Dimensional Analysis and the Pi Theorem

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ABSTRACT

A new version of the Buckingham pi theorem is presented which reveals the underlying mathematical structure of that classical result. In this context it becomes a theorem in linear algebra, and it is formulated without reference to physical quantities, units, dimensions, and so on. Also, the classical approach of Birkhoff is reviewed and some points in his proof are expanded.

1. INTRODUCTION

As all students of science and engineering know, equations must be dimensionally homogeneous; that is, all terms in an equation must have the same units—one cannot add apples and oranges. This simple observation forms the basis of what is called dimensional analysis. But it goes far deeper than that. The methods of dimensional analysis developed over the last century or so have led to important results in determining the nature of physical phenomena even when the governing equations were not known. This has been especially true in continuum mechanics, out of which the general methods of dimensional analysis evolved.

The most fundamental result in dimensional analysis is the pi theorem. Roughly it states that if there is a physical law which gives a relation among certain physical quantities, then there is an equivalent law which is expressed as a relation among certain dimensionless quantities (often denoted $\Pi_1, \Pi_2, \ldots$, and hence the name). One of the most famous examples of dimensional reasoning was the derivation, by G. I. Taylor [8], of the formula

$$ r = t^{2/5} \left(\frac{E}{\rho_0} \right)^{1/5} f(\gamma) \qquad (1) $$

which relates the radius r of a spherical blast wave produced by the release of a quantity of energy E, at a point in air of density ρ_0 and polytropic index γ, to the two-fifths power of the time t. Equation (1) follows from the assumption that there is a physical law of the form $g(t, r, \rho_0, E, \gamma) = 0$. The pi theorem guarantees that there is an equivalent physical law relating the dimensionless quantities in the problem. Here there are two dimensionless quantities,

$$ \Pi_1 = \gamma, \qquad \Pi_2 = \frac{r^5 \rho_0}{t^2 E}. \qquad (2) $$

Hence there is a physical law, equivalent to $g = 0$, relating Π_1 and Π_2. This new law is then of the form $F(\Pi_1, \Pi_2) = 0$, from which we obtain (1). The pi theorem is also widely used in computing "dimensionless groups," i.e., local Lie groups under which partial differential equations are invariant; these give rise to special classes of solutions called similarity solutions which play an important role in many applications (see Bluman and Cole [2]).

The pi theorem appears to have been first stated by A. Vaschy [9] in 1892. Later, in 1914, E. Buckingham [4] gave the first proof of the pi theorem for special cases, and now the theorem often carries his name. Riabouchinsky and Martinot-Lagarde [6] have given a more general proof, and G. Birkhoff [1] has clarified the proof still further. There has been much discussion of the formulation and applicability of the pi theorem. Bridgman [3] and Birkhoff [1], two standard references, can be consulted for further details and bibliography.

A difficulty with existing formulations and proofs of the pi theorem is that there appears to be a dependence on "physical" terminology which is not explained precisely. This tends to obscure the mathematical content of the theorem. No clear distinction is made between the mathematical content and the parts which serve to relate the mathematical result to the physical world.

We propose in this note to give a careful formulation and proof of the pi theorem. In Section 2 we present an algorithm which is an effective procedure for reducing a dimensionally homogeneous physical law involving dimensional quantities $Q_1, \ldots, Q_m$ to an equivalent law involving a (smaller) number of dimensionless variables $\Pi_{r+1}, \ldots, \Pi_m$. The algorithm is clear and easy to apply, and we illustrate it by an example. Then in Section 3 we formulate the abstract mathematical version of the pi theorem in a way that involves no discussion of dimension, physical quantities, etc. Our formulation, which is new, distinguishes between dimensional quantities and their real, numerical values. Bridgman emphasizes the desirability of such a distinction; however, neither he nor Birkhoff thoroughly accomplish this. In addition, we formulate precisely, in terms of linear algebra concepts, what we mean by a physical law. Although our definition coincides with that of Birkhoff, or more recently Evans [5], in meaning, it differs considerably in spirit, expression, and structure.

2. THE ALGORITHM

The following formulation and algorithm can essentially be found in Birkhoff [1]. We have included it for motivation for our abstract version of the pi theorem, and we have lengthened his proof in order to clarify some difficult points. Here we shall speak in physical terms and not distinguish between dimensionless quantities and the numerical values which these quantities assume.

