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Inequalities for C-S seminorms and Lieb functions

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Abstract

Let M_n be the space of $n \times n$ complex matrices. A seminorm $\|\cdot\|$ on M_n is said to be a *C-S seminorm* if $\|A^*A\| = \|AA^*\|$ for all $A \in M_n$ and $\|A\| \leqslant \|B\|$ whenever A, B, and B-A are positive semidefinite. If $\|\cdot\|$ is any nontrivial C-S seminorm on M_n, we show that $\||A|\|$ is a unitarily invariant norm on M_n, which permits many known inequalities for unitarily invariant norms to be generalized to the setting of C-S seminorms. We prove a new inequality for C-S seminorms that includes as special cases inequalities of Bhatia et al., for unitarily invariant norms. Finally, we observe that every C-S seminorm belongs to the larger class of Lieb functions, and we prove some new inequalities for this larger class.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and denote the matrix absolute value of any $A \in M_n$ by $|A| \equiv (A^*A)^{1/2}$. Horn and Mathias ([5,6]; see also [4,3.5,22]) gave two proofs of the following Cauchy–Schwarz inequality conjectured by Wimmer [11]

$$\|A^*B\|^2 \leqslant \|A^*A\|\|B^*B\| \quad \text{for all } A, B \in M_n \tag{1}$$

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and any unitarily invariant norm $\|\cdot\|$ on M_n, which can also be derived from (11) and (16) in [2]. See [3] in this connection.

For Hermitian matrices $A, B \in M_n$, $A \preceq B$ (equivalently, $B \succeq A$) means that $B - A$ is positive semidefinite. Every unitarily invariant norm $\|\cdot\|$ on M_n satisfies

$$\|A\| \leqslant \|B\| \quad \text{whenever } A, B \in M_n \text{ are Hermitian and } 0 \preceq A \preceq B \tag{2}$$

as well as

$$\|A^*A\| = \|AA^*\| \quad \text{for all } A \in M_n. \tag{3}$$

We say that a seminorm $\|\cdot\|$ on M_n is *a C-S seminorm* if it satisfies both (2) and (3); a seminorm $\|\cdot\|$ on M_n is *nontrivial* if there is some $A_0 \in M_n$ such that $\|A_0\| > 0$. For example, $\|A\|_{|\mathrm{tr}|} \equiv \sum_{i=1}^{n} |a_{ii}|$ is a nontrivial *C-S* seminorm that is not a norm and is not unitary similarity invariant. See [5, Examples 4.12 and 4.13] for examples of unitary similarity invariant norms that are not *C-S* seminorms. Any unitarily invariant norm is, of course, a nontrivial *C-S* norm. However, the norm $\|A\|_\infty \equiv \max\{|a_{ij}|: 1 \leqslant i, j \leqslant n\}$ for $A = [a_{ij}] \in M_n$ satisfies (1) but does not satisfy (3); there is no seminorm on M_n that satisfies (1) but not (2):

Lemma 1. *If a seminorm $\|\cdot\|$ on M_n satisfies* (1), *then it also satisfies* (2).

Proof. Let $U, P \in M_n$ be given with U unitary and P positive semidefinite. Setting $A = P$ and $B = UP$ in (1) gives

$$\|PUP\| \leqslant \|P^2\|. \tag{4}$$

Let $A, B \in M_n$ be positive semidefinite and assume B is nonsingular and $0 \preceq A \preceq B$. Then $C \equiv B^{-1/2}AB^{-1/2} \preceq I$ and $A = B^{1/2}CB^{1/2}$. Since every contraction is a convex combination of unitary matrices [4, Section 3.1, Problem 27] (in fact, it is the average of two unitary matrices), there are finitely many unitary matrices U_i and scalars $\alpha_i > 0$ with $\sum_i \alpha_i = 1$ such that $C = \sum_i \alpha_i U_i$. Using (4), we have

$$\|A\| = \|B^{1/2}CB^{1/2}\| = \left\|\sum_i \alpha_i B^{1/2}U_iB^{1/2}\right\| \leqslant \sum_i \alpha_i\|B\| = \|B\|.$$

