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Gevrey class regularity for solutions of micropolar fluid equations ☆

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ABSTRACT

In this paper we consider the solutions of micropolar fluid equations in space dimension two with periodic boundary condition. We show that the strong solutions are analytic in time with values in an appropriate Gevrey class of function, provided that external forces and moments are time-independent and are in a Gevrey class.

1. Introduction

In this article we consider the micropolar fluid equations, which in space dimension two have form (cf. e.g. [11])

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \operatorname{rot}\omega + f, \tag{1a}$$

$$\operatorname{div} u = 0, \tag{1b}$$

$$\frac{\partial \omega}{\partial t} - \alpha\Delta\omega + (u\cdot\nabla)\omega + 4\nu_r\omega = 2\nu_r \operatorname{rot} u + g, \tag{1c}$$

where $u = (u_1, u_2)$ is the velocity field, p is the pressure and ω is the microrotation field interpreted as the angular velocity of particles. In the two-dimensional case we assume that the axis of rotation of particles is perpendicular to the x_1, x_2 plane. The external forces $f = (f_1, f_2)$ and moments g are given. The positive constants ν, ν_r and α represent viscosity coefficients. Moreover

$$\operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad \operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \operatorname{rot}\omega = \left(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}\right).$$

We supplement Eqs. (1) with initial condition

$$u(x,0) = u_0(x), \qquad \omega(x,0) = \omega_0(x) \tag{2}$$

and periodic boundary conditions

$$u(x + Le_i, t) = u(x,t), \qquad \omega(x + Le_i, t) = \omega(x,t) \quad \forall x \in \mathbb{R}^2,\ \forall t > 0, \tag{3}$$

where e_1, e_2 is the usual basis of $\mathbb{R}^2$ and L is the period in the ith direction.

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We assume that u, ω, f and g are periodic in each direction with period L; we denote $Q = (0, L)^2$. For simplicity we assume that the average values of functions u, ω, f and g is zero:

$$\int_Q h(x,t)\,dx = 0, \quad \forall t > 0,\ h = u, \omega, f, g.$$

From a priori estimates derived in [12] it follows that the system of micropolar fluid equations possesses a strong solution i.e.

Theorem 1. *Let $(u_0, \omega_0) \in \mathcal{V}$, $(f, g) \in L^2_{loc}(0, \infty; \mathcal{H})$. Then there exists a strong solution of Eqs.* (1)*, that is a pair of function*

$$(u, \omega) \in L^\infty(0, T; \mathcal{V}) \cap L^2\big(0, T; D(A) \times D(A_1)\big)$$

such that $(u, v) \in C([0, T]; \mathcal{V})$, $u(x, 0) = u_0(x)$, $\omega(x, 0) = \omega_0(x)$. Moreover the equalities

$$\frac{d}{dt}u + (\nu + \nu_r)Au + B(u, u) = 2\nu_r \operatorname{rot}\omega + f$$

and

$$\frac{d}{dt}\omega + \alpha A_1\omega + B_1(u, \omega) + 4\nu_r\omega = 2\nu_r \operatorname{rot} u + g$$

are satisfied as equalities in $L^2(0, T; H)$ and $L^2(0, T; \dot{H}^0_{per})$, respectively. In addition the solution depends continuously in the topology of $\mathcal{V}$ on initial data.

The notation used in the above theorem will be introduced in the next section.

In this paper we show time analyticity and Gevrey class regularity in the space variable of strong solution of Eqs. (1). We follow the method used by Foias and Temam in [3].

2. Notations and main result

For every Banach space X we will denote by $\mathbb{X}$ the space $X \times X$ with a standard product norm.

L^q is the usual Lebesgue's space $L^q(Q)$ for $q \in [1, \infty]$. We denote the scalar product in L^2 by $(\cdot,\cdot)$ and the norm in L^2 by $|\cdot|$ when it does not lead to confusion.

H^m, $m \in \mathbb{N}$ are the usual Sobolev spaces $H^m(Q)$ of functions whose derivatives up to order m are square integrable, with the norm

$$\|u\|_m = \left(\sum_{|\alpha| \leqslant m} \int_Q \left|D^\alpha u\right|^2 dx\right)^{1/2}.$$

By $H^m_{per}(Q)$, $m \in \mathbb{N}$, we denote the space of real functions which are in $H^m_{loc}(\mathbb{R}^n)$ and periodic with the period L in each direction: $u(x + Le_i) = u(x)$, $i = 1, 2$. For an arbitrary $m \in \mathbb{N}$, $H^m_{per}(Q)$ is a Hilbert space with the scalar product

$$(u, v)_m = \sum_{|\alpha| \leqslant m} \int_Q D^\alpha u(x) D^\alpha v(x)\,dx$$

and the norm induced by it. The functions in $H^m_{per}(Q)$ are explicitly characterized by their Fourier series expansion

$$H^m_{per}(Q) = \left\{u:\ u = \sum_{k \in \mathbb{Z}^n} u_k e^{2i\pi k/L\cdot x},\ \bar{u}_k = u_{-k},\ |u|^2_m = \sum_{k \in \mathbb{Z}^n} |k|^{2m}|u_k|^2 < \infty\right\},$$

where $k/L = (k_1/L, k_2/L)$. We also set

$$\dot{H}^m_{per}(Q) = \big\{u \in H^m_{per}(Q),\ u_0 = 0\big\}.$$

By H and V we denote the divergence-free subsets of $\dot{\mathbb{H}}^0_{per}(Q)$ and $\dot{\mathbb{H}}^1_{per}(Q)$, respectively. We equip V with the scalar product and the Hilbert norm

$$((u, v)) = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right), \qquad \|u\| = \big\{((u, u))\big\}^{1/2}.$$

This norm is equivalent to that induced by $H^1_{per}(Q)$, and V is a Hilbert space for this scalar product.

One can check that $\dot{H}^{-m}_{per}$ is the dual space to $\dot{H}^m_{per}$, we also denote the dual space to V as V'.

Let $\mathcal{H}$ and $\mathcal{V}$ denote $H \times \dot{H}^0_{per}$ and $V \times \dot{H}^1_{per}$, respectively, with standard product norms.

We define the operators A (the Stokes operator) and A_1 on H and $\dot{H}^0_{per}$, respectively, as follows:

$$A = -\Delta, \qquad D(A) = \dot{\mathbb{H}}^2_{per} \cap H,$$
$$A_1 = -\Delta, \qquad D(A_1) = \dot{H}^2_{per} \cap \dot{H}^0_{per}.$$

Let $\bar{u} = (u, \omega) \in \mathcal{H}$. The operator $\mathcal{A}$ is defined as $\mathcal{A}\bar{u} = (Au, A_1\omega)$.

