On Asian option pricing for NIG Lévy processes

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Abstract

In this paper, we derive approximations and bounds for the Esscher price of European-style arithmetic and geometric average options. The asset price process is assumed to be of exponential Lévy type with normal inverse Gaussian (NIG) distributed log-returns. Numerical illustrations of the accuracy of these bounds as well as approximations and comparisons of the NIG average option prices with the corresponding Black–Scholes prices are given.

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1. Introduction

In recent years it has been realized that distributions of logarithmic asset returns can often be fitted extremely well by normal inverse Gaussian (NIG) distributions (see e.g. [3,4,23,24]). It is therefore of particular interest to study stochastic process models for stock prices and asset returns that capture this property. For a general survey on stochastic processes of normal inverse Gaussian type, we refer to Barndorff-Nielsen [5].

As a member of the family of generalized hyperbolic (GH) distributions, the normal inverse Gaussian distribution is infinitely divisible and thus generates a Lévy process \( (Z_t)_{t \geq 0} \), which gives rise to the following exponential Lévy model (see e.g. [11]). By setting

\[
S_t = S_0 \exp(Z_t),
\]

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where \((S_t)_{t \geq 0}\) denotes the asset price process over time, the log-returns of this model produce exactly a normal inverse Gaussian distribution. \((S_t)_{t \geq 0}\) is again a Lévy process. The applicability of such a Lévy model in practice is e.g. discussed in Geman [12].

Since this market model is incomplete (cf. [7]), there are many candidates of equivalent martingale measures for risk-neutral valuation of derivative securities. One mathematically tractable choice is the so-called Esscher equivalent measure, a concept which was introduced to mathematical finance by Madan and Milne [20]; see also Gerber and Shiu [13]. This particular choice of the pricing measure can be justified both within utility and equilibrium theory (cf. [6,14,15]).

In this paper we observe that the Esscher equivalent measure in the NIG model has a particularly simple structure and we exploit this property in various ways to obtain easy computable approximations and bounds for the Esscher price of arithmetic and geometric average options. For that purpose we adapt several techniques developed for the Black–Scholes setting to our situation.

In Section 2 we introduce various properties of the normal inverse Gaussian distribution needed for the development of the NIG asset price model and the derivation of the Esscher equivalent measure in Section 3. Section 4 uses the notion of stop-loss transforms, which is well-known to actuaries, to obtain upper bounds for arithmetic average option Esscher prices. Two approximation techniques for arithmetic average option prices are developed in Section 5. Section 6 contains approximation methods for geometric average rate options and gives bounds for the arithmetic average option prices in terms of the geometric price. Finally, in Section 7 we give numerical illustrations of these bounds and approximations and also compare the Asian Esscher option prices in the NIG model with the corresponding prices in the Black–Scholes world showing significant price differences, which indicates that a careful choice of the asset price model is an important issue in practice.

2. The normal inverse Gaussian distribution

The normal inverse Gaussian distribution is defined by the density

\[
f_{\text{NIG}}(x, \beta, \delta, \mu)(x) = f_{\text{NIG}}(x) = c \exp(\beta(x - \mu)) \frac{K_1(\beta \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}}
\]

with \(0 \leq |\beta| \leq x, \ \delta \geq 0, \ \mu \in \mathbb{R}\) and

\[
c = \frac{2\delta}{\pi} \exp(\delta \sqrt{x^2 - \beta^2}).
\]

Here \(K_1(x)\) denotes the modified Bessel function of the third kind of order 1. In general, for a real number \(v\), the function \(K_v(x)\) satisfies the differential equation

\[
x^2 y'' + xy' - (x^2 + v^2)y = 0.
\]

(cf. [1]). The moment generating function of (1) is given by

\[
M_{\text{NIG}}(u) = \exp(\delta(\sqrt{\beta^2 - \beta^2} - \sqrt{x^2 - (\beta + u)^2}) + \mu u)
\]

from which one can deduce the following important convolution property:

\[
f_{\text{NIG}}(x, \beta, \delta_1, \mu_1) \ast f_{\text{NIG}}(x, \beta, \delta_2, \mu_2) = f_{\text{NIG}}(x, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2).
\]
The NIG distribution is the special case \( \lambda = -\frac{1}{2} \) of the generalized hyperbolic distribution given by the density
\[
f_{\text{GH}}(x) = \frac{\sqrt{2} \Gamma(\lambda)}{\sqrt{2\pi} \delta^{2\lambda-1} \delta ^{2\lambda} K_{\lambda-1/2}(\delta x)} e^{\beta(x-\mu)} \eta(x)^{\lambda-1/2} K_{\lambda-1/2}(\eta(x)),
\]
with \( \delta = \sqrt{\alpha^2 - \beta^2} \) and \( \eta(x) = \alpha \delta^2 + (x - \mu)^2 \), which was introduced by Barndorff-Nielsen [2].

