t^{t+1} \| f(s)\|_{L^{\gamma+2/\gamma+1}}^{\gamma+2/\gamma+1} ds\right)^{\gamma+1/\gamma+2}.$$

In [1, Part II], Amerio and Prouse investigated the bounded and the almost periodic solutions for the problem (1)–(2). In particular, concerning the asymptotic stability of the bounded solution $u(t)$ on $R$, it was proved that if $v(t)$ is any solution of the problem (1)–(2) we have

$$\lim_{t\to\infty} \| u(t) - v(t)\|_E = 0,$$

where $\| \cdot \|_E$ denotes the total energy norm.

Since the almost periodic solution (with respect to $\| \cdot \|_E$) is, of course, bounded solution, it satisfies the same asymptotic property.

The object of this paper is to show that the bounded solution or the almost periodic solution $u(t)$ satisfies, in fact, for any solution $v(t)$

$$\| u(t) - v(t)\|_E \leqslant \text{const. } t^{-(1/\gamma)} \qquad \text{if} \qquad \gamma > 0$$

and

$$\| u(t) - v(t)\|_E < \text{const. } e^{-kt} \qquad \text{if} \qquad \gamma = 0$$

with a certain constant $k > 0$.

The precise statement of the result will be given in Section 4.

## 2. Definitions

For the definitions of function spaces see [1] or [3]. Let $A$ be the operator from $H_0^1$ to $H^{-1}$ defined by

$$\langle Au, v\rangle \equiv \int_\Omega \left(\sum_{i,j=1}^n a_{ij} \frac{\partial}{\partial x_j} u \frac{\partial}{\partial x_i} v + a_0(x)\, uv\right) dx.$$

Then the problem (1)–(2) is written formally as follows:

$$u''(t) + Au(t) + \beta(u'(t)) = f(t), \qquad u \in H_0^1. \tag{3}$$

DEFINITION I. A measurable function $u(x, t)$ on $\Omega \times R^+$ is said to be a solution of (3) with initial data $(u_0, u_1) \in H_0^1 \times L^2$ if

(i) $u(t) \in L^\infty(R^+; E) \cap C(R^+; E)$, $u'(t) \in L^{\gamma+2}_{\mathrm{loc}}(R^+; L^{\gamma+2}(\Omega))$, where

$$\| u \|_E^2 = \| u' \|_{L^2}^2 + \| u \|_{H_0^1}^2 \qquad \text{and} \qquad \| u \|_{H_0^1}^2 = \langle Au, u \rangle,$$

(ii) $u(t)$ satisfies the variational equation

$$\int_0^\infty \{-(u'(t), h'(t)) + \langle Au(t), h(t)\rangle + (\beta(u'(t)), h(t)) - (f(t), h(t))\}\, dt = 0 \quad (4)$$

for $\forall$ test function $h(t) \in L^1(R^+; E) \cap L^{\gamma+2}(R^+; L^{\gamma+2})$ with compact support in $(0, \infty)$, where $(v, w)$ denotes

$$(v, w) = \int_\Omega vw\, dx$$

if the right-hand side is meaningful, and

(iii) $$u(0) = u_0, \qquad u'(0) = u_1.$$

DEFINITION II. $u(x, t)$ is said to be a bounded solution of (3) on $R$ if the conditions (i), (ii) are satisfied with $R^+$ replaced by $R$ (in (4) $(0, \infty)$ is replaced by $(-\infty, \infty)$).

Let us assume $H_1$, $H_2$, $H_3$. In [1] it was proved that problem (3) has a unique solution for each initial data $(u_0, u_1) \in H_0^1 \times L^2$ in the sense of Definition I and that if $f(t)$ is $\| \cdot \|_S$-uniformly continuous (3) has a unique bounded solution $u(t)$ in the sense of Definition II. Moreover, the bounded solution $u(t)$ was shown to be $E$-almost periodic if $f(t)$ is $\| \cdot \|_S$-almost periodic.

In what follows we shall study the asymptotic property of the bounded or almost periodic solution $u(t)$.