The operator A^{-1} is linear, continuous from H into $D(A)$. Since the injection of $D(A)$ in H is compact, we can consider A^{-1} as a compact operator in H. A^{-1} is also self-adjoint as an operator in H. Hence it possesses a sequence of eigenfunctions w_j, $j \in \mathbb{N}$ which form an orthonormal basis of H

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A),$$
$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_j \to \infty \text{ for } j \to \infty.$$

The operator A_1 has the same properties as the Stokes operator: it is one-to-one from $D(A_1)$ onto $\dot{H}^0_{per}$. The operator A_1^{-1} is linear, continuous from $\dot{H}^0_{per}$ onto $D(A_1)$ and compact as an operator in $\dot{H}^0_{per}$. Although the eigenvalues of A and A_1 are the same the eigenfunctions are different, because ω is a scalar function. We denote the eigenfunctions of A_1 by ρ_k.

We can express every element $u \in H$ and $\omega \in \dot{H}^0_{per}$ as

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) w_k(x), \qquad \omega(x,t) = \sum_{k=1}^{\infty} \omega_k(t)\rho_k(x). \tag{4}$$

The Galerkin projectors corresponding to the first m modes are:

$$P_m u(x,t) = \sum_{k=1}^{m} u_k(t) w_k(x), \qquad P^1_m \omega_i(x,t) = \sum_{k=1}^{m} \omega_k(t)\rho_k(x).$$

Moreover for $\bar{u} = (u, \omega)$ we define

$$\mathbb{P}_m \bar{u} = \left(P_m u, P^1_m \omega\right).$$

We define the trilinear forms b and b_1 as follows

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_Q u_i \frac{\partial v_j}{\partial x_i} w_j \, dx$$

for all $u, v, w \in V$, and

$$b_1(u, \omega, \psi) = \sum_{i=1}^{2} \int_Q u_i \frac{\partial \omega}{\partial x_i} \psi \, dx$$

for all $u \in V$ and all scalar functions $\omega, \psi \in \dot{H}^1_{per}(Q)$.

The functionals $B : V \times V \to V'$ and $B_1 : V \times \dot{H}^1_{per} \to \dot{H}^{-1}_{per}$ are defined as follows

$$\left(B(u, v), w\right) = b(u, v, w)$$

(we will write $B(u)$ instead of $B(u, u)$ for short) and

$$\left(B_1(u, \omega), \vartheta\right) = b_1(u, \omega, \vartheta).$$

The operators $\mathcal{B}$ and $\mathcal{R}$ are defined by

$$\mathcal{B}(\bar{u}) = \left(B(u), B_1(u, \omega)\right), \qquad \mathcal{R}(\bar{u}) = (-2\nu_r \operatorname{rot}\omega, 4\nu_r\omega - 2\nu_r \operatorname{rot} u).$$

In order to extend the solutions of the micropolar fluid equations to complex times, we have to complexify the function spaces H, V, etc. The elements of complex equivalent of real space are of the form $u = u_1 + iu_2$, where u_1 and u_2 belong to the corresponding real space and i is the imaginary unit. The complexified spaces are denoted by adding the subscript $\mathbb{C}$ e.g. $H_{\mathbb{C}}$ is the complexified H. The inner product in $H_{\mathbb{C}}$ takes the form

$$(u, v)_{\mathbb{C}} = (u_1 + iu_2, v_1 + iv_2) = (u_1, v_1) + (u_2, v_2) + i\left[(u_2, v_1) - (u_1, v_2)\right]$$

and similarly for other spaces.

The operators are extended to the complexified spaces, they take form e.g.

$$A_{\mathbb{C}} u = Au_1 + iAu_2,$$
$$B(u, v)_{\mathbb{C}} = B(u_1, v_1) + B(u_2, v_2) + i\left[B(u_1, v_2) + B(u_2, v_1)\right]$$

and analogously for other operators.

The operator $\mathcal{A}_{\mathbb{C}}$ is linear self-adjoint operator in $\mathcal{H}_{\mathbb{C}}$, with $D(\mathcal{A}_{\mathbb{C}}) = D(\mathcal{A})_{\mathbb{C}}$. The operators $B_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to V'_{\mathbb{C}}$ and $B_{1\mathbb{C}} : V_{\mathbb{C}} \times \dot{H}^1_{per\mathbb{C}} \to \dot{H}^{-1}_{per\mathbb{C}}$ are still bilinear.

Now we can write Eqs. (1) in a functional form

$$\frac{du}{dt} + (\nu + \nu_r)Au + B(u) = 2\nu_r \operatorname{rot}\omega + f, \tag{5a}$$

$$\frac{d\omega}{dt} + \alpha A_1\omega + B_1(u,\omega) + 4\nu_r\omega = 2\nu_r \operatorname{rot} u + g. \tag{5b}$$

Using the Fourier series expansions of velocity and rotation of fluid

$$u(x,t) = \sum_{k\in\mathbb{Z}^2} e^{ik/L\cdot x} u_k(t),$$
$$\omega(x,t) = \sum_{k\in\mathbb{Z}^2} e^{ik/L\cdot x} \omega_k(t), \tag{6}$$

where $k/L = (k_1/L, k_2/L)$, we define the operators $e^{\tau A^{1/2}}$ and $e^{\tau A_1^{1/2}}$ in the following way:

$$e^{\tau A^{1/2}} u(x,t) = \sum_{k\in\mathbb{Z}^2} e^{ik/L\cdot x + \tau|k|} u_k(t),$$
$$e^{\tau A_1^{1/2}} \omega(x,t) = \sum_{k\in\mathbb{Z}^2} e^{ik/L\cdot x + \tau|k|} \omega_k(t).$$

We refer the reader interested how expansions (4) and (6) are connected to [13].

