Ergodic Characterizations of Reflexivity of Banach Spaces

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Let $X$ be a Banach space with a basis. We prove the following characterizations:

(i) $X$ is finite-dimensional if and only if every power-bounded operator is uniformly ergodic.

(ii) $X$ is reflexive if and only if every power-bounded operator is mean ergodic.

(iii) $X$ is quasi-reflexive of order one if and only if for every power-bounded operator $T$, $T$ or $T^*$ is mean ergodic.

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1. INTRODUCTION

Using the spectral theorem, von-Neumann (1931) proved that for every unitary operator $T$ in a complex Hilbert space,

$$P_x := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} T^k x$$

exists $\forall x$.  \hfill (*)

A linear operator $T$ on a (real or complex) Banach space $X$ is called mean ergodic if (*) is satisfied, and uniformly ergodic if the convergence in
is uniform on the unit ball, i.e., \( \lim_{n \to \infty} \| \frac{1}{n} \sum_{k=1}^{n} T^k - P \| = 0 \). A Banach space \( X \) will be called mean ergodic if every power-bounded operator \( T \in B(X) \) satisfies (\( * \)).

A simple proof of von-Neumann’s mean ergodic theorem, due to F. Riesz, appeared in 1937 in Hopf’s *Ergodentheorie*, and was followed by more general results: Riesz (1938) showed that the \( L_p \) spaces (\( 1 < p < \infty \)) are mean ergodic, and Lorch (1939) proved that all reflexive Banach spaces are mean ergodic. In general Banach spaces, Kakutani (1938) and Yosida (1938) obtained characterizations of the convergence of the sequence \( \frac{1}{n} \sum_{k=1}^{n} T^k x \) for a given \( x \in X \). A power-bounded \( T \) in a Banach space is mean ergodic if and only if \( X \) has the following ergodic decomposition

\[
X = \{ y \in X : Ty = y \} \oplus (I-T)X. \tag{**}
\]

In general, the right-hand side of (\( ** \)) is precisely the set of \( x \in X \) for which the sequence \( \frac{1}{n} \sum_{k=1}^{n} T^k x \) converges. We denote by \( F(T) \) the set of fixed points of the linear operator \( T \). Sine (1970) proved that a power bounded \( T \) is mean ergodic if and only if \( F(T) \) separates \( F(T^*) \). We refer the reader to [K] for the proofs and for the references of the above results. Note that since \( F(T) \cap (I-T)X = \{ 0 \} \), the Hahn–Banach Theorem yields that \( F(T^*) \) always separates \( F(T) \).

Brunel and Sucheston [BSu_1], [BSu_2] proved that a Banach space \( X \) is super-ergodic if and only if it is super-reflexive. For the definitions see those papers. It is known [Da, p. 169] that super-reflexivity characterizes the existence of an equivalent norm, in which the space is uniformly convex, and thus also super-ergodicity characterizes that property. It follows from the work of Brunel and Sucheston that super-ergodicity with respect to contractions is the same as super-ergodicity with respect to power-bounded operators (operators which are contractions in some equivalent norm).

Sucheston [Su] posed the following question, concerning the converse of Lorch’s result: *If every contraction in a Banach space \( X \) is mean ergodic, is \( X \) reflexive?* The weaker assumption, that only all isometries are mean ergodic, is not sufficient for reflexivity, since Davis [D] had constructed an equivalent norm on the real \( \ell_1 \), for which the only isometries are \( I \) and \(-I\). Even under the stronger assumption, that all power-bounded operators are mean ergodic, i.e., \( X \) is mean ergodic, the problem is still unsolved.

In this paper, we obtain a positive solution to this last problem for Banach spaces with bases (throughout this paper, a basis means a Schauder basis). From this result we conclude that a Banach space \( X \) is reflexive if and only if every closed subspace is mean ergodic. Our construction also yields that a Banach space with basis is finite-dimensional if and only if every power-bounded operator is uniformly ergodic. We show that a non-reflexive Banach space with basis is 1-quasi-reflexive if and only if for every
power-bounded $T$, $T$ or $T^*$ is mean ergodic, and such a space is not mean ergodic.

Recently, Emel’yanov and Wolff [EW] have proved that on any (not necessarily separable) Banach space $X$ which contains $c_0$ there is a power-bounded operator which is not mean ergodic. Our methods yield a different proof of this result.

We mention that Eeml’yanov [E] proved that if every power-bounded operator on a Banach lattice $E$ is mean ergodic, then $E$ is reflexive. For a dual Banach lattice, Zaharopol [Z] proved that if all power-bounded positive operators are mean ergodic, then the Banach lattice is reflexive.

2. ERGODIC CHARACTERIZATIONS OF REFLEXIVITY AND 1-QUASI-REFLEXIVITY

**Definition 1.** A Schauder decomposition of a Banach space $X$ is an infinite sequence $\{E_k\}_{k=1}^\infty$ of closed subspaces $\{0\} \neq E_k \subset X$ such that each $x \in X$ has a unique representation $x = \sum_{k=1}^\infty x_k$, with $x_k \in E_k$, $(k = 1, 2, \ldots)$. We denote it by $X = \sum_{k} E_k$.

Note that the definition (see [S, vol. II, pp. 485–489]) does not require the spaces $E_k$ to be finite-dimensional.