*Remark* 1. In [1] $\gamma$ is assumed for a technical reason to be such that

$$0 \leqslant \gamma \leqslant 4/(n-1).$$

But this is easily improved as in $H_2$ (cf. [2]).

## 3. LEMMAS

LEMMA 1 (Sobolev). *Let $\gamma$ be as in* $H_2$. *Then we have*

$$\| u \|_{L^{\gamma+2}} \leqslant S_{\gamma+2} \| u \|_{H_0^1} \qquad \text{for } u \in H_0^1, \tag{5}$$

*where $S_\gamma$ is a constant depending on $\Omega$, $\nu$, and $\gamma$.*

The following lemmas are elementary but essential for our purpose.

LEMMA 2. *Let $\phi(t)$ be a bounded positive function on $R^+$ satisfying, for some constants $k$ and $\alpha > 0$,*

$$k\phi(t)^{\alpha+1} \leqslant \phi(t) - \phi(t+1) \qquad \text{for } \forall t \geqslant 0. \tag{6}$$

*Then we have*

$$\phi(t) \leqslant (\alpha k(t-1) + M^{-\alpha})^{-(1/\alpha)} \qquad \text{for } \forall t \geqslant 1, \tag{7}$$

*where*

$$M = \max_{t\in[0,1]} \phi(t).$$

*Proof.* Put $\phi(t)^{-\alpha} = y(t)$. Then

$$\begin{aligned} y(t+1) - y(t) &= \int_0^1 \frac{d}{d\theta} (\theta\phi(t+1) + (1-\theta)\,\phi(t))^{-\alpha}\, d\theta \\ &= -\alpha \int_0^1 (\theta\phi(t+1) + (1-\theta)\,\phi(t))^{-\alpha-1}\, d\theta(\phi(t+1) - \phi(t)) \\ &\geqslant \alpha k\phi(t)^{\alpha+1} \int_0^1 (\phi(t))^{-\alpha-1}\, d\theta \qquad \text{(by (6))} \\ &= \alpha k. \end{aligned}$$

For $\forall t \geqslant 1$, choose the integer $n$ as $n \leqslant t < n+1$, and we have from above

$$y(t) \geqslant y(t-n) + n\alpha k \geqslant y(t-n) + (t-1)\,\alpha k,$$

and hence

$$\phi(t)^{-\alpha} \geqslant (t-1)\,\alpha k + \phi(t-n)^{-\alpha}$$

or

$$\begin{aligned} \phi(t) &\leqslant (\alpha k(t-1) + \phi(t-n)^{-\alpha})^{-1/\alpha} \\ &\leqslant (\alpha k(t-1) + M^{-\alpha})^{-1/\alpha}. \end{aligned} \qquad \text{Q.E.D.}$$

LEMMA 3. *Let $\phi(t)$ be as in Lemma 2, which satisfies (6) with $\alpha = 0$. Then we have*

$$\phi(t) \leqslant Me^{-k't} \qquad \text{for } t \geqslant 1, \tag{8}$$

*where $k' = -\log(1-k) > 0$.*

*Proof.* By (6) with $\alpha = 0$,

$$\phi(t+1) \leqslant (1-k)\,\phi(t) \qquad \text{(which implies } k < 1).$$

Therefore, if $t \geqslant 1$, we have for the integer $n$ with $n \leqslant t < n+1$

$$\phi(t) \leqslant \frac{1}{1-k}\phi(t-1) \leqslant \left(\frac{1}{1-k}\right)^n \phi(t-n)$$
$$\leqslant M(1-k)^{-t} = Me^{t \log(1-k)},$$

which proves the lemma. Q.E.D.

## 4. Theorem

In this section we shall prove our theorem.