For given $\tau, s > 0$ we define the Gevrey class $D(e^{\tau\mathcal{A}^s})$ by

$$D\big(e^{\tau\mathcal{A}^s}\big) = D\big(e^{\tau A^s}\big) \times D\big(e^{\tau A_1^s}\big),$$

where

$$D\big(e^{\tau A^s}\big) = \bigg\{u \in H\colon \ \big|e^{\tau A^s}u\big|^2 = \sum_{j\in\mathbb{Z}^2} e^{2\tau|j|^{2s}}|u_j|^2 < \infty\bigg\},$$
$$D\big(e^{\tau A_1^s}\big) = \bigg\{\omega \in \dot{H}^0_{per}\colon \ \big|e^{\tau A_1^s}\omega\big|^2 = \sum_{j\in\mathbb{Z}^2} e^{2\tau|j|^{2s}}|\omega_j|^2 < \infty\bigg\}.$$

We endow the space $D(e^{\tau A^{1/2}})$ with a scalar product

$$(u,v)_\tau = \big(e^{\tau A^{1/2}}u, e^{\tau A^{1/2}}v\big)$$

and a norm $|u|_\tau = (u,u)_\tau^{1/2}$. We introduce the norm and the scalar product in $D(e^{\tau A_1^{1/2}})$ in the same way and we will denote them as the norm and scalar product in $D(e^{\tau A^{1/2}})$.

For given $\tau, s, \alpha > 0$ we define the spaces $D(A^\alpha e^{\tau A^s})$ and $D(A_1^\alpha e^{\tau A_1^s})$ of functions from $D(e^{\tau A^s})$ and $D(e^{\tau A^s})$, respectively, satisfying

$$\big|A^\alpha e^{\tau A^s}u\big|^2 = \sum_{k\in\mathbb{Z}^2} |k|^{2\alpha} e^{2\tau|k|^{2s}}|u_k|^2 < \infty,$$
$$\big|A_1^\alpha e^{\tau A_1^s}\omega\big|^2 = \sum_{k\in\mathbb{Z}^2} |k|^{2\alpha} e^{2\tau|k|^{2s}}|\omega_k|^2 < \infty.$$

We endow the space $D(A^{1/2}e^{\tau A^{1/2}})$ with a scalar product

$$((u,v))_\tau = \big(A^{1/2}e^{\tau A^{1/2}}u, A^{1/2}e^{\tau A^{1/2}}v\big)$$

and norm $\|\cdot\|_\tau$ induced by it. We define the scalar product and norm in $D(A_1^{1/2}e^{\tau A_1^{1/2}})$ analogously and denote them $((\cdot,\cdot))$ and $\|\cdot\|_\tau$. Moreover the norm and scalar product in $D(e^{\tau A^{1/2}}) \times D(e^{\tau A_1^{1/2}})$ is denoted by $|\cdot|_\tau$ and $(\cdot,\cdot)_\tau$. The notation seems to be confusing but it will always be clear if we consider the scalar product or norm in $D(e^{\tau A^{1/2}})$, $D(e^{\tau A_1^{1/2}})$ or $D(e^{\tau A^{1/2}}) \times D(e^{\tau A_1^{1/2}})$.

Since we consider the space-periodic case, the operators A, $e^{\tau A^{1/2}}$, P_m and A_1, $e^{\tau A_1^{1/2}}$, P_m^1 commute.

The following theorem is the main result of this paper.

Theorem 2. *Let $\bar{u}(0) \in \mathcal{V}$, $G = (f, g) \in D(e^{\sigma_1 \mathcal{A}^{1/2}})$ for some $\sigma_1 > 0$. Then there exists $T_* > 0$ that depend only on data and $\|\bar{u}(0)\|$ such that the following holds*:

(i) *Eqs.* (1) *posses on $(0, T_*)$ a unique regular solution $\bar{u}(t)$, continuous from $[0, T_*]$ into $\mathcal{V}$, such that the map $t \to e^{\psi(t)\mathcal{A}^{1/2}} \mathcal{A}^{1/2}\bar{u}(t)$, where $\psi(t) = \min(t, \sigma_1, T_*)$, is analytic on $(0, T_*)$ with values in $\mathcal{H}$.*
(ii) *The solution $\bar{u}(t)$ is analytic on (T_*, ∞) with values in $D(\mathcal{A}^{1/2} e^{\sigma \mathcal{A}^{1/2}})$ for some $\sigma > 0$ and T_* as before.*

The following lemma gives the proper estimate of the nonlinear term.

Lemma 1. *(See [3].) Let u, v, w be given in $D(Ae^{\tau A^{1/2}})$ and ω, ρ in $D(A_1 e^{\tau A_1^{1/2}})$, $\tau > 0$. Then $B(u, v) \in D(e^{\tau A^{1/2}})$, $B_1(u, \omega) \in D(e^{\tau A_1^{1/2}})$ and the following estimates hold for an appropriate constant c_2:*

$$\left|\left(e^{\tau A^{1/2}} B(u, v), e^{\tau A^{1/2}} Aw\right)\right| \leqslant c_2 \left|e^{\tau A^{1/2}} A^{1/2} u\right|^{1/2} \cdot \left|e^{\tau A^{1/2}} Au\right|^{1/2} \cdot \left|e^{\tau A^{1/2}} A^{1/2} v\right| \cdot \left|e^{\tau A^{1/2}} Aw\right|,$$
$$\left|\left(e^{\tau A_1^{1/2}} B_1(u, \omega), e^{\tau A_1^{1/2}} A_1\rho\right)\right| \leqslant c_2 \left|e^{\tau A^{1/2}} A^{1/2} u\right|^{1/2} \cdot \left|e^{\tau A^{1/2}} Au\right|^{1/2} \cdot \left|e^{\tau A_1^{1/2}} A_1^{1/2}\omega\right| \cdot \left|e^{\tau A_1^{1/2}} A_1\rho\right|.$$

The proof for B_1 is similar to that for B.

Proof. In order to obtain analyticity in time we complexify our equations. The proof is composed of five steps including: a priori estimates, basis of Galerkin approximation, showing that Galerkin approximations are analytic and passing to the limit.

For simplicity of notation we will denote the complex operators and spaces as their real equivalents.

Eqs. (5) are rewritten

$$\frac{du}{d\zeta} + (\nu + \nu_r)Au + B(u) = 2\nu_r \operatorname{rot}\omega + f, \tag{7a}$$
$$\frac{d\omega}{d\zeta} + \alpha A_1\omega + B_1(u, \omega) + 4\nu_r\omega = 2\nu_r \operatorname{rot} u + g, \tag{7b}$$

where $\zeta = se^{i\theta}$, $s > 0$ and $\cos\theta > 0$ (thus $\operatorname{Re}\zeta > 0$).