When $X = \sum_{k} E_k$, the corresponding “coordinate” projectors $Q_k: X \to E_k$ are defined for $x = \sum_{k} x_k$ $(x_k \in E_k)$ by $Q_kx = x_k$ $(k = 1, 2, \ldots)$. The “partial sum” operators $P_n = \sum_{k=1}^n Q_k$ $(n = 1, 2, \ldots)$ satisfy $\lim_{n} P_n x = x$ for every $x$. An adaptation of the proof given in [S, vol. I, pp. 18–20] for bases shows that the partial sums operators are continuous and uniformly bounded (see [S, vol. II, pp. 499]). Hence also the coordinate projectors are continuous and uniformly bounded. By introducing the following norm (which is equivalent to the original one)

$$||x|| = \sup \{||Q_kx||, ||P_kx|| : k = 1, 2, \ldots\},$$

we get

$$||Q_k|| = ||P_k|| = 1 \quad \forall k \geq 1.$$  \hfill (1)

Since power-boundedness of a linear operator is the same in all equivalent norms, whenever necessary we may assume that the original norm $||\cdot||$ satisfies (1).
Definition 2. A Schauder decomposition \( X = \sum_{k=1}^{\infty} E_k \) is called shrinking if for each \( f \in X^* \) we have \( \lim_{n \to \infty} \| f \sum_{k=n}^{\infty} x_k \| = 0 \).

Note that if each subspace \( E_k \) of a Schauder decomposition is spanned by one vector \( e_k \), then \( \{e_k\} \) is a basis of \( X \), and when this decomposition is shrinking we call \( \{e_k\} \) a shrinking basis.

Lemma. Let \( X = \sum_k X_k \) be a non-shrinking Schauder decomposition of a Banach space \( X \). Then there is a Schauder decomposition \( X = \sum_k E_k \) with the following property: there exist a linear functional \( h \in X^* \) and a sequence \( \{e_k\} \), such that for every \( k \geq 1 \) we have \( e_k \in E_k, \|e_k\| \leq 1 \), and \( h(e_k) = 1 \).

Proof. Since the decomposition \( X = \sum_k X_k \) is not shrinking, there is a functional \( f \in X^* \) with \( \|f\| = 1 \) and \( \limsup_n \| f \sum_{k=n}^{\infty} x_k \| = a > 0 \) (obviously \( a \leq 1 \)). Take a vector \( y_1 \) such that
\[
y_1 = \sum_{k=n_1+1}^{\infty} a_k^{(1)} x_k^{(1)}, \quad x_k^{(1)} \in X_k, \quad \|y_1\| = 1, \quad |f(y_1)| > \frac{a}{2}.
\]

Find \( n_2 > n_1 \) with \( \| \sum_{k=n_2+1}^{\infty} a_k^{(1)} x_k^{(1)} \| < a/4 \), and take a vector \( y_2 \) such that
\[
y_2 = \sum_{k=n_2+1}^{\infty} a_k^{(2)} x_k^{(2)}, \quad x_k^{(2)} \in X_k, \quad \|y_2\| = 1, \quad |f(y_2)| > \frac{a}{2}.
\]

We continue inductively and obtain a strictly increasing sequence of integers \( \{n_j\} \) and a sequence of vectors \( \{y_j\} \), such that for each \( j \),
\[
y_j = \sum_{k=n_j+1}^{\infty} a_k^{(j)} x_k^{(j)}, \quad x_k^{(j)} \in X_k, \quad \|y_j\| = 1, \quad |f(y_j)| > \frac{a}{2},
\]
and \( \| \sum_{k=n_j+1}^{\infty} a_k^{(j-1)} x_k^{(j-1)} \| < \frac{a}{4} \).

Define \( E_1 = \sum_{k=1}^{n_1} X_k \), and \( E_j = \sum_{k=n_j+1}^{n_{j+1}} X_k \) for \( j \geq 2 \). Clearly \( \{E_j\} \) is a Schauder decomposition. Put \( z_j = \sum_{k=n_j+1}^{n_{j+1}} a_k^{(j)} x_k^{(j)} \). Then \( z_j \in E_j \), and, by the construction, \( 1 - \alpha/4 \leq \|z_j\| \leq 1 + \alpha/4 \) and \( \alpha/4 \leq |f(z_j)| \leq 1 + \alpha/4 \). Finally, let \( h = \frac{\alpha}{1 + \alpha/4} f \), and define \( e_j = \frac{\alpha}{1 + \alpha/4} z_j \). Then \( \|e_j\| \leq 1 \), and \( h(e_j) = 1 \) for every \( j \).

Theorem 1. If a Banach space \( X \) admits a non-shrinking Schauder decomposition, then there exists a power bounded linear operator \( T \in \mathcal{B}(X) \) which is not mean ergodic.
Proof. Let \( X = \sum_{k=1}^{\infty} E_k \) be the decomposition given by the Lemma, so we have \( h \in X^* \) and a sequence \( \{e_k\} \) such that \( e_k \in E_k \), \( h(e_k) = 1 \), \( \|e_k\| \leq 1 \), \( k = 1, 2, \ldots \). The change to a norm satisfying (1) yields that \( \|e_k\| \leq M \), so replace \( e_k \) by \( M^{-1} e_k \) and \( h \) by \( M h \). Thus, we can assume that the norm satisfies (1) (for the projectors defined by \( \{E_k\} \)).