Theorem. *Let $u(t)$, $v(t)$ be any two solutions satisfying* (i) *and* (ii). *Then under the hypotheses* $H_1$, $H_2$, $H_3$ *we have*

$$\| u(t) - v(t)\|_E \leqslant ((\gamma/2)\, K(t-1) + M^{-\gamma})^{-(1/\gamma)} \qquad \text{for } t \geqslant 1 \quad \text{if} \quad \gamma > 0, \tag{9}$$

*and*

$$\| u(t) - v(t)\|_E \leqslant Me^{-K't} \qquad \text{for } t \geqslant 1 \quad \text{if} \quad \gamma = 0 \tag{10}$$

$(K' = -\frac{1}{2}\log(1-K))$, *where*

$$M = \max_{t\in[0,1]} \| u(t) - v(t)\|_E \qquad \text{and} \qquad K \text{ is a positive constant}$$

*depending on* $\max_{t\in[0,1]} \| u(t)\|_E$, $\max_{t\in[0,1]} \|_E v(t)\|$, $\sup_{t\in R} \| f(t)\|_S$ *and* $\gamma$.

*Remark* 2. The precise value of $K$ will be given in the proof.

*Proof of Theorem.* Putting $w(t) = u(t) - v(t)$, we have

$$w''(t) + Aw(t) + \beta(u'(t)) - \beta(v(t)) = 0. \tag{11}$$

By Strauss [4] we know

$$\| w(t_2)\|_E^2 - \| w(t_1)\|_E^2 + \int_{t_1}^{t_2} (\beta(u'(s)) - \beta(v'(s)), w'(s))\, ds = 0$$
$$\text{for any } t_1, t_2 \geqslant 0. \tag{12}$$

Hence by our assumption,

$$k_1 \int_t^{t+1} \| w'(s)\|_{L^{\gamma+2}}^{\gamma+2} ds \leqslant \| w(t)\|_E^2 - \| w(t+1)\|_E^2 \equiv k_1 A(t)^{\gamma+2}. \tag{13}$$

Thus we find that there exist two points $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\| w'(t_i)\|_{L^{\gamma+2}} \leqslant 2A(t), \qquad i = 1, 2. \tag{14}$$

Since $u(t)$ is a solution of (3) we have

$$k_1 \int_t^{t+1} \| u'(t)\|_{L^{\gamma+2}}^{\gamma+2} ds \leqslant \| u(t)\|_E^2 - \| u(t+1)\|_E^2 + \int_t^{t+1} (f(s), u'(s))\, ds,$$

and with the aid of Young's inequality

$$\int_t^{t+1} \| u'(s)\|_{L^{\gamma+2}}^{\gamma+2} ds \leqslant M_1^{\gamma+2}, \tag{15}$$

where

$$M_1^{\gamma+2} = \frac{2}{k_1} \sup_{t\in R^+} \left(\| u(t)\|_E^2 + \frac{\gamma+1}{\gamma+2}\left(\frac{2k_1}{\gamma+2}\right)^{1/\gamma+1} \| f(t)\|_S^{\gamma+2/\gamma+1}\right)$$

Similarly we have

$$\int_t^{t+1} \| v'(s)\|_{L^{\gamma+2}}^{\gamma+2} ds \leqslant M_2^{\gamma+2}, \tag{16}$$

where $M_2 = M_1$ with $u(t)$ replaced by $v(t)$.

Now, multiplying (11) by $w(t)$ and integrating over $(t_1, t_2)$ we have

$$\int_{t_1}^{t_2} \| w(s)\|_{H_0^1}^2 ds \leqslant |(w'(t_1), w(t_1)| + |(w'(t_2), w(t_2))| + \int_{t_1}^{t_2} \| w'(s)\|_{L^2}^2 ds$$
$$+ \int_{t_1}^{t_2} (\beta(u'(s)) - \beta(v'(s)), w(s))\, ds. \tag{17}$$

By (14) and (5),

$$| w'(t_1)), w(t_1))| + |(w'(t_2), w(t_2))| \leqslant 4S_2 \operatorname{mes}(\Omega)^{\gamma/\gamma+2} A(t) \max_{s\in[t,t+1]} \| w(s)\|_E .$$

By (13),

$$\int_{t_1}^{t_2} \| w'(s)\|_{L^2}^2 ds \leqslant \operatorname{mes}(\Omega)^{\gamma/\gamma+2} A(t)^2.$$