Step 1 (a priori estimates). The a priori estimates we derive formally can be obtained in a rigorous way for Galerkin approximations. Let $\varphi(t) = \min(t, \sigma_1)$. We will use in the course of proof the fact that $0 < \cos\theta \leqslant 1$. We take the scalar product of (7a) with $Au(se^{i\theta})$ in $D(e^{\varphi(s\cos\theta)A^{1/2}})$, multiply the equation we obtain by $e^{-i\theta}$ and take the real part. We have

$$\begin{aligned}
&\operatorname{Re} e^{-i\theta}\left(e^{\varphi(s\cos\theta)A^{1/2}} \frac{du}{d\zeta}\left(se^{i\theta}\right), e^{\varphi(s\cos\theta)A^{1/2}} Au\left(se^{i\theta}\right)\right) \\
&\quad = \operatorname{Re}\left(A^{1/2} \frac{d}{ds}\left(e^{\varphi(s\cos\theta)A^{1/2}} u\left(se^{i\theta}\right)\right) - e^{-i\theta}\varphi'(s\cos\theta)Au\left(se^{i\theta}\right), e^{\varphi(s\cos\theta)A^{1/2}} A^{1/2}u\left(se^{i\theta}\right)\right) \\
&\quad \geqslant \frac{1}{2}\frac{d}{ds}\left|A^{1/2}u\left(se^{i\theta}\right)\right|^2_{\varphi(s\cos\theta)} - \cos^2\theta\varphi'(s\cos\theta)\left|Au\left(se^{i\theta}\right)\right|_{\varphi(s\cos\theta)} \cdot \left|A^{1/2}u\left(se^{i\theta}\right)\right|_{\varphi(s\cos\theta)}.
\end{aligned}$$

Using the Young inequality we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left|A^{1/2}u\left(se^{i\theta}\right)\right|_{\varphi(s\cos\theta)} - \cos^2\theta\varphi'(s\cos\theta)\left|Au\left(se^{i\theta}\right)\right|_{\varphi(s\cos\theta)}\left|A^{1/2}u\left(se^{i\theta}\right)\right|_{\varphi(s\cos\theta)} \\
&\quad \geqslant \frac{1}{2}\frac{d}{ds}\left\|u\left(se^{i\theta}\right)\right\|^2_{\varphi(s\cos\theta)} - \frac{\nu\cos\theta}{4}\left|Au\left(se^{i\theta}\right)\right|^2_{\varphi(s\cos\theta)} - \frac{1}{\nu\cos\theta}\left\|u\left(se^{i\theta}\right)\right\|^2_{\varphi(s\cos\theta)}.
\end{aligned} \tag{8}$$

Moreover

$$\operatorname{Re} e^{-i\theta}\left(e^{\varphi(s\cos\theta)A^{1/2}} Au\left(se^{i\theta}\right), e^{\varphi(s\cos\theta)A^{1/2}} Au\left(se^{i\theta}\right)\right) = \cos\theta|Au|^2_{\varphi(s\cos\theta)}.$$

Hence, omitting the dependence of u on s when possible, and writing φ instead of $\varphi(s\cos\theta)$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\|u\|^2_\varphi + \left(\frac{3}{4}\nu + \nu_r\right)\cos\theta|Au|^2_\varphi - \frac{1}{\nu\cos\theta}\|u\|^2_\varphi \leqslant{}& \operatorname{Re} 2\nu_r e^{-i\theta}(\operatorname{rot}\omega, Au)_\varphi + \operatorname{Re} e^{-i\theta}(f, Au)_\varphi \\
&+ \operatorname{Re} e^{-i\theta}\big(B(u), Au\big)_\varphi.
\end{aligned} \tag{9}$$

We majorize every term in the RHS of (9) using the Cauchy–Schwartz inequality. Then we use Lemma 1 and the Young inequality to get

$$2\nu_r\cos\theta|\operatorname{rot}\omega|_\varphi|Au|_\varphi+\cos\theta|f|_\varphi|Au|_\varphi+c_2\cos\theta\|u\|_\varphi^{3/2}|Au|_\varphi^{3/2}\leqslant\nu_r\cos\theta|Au|_\varphi^2+\nu_r\cos\theta\|\omega\|_\varphi^2+\frac{\nu\cos\theta}{8}|Au|_\varphi^2$$
$$+\frac{2\cos\theta}{\nu}|f|_\varphi^2+\frac{7c_2^4}{(\nu\cos\theta)^3}\|u\|_\varphi^6+\frac{\nu\cos\theta}{4}|Au|_\varphi^2.$$

Inserting the above estimate into (9) we obtain

$$\frac{1}{2}\frac{d}{ds}\|u\|_\varphi^2+\frac{3}{8}\nu\cos\theta|Au|_\varphi^2\leqslant\frac{1}{\nu\cos\theta}\|u\|_\varphi^2+\nu_r\cos\theta\|\omega\|_\varphi^2+\frac{2\cos\theta}{\nu}|f|_\varphi^2+\frac{7c_2^4}{(\nu\cos\theta)^3}\|u\|_\varphi^6. \tag{10}$$

Now we devote our attention to the microrotation equation. We multiply the scalar product of Eq. (7b) with $A_1\omega$ in $D(e^{\varphi(s\cos\theta)A_1^{1/2}})$ by $e^{-i\theta}$ and take the real part. Proceeding in a similar way as above we conclude that

$$\operatorname{Re}e^{-i\theta}\left(e^{\varphi(s\cos\theta)A_1^{1/2}}\frac{d\omega}{d\zeta}\big(se^{i\theta}\big),e^{\varphi(s\cos\theta)A_1^{1/2}}A_1\omega\big(se^{i\theta}\big)\right)\geqslant\frac{1}{2}\frac{d}{ds}\big\|\omega(s\cos\theta)\big\|_{\varphi(s\cos\theta)}^2-\frac{\alpha\cos\theta}{4}\big|A_1\omega(s\cos\theta)\big|_{\varphi(s\cos\theta)}^2$$
$$-\frac{1}{\alpha\cos\theta}\big\|\omega(s\cos\theta)\big\|_{\varphi(s\cos\theta)}^2.$$

Here we again omit the dependence on s, for short

$$\frac{1}{2}\frac{d}{ds}\|\omega\|_\varphi^2+\frac{3}{4}\alpha\cos\theta|A_1\omega|_\varphi^2+\left(4\nu_r\cos\theta-\frac{1}{\alpha\cos\theta}\right)\|\omega\|_\varphi^2\leqslant\operatorname{Re}2\nu_re^{-i\theta}(\operatorname{rot}u,A_1\omega)_\varphi+\operatorname{Re}e^{-i\theta}(g,A_1\omega)_\varphi$$
$$+\operatorname{Re}e^{-i\theta}\big(B_1(u,\omega),A_1\omega\big)_\varphi. \tag{11}$$

