Some Results of Numerical Investigation of the Carleman System

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Abstract

The Cauchy problem for one-dimensional Carleman system with stationary normal stochastic processes as initial data is considered. The Carleman system is a model system of the kinetic theory of gases. Results of numerical investigation of the Cauchy problem for the studied system are presented and discussed.

Keywords: Carleman system; Stochastic process; Solution stabilization

1. Introduction

The Cauchy problem of one-dimensional Carleman system is considered. The Carleman system is a discrete kinetic model of one-dimensional gas consisting of identical molecules. There are two groups of molecules. Molecules of the same group have equal speed. Speeds of different groups have equal values and opposite directions. The Carleman system is a model problem of kinetic theory of gasses. It has the main properties inhered to the Boltzmann equation [1]. The Carleman system is a nonlinear hyperbolic system of partial differential equations. It has been well studied [2-4] for non-stochastic initial data. In general case there is no an analytical solution of the system. It explains the importance of numerical investigation of the Cauchy problem to Carleman system with stochastic processes as initial conditions. As a rule, the practical interest is not only stochastic solution but its statistical characteristics such as mathematical expectation, mean square deviation, distribution, correlation function.

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2. Formulation of the problem

We consider the Cauchy problem for the Carleman system

\[
\begin{align*}
\partial_t u + \partial_x u &= \frac{1}{\varepsilon}(w^2 - u^2), \
\partial_t w - \partial_x w &= -\frac{1}{\varepsilon}(w^2 - u^2),
\end{align*}
\]

(1)

with initial data

\[
\begin{align*}
u|_{t=0} &= u^0(x), \\w|_{t=0} &= w^0(x).
\end{align*}
\]

(2)

Here \(\varepsilon\) is a small parameter which is interpreted as the Knudsen number in kinetics. Initial data \(u^0(x)\) and \(w^0(x)\) are stationary normal stochastic processes. In this case the solution of problem (1), (2) is a pair of stationary processes for fixed time value \(t\). Thus, for any fixed time variable \(t\) mathematical expectations and mean square deviations of \(u(t,x)\) or \(w(t,x)\) are constants. For numerical investigation of statistical characteristics of the solution was applied the method described in [5, 6].

3. Results of numerical investigation

Consider the Cauchy problem (1), (2) for the initial conditions, which are stationary normal random processes with different mathematical expectations and the same correlation function. The mathematical expectation of \(u^0(x)\) \(\mathbb{M}u^0(x) = 1,2\) and the mathematical expectation of \(w^0(x)\) \(\mathbb{M}w^0(x) = 1\). The correlation function \(R(t)\) of initial processes \(u^0(x)\) and \(w^0(x)\) is equal \(\alpha e^{\alpha t}\). Figure 1 shows initial random processes \(u^0(x)\) and \(w^0(x)\). Figures 2, 3 shows the results of numerical solutions of problem (1), (2) \(u(t, x)\) and \(w(t, x)\) for the values of the time variable \(t_1=0,02, t_2=0,08, \varepsilon=0,05, \alpha=\varepsilon\). In the initial time period \(0 < t < t^*\) there is a rapid redistribution of molecules between the two groups (Fig. 2). At \(t=t^*\) the mathematical expectation of \(u(t, x)\) and \(w(t, x)\) are equal. At \(t \geq t^*\) \(u(t, x)\) and \(w(t, x)\) are almost the same \(u(t, x) \approx w(t, x+\Delta x)\) (Fig. 3). However, there is some shift between \(u(t, x)\) and \(w(t, x)\). The shift decreases with decreasing small parameter \(\varepsilon\).

Figure 4 presents the dependence of the mathematical expectation of \(u(t, x)\) and \(w(t, x)\) from time variable \(t\). For \(0 < t < t^*\) there are rapid decreasing \(\mathbb{M}u(t, x)\) and increasing \(\mathbb{M}w(t, x)\). Graphs of mathematical expectations \(\mathbb{M}u(t, x)\) and \(\mathbb{M}w(t, x)\) are symmetric about the equilibrium state \(u^*(x) = w^*(x) = 0,5(\mathbb{M}u^0(x)+\mathbb{M}w^0(x))=1,1\). For \(t \geq t^*\) mathematical expectations of \(u(t, x)\) and \(w(t, x)\) are equal \(\mathbb{M}(u(t, x) = w(t, x) = 1,1, \ t \geq t^*)\).

For the periodic initial data solution stabilization to the equilibrium state was proved [2]. There is the solution stabilization for stochastic initial data. For \(t \geq t^*\) mathematical expectations of \(u(t, x)\) is equal \(u^*(x)\) and mathematical expectation of \(w(t, x)\) is equal \(w^*(x)\). Figure 5 presents the dependence of mean square deviations of \(u(t, x)\) and \(w(t, x)\) from equilibrium state \(u^*(x) = 1,1, \ w^*(x) = 1,1\).

So, we can see solution stabilization.
Fig. 1. Initial random processes $u(0, x) = u_0(x)$ and $w(0, x) = w_0(x)$.

Fig. 2. Numerical solution $u(t_1, x)$ and $w(t_1, x)$, $t_1 = 0.02$.

Fig. 3. Numerical solution $u(t_2, x)$ and $w(t_1, x)$, $t_2 = 0.08$. 
4. Conclusion

We present the results of numerical investigation of the solution of the Cauchy problem for the Carleman system with stochastic initial conditions. Based on numerical investigation we can conclude that in the case of considered stationary stochastic initial conditions there is a stabilization of the solution to the equilibrium state.

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References