Take an arbitrary sequence \( a = \{a_j\}_{j=1}^{\infty} \) of positive numbers with
\[
\sum_{j=1}^{\infty} a_j = 1, \quad a_j > 0, \quad j = 1, 2, \ldots, \tag{2}
\]
and denote \( A_n = \sum_{j=1}^{n} a_j \). For \( x \in X \) and \( m > n \geq 2 \) we then have
\[
\sum_{k=n}^{m} A_k Q_k x = \sum_{k=n}^{m} Q_k \left( \sum_{j=n}^{k-1} a_j + \sum_{j=n}^{k} a_j \right) x
= \left( \sum_{j=1}^{n-1} a_j \right) \left( \sum_{k=n}^{m} Q_k x \right) + \sum_{j=n}^{m} a_j \left( \sum_{k=j}^{m} Q_k x \right).
\]
Since \( \sum_k Q_k x \) converges, we see that \( \{\sum_{k=1}^{m} A_k Q_k x\}_m \) is a Cauchy sequence in the norm, hence converges. Denoting \( P_0 = 0 \), we obtain by (1) that
\[
\left\| \sum_{k=1}^{m} A_k Q_k x \right\| = \left\| \sum_{j=1}^{m} a_j \left( \sum_{k=j}^{m} Q_k x \right) \right\| = \left\| \sum_{j=1}^{m} a_j (P_m - P_{j-1}) x \right\| \leq 2 \|x\|. \tag{3}
\]

We now define an operator \( T_a : X \to X \) by
\[
T_a x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{j=2}^{\infty} h(P_{j-1} x) a_j e_j. \tag{4}
\]

Since \( \|e_j\| \leq 1 \) for every \( j \), (1), (2) and (3) yield \( \|T_a x\| \leq (2 + \|h\|) \|x\| \), so
\[
\|T_a\| \leq 2 + \|h\|. \tag{5}
\]

The bound \((2 + \|h\|)\) for \( \|T_a\| \) does not depend on the choice of the sequence \( \{a_j\} \) satisfying (2), so in order to prove that the operator \( T_a \) is power-bounded, it is enough to show that for sequences \( a \) and \( b \) satisfying (2), the composition \( T_a T_b \) is of the same type (say \( T_c \)). We formulate it precisely:

**Claim.** Let the sequences \( a = \{a_j\} \) and \( b = \{b_j\} \) satisfy (2), and define the operators \( T_a \) and \( T_b \) by (4) (with \( B_0 = 0 \) and \( B_n = \sum_{j=1}^{n} b_j \)). Then the sequence \( c = \{c_j\} \), defined by \( c_j = A_j b_j + B_{j-1} a_j \), \( j = 1, 2, \ldots \), satisfies (2), and the composition satisfies \( T_a T_b = T_c \).
Proof. Clearly $C_1 = A_1 B_1$. We obtain that $c$ satisfies (2), since for $n \geq 2$ we have
\[
C_n = \sum_{j=1}^{n} c_j = \sum_{j=1}^{n} (B_{j-1} a_j + b_j A_j) = \sum_{k=1}^{n-1} b_k \sum_{j=k+1}^{n} a_j + \sum_{k=1}^{n} b_k \sum_{j=1}^{k} a_j
\]
\[= \sum_{k=1}^{n-1} b_k A_n + b_n A_n = A_n B_n.
\]

Now we show that $T_a T_b = T_c$. In view of the decomposition $X = \sum_k E_k$, it is enough to show that $T_a T_b e = T_c e$ for each vector $e \in E_k$, $k = 1, 2, \ldots$ Fix $k$, and take $x_k \in E_k$. The definition (4) yields
\[
T_a x_k = A_k x_k + \sum_{j=k+1}^{\infty} h(x_k) a_j e_j = A_k x_k + h(x_k) \sum_{j=k+1}^{\infty} a_j e_j.
\]
(6)

We apply (6) to $T_b$ and to $T_a$, and obvious computations yield
\[
T_a (T_b x_k) = T_a \left( B_k x_k + h(x_k) \sum_{j=k+1}^{\infty} b_j e_j \right)
\]
\[= A_k B_k x_k + h(x_k) \sum_{j=k+1}^{\infty} (B_{j-1} a_j + b_j A_j) e_j.
\]

Since $A_k B_k = C_k$, an application of (6) to $c$ yields $T_a (T_b x_k) = T_c x_k$, and the claim is proved.

To prove that the power-bounded operator $T_a$ is not mean ergodic, it is enough to show (by the above mentioned Sine's criterion [K]) that the non-zero functional $h$ is a fixed point for $T_a^*$, while zero is the only fixed point for $T_a$.

Suppose that $T_a x = x$. Using the definition (4), we have
\[
\sum_{k=1}^{\infty} Q_k x = x = T_a x = \sum_{k=1}^{\infty} A_k Q_k x + \sum_{k=2}^{\infty} h(P_{k-1} x) a_k e_k.
\]

We look at the components in each $E_k$. For $k = 1$ we have $Q_1 x = A_1 Q_1 x$, so $Q_1 x = 0$ since $A_1 = a_1 < 1$. For $k > 1$ we obtain $(1 - A_k) Q_k x = h(P_{k-1} x) a_k e_k$. Assume now that $Q_j x = 0$ for every $j < k$; then $P_{k-1} x = 0$, and thus $(1 - A_k) Q_k x = 0$, yielding $Q_k x = 0$. Hence by induction we have $Q_k x = 0$ for every $k \geq 1$, so $T_a x = x$ implies $x = 0$. 
Fix an arbitrary $k \geq 1$ and take an arbitrary $e \in E_k$. Applying $h$ to (6) and using $h(e_j) = 1$ for every $j$, we obtain

$$(T^*_k h)(e) = h(T_a e) = h\left(A_x e + h(e) \sum_{j=k+1}^{\infty} a_j e_j\right)$$

$$= A_x h(e) + h(e) \sum_{j=k+1}^{\infty} a_j = h(e).$$

In view of the decomposition $X = \sum_k E_k$, we have $T^*_k h = h$. The Theorem is now proved.