First we consider a "physical law" $f(Q_1, \ldots, Q_m) = 0$ relating dimensional quantities $Q_1, \ldots, Q_m$. For example, in Taylor's blast wave problem these quantities are t, r, E, ρ_0, and γ. Presently, the only assumption concerning f is that it is defined for $Q_i > 0$, and it gives a single, well-defined relation among $Q_1, \ldots, Q_m$. Later we shall require an additional assumption.

The dimensions of a dimensional quantity Q_i can be written in a natural way in terms of certain fundamental dimensions $q_1, \ldots, q_n$, appropriate to the problem being studied. For instance, in the blast wave problem, time T, length L, and mass M are the fundamental dimensions and the dimensions of each quantity can be expressed in terms of T, L, and M; thus the dimensions of energy E are ML^2T^{-2}. In general, we make the following definition.

DEFINITION 1. The dimension of each dimensional quantity Q, expressed as a monomial with real exponents in the $q_1, \ldots, q_n$, is called the dimension monomial of Q and denoted $[Q]$. Thus $[Q] = q_1^{r_1} q_2^{r_2} \cdots q_n^{r_n}$ for some choice of

$r_1, \ldots, r_n$. We say Q is *dimensionless* if $[Q] = q_1^0 q_2^0 \cdots q_n^0$, and we write $[Q] = 1$.

We multiply monomials in the usual way,

$$\left(q_1^{r_1} q_2^{r_2} \cdots q_n^{r_n}\right)\left(q_1^{s_1} \cdots q_n^{s_n}\right) = q_1^{r_1+s_1} q_2^{r_2+s_2} \cdots q_n^{r_n+s_n}$$

Then the basic property of the correspondence $Q \to [Q]$ can be stated as $[Q_1 Q_2] = [Q_1][Q_2]$.

We now present an *algorithm* for effectively determining the dimensionless quantities which can be formed among $Q_1, \ldots, Q_m$. For each of the dimensional quantities Q, we have

$$\left[Q_j\right] = q_1^{a_{1j}} q_2^{a_{2j}} \cdots q_n^{a_{nj}}$$

The powers in the dimension-monomial define a matrix $A = (a_{ij})$, called the *dimension matrix*. Then, a quantity Q formed from $Q_1, \ldots, Q_m$ by

$$Q = Q_1^{\alpha_1} Q_2^{\alpha_2} \cdots Q_m^{\alpha_m}$$

is dimensionless if, and only if,

$$A\alpha = 0, \tag{5}$$

where $\alpha = [\alpha_1, \ldots, \alpha_m]^T$, and 0 is the zero vector. Letting a_j denote the jth column of A, we see that (5) is equivalent to

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m = 0. \tag{6}$$

Let A have rank r. We may reorder the Q_j so that the columns $a_1, \ldots, a_r$ are linearly independent. Then $a_{r+1}, \ldots, a_m$ are linear combinations of $a_1, \ldots, a_r$, and we may write

$$a_k = c_{k1} a_1 + \cdots + c_{kr} a_r, \qquad k = r+1, \ldots, m. \tag{7}$$

Now, define Π_k for $k > r$ by

$$\Pi_k = Q_1^{-c_{k1}} Q_2^{-c_{k2}} \cdots Q_r^{-c_{kr}} Q_k. \tag{8}$$

Each Π_k is dimensionless, since the vector

$$\alpha_k = \left[-c_{k1}, \ldots, -c_{kr}, 0, \ldots, 1, \ldots, 0 \right]^T$$

(1 is in position k) satisfies (6) by (7). Consequently, among the m dimensional quantities $Q_1, \ldots, Q_m$, we have shown that $m - r$ dimensionless quantities can be formed, where $r = \operatorname{rank} A$.

We now show, under certain assumptions, that the physical law $f(Q_1, \ldots, Q_m) = 0$ is equivalent to a physical law written only in terms of the dimensionless quantities $\Pi_{r+1}, \ldots, \Pi_m$.