The general case in which B can be singular now follows by continuity. □

Nontriviality for a seminorm is equivalent to its nontriviality on positive definite matrices:

Lemma 2. *Let $\|\cdot\|$ be a given seminorm on M_n. Then $\|\cdot\|$ is nontrivial if and only if there is some positive definite $P \in M_n$ such that $\|P\| > 0$.*

Proof. Using the Cartesian decomposition, one can write any square complex matrix as a linear combination of two Hermitian matrices, each of which can be written as a difference of two positive definite matrices. Thus, for each $A \in M_n$ there are positive definite $P_1, \ldots, P_4 \in M_n$ such that $A = P_1 - P_2 + i(P_3 - P_4)$ and

$$\|A\| = \|P_1 - P_2 + i(P_3 - P_4)\| \leqslant \|P_1\| + \|P_2\| + \|P_3\| + \|P_4\|.$$

Thus, $\|\cdot\|$ is nontrivial if and only if there is some positive definite $P \in M_n$ such that $\|P\| > 0$. □

We shall develop basic properties of *C-S* seminorms, generalize (1) and other inequalities for unitarily invariant norms to *C-S* seminorms, prove a new inequality for *C-S* seminorms, discuss the Lieb functions, and prove some new inequalities for Lieb functions.

2. C-S Seminorms

For a positive semidefinite $P \in M_n$, let $\Lambda(P) \equiv \mathrm{diag}(\lambda_1(P), \ldots, \lambda_n(P))$, where $\lambda_1(P) \geqslant \cdots \geqslant \lambda_n(P)$ are the decreasingly ordered eigenvalues of P. For any $A \in M_n$, let $\sigma_1(A) \geqslant \cdots \geqslant \sigma_n(A)$ denote the decreasingly ordered singular values of A. Let $E_i \in M_n$ be the matrix whose only nonzero entry is a 1 in position (i, i). We first establish some basic properties of *C-S* seminorms.

Theorem 1. *Let $\|\cdot\|$ be a C-S seminorm on M_n and let $P \in M_n$ be positive semidefinite. Then*

(a) $\|U^*PU\| = \|P\|$ *for all unitary $U \in M_n$. In particular,* $\|P\| = \|\Lambda(P)\|$.
(b) $\lambda_1(P)\|E_1\| \leqslant \|P\| \leqslant \lambda_1(P)\|I\| \leqslant n\lambda_1(P)\|E_1\|$.
(c) *If $\|\cdot\|$ is nontrivial and $P \neq 0$ then* $\|P\| > 0$.
(d) $\||ABC|\| \leqslant \sigma_1(A)\||B|\|\sigma_1(C)$ *for all $A, B, C \in M_n$.*

Proof. (a) Using (3), we have

$$\|U^*PU\| = \|(P^{1/2}U)^*(P^{1/2}U)\| = \|P^{1/2}UU^*P^{1/2}\| = \|P\|.$$

(b) Let Q_i denote the permutation matrix obtained by interchanging the first and ith rows of the identity matrix I. Using (a), we have $\|E_i\| = \|Q_iE_1Q_i^{\mathrm{T}}\| = \|E_1\|, i = 1, \ldots, n$, so $\|I\| = \|\sum_{i=1}^n E_i\| \leqslant \sum_{i=1}^n \|E_i\| = n\|E_1\|$. Since $0 \preceq \lambda_1(P)E_1 \preceq \Lambda(P) \preceq \lambda_1(P)I$, (a) and (2) imply (b).

(c) Lemma 2 and (b) ensure that $\|E_1\| > 0$. If $P \neq 0$, then $\lambda_1(P) > 0$ and (b) gives $\|P\| \geqslant \lambda_1(P)\|E_1\| > 0$.