**Remarks.** 1. Clearly, $\sum_{j=1}^{\infty} a_j P_{j-1} x$ converges in norm for $\{a_j\}$ satisfying (2), and the equality of the vector sums appearing in the first and third terms of (3) yields

$$\sum_{k=1}^{\infty} A_k Q_k x = \sum_{j=1}^{\infty} a_j (I - P_{j-1}) x = x - \sum_{j=1}^{\infty} a_j P_{j-1} x \quad \forall x \in X. \quad (7)$$

2. In fact, the functional $h$ of the previous proof is the only fixed point for $T^*_k$ (up to a scalar multiplier). We now prove this fact, though not needed for Theorem 1, since it will be important for Theorem 4.

So, we assume that $T^*_k f = f$, and prove that $f = th$ for some scalar $t$. With $P_0 = 0$, we can write (4) as

$$T_a x = \sum_{m=1}^{\infty} (A_m Q_m x + h(P_{m-1} x) a_m e_m).$$

Duality yields (with $w^*$-convergence of the series)

$$T^*_a f = \sum_{m=1}^{\infty} (A_m Q^*_m f + a_m f(e_m) P^*_m h). \quad (8)$$

Since $f = \sum_{m=1}^{\infty} Q^*_m f$ (again, $w^*$-convergence of the series), the assumption $T^*_a f = f$ and (8) yield

$$\sum_{m=1}^{\infty} [(1 - A_m) Q^*_m f - a_m f(e_m) P^*_m h] = 0. \quad (9)$$
Now fix an integer \( n \), and apply the functional of the left side of (9) to a vector \( z_n \in E_n \):

\[
\sum_{m=1}^{\infty} \left[ (1-A_m)(Q_m^*f)(z_n) - a_m f(e_m)(P_{m-1}h)(z_n) \right]
\]

\[
= \sum_{m=1}^{\infty} \left[ (1-A_m) f(Q_m z_n) - a_m f(e_m) h(P_{m-1}z_n) \right]
\]

\[
= (1-A_n) f(z_n) - \sum_{m=n+1}^{\infty} a_m f(e_m) h(z_n) = 0.
\]  \( \text{(10)} \)

Since \( h(e_n) = 1 \), (10) with \( z_n = e_n \) yields the following system of linear equations in the unknowns \( t_m = f(e_m) m = 1, 2, \ldots \):

\[
(1-A_n) t_n = \sum_{m=n+1}^{\infty} a_m t_m \quad n = 1, 2, \ldots. \]  \( \text{(11)} \)

Subtraction of equation number \( n \) from equation number \( (n+1) \) shows that the only solution of the system (11) is \( t_1 = t_2 = \cdots = t \), so

\[
f(e_n) = t = th(e_n), \quad n = 1, 2, \ldots. \]  \( \text{(12)} \)

In order to prove that \( f = th \), we show the equality on each \( E_n \). Fix \( x \in E_n \). Then \( h(x-h(x) e_n) = h(x) - h(x) h(e_n) = 0 \) since \( h(e_n) = 1 \). Denote \( z = x - h(x) e_n \). By (10) with \( z_n = z \) we have

\[
(1-A_n) f(z) - \sum_{m=n+1}^{\infty} a_m f(e_m) h(z) = 0.
\]

Since \( h(z) = 0 \), this yields \( f(z) = 0 \), so \( f(x) = h(x) f(e_n) = th(x) \).

**Corollary 1.** Let \( X \) be a (separable) Banach space with a basis. Then \( X \) is reflexive if and only if every power-bounded operator in \( X \) is mean ergodic.

**Proof.** Zippin [Zi] proved that if a non-reflexive Banach space has a basis, then it has a non-shrinking basis. Thus, if \( X \) is not reflexive, Theorem 1 yields a power-bounded operator which is not mean ergodic. If \( X \) is reflexive, apply Lorch’s Theorem.

**Corollary 2.** For every Banach space \( X \) the following assertions are equivalent:
(i) $X$ is reflexive.

(ii) Every closed subspace of $X$ is mean ergodic (i.e., each power bounded operator defined on a closed subspace is mean ergodic).

Proof. (ii) $\Rightarrow$ (i): Suppose that $X$ is non-reflexive. By a result of Pelczynski [Di, p. 54], $X$ has a non-reflexive (separable) closed subspace with a basis, and Corollary 1 yields a contradiction. (i) $\Rightarrow$ (ii) follows from Lorch’s Theorem, since a closed subspace of a reflexive Banach space is reflexive.

**Theorem 2.** If an infinite-dimensional Banach space $X$ admits a Schauder decomposition, then there is a mean ergodic power-bounded operator $T \in B(X)$ which is not uniformly ergodic.