Let $R_+^m = \{(x_1, \ldots, x_m) \in R^m \mid \text{each } x_i > 0\}$. Having defined the Π_k in (8), we now define a transformation $\Phi: R_+^m \to R_+^m$ by $\Phi(Q_1, \ldots, Q_m) = (Q_1, \ldots, Q_r, \Pi_{r+1}, \ldots, \Pi_m)$. Clearly Φ is one to one and onto. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in R_+^n$, and let L be a dimensional quantity with $[L] = q_1^{b_1} \cdots q_n^{b_n}$. Define $S_\lambda(L) = \lambda_1^{b_1} \cdots \lambda_n^{b_n} L$. We regard the λ_i as dimensionless (in practice they are just conversion factors), so that $[S_\lambda(L)] = [L]$.

Now, consider the physical law

$$f(Q_1, \ldots, Q_m) = 0.$$

For each $\lambda \in R^n$ define a new law

$$(S_\lambda f)(Q_1, \ldots, Q_m) = 0$$

where

$$(S_\lambda f)(Q_1, \ldots, Q_m) = f(S_\lambda(Q_1), \ldots, S_\lambda(Q_m)).$$

DEFINITION 2.   The law $f(Q_1, \ldots, Q_m) = 0$ is *unit free* if for all λ the laws $f = 0$ and $S_\lambda(f) = 0$ are equivalent, i.e., $f(Q_1, \ldots, Q_m) = 0$ if, and only if, $(S_\lambda f)(Q_1, \ldots, Q_m) = 0$.

This is a reasonable definition. $S_\lambda(Q)$ is just Q "measured in different units," so it expresses the fact that a physical law should not depend on the units to express the various quantities. Given $f: R_+^m \to R$, let $g = f \circ \Phi^{-1}: R_+^m \to R$. So $f(Q_1, \ldots, Q_m) = 0$ if, and only if, $g(Q_1, \ldots, Q_r, \Pi_{r+1}, \ldots, \Pi_m) = 0$. Thus, beginning with the law $f(Q_1, \ldots, Q_m) = 0$, we can construct the law $g(Q_1, \ldots, Q_r, \Pi_{r+1}, \ldots, \Pi_m) = 0$ which is, in an obvious sense, an equivalent law.

Having constructed dimensionless quantities and now an equivalent physical law, we prove the following:

LEMMA 1.   *If the law $f(Q_1, \ldots, Q_m) = 0$ is unit free, so is the law* $g(Q_1, \ldots, Q_r, \Pi_{r+1}, \ldots, \Pi_m) = 0$.

Proof. Let us pick a λ. We must show that $g(Q_1,\ldots,Q_r,\Pi_{r+1},\ldots,\Pi_m)$ $= 0$ if and only if $g(S_\lambda(Q_1),\ldots,S_\lambda(\Pi_m)) = 0$. We know $g(Q_1,\ldots,\Pi_m) =$ $f(Q_1,\ldots,Q_m)$. If we show that

$$g(S_\lambda(Q_1),\ldots,S_\lambda(\Pi_m)) = f(S_\lambda(Q_1),\ldots,S_\lambda(Q_m)), \qquad (9)$$

we shall be done, since $f = 0$ is assumed unit free. But one can easily check that for $k = r+1,\ldots,m$ we have

$$S_\lambda(\Pi_k) = \Pi_k = S_\lambda(Q_1)^{-c_{k1}} \cdots S_\lambda(Q_r)^{-c_{kr}} S_\lambda(Q_k), \qquad (10)$$

and (9) is immediate from (10). This proves Lemma 1.   ∎

In the next lemma we show that the physical law $f(Q_1,\ldots,Q_m) = 0$ is equivalent to an equation relating only the dimensionless variables. This lemma is the content of the pi theorem.

LEMMA 2.   *If the law $f(Q_1,\ldots,Q_m) = 0$ is unit free, then it is equivalent to a law of the form $\phi(\Pi_{r+1},\ldots,\Pi_m) = 0$.*

Proof.   Given $Q_1,\ldots,Q_r$, there is a λ such that

$$S_\lambda(Q_j) = 1, \qquad j = 1,\ldots,r. \qquad (11)$$

This is because (9) states

$$\lambda_1^{a_{1j}} \cdots \lambda_n^{a_{nj}} Q_j = 1, \qquad j = 1,\ldots,r,$$

or, equivalently,

$$a_{1j}\ln\lambda_1 + \cdots + a_{nj}\ln\lambda_n + \ln Q_j = 0, \qquad j = 1,\ldots,r.$$