(d) The key observation is that $\sigma_i(ABC) \leqslant \sigma_1(A)\sigma_i(B)\sigma_1(C)$ for all $i = 1, \ldots, n$ [4, 3.3.18], which ensures that $0 \leqslant \Lambda(|ABC|) \leqslant \sigma_1(A)\Lambda(|B|)\sigma_1(C)$. Now use (2) again to compute

$$|||ABC||| = \|A(|ABC|)\| \leqslant \sigma_1(A)\|A(|B|)\|\sigma_1(C) = \sigma_1(A)\||B|\|\sigma_1(C). \quad \square$$

Remark 1. The restricted unitary similarity invariance property in Theorem 1(a) clearly implies the property (3), so these two properties of a seminorm are equivalent.

Corollary 1. *Let $\|\cdot\|$ be a given nontrivial C-S seminorm on M_n. The following are equivalent:*

(a) $\|E_1\| \geqslant 1$.

(b) $\|P\| \geqslant \lambda_1(P)$ *for every positive semidefinite* $P \in M_n$.

(c) $\||PQ|\| \leqslant \|P\|\|Q\|$ for *all positive semidefinite* $P, Q \in M_n$.

Proof. (a) $\Rightarrow$ (b). If P is positive semidefinite, Theorem 1(b) ensures that $\|P\| \geqslant \lambda_1(P)\|E_1\| \geqslant \lambda_1(P)$.

(b) $\Rightarrow$ (c). If P and Q are positive semidefinite, then use Theorem 1(d) to compute $\||PQ|\| \leqslant \lambda_1(P)\|Q\| \leqslant \|P\|\|Q\|$.

(c) $\Rightarrow$ (a). Using $\|E_1\| = \|E_1E_1\| = \||E_1E_1|\| \leqslant \|E_1\|^2$, Theorem 1(c) permits us to deduce that $1 \leqslant \|E_1\|$. $\square$

We now have in hand several examples of how nontrivial C-S seminorms act on the positive semidefinite matrices the way unitarily invariant norms act on all of M_n: Theorem 1(c) says that a nontrivial C-S seminorm acts like a norm on the positive semidefinite matrices; Theorem 1(d) is an analog of the fact that unitarily invariant norms are *symmetric* [4, problem 4, p. 211]; and Corollary 1 corresponds to the fact that a unitarily invariant norm $\|\cdot\|$ on M_n is submultiplicative if and only if $\|A\| \geqslant \sigma_1(A)$ for all $A \in M_n$ [4, problem 3, p. 211]. The following theorem provides a basic explanation for these examples.

Definition 1. For any seminorm $\|\cdot\|$ on M_n, define $\|\cdot\|_{\text{abs}} : M_n \to \mathbb{R}_+$ by $\|A\|_{\text{abs}} \equiv \||A|\|$ for all $A \in M_n$.

Theorem 2. *If $\|\cdot\|$ is a nontrivial C-S seminorm on M_n, then $\|\cdot\|_{\text{abs}}$ is a unitarily invariant norm on M_n.*

Proof. Let $A, B \in M_n$ and $c \in \mathbb{C}$ be given. We are assured that $\|A\|_{\text{abs}} \geqslant 0$; if $A \neq 0$ then $|A| \neq 0$, so Theorem 1(c) ensures that $\|A\|_{\text{abs}} = \||A|\| > 0$. Homogeneity is easily checked: $\|cA\|_{\text{abs}} = \||cA|\| = \||c||A|\| = |c|\||A|\| = |c|\|A\|_{\text{abs}}$. The triangle inequality for $\|\cdot\|_{\text{abs}}$ follows easily from Robert Thompson's matrix-valued triangle inequality ([10] or [4, (3.1.15)]: There are unitary $U, V \in M_n$ such that $|A+B| \preceq U|A|U^* + V|B|V^*$, so Theorem 1(a) and (2) permit us to compute

$$\|A+B\|_{\text{abs}} = |||A+B||| \leqslant \|U|A|U^* + V|B|V^*\|$$
$$\leqslant \|U|A|U^*\| + \|V|B|V^*\| = |||A||| + |||B||| = \|A\|_{\text{abs}} + \|B\|_{\text{abs}}.$$