We estimate the linear terms in the RHS of (11) as follows

$$\operatorname{Re}2\nu_re^{-i\theta}(\operatorname{rot}u,A_1\omega)_\varphi\leqslant2\nu_r\cos\theta|\operatorname{rot}u|_\varphi|A_1\omega|_\varphi\leqslant\frac{\alpha\cos\theta}{8}|A_1\omega|_\varphi^2+\frac{8\nu_r^2\cos\theta}{\alpha}\|u\|_\varphi^2,$$
$$\operatorname{Re}e^{-i\theta}(g,A_1\omega)_\varphi\leqslant\cos\theta|g|_\varphi|A_1\omega|_\varphi\leqslant\frac{\alpha\cos\theta}{8}|A_1\omega|_\varphi^2+\frac{2\cos\theta}{\alpha}|g|_\varphi^2$$

and the form B_1 using Lemma 1 and Young's inequality

$$\begin{aligned}\cos\theta\big(B_1(u,\omega),A_1\omega\big)_\varphi&\leqslant c_2\cos\theta\|u\|_\varphi^{1/2}|Au|_\varphi^{1/2}\|\omega\|_\varphi|A_1\omega|_\varphi\\&\leqslant\frac{\alpha\cos\theta}{4}|A_1\omega|_\varphi^2+\frac{c_2^2\cos\theta}{\alpha}\|u\|_\varphi|Au|_\varphi\|\omega\|_\varphi^2\\&\leqslant\frac{\alpha\cos\theta}{4}|A_1\omega|_\varphi^2+\frac{\nu\cos\theta}{8}|Au|_\varphi^2+\frac{2c_2^4}{\nu\alpha^2\cos^3\theta}\|u\|_\varphi^2\|\omega\|_\varphi^4\\&\leqslant\frac{\alpha\cos\theta}{4}|A_1\omega|_\varphi^2+\frac{\nu\cos\theta}{8}|Au|_\varphi^2+\frac{2c_2^4}{\nu\alpha^2\cos^3\theta}\big(\|u\|_\varphi^6+\|\omega\|_\varphi^6\big).\end{aligned}$$

Combining above estimates with (11) we infer that

$$\frac{1}{2}\frac{d}{ds}\|\omega\|_\varphi^2+\frac{\alpha\cos\theta}{4}|A_1\omega|_\varphi^2-\frac{\nu\cos\theta}{8}|Au|_\varphi^2+4\nu_r\cos\theta\|\omega\|_\varphi^2\leqslant\frac{2\cos\theta}{\alpha}|g|_\varphi^2+\frac{8\nu_r^2\cos\theta}{\alpha}\|u\|_\varphi^2+\frac{1}{\alpha\cos\theta}\|\omega\|_\varphi^2$$
$$+\frac{2c_2^4}{\alpha^2\nu\cos^3\theta}\big(\|u\|_\varphi^6+\|\omega\|_\varphi^6\big). \tag{12}$$

Adding (10) and (12) we obtain

$$\frac{1}{2}\frac{d}{ds}\big(\|u\|_\varphi^2+\|\omega\|_\varphi^2\big)+\frac{k_1\cos\theta}{4}\big(|Au|_\varphi^2+|A_1\omega|_\varphi^2\big)\leqslant\frac{2\cos\theta}{k_1}\big(|f|_\varphi^2+|g|_\varphi^2\big)+\frac{8\nu_r^2+1}{k_1\cos\theta}\big(\|u\|_\varphi^2+\|\omega\|_\varphi^2\big)$$
$$+\frac{9c_2^4}{(k_1\cos\theta)^3}\big(\|u\|_\varphi^6+\|\omega\|_\varphi^6\big), \tag{13}$$

where $k_1=\min(\nu,\alpha)$. Using the Young inequality we estimate the term with $\|u\|_\varphi^2+\|\omega\|_\varphi^2$ as follows

$$\begin{aligned}\frac{8\nu_r^2+1}{k_1\cos\theta}\big(\|u\|_\varphi^2+\|\omega\|_\varphi^2\big)&\leqslant\frac{8\nu_r^2+1}{k_1\cos\theta}\left(1+\frac{2}{3}\|u\|_\varphi^6+\frac{2}{3}\|\omega\|_\varphi^6\right)\\&\leqslant\frac{8\nu_r^2+1}{k_1\cos\theta}\big(1+\|u\|_\varphi^6+\|\omega\|_\varphi^6\big).\end{aligned}$$

Inserting the above estimate into (13) we obtain

$$\frac{d}{ds}\left(\|u\|_\varphi^2+\|\omega\|_\varphi^2\right)\leqslant\frac{4\cos\theta}{k_1}\left(|f|_\varphi^2+|g|_\varphi^2\right)+\frac{16\nu_r^2+2}{k_1\cos\theta}+\frac{18c_2^4+2k_1^2(8\nu_r^2+1)}{(k_1\cos\theta)^3}\left(\|u\|_\varphi^2+\|\omega\|_\varphi^2\right)^3. \tag{14}$$

Let us restrict θ to the interval $(-\pi/4,\pi/4)$ and denote

$$y(s)=1+\|u\|_\varphi^2+\|\omega\|_\varphi^2=1+\left|e^{\varphi(s\cos\theta)\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}\left(se^{i\theta}\right)\right|^2,$$

$$K_1=\frac{4}{k_1}\left(|f|_\varphi^2+|g|_\varphi^2\right)+\frac{8(18c_2^4+4k_1^2(8\nu_r^2+1))}{(k_1\sqrt{2})^3}.$$

Then we can rewrite (14) in the form

$$\frac{dy}{ds}\leqslant K_1y^3. \tag{15}$$

Inequality (15) implies $y(s)\leqslant 2y(0)$, i.e.