By $H_2$, (5), (13), (15), and (16), we have

$$\begin{aligned}
&\int_{t_1}^{t_2} (\beta(u'(s)) - \beta(v'(s)), w(s))\, ds \\
&\leqslant k_0 \int_{t_1}^{t_2}\int_\Omega (1 + | u' |^\gamma + | v' |^\gamma) \, | w' | \, | w | \, dx\, ds \\
&\leqslant k_0 \int_{t_1}^{t_2} (\operatorname{mes}(\Omega)^{\gamma/\gamma+2} + \| u' \|_{L^{\gamma+2}}^\gamma + \| v' \|_{L^{\gamma+2}}^\gamma) \| w' \|_{L^{\gamma+2}} \| w \|_{L^{\gamma+2}} ds \\
&\leqslant k_0 S_{\gamma+2}(\operatorname{mes}(\Omega)^{\gamma/\gamma+2} + M_1^\gamma + M_2^\gamma)\, A(t) \max_{s\in[t,t+1]} \| w(s)\|_E .
\end{aligned} \tag{18}$$

From (17) and the above estimates we obtain

$$\int_{t_1}^{t_2} \| w(s)\|_{H_0^1}^2 \, ds \leqslant C_0 A(t)^2 + C_1 A(t) \max_{s\in[t,t+1]} \| w(s)\|_E , \tag{19}$$

where

$$C_0 = \mathrm{mes}(\Omega)^{\gamma/\gamma+2}$$

and

$$C_1 = 4S_2 \, \mathrm{mes}(\Omega)^{\gamma/\gamma+2} + k_0 S_{\gamma+2}(\mathrm{mes}(\Omega)^{\gamma/\gamma+2} + M_1^{\gamma} + M_2^{\gamma}).$$

Thus from (13) and (19) we obtain

$$\int_{t_1}^{t_2} \| w(s)\|_E^2 \, ds \leqslant \mathrm{mes}(\Omega)^{\gamma/\gamma+2} \left( \int_{t_1}^{t_2} \| w'(s)\|_{L^{\gamma+2}}^{\gamma+2} \, ds \right)^{2/\gamma+2} + \int_{t_1}^{t_2} \| w(s)\|_{H_0^1}^2 \, ds$$
$$\leqslant 2C_0 A(t)^2 + C_1 A(t) \max_{s\in[t,t+1]} \| w(s)\|_E ,$$

and hence there exists a point $t^* \in [t_1, t_2]$ such that

$$\| w(t^*)\|_E^2 \leqslant 4C_0 A(t)^2 + 2C_1 A(t) \max_{s\in[t,t+1]} \| w(s)\|_E . \tag{20}$$

Therefore, as in (12) and (18),

$$\max_{s\in[t,t+1]} \| w(s)\|_E^2 \leqslant \| w(t^*)\|_E^2 + \int_t^{t+1} (\beta(u'(s)) - \beta(u'(s)), w'(s)) \, ds$$
$$\leqslant 4C_0 A(t)^2 + 2C_1 A(t) \max_{s\in[t,t+1]} \| w(s)\|_E + C_2 A(t)^2,$$

where

$$C_2 = C_1 - 2S_2 C_0 .$$

Recalling the definition of $A(t)$ and using Young's inequality we have from above

$$\max_{s\in[t,t+1]} \| w(s)\|_E^{\gamma+2} \leqslant K^{-1}(\| w(t)\|_E^2 - \| w(t+1)\|_E^2), \tag{21}$$

where

$$K = k_1/(8C_0 + 2C_2 + 4C_1^2)^{\gamma+2/2}.$$

If $\| w(\bar{t})\|_E = 0$ for some $\bar{t}$, we know $w(t) \equiv 0$ by the uniqueness of solution of initial-value problem for (3), and (9), (10) are, of course, valid. If $\| w(t)\|_E \neq 0$ for all $t \geqslant 0$, we can apply Lemmas 2 and 3 to (21) with $\phi = \| w(t)\|_E^2$ to obtain the desired result. Q.E.D.

From the above theorem we immediately get the following

COROLLARY. *Let $u(t)$ be the bounded or E-almost periodic solution and $v(t)$ be any solution on $[\gamma_0, \infty)$. Then* (9) *or* (10) *is valid for $t \geqslant \gamma_0 + 1$ with trivial modifications.*

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