Thus, $\|\cdot\|_{\text{abs}}$ is a norm on M_n; its unitary invariance follows from Theorem 1(a): For any unitary $U, V \in M_n$, $\|UAV\|_{\text{abs}} = |||UAV||| = \|V^*|A|V\| = |||A||| = \|A\|_{\text{abs}}$. □

Implicit in this result is the following principle: *any theorem about unitarily invariant norms that involves only positive semidefinite matrices must hold for nontrivial C-S seminorms as well.* We offer several examples as corollaries to Theorem 2; in each case the proof is the same: apply a known result about unitarily invariant norms to the unitarily invariant norm $\|\cdot\|_{\text{abs}}$.

Corollary 2. *Let $L, M, X \in M_n$ be given and suppose*

$$\begin{bmatrix} L & X \\ X^* & M \end{bmatrix} \geqslant 0.$$

Then

$$\||X|^r\| \leqslant \|L^{pr/2}\|^{1/p} \|M^{qr/2}\|^{1/q} \tag{5}$$

for every C-S seminorm $\|\cdot\|$ on M_n and all positive p, q, and r such that $p^{-1} + q^{-1} = 1$. In particular,

$$|||X||| \leqslant \|L^{p/2}\|^{1/p} \|M^{q/2}\|^{1/q}. \tag{6}$$

Proof. Apply Theorem 2 to [6, (2.11)]. □

Setting $L = AA^*$, $M = B^*B$, and $X = AB$ in Corollary 2 and using Theorem 1(a) gives

Corollary 3. *Let $A, B \in M_n$ be given, let $\|\cdot\|$ be a given C-S seminorm on M_n, let $p, q, r \in (0, \infty)$ be given, and suppose $p^{-1} + q^{-1} = 1$. Then*

$$\||AB|^r\| \leqslant \||A|^{pr}\|^{1/p} \||B|^{qr}\|^{1/q}; \tag{7}$$

in particular,

$$|||AB||| \leqslant \||A|^p\|^{1/p} \||B|^q\|^{1/q}. \tag{8}$$

One obtains (1) by setting $p = q = 2$ and $r = 1$ in (7), which is therefore a generalization of (1) to the larger class of C-S seminorms.

Corollary 4. *Let $\|\cdot\|$ be a given seminorm on M_n and define the function $v : M_n \to \mathbb{R}_+$ by $v(A) \equiv \|A^*A\|^{1/2}$. If $\|\cdot\|$ is a nontrivial C-S seminorm, then $v(\cdot)$ is a unitarily invariant norm.*

Proof. The function ν is clearly homogeneous. Unitary invariance and positivity are ensured by Theorem 1(a,c). The triangle inequality is a straightforward computation using (1). □

Note that by Theorem 2, ν in Corollary 4 is a Q-norm [1, p. 95].

Using the trace norm in (8) and the fact that $|\mathrm{tr}X| \leqslant \|X\|_{\mathrm{tr}}$ for any $X \in M_n$ gives the known inequality [9, Theorem 6]

$$\mathrm{tr}\left(A^{\alpha}B^{1-\alpha}\right) \leqslant (\mathrm{tr}A)^{\alpha}(\mathrm{tr}B)^{1-\alpha}$$

for all positive semidefinite A and B and all $\alpha \in (0,1)$.