Proof. We may assume that the norm satisfies (1). For a sequence $\{a_j\}$ satisfying (2), let $T_\alpha x = \sum_{k=1}^{\infty} A_k Q_k x$ ($T_\alpha$ is defined as in (4) with $h = 0$). By the proof of Theorem 1, $T_\alpha$ is power-bounded and has no fixed points except 0 (this part of the proof did not require the special properties of $h$, which were used only to show that $T_\alpha^*$ had $h$ as fixed point).

Let $f \in X^*$ satisfy $T_\alpha^* f = f$. Then for $z_n \in E_n$, we have

$$f(z_n) = f(T_\alpha z_n) = \sum_{k=1}^{\infty} A_k f(Q_k z_n) = A_n f(z_n).$$

Since $A_n < 1$ for each $n$, we have $f(z_n) = 0$ for any $z_n \in E_n$. Hence $T_\alpha^* f = f$ implies $f = 0$, which yields (Hahn–Banach) that $(I-T_\alpha) X = X$, so $T_\alpha$ is mean ergodic.

Since $T_\alpha$ has no non-zero fixed points, it is uniformly ergodic if and only if $I-T_\alpha$ is invertible on $X [L_2]$. By definition,

$$(I-T_\alpha) x = x - \sum_{k=1}^{\infty} A_k Q_k x = \sum_{k=1}^{\infty} (1-A_k) Q_k x = \sum_{k=1}^{\infty} \left( \sum_{j=k+1}^{\infty} a_j \right) Q_k x.$$

We now take $a_j = 2^{-j}$ for $j \geq 1$, and put $T = T_\alpha$. Then $(I-T) x = \sum_{k=1}^{\infty} 2^{-k} Q_k x$. Take a sequence $e_k \in E_k$ with $\|e_k\| = 1$ for every $k$. Then $\sum_{k=1}^{\infty} \frac{1}{k^j} e_k$ converges, say to $y$. When we try to solve $(I-T) x = y$, we obtain the equations $Q_k x = k^{-j} e_k$, which imply $\|Q_k x\| \to \infty$. Since for $x \in X$ we have $Q_k x \to 0$, there is no $x \in X$ with $(I-T) x = y$, so $I-T$ is not invertible, and therefore $T$ is not uniformly ergodic.
Remark. The existence of $T$ which is not uniformly ergodic in $X$ with an unconditional basis was proved in [FLR].

**Corollary 3.** Let $X$ be a Banach space with basis. Then the following conditions are equivalent:

1. $X$ is finite-dimensional.
2. Every power-bounded operator is uniformly ergodic.
3. Every mean ergodic power-bounded operator is uniformly ergodic.

**Proof.** Clearly, if $X$ is finite-dimensional, every power-bounded $T \in B(X)$ is uniformly ergodic. Obviously, (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i) follows from Theorem 2.

The ideas in the proof of Theorem 1 can be used to obtain the following generalization, which applies also to non-separable spaces:

**Theorem 3.** Let a Banach space $X$ admit a sequence of projectors $\{P_n\}$ such that

1. $\sup \|P_n\| < \infty$
2. $P_n P_m = P_{\min(m, n)}$
3. There exists a functional $h \in X^*$ such that for each $n \geq 1$ there is a vector $e_n \in (P_n - P_{n-1}) X$ with $\|e_n\| \leq 1$ and $h(e_n) = 1$.

Then, for a sequence $\{a_n\}$ which satisfies (2), the operator

$$Sz = x - \sum_{n=2}^{\infty} a_n P_{n-1} x + \sum_{n=2}^{\infty} a_n h(P_{n-1} x) e_n$$

is power-bounded and not mean ergodic.

**Proof.** It is immediate from the assumptions that $S$ is well defined. Denote $Y_n = P_n X$. By (ii), $\{Y_n\}$ is an increasing sequence of subspaces, and $Y = \bigcup_{n \geq 1} Y_n$ is a $S$-invariant subspace. By (ii), $\lim_n P_n y = y$ for $y \in Y_n$, so by (i) $\lim_n P_n y = y$ for every $y \in Y$. Let $Q_1 = P_1$, and $Q_k = P_k - P_{k-1}$ for $k \geq 2$. It is easily checked, using (ii), that each $Q_k$ is a projection, and $Q_k Q_j = 0$ for $j \neq k$. Since $\sum_{k=1}^{\infty} Q_k y = P_n y \rightarrow y$ for every $y \in Y$, the sequence $\{E_k\}$ with $E_k = Q_k X = Q_k Y$ is a Schauder decomposition of $Y$.

Assumption (iii) allows us to apply the proof of Theorem 1 to $Y$ — the restriction of $S$ to $Y$ is the operator $T_0$ constructed in that proof, when we substitute (7) into (4). Hence there is a vector $y \in Y$ such that the sequence $\{\frac{1}{n} \sum_{k=1}^{n} T_0^k y\}$ does not converge, which shows that $S$ is not mean ergodic.
To complete the proof, we have to show that $S$ is power-bounded on all of $X$ (this does not follow from the proof of Theorem 1, since $Y$ is not necessarily complemented in $X$).