$$1+\left|e^{\varphi(s\cos\theta)\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}\left(se^{i\theta}\right)\right|^2\leqslant 2+2\left|\mathcal{A}^{1/2}\bar{u}(0)\right|^2 \tag{16}$$

or equivalently

$$1+\left|\mathcal{A}^{1/2}\bar{u}\left(se^{i\theta}\right)\right|_\varphi^2\leqslant 2+2\left|\mathcal{A}^{1/2}\bar{u}(0)\right|^2$$

for $\zeta=se^{i\theta}\in\Delta(\bar{u}_0)$, where $\Delta(\bar{u}_0)$ is the set of $(s,\theta)\in\mathbb{R}^+\times(-\pi/4,\pi/4)$, such that

$$0\leqslant s\leqslant T_1\left(\left|\mathcal{A}^{1/2}\bar{u}(0)\right|\right)=\frac{1}{4K_1}\left(1+\left|\mathcal{A}^{1/2}\bar{u}(0)\right|^2\right)^{-2},\qquad \cos\theta\in\left(\frac{\sqrt{2}}{2},1\right). \tag{17}$$

This shows that even if $\bar{u}(0)$ belongs only to $\mathcal{V}$, $\bar{u}(se^{i\theta})\in D(e^{\varphi(s\cos\theta)\mathcal{A}^{1/2}}\mathcal{A}^{1/2})$ in the subset of $\mathbb{C}$ given by (17).

Step 2 (Galerkin approximation). Notice that the operator $\mathcal{A}$ possesses an orthonormal family of eigenfunctions $\bar{w}_1,\bar{w}_2,\dots$ with corresponding eigenvalues $\lambda_1,\lambda_2,\dots$ such that

$$0<\lambda_1\leqslant\lambda_2\leqslant\cdots\quad\to\infty,$$
$$\bar{w}_j\in D(\mathcal{A}).$$

We consider the following complex differential system in the space $W_m=span\{\bar{w}_1,\dots,\bar{w}_m\}$ spanned by the first m eigenvectors of the $\mathcal{A}$:

$$\begin{aligned}&\frac{d}{d\zeta}\bar{u}_m+\mathcal{A}\bar{u}_m+\mathbb{P}_m\mathcal{B}(\bar{u}_m)+\mathcal{R}\bar{u}_m=\mathbb{P}_mG,\\&\bar{u}_m(0)=\mathbb{P}_m\bar{u}(0),\end{aligned} \tag{18}$$

where $\mathbb{P}_m:\mathcal{H}_\mathbb{C}\to W_m$ is the orthogonal projection and $G=(f,g)$ as in the statement of Theorem 2. The Galerkin approximation (18) has an analytic solution in a complex neighborhood of the complex origin, because it is a complex system of ODEs with polynomial nonlinearity (Theorem 13.4.1 in [7]). By estimate (16)

$$1+\left|e^{\varphi(s\cos\theta)\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}_m\left(se^{i\theta}\right)\right|^2\leqslant 2+2\left|\mathcal{A}^{1/2}\bar{u}(0)\right|^2 \tag{19}$$

if $(s,\theta)\in\Delta(\bar{u}_0)$.

Step 3 (proof of (ii)). Denote $\tau=\varphi(T_1/\sqrt{2})$. Since the norm of solution is bounded in $\mathcal{V}$ for every $t>0$, we can repeat the above argument (Eqs. (16)–(17)) at any time $t_0>0$. From (19) it follows that

$$1+\left|e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}_m\left(T_2e^{i\theta}+t_0\right)\right|^2\leqslant 2\left(1+M_2^2\right)$$

for $t_0>0$, $\cos\theta\in[\sqrt{2}/2,1]$ and

$$T_2=T_2(M_2)=\frac{1}{4K_1}\left(1+M_2^2\right)^{-2}.$$

Hence we have

$$\left|e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}_m(\zeta)\right|^2\leqslant 2\left(1+M_2^2\right)$$

for ζ in the set

$$\Delta(M_2)=\left\{\zeta\in\mathbb{C}:\ \operatorname{Re}\zeta>\frac{T_2}{\sqrt{2}},\ |\operatorname{Im}\zeta|\leqslant\frac{T_2}{\sqrt{2}}\right\}.$$

Notice, that $\Delta(M_2)$ is the domain of analyticity of every $\bar{u}_m(\zeta)$ ($\bar{u}_m$ are analytic as solutions of a finite-dimensional system of ODEs with polynomial nonlinearity and they exist in $\Delta(M_2)$ because of a priori estimates). Denote

$$\tilde{u} = e^{\tau A^{1/2}} u, \qquad \tilde{\omega} = e^{\tau A_1^{1/2}} \omega, \qquad \tilde{\bar{u}} = e^{\tau \mathcal{A}^{1/2}} \bar{u},$$

for $u \in D(e^{\tau A^{1/2}})$, $\omega \in D(e^{\tau A_1^{1/2}})$ and $\bar{u} \in D(e^{\tau \mathcal{A}^{1/2}})$. The Cauchy integral formula gives

$$\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m(\zeta) = \frac{k!}{2\pi i} \int\limits_{|z-\zeta|=d/2} \frac{\tilde{\bar{u}}_m(z)}{(z-\zeta)^{k+1}}\, dz,$$

where $d = \operatorname{dist}(\zeta, \partial\Delta(M_2))$. Hence for $\zeta \in \Delta(M_2)$

$$\left|\mathcal{A}^{1/2}\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m(\zeta)\right| \leqslant \frac{k!2^k}{d^k} \sup_{z\in\Delta(M_2)} \left|\mathcal{A}^{1/2}\tilde{\bar{u}}(z)\right| \leqslant \frac{k!2^{k+1}}{d^k}\left(1+M_2^2\right).$$

Therefore for any set $K \subset\subset \Delta(M_2)$ we have

$$\left|\mathcal{A}^{1/2}\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m(\zeta)\right| \leqslant \frac{k!2^{k+1}}{[\operatorname{dist}(K, \partial\Delta(M_2))]^k}\left(1+M_2^2\right), \quad \text{for all } \zeta \in K,$$

which implies

$$\sup_{\zeta\in K}\left|\mathcal{A}^{1/2}\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m(\zeta)\right| \leqslant C_1(K, M_2).$$

Let us return to Eq. (18). Since

$$\frac{d}{d\zeta}\tilde{\bar{u}}_m + \mathcal{A}\tilde{\bar{u}}_m + \mathbb{P}_m\widetilde{\mathcal{B}(\bar{u}_m)} + \mathcal{R}\tilde{\bar{u}}_m = \mathbb{P}_m\tilde{G},$$

then

$$|\mathcal{A}\tilde{\bar{u}}_m| \leqslant \left|\frac{d}{d\zeta}\tilde{\bar{u}}_m\right| + \left|\widetilde{\mathcal{B}(\bar{u}_m)}\right| + |\mathcal{R}\tilde{\bar{u}}_m| + |\tilde{G}|. \tag{20}$$