Now let $\|\cdot\|$ be any given unitarily invariant norm on M_n. Bhatia and Davis [3] (see also [1, Theorem IX.5.2]) showed that

$$\||AXB|^r\|^2 \leqslant \||A^*AX|^r\|\||XBB^*|^r\| \tag{9}$$

for all $A, B, X \in M_n$ and all $r > 0$, which is equivalent to the same inequality with A and B restricted to be positive semidefinite. Kittaneh [7] (see [3] for another proof) proved that

$$\|AXB\| \leqslant \|A^pX\|^{1/p}\|XB^q\|^{1/q} \tag{10}$$

for all positive semidefinite $A, B \in M_n$ and all positive p and q such that $p^{-1} + q^{-1} = 1$. Our next theorem includes both (9) and (10), generalized to the setting of C-S seminorms. Our proof makes use of the following lemma, whose elegant proof is in [3]. We write $x \prec_w y$ to denote weak (additive) majorization between nonnegative vectors [4, 3.2.9].

Lemma 3. *Let $A, B \in M_n$ be positive semidefinite and suppose $0 < s \leqslant t$. Then*

$$\prod_{i=1}^{k}\lambda_i^{1/s}(A^sB^s) \leqslant \prod_{i=1}^{k}\lambda_i^{1/t}(A^tB^t), \quad k = 1, 2, \ldots, n.$$

Consequently,

$$\left[\lambda_i^{r/s}(A^sB^s)\right]_{i=1}^{n} \prec_w \left[\lambda_i^{r/t}(A^tB^t)\right]_{i=1}^{n} \quad \textit{for all } r > 0. \tag{11}$$

Theorem 3. *Let $A, B, X \in M_n$ be given with A and B positive semidefinite. Then*

$$\||AXB|^r\| \leqslant \||A^pX|^r\|^{1/p}\||XB^q|^r\|^{1/q} \tag{12}$$

for every C-S seminorm $\|\cdot\|$ on M_n and all positive p, q, and r such that $p^{-1} + q^{-1} = 1$.

Proof. Let $X = UP$ be a polar decomposition of X (with U unitary and P positive semidefinite), write $AXB = \left(AUP^{1/p}\right)\left(P^{1/q}B\right)$, and use (7) to obtain

$$\||AXB|^r\| \leqslant \|(P^{1/p}U^*A^2UP^{1/p})^{pr/2}\|^{1/p}\|(BP^{2/q}B)^{qr/2}\|^{1/q}. \tag{13}$$

Since the eigenvalues of YZ and ZY are the same for all $Y, Z \in M_n$, (11) ensures that

$$\begin{aligned}\lambda_i^{pr/2}(P^{1/p}U^*A^2UP^{1/p}) &= \lambda_i^{pr/2}((A^{2p})^{1/p}(UP^2U^*)^{1/p}) \\ &\prec_w \lambda_i^{r/2}(A^{2p}UP^2U^*) \quad (\text{since } p^{-1} < 1) \\ &= \lambda_i^{r/2}(A^{2p}XX^*) = \lambda_i^{r/2}((A^pX)^*(A^pX)) \\ &= \sigma_i^r(A^pX) \end{aligned} \tag{14}$$

and

$$\begin{aligned}\lambda_i^{qr/2}(BP^{2/q}B) &= \lambda_i^{qr/2}((P^2)^{1/q}(B^{2q})^{1/q}) \\ &\prec_w \lambda_i^{r/2}(P^2B^{2q}) \quad (\text{since } q^{-1} < 1) \\ &= \lambda_i^{r/2}(X^*XB^{2q}) = \lambda_i^{r/2}((XB^q)^*(XB^q)) \\ &= \sigma_i^r(XB^q). \end{aligned} \tag{15}$$

The Fan Dominance Theorem [4, Corollary 3.5.9] now permits us to conclude from Eqs. (14) and (15) that

$$\left\|(P^{1/p}U^*A^2UP^{1/p})^{pr/2}\right\| \leqslant \||A^pX|^r\| \quad \text{and} \quad \left\|(BP^{2/q}B)^{qr/2}\right\| \leqslant \||XB^q|^r\|.$$

Combining these inequalities with (13) gives (12). □

Kittaneh's inequality (10) is not valid for all C-S seminorms. Consider the C-S norm $\|A\| \equiv \|A\|_F + \|A\|_{|\mathrm{tr}|} = (\mathrm{tr}A^*A)^{1/2} + \sum_{i=1}^n |a_{ii}|$, the matrices

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad B = I_2,$$

and $p = q = 2$. Then $\|AXB\|^2 = 3 + 2\sqrt{2} > 2 + 2\sqrt{2} = \|A^2X\|\|XB^2\|$.