Denote $S$ by $S_a$ to indicate the dependence on $\{a_j\}$ (which satisfies (2)). Clearly

$$\|S_a x\| \leq \|x\| + \|x\| \left(1 + \|h\|\right) \sup_n \|P_n\|$$

so we have an estimate of the norm of $S_a$, which is independent of $a$. As in the proof of Theorem 1, the power-boundedness follows from the following claim.

Claim. Let the sequences $a = \{a_j\}$ and $b = \{b_j\}$ satisfy (2), and define the sequence $c = \{c_j\}$ by $c_j = A_j b_j + B_{j-1} a_j$, $j = 1, 2, \ldots$. Then $\{c_j\}$ satisfies (2), and the operators $S_a$, $S_b$ and $S_c$ defined by (13) satisfy $S_a S_b = S_c$.

Proof. $\{c_j\}$ satisfies (2) by the claim in the proof of Theorem 1. Apply property (ii) to (13), to obtain

$$P_n(S_b x) = P_n x - \sum_{i=2}^{\infty} b_i P_{n-1} x + \sum_{i=2}^{\infty} b_i h(P_{n-1} x) P_n e_i$$

$$= P_n x - \sum_{i=1}^{\infty} b_i P_{n-1} x - (1-B_n) P_n x + \sum_{i=1}^{\infty} b_i h(P_{n-1} x) e_i.$$ 

We substitute this into

$$S_a(S_b x) = S_b x - \sum_{n=2}^{\infty} a_n P_{n-1}(S_b x) + \sum_{n=2}^{\infty} a_n h(P_{n-1}(S_b x)) e_n,$$

and some straightforward (tedious) calculations prove the claim.

Corollary 4. Let $X$ be a Banach space which contains a closed subspace isomorphic to $c_0$. Then there exists a power-bounded $T \in B(X)$ which is not mean ergodic.

Proof. Let $Y$ be a closed subspace of $X$ isomorphic to $c_0$, and let $y_n \in Y$ be the image of the of the unit vector $e_n \in c_0$. Then $\{y_n\}$ is a basis of $Y$, and there is $K > 0$ such that $\|\sum_{j=1}^{\infty} a_j y_j\| \leq K \sup_j |a_j|$. Let $\{y^*_n\} \subset Y^*$ be the coefficient functionals, which are uniformly bounded, and take $f_n \in X^*$ a Hahn–Banach extension of $y^*_n$. We now define $x_n = \sum_{j=1}^{\infty} y_j$ and $g_n = f'_n - f_{n+1}$. Then $g_n(x_n) = \delta_{kn}$, and the operators $P_n x = \sum_{k=1}^{n} g_k(x) x_k$ are commuting projections satisfying assumption (ii) of Theorem 3. The
functional \( h = f_1 \) satisfies assumption (iii) since \( h(x_n) = 1 \). Finally, the isomorphism of \( Y \) and \( c_0 \) yields that \( \sup_n \| P_n \| \leq 2K \sup_n \| f_n \| \), since

\[
P_n x = f_1(x) y_1 - f_{n+1}(x) x_n + \sum_{k=2}^{n} f_k(x)(x_k - x_{k-1})
\]

\[
= \sum_{k=1}^{n} [f_k(x) - f_{n+1}(x)] y_k.
\]

**Remark.** The Corollary was first proved in [EW] using a different method. Note that if \( X \) is separable (as any space with a basis is) and contains \( c_0 \), then (even without a basis), we easily obtain a power-bounded operator \( T \in B(X) \) which is not mean ergodic, since \( c_0 \) is complemented in \( X \) [Di, p. 71], and \( T_0 \in B(c_0) \) defined by \( T_0(a_1, a_2, a_3, \ldots) = (a_1, a_1, a_2, \ldots) \) (i.e., \( T_0(\sum_{j=1}^{\infty} a_j e_j) = a_1 e_1 + \sum_{j=2}^{\infty} a_{j-1} e_j \)) in terms of the standard basis \( \{e_j\} \) is power-bounded and not mean ergodic (\( T_0 \) has no non-zero fixed points in \( c_0 \), but \( T_0 e_1 = e_1 \) in \( \ell_1 \)). Thus, the novelty of the result is for non-separable spaces, in which \( c_0 \) need not be complemented.

For a basis \( \{x_i\} \) of a Banach space \( X \), we denote by \( \{x^*_i\} \) the associated coefficient functionals. Recall [S, vol. I p. 268] that a basis \( \{x_i\} \) is called \( k \)-shrinking if \( \text{codim} \{y_j\}_{j=1}^{\infty} = k \) (where \( \{y_j\}_{j=1}^{\infty} \) denotes the closed linear manifold generated by the sequence \( \{y_j\}_{j=1}^{\infty} \)). It is well known [Da], [S, vol. I p. 272] that a basis is 0-shrinking if and only if it is shrinking in the sense of Definition 2.

**Definition 3.** A Banach space \( X \) is called quasi-reflexive of order \( k \) if \( \dim X^{**}/X < k < \infty \) (we identify \( X \) with its natural embedding in \( X^{**} \)). The original construction of the James space [Ja], valid over the real or complex field, yields an example of a Banach space with basis which is quasi-reflexive of order 1.

**Theorem 4.** Let \( X \) be a Banach space with a basis, such that \( \dim X^{**}/X \geq 2 \). Then there exists a power-bounded operator \( T \in B(X) \) such that neither \( T \) nor \( T^* \) are mean ergodic.