We estimate the RHS of the above inequality by using Lemma 1, the Young and the Poincaré inequalities

$$\begin{aligned}\left|\widetilde{\mathcal{B}(\bar{u}_m)}\right| &\leqslant c_2\left(\left|A^{1/2}\tilde{u}_m\right|^{3/2}|A\tilde{u}_m|^{1/2} + \left|A^{1/2}\tilde{u}_m\right|^{1/2}|A\tilde{u}_m|^{1/2}\left|A_1^{1/2}\tilde{\omega}_m\right|\right)\\ &\leqslant \frac{1}{2}|\mathcal{A}\tilde{\bar{u}}_m| + c_3\left|\mathcal{A}^{1/2}\tilde{\bar{u}}_m\right|^3,\\ |\mathcal{R}\tilde{\bar{u}}_m| &\leqslant 2\nu_r|\operatorname{rot}\tilde{\omega}_m| + 2\nu_r|\operatorname{rot}\tilde{u}_m| + 4\nu_r|\tilde{\omega}_m|\\ &\leqslant 4\nu_r\left|\mathcal{A}^{1/2}\tilde{\bar{u}}_m\right| + \frac{4\nu_r}{\sqrt{\lambda_1}}\left|A_1^{1/2}\tilde{\omega}_m\right|\\ &\leqslant c_4\left|\mathcal{A}^{1/2}\tilde{\bar{u}}_m\right|.\end{aligned}$$

Inserting above estimate into (20) we obtain

$$\sup_K|\mathcal{A}\tilde{\bar{u}}_m| \leqslant c_5(K, M_2) \tag{21}$$

for all $K \subset\subset \Delta(M_2)$. It follows from estimate (21) that for each $\zeta \in K$

$$\left|\mathcal{A}\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m(\zeta)\right| \leqslant \frac{k!}{2\pi}\int\limits_{|z-\zeta|=d/2}\frac{|\mathcal{A}\tilde{\bar{u}}_m|}{|z-\zeta|^{k+1}}\,dz \leqslant \frac{2^k k!}{[\operatorname{dist}(K, \partial\Delta(M_2))]^k}\sup_{z\in K'}\left|\mathcal{A}\tilde{\bar{u}}(z)\right|,$$

where $d = \operatorname{dist}(\zeta, \partial\Delta(M_2))$ and

$$K' = \left\{\zeta \in \Delta(M_2)\colon \operatorname{dist}\left(\zeta, \partial\Delta(M_2)\right) \leqslant \frac{1}{2}\operatorname{dist}\left(K, \partial\Delta(M_2)\right)\right\}.$$

Thus

$$\sup_K\left|\mathcal{A}\frac{d^k}{d\zeta^k}\tilde{\bar{u}}_m\right| \leqslant c_6(K, M_2), \tag{22}$$

which implies that $\tilde{\bar{u}}_m$ is analytic function with values in $D(\mathcal{A})$ (cf. e.g. [1,2]).

Step 4 (passing to the limit). Let $\{\zeta_k\}$ be a countable dense subset of $\Delta(M_2)$. Estimate (22) implies that for every ζ_k the sequence $\bar{u}_m(\zeta_k)$ is bounded in $D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A})$. Since the embedding $D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A}) \hookrightarrow D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2})$ is compact, we can choose for every k a subsequence $\bar{u}^k_{m_n}$, which is convergent in $D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2})$. Using the diagonal method we can extract a subsequence (denoted by $\bar{u}_m$) which is convergent in every point ζ_k. By virtue of the vector version of Vitali's theorem (Theorem 3.14.1 in [8]), we conclude that the convergence is uniform with respect to ζ on every compact subset of $\Delta(M_2)$ and the limit function $\bar{u}$ is analytic with values in $D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2})$.

Observe that the restriction of $\bar{u}(\zeta)$ to real axis is the strong solution of Eqs. (1). Hence $\bar{u}(\zeta)$ is a unique analytic continuation of $\bar{u}(t)$, for $t \in \mathbb{R}$. Consequently $\bar{u}(t)$ is analytic function of real time variable t with values in $D(e^{\tau\mathcal{A}^{1/2}}\mathcal{A}^{1/2})$ for $t > T_2(M_2)/\sqrt{2}$.

Step 5 (proof of (i)). From (16) it follows that the sequence $e^{\varphi(s)\mathcal{A}^{1/2}}\mathcal{A}^{1/2}\bar{u}_m(se^{i\theta})$ is bounded uniformly with respect to $se^{i\theta}$ in $\Delta(\bar{u}_0)$. As above we can choose a dense countable subset $s_k e^{i\theta_k}$ of $\Delta(\bar{u}_0)$. Since the embedding $\mathcal{V} \hookrightarrow \mathcal{H}$ is compact, we can use the diagonal method to extract a subsequence (denoted by $\bar{u}_m$) such that

$$e^{\varphi(s_k)\mathcal{A}^{1/2}}\bar{u}_m\big(s_k e^{i\theta_k}\big) \to e^{\varphi(s)\mathcal{A}^{1/2}}\bar{u}\big(se^{i\theta}\big) \quad \text{in } \mathcal{H}$$

for every k. It results from the Vitali theorem that

$$e^{\varphi(s)\mathcal{A}^{1/2}}\bar{u}_m\big(se^{i\theta}\big) \to e^{\varphi(s)\mathcal{A}^{1/2}}\bar{u}\big(se^{i\theta}\big) \quad \text{in } \mathcal{H}$$

uniformly on compact subsets $\Delta(\bar{u}_0)$ and $e^{\varphi(s)\mathcal{A}^{1/2}}\bar{u}(se^{i\theta})$ is an analytic function with values in $\mathcal{H}$.

The result follows with $T_* = T_2$ and $\sigma = \min(\sigma_1, T_1(\mathcal{A}^{1/2}\bar{u}_0)/\sqrt{2})$. □

Remark 1. The pressure gradient is absent in Eqs. (5) so there arise a question how to recover it. From Eq. (1a) we have

$$p_m = \nabla^{-1}\left(-\frac{d}{d\zeta}u_m + (\nu+\nu_r)\Delta u_m - (u_m\cdot\nabla)u_m + 2\nu_r \operatorname{rot}\omega_m + f_m\right),$$

where u_m is $P_m u$, etc.