There is a special subclass of the C-S seminorms that satisfy a pair of conditions that is stronger than (2) and (3):

Theorem 4. *If a seminorm* $\|\cdot\|$ *satisfies* (3) *and*

$$\|A\| \leqslant \||A|\| \text{ for all } A \in M_n, \tag{16}$$

then it is a C-S seminorm.

Proof. We must show that conditions (16) and (3) imply (2). Since $0 \preceq A(X) \preceq A(Y)$ whenever X and Y are Hermitian and $0 \preceq X \preceq Y$, it suffices to consider nonnegative diagonal matrices. Let $D = \mathrm{diag}(d_1, \ldots, d_n) \geqslant 0$. For $\alpha \in [0, 1]$, denote $D(i, \alpha) \equiv \mathrm{diag}(d_1, \ldots, d_{i-1}, \alpha d_i, d_{i+1}, \ldots, d_n)$. Then (16) ensures that

$$\|D(i,\alpha)\| = \left\| \frac{1+\alpha}{2} D + \frac{1-\alpha}{2} \operatorname{diag}(d_1,\ldots,d_{i-1},-d_i,d_{i+1},\ldots,d_n) \right\|$$
$$\leqslant \frac{1+\alpha}{2}\|D\| + \frac{1-\alpha}{2}\|\operatorname{diag}(d_1,\ldots,d_{i-1},-d_i,d_{i+1},\ldots,d_n)\|$$
$$\leqslant \frac{1+\alpha}{2}\|D\| + \frac{1-\alpha}{2}\|D\| = \|D\|.$$

Using Remark 1 and this fact successively for $i = 1,\ldots,n$ we deduce that $\|X\| = \|A(X)\| \leqslant \|A(Y)\| = \|Y\|$. □

The class of seminorms satisfying conditions (3) and (16) is not the entire class of C-S seminorms. For an example of a C-S seminorm that does not satisfy (16), see [5, Example 4.12].

From Theorem 4 and Corollary 3 we know that if a seminorm satisfies (3) and (16) then it satisfies the Cauchy–Schwarz inequality (1).

Although every unitarily invariant norm on M_n is self-adjoint, there are C-S seminorms on M_n that are not self-adjoint: On M_2 consider

$$\|A\| \equiv \max\{|a_{11}| + |a_{22}|, |a_{12}|\}.$$

3. Functions in Lieb's class $\mathscr{L}$

Lieb [8] introduced the class $\mathscr{L}$ of continuous complex-valued functions f on M_n that satisfy the following two conditions:

$$f(A) \geqslant f(B) \geqslant 0 \text{ whenever } A \succeq B \succeq 0, \tag{17}$$

and

$$|f(A^*B)|^2 \leqslant f(A^*A)f(B^*B) \text{ for all } A,B \in M_n. \tag{18}$$

Examples of functions in $\mathscr{L}$ are the determinant, permanent, spectral radius, any elementary symmetric function of the eigenvalues, and any unitarily invariant norm.

The hypothesis (2) and (8) with $p=q=2$ show that the set of all C-S seminorms is contained in $\mathscr{L}$, but this containment is proper. The following example shows that there is a function in Lieb's class $\mathscr{L}$ that does not satisfy (6) and (8) for some p, q. The function f_C and the matrices in the following example are taken from [8, pp. 175 and 177], where they serve another purpose.