**Proof.** According to Zippin’s result [Zi] mentioned above, the (non-reflexive) space \( X \) has a non-shrinking basis, say \( \{u_i\} \); that is, \( \{u_i\} \) is a basis which is not 0-shrinking. If \( \{u_i\} \) is \( k \)-shrinking with \( k \geq 2 \), we keep it. If \( \{u_i\} \) is 1-shrinking, we use Theorem 1 of [DeLS]: Let \( X \) be a Banach space which is not quasi-reflexive of order \( k \) (in our case \( \dim X^{**}/X \geq 2 \), so \( X \) is not quasi-reflexive of order 1). If \( X \) has a \( k \)-shrinking basis, then \( X \) has a
Thus, we have established that there exists in $X$ a basis $\{x_i\}$ such that

$$\text{codim } [x_i^*]_{i-1}^{\infty} \geq 2.$$  \hfill (14)

Since this basis $\{x_i\}$ is not shrinking, the Lemma (with $X_k = \{tx_k : t \in \mathbb{R}\}$) yields a Schauder decomposition $X = \sum_k E_k$ with the following property: there exist a functional $h \in X^*$ and a sequence $\{e_k\}$, $e_k \in E_k$, $k = 1, 2, \ldots$ such that $h(e_k) = 1$, $\|e_k\| \leq 1$, $k = 1, 2, \ldots$ By the construction in the proof of the Lemma, each $E_k$ is finite-dimensional, and the decomposition $X = \sum_k E_k$ has the following additional property: the “partial sum” operators $P_m$ are of the form

$$P_m x = \sum_{i=1}^{n_m} x_i^*(x) x_i, \quad x \in X, \quad m = 1, 2, \ldots$$

This yields $P_m f = \sum_{i=1}^{n_m} f(x_i) x_i^*$ for $f \in X^*$, and so, $\bigcap_m \ker P_m^* = [x_i^*]_{i-1}^{\infty}$. By (14),

$$\dim \bigcap_{m=1}^{\infty} \ker P_m^* = \dim(X^*/[x_i^*]_{i-1}^{\infty})^* \geq \text{codim } [x_i^*]_{i-1}^{\infty} \geq 2.$$  \hfill (15)

We now proceed as in the proof of Theorem 1. For a sequence $a = \{a_j\}$ satisfying (2), define the operator $T_a$ according to (4). It was shown that $T_a$ is power-bounded and not mean ergodic, $F(T_a) = \{0\}$, and $F(T_a^*) = \{th\}$. We will choose $\{a_j\}$ satisfying (2) such that $\sum_{n=1}^{\infty} (1 - A_n) < \infty$ (e.g., $a_j = 2^{-j}$). Since $\dim F(T_a^*) = 1$, to prove that the operator $T_a^*$ is not mean ergodic we have to show (by Sine’s criterion) that $\dim F(T_a^*) \geq 2$. By (15), it is enough to show

$$F(T_a^*) = \bigcap_{m=1}^{\infty} \ker P_m^*.$$  \hfill (16)

From (4), (1), and the condition $\sum (1 - A_n) < \infty$, it follows that

$$T_a^* \psi = \psi + \sum_{m=1}^{\infty} (A_m - 1) Q_m^* \psi + \sum_{m=2}^{\infty} a_m P_m^* \psi(e_m).$$

Hence $\psi \in F(T_a^*)$ is equivalent to

$$\sum_{m=1}^{\infty} (1 - A_m) Q_m^* \psi = \sum_{m=2}^{\infty} a_m P_m^* \psi(e_m).$$  \hfill (17)
If $P^*_n \psi = 0$ for every $n \geq 1$, then clearly (17) holds, so $\psi \in F(T^{**})$. Suppose now that $\psi \in F(T^{**})$; we apply the operators $Q^*_n$ to both sides of (17), and obtain the equations $(1 - A_n) Q^*_n \psi = a_n P^*_n \psi \mathbf{e}_n$, $n = 1, 2, \ldots$. Solving successively, we obtain $Q^*_n \psi = 0$ for $n \geq 1$, which proves (16) and completes the proof of the theorem.

**Remark.** Every operator $T$ on $X$ is the restriction of $T^{**}$ to its invariant subspace $X$, so if $T^{**}$ is mean ergodic, so is $T$. Hence, if both operators $T$ and $T^{*}$ are not mean ergodic, then automatically all the next conjugates $(T^{**}, T^{***}, \ldots)$ are not mean ergodic.

**Theorem 5.** Let $X$ be a non-reflexive Banach space with a basis. Then the following assertions are equivalent:

(i) $X$ is quasi-reflexive of order one.

(ii) For each power-bounded operator $T \in B(X)$, $T$ or $T^{*}$ is mean ergodic.