The fact that $a_1,\ldots,a_r$ are linearly independent implies that the system

$$a_{1j}z_1 + \cdots + a_{nj}z_n = -\ln Q_j, \qquad j = 1,\ldots,r,$$

has solutions $(z_1,\ldots,z_n)$. Therefore, if we set $\lambda_i = e^{z_i}$, then (11) is satisfied. But then $g(Q_1,\ldots,Q_r,\Pi_{r+1},\ldots,\Pi_m) = 0$ if, and only if,

$g(1,\ldots,1,\Pi_{r+1},\ldots,\Pi_m)=0$. Thus, if we define

$$\phi(\Pi_{r+1},\ldots,\Pi_m)=g(1,\ldots,1,\Pi_{r+1},\ldots,\Pi_m),$$

then the law $f(Q_1,\ldots,Q_m)=0$ is equivalent to the law $\phi(\Pi_{r+1},\ldots,\Pi_m)=0$. This completes the proof. ∎

EXAMPLE (Falling body). We now work through the procedure for a specific example. The usual law governing how far an object falls (neglecting air resistance) is

$$x=\tfrac{1}{2}gt^2.$$

We write $Q_1=t$, $Q_2=x$, $Q_3=g$. The law is

$$f(Q_1,Q_2,Q_3)=Q_2-\tfrac{1}{2}Q_3Q_1^2=0.$$

We have fundamental dimensions

$$q_1=\text{time}=T, \qquad q_2=\text{length}=L.$$

Then we have the 2×3 matrix

$$(a_{ij})=\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 1\end{bmatrix}$$

We have chosen the notation so that

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

is nonsingular. Since $a_3=-2a_1+a_2$, we define

$$\Pi_3=Q_1^2Q_2^{-1}Q_3=\frac{t^2g}{x}.$$

The equivalent law, $g(Q_1,Q_2,\Pi_3)=0$, is given by $g(Q_1,Q_2,\Pi_3)=Q_2-\tfrac{1}{2}(\Pi_3Q_2Q_1^{-2})Q_1^2=0$. This is equivalent to the law $g(1,1,\Pi_3)=1-\tfrac{1}{2}\Pi_3=0$. We remark that in practice the pi theorem is not usually used to reduce

known laws to dimensionless form but rather to obtain the form of an unknown law in terms of dimensionless variables, much the same as was indicated by the blast wave example in Section 1. The above algorithm clearly gives an explicit method for calculating all the dimensionless quantities in a given problem.

3. THE PI THEOREM

In this section we give a rigorous, abstract theorem which reveals the mathematical content of the previous section. We include a series of remarks intended to help the reader relate this section to Section 2.

Let V be an m-dimensional real vector space and $T: V \to R^n$ a linear transformation of rank r. Let F be the set of all ordered bases (frames) of V.

REMARK A. V is the vector space of "quantities" for the problem at hand. If Q is a quantity in V, then $T(Q) = (\delta_1, \ldots, \delta_n)$ corresponds to the dimension monomial of Section 2,

$$(\delta_1, \ldots, \delta_n) \leftrightarrow q_1^{\delta_1} q_2^{\delta_2} \cdots q_n^{\delta_n}.$$

The assumption that T is linear corresponds to the property of dimensions that $[Q_1 Q_2] = [Q_1][Q_2]$ and that $[Q^\beta] = [Q]^\beta$. The usual product $Q_1 Q_2$ is expressed in the vector space V as $Q_1 + Q_2$, and Q^c is expressed as cQ. A frame in F is a choice of independent quantities $(Q_1, \ldots, Q_m)$. The linear combination $\alpha_1 Q_1 + \cdots + \alpha_m Q_m$ corresponds to $Q_1^{\alpha_1} \cdots Q_m^{\alpha_m}$. We note that a dimensionless quantity Q is one for which $T(Q) = 0$.

For each choice of a frame $e = (e_1, \ldots, e_m)$ we define an action $*e$ of the additive group R^n on R_+^m by

$$\lambda_e^* v = \left(e^{\lambda \cdot Te_1} v_1, \ldots, e^{\lambda \cdot Te_m} v_m \right).$$

Let $B = (b_{ij})$ be a nonsingular $m \times m$ matrix. Then we define $\tau_B: R_+^m \to R_+^m$ by $\tau_B(v_1, \ldots, v_m) = (v_1^{b_{11}} v_2^{b_{21}} \cdots v_m^{b_{m1}}, \ldots, v_1^{b_{1m}} v_2^{b_{2m}} \cdots v_m^{b_{mm}})$. Then τ_B is invertible and $\tau_B^{-1} = \tau_{B^{-1}}$.