Example 1. For any fixed positive semidefinite $C \in M_n$, the function $f_C(X) \equiv \det(C+X)$ is in Lieb's class $\mathscr{L}$ on M_n. Let

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = I_2.$$

Then $f_C(AB) = 2$, and $[f_C(|A|^p)]^{1/p}[f_C(|B|^q)]^{1/q} \to \sqrt{2}$ as $p \to 1$ and $q \to \infty$. Thus this Lieb function does not satisfy (8). Moreover, f_C does not satisfy (6): just set $L = A^*A$, $M = B^*B$, $X = A^*B$ and let $p \to 1$, $q \to \infty$.

Lemma 1 shows that the properties (18) and (17) are not independent for seminorms.

Bhatia [1, p. 270] gave the following characterization of the class $\mathscr{L}$.

Theorem 5. *Let $f: M_n \to \mathbb{C}$ be continuous. Then $f \in \mathscr{L}$ if and only if*

$$f(A) \geqslant 0 \text{ for all } A \succeq 0$$

and

$$|f(C)|^2 \leqslant f(A)f(B) \text{ for all } A,B,C \text{ such that } \begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \succeq 0. \tag{19}$$

Bhatia's characterization leads to a simple proof [1, p. 270] of the following result due to Lieb [8]. Bhatia observes that a sum of block matrices of the type in (19) is a positive semidefinite block matrix, to whose blocks the function f can then be applied to obtain an inequality of the type in (19).

Theorem 6. *Let $A_i, B_i \in M_n, i = 1, \ldots, m$. Then for any $f \in \mathscr{L}$*

$$\left|f\left(\sum_{i=1}^m A_i^*B_i\right)\right|^2 \leqslant f\left(\sum_{i=1}^m A_i^*A_i\right) f\left(\sum_{i=1}^m B_i^*B_i\right) \tag{20}$$

and

$$\left|f\left(\sum_{i=1}^m A_i\right)\right|^2 \leqslant f\left(\sum_{i=1}^m |A_i|\right) f\left(\sum_{i=1}^m |A_i^*|\right). \tag{21}$$

If each A_i is normal, (21) *reduces to*

$$\left|f\left(\sum_{i=1}^m A_i\right)\right| \leqslant f\left(\sum_{i=1}^m |A_i|\right). \tag{22}$$

Since the Hadamard product (denoted by $A \circ B$) of block matrices of the type in (19) is a positive semidefinite block matrix (the Schur product theorem [4, Theorem 5.2.1], applying Bhatia's observation to Hadamard products instead of sums gives the following theorem.

Theorem 7. *Let $A_i, B_i \in M_n, i = 1, \ldots, m$. Then for any $f \in \mathscr{L}$ on M_n*

$$\begin{aligned} |f[(A_1^*B_1) \circ \cdots \circ (A_m^*B_m)]|^2 \leqslant\ & f[(A_1^*A_1) \circ \cdots \circ (A_m^*A_m)] \\ & f[(B_1^*B_1) \circ \cdots \circ (B_m^*B_m)] \end{aligned} \tag{23}$$

and

$$|f(A_1 \circ \cdots \circ A_m)|^2 \leqslant f(|A_1| \circ \cdots \circ |A_m|) f(|A_1^*| \circ \cdots \circ |A_m^*|). \tag{24}$$

If each A_i is normal, (24) *reduces to*

$$|f(A_1 \circ \cdots \circ A_m)| \leqslant f(|A_1| \circ \cdots \circ |A_m|). \tag{25}$$

The special case of (25) when $m = 2$ and f is a unitarily invariant norm was observed by Horn and Mathias [6, p. 76], where an example was given to show that the hypothesis of normality is essential.

A linear map $\phi : M_n \to M_m$ is said to be *positive* if $\phi(A) \succeq 0$ whenever $A \succeq 0$. We have the following theorem.