**Proof.** (ii) $\Rightarrow$ (i): If dim $X^{**}/X \geq 2$, then Theorem 4 yields a contradiction to (ii).

(i) $\Rightarrow$ (ii): Let $T$ be a power-bounded operator on $X$ which is not mean ergodic. By Sine’s criterion, $F(T)$ does not separate $F(T^{*})$, so there is $f_0 \in F(T^{*})$ such that $f_0(y) = 0$ for every $y \in F(T)$. To show that $T^{*}$ is mean ergodic, we will prove that $F(T^{*})$ separates $F(T^{**})$. As mentioned in the introduction, $F(T^{*})$ always separates $F(T)$. Hence $F(T^{**})$ separates $F(T^{*})$, so there is $\psi_0 \in F(T^{**})$ such that $\psi_0(f_0) \neq 0$. By the definition of $f_0$, $\psi_0$ is not in $F(T)$, so $\psi_0 \notin X$. Since dim $X^{**}/X = 1$, every $\phi \in X^{**}$ is of the form $\phi = ax_0 + x$ with $x \in X$, so each $\psi \in F(T^{**})$ is of the form $\psi = ax_0 + y$ with $y \in F(T)$. We then have $\psi(f_0) = ax_0(f_0) \neq 0$ for $\psi \in F(T^{**})$ with $a \neq 0$. If $a = 0$, then $\psi$ is in $F(T)$, and the separation of $F(T)$ by $F(T^{*})$ provides an $f \in F(T^{*})$ with $\psi(f) = f(\psi) \neq 0$. Hence $F(T^{*})$ separates $F(T^{**})$, so $T^{*}$ is mean ergodic by Sine’s criterion.

**Remark.** The implication (i) $\Rightarrow$ (ii) does not require a basis for $X$.

If $T \in B(X)$ is power-bounded, then it is easily shown that

$$\left( I - T \right) X \subset \left\{ y : \sup_n \left\| \sum_{k=1}^n T^k y \right\| < \infty \right\} \subset \left( I - T \right) X.$$  

When $T$ is uniformly ergodic, then $[L_2] \left( I - T \right) X$ is closed, which yields

$$\left( I - T \right) X = \left\{ y : \sup_n \left\| \sum_{k=1}^n T^k y \right\| < \infty \right\}.  \quad (18)$$
If $X$ is a dual space and $T$ is a power-bounded dual operator, then (18) holds \([L_1]\). It now follows from Theorem 2 that in every infinite-dimensional reflexive Banach space $X$ with a basis there is a power-bounded $T$ which is not uniformly ergodic, but satisfies (18). It was shown in [FLR] that if $X$ is a separable Banach space which does not contain infinite-dimensional dual spaces, then (18) implies uniform ergodicity. This result is true also in complex Banach spaces, since the needed result of [F$_3$], stated for real spaces, is valid also in complex spaces, with the same proof.

**Proposition.** Let $Z$ be an infinite-dimensional Banach space which is the dual of a separable Banach space (e.g., $Z$ is a separable dual space). Then there exists an infinite-dimensional Banach space $E$ with a basis such that $E^*$ is isomorphic to a closed subspace of $Z$.

**Proof.** This proposition is an immediate consequence of the results of \([JR]\): Let $F$ be separable, with $F^* = Z$. Since the unit ball of $Z$ is compact in the weak-* topology and not in the norm, there is a sequence $\{y_n\}$ in $Z$ which is weak-* convergent to 0, such that $\limsup_n \|y_n\| > 0$. Combining Theorem III.1 and Proposition II.1(a) of \([JR]\), we obtain a separable Banach space $E$ with a basis, such that $E^*$ is isometrically isomorphic to the weak-* closed subspace generated in $Z$ by a subsequence $\{y_{n_k}\}$.

**Theorem 6.** Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ does not contain an infinite-dimensional closed subspace isomorphic to the dual of a separable Banach space.

(ii) Every power-bounded operator $T$ defined on a closed subspace $Y$, which satisfies $F(T) = \{0\}$ and $(I-T)Y = \{y \in Y : \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$, is uniformly ergodic.

If $X$ is separable, each of the previous conditions is equivalent to

(iii) Every power-bounded operator $T$ defined on a closed subspace $Y$ which satisfies $(I-T)Y = \{y \in Y : \sup_n \|\sum_{k=1}^n T^k y\| < \infty\}$ is uniformly ergodic.

**Proof.** The proof of (i) $\Rightarrow$ (ii) is the same as that of Corollary 3.4(ii) of \([FLR]\), noting that the results of \([F_1]\), \([F_2]\] used there yield a dual of a separable space. For the complex case, we observe that the proof of Proposition 6.7 in \([FLiP]\) is valid also for complex Banach spaces, and it implies the required result of \([F_1]\).

We now assume that (ii) holds. If (i) does not hold, then $X$ has an infinite-dimensional closed subspace $Z$ which is isomorphic to the dual $F^*$.
of a separable Banach space $F$. By the Proposition, there is an infinite-dimensional Banach space $E$ with a basis, such that $E^*$ is isomorphic to a subspace of $F^*$. Hence $E^*$ is isomorphic to a closed subspace of $Z$, say $Y$. By Theorem 2, there is a power-bounded $S \in B(E)$ which is not uniformly ergodic, with $F(S^*) = \{0\}$. Let $T \in B(Y)$ correspond to $S^*$. Then \((I - T) Y = \{ y \in Y : \sup_n \| \sum_{k=1}^n T^k y \| < \infty \}\) by [L_1], but $T$ is not uniformly ergodic since $S$ is not contradicting (ii). Hence (i) must hold.

When $X$ is separable, the proof of (i) $\Rightarrow$ (iii) runs along the lines of the proof of Theorem 3.3 of [FLR], applied to any closed subspace $Y$ (which also satisfies (i)). For the complex case, in the proof of [FLR] we should replace $[F_1]$ by Theorem 3.2 of [FLi], the proof of which is valid also for complex Banach spaces. Clearly (iii) $\Rightarrow$ (ii).

REFERENCES