In the course of proof and in the Lemma 1 we have shoved that:

$$\frac{d}{d\zeta}u_m \in D\big(e^{\varphi(s\cos\theta)A^{1/2}}A\big), \qquad \Delta u_m \in D\big(e^{\varphi(s\cos\theta)A^{1/2}}\big),$$
$$(u_m\cdot\nabla)u_m \in D\big(e^{\varphi(s\cos\theta)A^{1/2}}\big), \qquad \operatorname{rot}\omega_n \in D\big(e^{\varphi(s\cos\theta)A_1^{1/2}}A_1^{1/2}\big),$$

moreover $f_m \in D(e^{\sigma_1 A^{1/2}})$. Then $p_m \in D(e^{\varphi(s\cos\theta)A_1^{1/2}}A_1^{1/2})$ and we can pass to the limit similar as in the steps 4 and 5 to get

(1) The function $t \to e^{\psi(t)A_1^{1/2}}p(t)$, where $\psi(t) = \min(t, \sigma_1, T_*)$, is analytic on $(0, T_*)$ with values in $\mathcal{H}$.
(2) The function $p(t)$ is analytic on (T_*, ∞) with values in $D(e^{\sigma A_1^{1/2}})$ for $T_* = T_2$ and $\sigma = \min(\sigma_1, T_1(\mathcal{A}^{1/2}\bar{u}_0)/\sqrt{2})$.

3. Properties of Gevrey class solutions

It is shown in [12] that a sufficient number of "nodal values" determine the asymptotic behavior of solutions of 2D micropolar fluid equations in particular if $\bar{u}_1 = (u_1, \omega_1)$ and $\bar{u}_2 = (u_2, \omega_2)$ are two solutions of Eqs. (1) with periodic boundary conditions then there exists a number N such that if Q is covered by N identical squares Q^i, $1 \leqslant i \leqslant N$, and there is one and only one point $x^i \in Q^i$ and

$$\sup_{1\leqslant i\leqslant N}\big|\bar{u}_1\big(x^i, t\big) - \bar{u}_2\big(x^i, t\big)\big| \to 0 \quad \text{as } t \to \infty,$$

one has

$$\sup_{x\in Q}\big|\bar{u}_1\big(x^i, t\big) - \bar{u}_2\big(x^i, t\big)\big| \to 0 \quad \text{as } t \to \infty.$$

Since the asymptotic behavior of solutions on these nodes fixes the asymptotic behavior of whole solutions they have been called the "determining nodes." However, this result does not say if these nodes determine instantaneously the values of $\bar{u}(x, t)$ in the whole domain. In general there is no reason to believe that the behavior of solution in the transient period of time is determined by a finite number of parameters.

In this section we show that the nodal values determine uniquely the elements of the global attractor $\mathbb{A}_{\nu_r}$, provided that we consider a flow driven by forces and moments belonging to a Gevrey class. We prove the existence of a set of instantaneously determining nodes that is the set of k points $\mathbf{x} = \{x_1, \ldots, x_k\}$ in the domain of flow Q, such that there exists a $1-1$ mapping from $\mathbb{A}_{\nu_r}$ to $\mathbb{R}^{3k}$ given by

$$E_{\mathbf{x}} : \bar{u} \in \mathbb{A} \to \big(\bar{u}(x_1), \bar{u}(x_2), \ldots, \bar{u}(x_k)\big) \in \mathbb{R}^{3k}$$

between attractor $\mathbb{A}_{\nu_r}$ and its image. Roughly speaking the value of solution at a finite number of points of Q determine uniquely elements lying on the attractor. To this end we use the following theorem by Fritz and Robinson

Theorem 3. *(See [4].) Let Ω be a periodic domain in $\mathbb{R}^n$, and $\mathbb{A}$ a finite-dimensional compact subset of $[L^2(\Omega)]^m$ which is bounded in $D(e^{\tau A^{1/2}})$ for some $\tau > 0$. Then, provided that $k > 16nd_f(\mathbb{A})$, the map from $\mathbb{A}$ into $\mathbb{R}^{mk}$ given by*

$$E_{\mathbf{x}} : u \mapsto \big(u(x_1), \dots, u(x_k)\big)$$

is $1-1$ between $\mathbb{A}$ and its image for almost every $\mathbf{x} = (x_1, \dots, x_k)$ in Ω^k (with respect to nk-dimensional Lebesgue measure).

What we need is the existence of the global attractor and the estimate of its fractal dimension. The former was proved in [11] and the latter in [12]. The fractal dimension of the global attractor $\mathbb{A}_{\nu_r}$ is estimated by

$$d_f(\mathbb{A}_{\nu_r}) \leqslant c\big(|f|^2_{\mathbb{L}^2(Q)} + |g|^2\big)^{1/2} \tag{23}$$

for an appropriate constant c.

Now we have everything we need to prove the following theorem.

Theorem 4. *Let $\mathbb{A}_{\nu_r}$ be the global attractor for a semigroup $\{S(t)\}_{t \geqslant 0}$ associated with the micropolar fluid equations (1) with periodic boundary conditions (3). Assume that $k > 32c(|f|^2_{\mathbb{L}^2(Q)} + |g|^2)^{1/2}$.*

Then for almost every (with respect to 2k-dimensional Lebesgue measure) set of k nodes

$$\mathbf{x} = (x_1, \dots, x_k), \quad x_j \in Q,$$

values $\bar{u}(x_j)$ determine uniquely the function $\bar{u} \in \mathbb{A}_{\nu_r}$.

Proof of Theorem 4. The existence of the global attractor $\mathbb{A}_{\nu_r}$ is shown in [11], the fractal dimension of $\mathbb{A}_{\nu_r}$ is estimated in [12] as follows $d_f(\mathbb{A}_{\nu_r}) \leqslant c(|f|^2_{\mathbb{L}^2(Q)} + |g|^2)^{1/2}$ - this is a finite-dimensional compact subset of $[L^2(Q)]^3$. In the course of proof of Theorem 2 we have shown that the global attractor $\mathbb{A}_{\nu_r}$ is bounded in $D(e^{\tau \mathcal{A}^{1/2}})$ (step 3 of the proof). We can now use Theorem 3 with $n = 2, m = 3$ since $Q \subset \mathbb{R}^2$ and $\bar{u}(x,t) \in \mathbb{R}^3$. That ends the proof. □

Remark 2. The solutions of Navier–Stokes (and other) equations in Gevrey classes was studied intensively by Robinson, Kukavica, Fritz and Langa in a series of papers [4–10].

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