Theorem 8. *Let $\phi : M_n \to M_m$ be any positive linear map. Then*

$$|f[\phi(A)]| \leqslant f[\phi(|A|)] \tag{26}$$

for all $f \in \mathscr{L}$ on M_m and any normal $A \in M_n$. Conversely, if a nonsingular A satisfies (26) *for some pre-norm f on M_m and all positive linear maps $\phi : M_n \to M_m$, then A is normal.*

Proof. Suppose A is normal and $\phi : M_n \to M_m$ is a positive linear map. Let $A = UDU^*$ be a spectral decomposition with $U = (u_1, \ldots, u_n)$ unitary and $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then $A = \sum_{i=1}^n \lambda_i u_i u_i^*$, $|A| = \sum_{i=1}^n |\lambda_i| u_i u_i^*$, $\phi(A) = \sum_{i=1}^n \lambda_i \phi(u_i u_i^*)$, and $\phi(|A|) = \sum_{i=1}^n |\lambda_i| \phi(u_i u_i^*)$. Each $\phi(u_i u_i^*) \succeq 0$ since ϕ is positive and $u_i u_i^* \succeq 0$, so (22) with $A_i = \lambda_i \phi(u_i u_i^*)$ yields (26).

Conversely, let $A \in M_n$ be nonsingular and satisfy (26) for some pre-norm f on M_m and all positive linear maps $\phi : M_n \to M_m$. Let $U \in M_n$ be a unitary matrix such that $U^*AU = T$ is upper triangular. Given $B = [b_{ij}] \in M_n$, denote $D_i(B) \equiv \mathrm{diag}(b_{ii}, 0, \ldots, 0) \in M_m$. Define $\phi_i : M_n \to M_m$ by $\phi_i(X) \equiv D_i(U^*XU)$, $1 \leqslant i \leqslant n$. Then ϕ_i is a positive linear map, $\phi_i(A) = D_i(T)$, and $\phi_i(|A|) = D_i(|T|)$.

Write $T = [t_{ij}]$ and $|T| = [p_{ij}]$. We first consider ϕ_1. Since $|T|^2 = T^*T$, the Euclidean lengths of corresponding columns of $|T|$ and T are equal. Examining the respective first columns gives

$$\sum_{i=1}^n |p_{i1}|^2 = |t_{11}|^2. \tag{27}$$

On the other hand, (26) ensures that $f(\phi_1(A)) \leqslant f(\phi_1(|A|))$, so $|t_{11}| \leqslant p_{11}$, which together with (27) gives $p_{11} = |t_{11}|$ and $p_{21} = \cdots = p_{n1} = 0$. Thus $|T| = p_{11} \oplus P_{n-1}$. From $|T|^2 = T^*T$ we know that each non-diagonal entry in the first row of T^*T equals zero, i.e., $\bar{t}_{11} t_{1j} = 0, j = 2, \ldots, n$. But $t_{11} \neq 0$ since T is nonsingular, so $t_{1j} = 0, j = 2, \ldots, n$. Hence $T = t_{11} \oplus T_{n-1}$. Continuing this argument with $\phi_2, \ldots, \phi_{n-1}$ successively shows that T is diagonal and hence A is normal. $\square$

Given $\alpha \subseteq \{1, 2, \ldots, n\}$ and $A \in M_n$, let $|\alpha|$ denote the cardinality of α and let $A[\alpha]$ be the principal submatrix of A indexed by α. The map $\phi : M_n \to M_{|\alpha|}$ given by $\phi(A) \equiv A[\alpha]$ is linear and positive. Applying Theorem 8 gives

Corollary 5. *Let $A \in M_n$ be normal and $\alpha \subseteq \{1, 2, \ldots, n\}$. Then*

$$|f(A[\alpha])| \leqslant f(|A|[\alpha]) \tag{28}$$

for all $f \in \mathscr{L}$ on $M_{|\alpha|}$.

The special case of Corollary 5 when f is a unitarily invariant norm and A is Hermitian is in [12, Lemma 3].

Open Question: Can the hypothesis of nonsingularity of A in the second part of Theorem 8 be removed?

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