DEFINITION 2. A law L on V compatible with T is an assignment, to each frame, e, of a nonempty set $L_e \subset R_+^m$ such that

(i) $R^n *_e L_e = L_e$ for all $e \in F$,
(ii) If $e, \bar{e} \in F$ with $\bar{e}_j = \sum_{i=1}^m b_{ij} e_i$, then $L_{\bar{e}} = \tau_B(L_e)$.

REMARK B.   The choice of frame e corresponds to a choice of indepen-
dent quantities $(Q_1,\ldots,Q_m)$ to describe the law. A point $(x_1,\ldots,x_m)$ in L_e is
interpreted as a set of values for the quantities $(Q_1,\ldots,Q_m)$, respectively.
Requirement (i) of the above definition corresponds to the intuitive idea that
changes of units change the numbers $(x_1,\ldots,x_m)$, but the new values still
"obey the law." Thus (i) is the requirement that the law be unit free (see
Definition 2). A different choice of quantities, say $(\bar{Q}_1,\ldots,\bar{Q}_m)$, corresponds
to a new frame $\bar{e}$, and property (ii) specifies which values of the new
quantities obey the law.

THEOREM (Pi theorem).   *Let L be a law on V compatible with T. Then
there exist frames e such that $T(e_k) = 0$ for $k = r+1,\ldots,m$, and for any such
frame we have $L_e = R^r_+ \times \tilde{L}_e$ for some $\tilde{L}_e \subset R^{m-r}_+$.*

REMARK C.   Let $(Q_1,\ldots,Q_r, \Pi_{r+1},\ldots,\Pi_m)$ be the quantities making up a
frame e, as in the theorem. Then, since $T(\Pi_k) = 0$ for $k = r+1,\ldots,m$, we see
the Π_k are dimensionless. The conclusion $L_e = R^r_+ \times \tilde{L}_e$ means that in order
to obey the law, the values of $Q_1,\ldots,Q_r$ are unrestricted while the values of
$(\Pi_{r+1},\ldots,\Pi_m)$ must lie in a subset of R^{m-r}_+. We say the law is a relationship
among the Π_k's.

Proof of the pi theorem.   The transformation T has rank r, as there exist
frames $e = (e_1,\ldots,e_m)$ such that $e_{r+1},\ldots,e_m$ lie in $\ker T$ and $Te_1,\ldots,Te_r$
are linearly independent. Given real numbers $y_1,\ldots,y_r$, we can find
$\lambda \in R^n$ such that $\lambda \cdot Te_i = y_i$, $i = 1,\ldots,r$. This is a simple consequence of
the linear independence of $Te_1,\ldots,Te_r$. Now we have $\lambda^*_e v =$
$(e^{\lambda \cdot Te_1} v_1,\ldots,e^{\lambda \cdot Te_r} v_r, v_{r+1},\ldots,v_m)$ for all $v \in R^m_+$. Now define $\tilde{L}_e =$
$\{(z_{r+1},\ldots,z_m)|(1,\ldots,1, z_{r+1},\ldots,z_m) \in L_e\}$. Suppose $v \in L_e$. Choose λ so
that $e^{\lambda \cdot Te_i} = 1/v_i$ for $i = 1,\ldots,r$. Then $\lambda^*_e v = (1,\ldots,1, v_{r+1},\ldots,v_m) \in L_e$.
Therefore $L_e \subset R^r_+ \times \tilde{L}_e$. Conversely let $v \in R^r_+ \times \tilde{L}_e$. Then $(1,\ldots,1,$
$v_{r+1},\ldots,v_m)$ is in L_e. Choose λ so that $e^{\lambda \cdot Te_i} = v_i$, $i = 1,\ldots,r$. Then
$\lambda^*_e(1,\ldots,1, v_{r+1},\ldots,v_m) = v$, which belongs to L_e. This completes the proof.   ∎

In conclusion, we have presented a theorem which reveals the underlying
mathematical content of the classical pi theorem. Although this formulation
does not yield new examples of application of the theorem, it does give insight
into its linear, algebraic structure; in this context, the proof is nearly trans-
parent.

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