3. The NIG Lévy asset price model

As already mentioned in the introduction, the NIG distribution is infinitely divisible and hence generates a Lévy process \((Z_t)_{t \geq 0}\) (i.e. a stochastic process with stationary and independent increments, \(Z_0 = 0\) a.s. and \(Z_1\) is NIG-distributed; from the convolution property (2) it follows that the increments are NIG-distributed for arbitrary time intervals).

Let now \(S_t\) for \(t \geq 0\) denote the price of a nondividend-paying stock at time \(t\) and consider the following dynamics for the stock price process (see [9])
\[
dS_t = S_t \left( dZ_t + e^{\Delta Z} - 1 - \Delta Z_t \right), \tag{3}
\]
where \((Z_t)_{t \geq 0}\) denotes the NIG Lévy motion, \(Z_t\) - the left hand limit of the path at time \(t\) and \(\Delta Z_t = Z_t - Z_t^-\) the jump at time \(t\). Then, the solution of the stochastic differential equation (3) is given by
\[
S_t = S_0 \exp(Z_t)
\]
and it follows that the log-returns \(\ln(S_t/S_{t-1})\) are indeed NIG-distributed.

Since our aim is the risk-neutral valuation of derivative securities in this model and since the model is incomplete, we have to choose an equivalent martingale measure. In this paper we choose the method of Esscher transforms to find such a measure. This approach is applicable, whenever the stochastic process \((Z_t)_{t \geq 0}\) has stationary and independent increments (see [10,13]). Apart from its mathematical simplicity, this particular choice can also be economically justified (see [21] for a survey on this issue).

From (2) it follows that the density of \(Z_t\) is given by
\[
f_{\text{NIG}}^{\ast t}(x) = f_{\text{NIG}}(\theta, \beta, \delta, \mu)(x).
\]
For a real number \(\theta\) let us consider the Esscher transform
\[
f_{\text{NIG}}^{\ast t}(x; \theta) = \frac{e^{\theta x} f_{\text{NIG}}^{\ast t}(x)}{\int_{-\infty}^{\infty} e^{\theta y} f_{\text{NIG}}^{\ast t}(y) dy} = \frac{e^{\theta x}}{M_{\text{NIG}}(\theta)} f_{\text{NIG}}^{\ast t}(x), \tag{4}
\]
of the one-dimensional marginal distributions \(f_{\text{NIG}}^{\ast t}(x)\) of \((Z_t)_{t \geq 0}\). For any Lévy process \((Z_t)_{t \geq 0}\) (on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})\)) it is now possible to define a locally equivalent probability measure \(\mathbb{P}^\theta\) through
\[
d\mathbb{P}^\theta = \exp(\theta Z_t - t \log M_{\text{NIG}}(\theta)) d\mathbb{P},
\]
such that \((Z_t^0)_{t \geq 0}\) defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P}^\theta)\) is again a Lévy process and the one-dimensional marginal distributions of \((Z_t^0)_{t \geq 0}\) are the Esscher transforms of the corresponding marginals of \((Z_t)_{t \geq 0}\) (see e.g. [22]). \(\mathbb{P}^\theta\) is called the Esscher equivalent measure.
The parameter $\theta$ can now be chosen in such a way, that the discounted stock price process $(e^{-rt}S_t)_{t \geq 0}$ is a $\mathbb{P}^\theta$-martingale, namely if $\theta$ is the (unique) solution of
\begin{equation}
    r = \mu + \delta(\sqrt{x^2 - (\beta + \theta)^2} - \sqrt{x^2 - (\beta + \theta + 1)^2}),
\end{equation}
which can be derived explicitly. Here $r$ is the constant daily interest rate. For further details concerning the construction of the Esscher equivalent martingale measure we refer to Gerber and Shiu [13].

The following observation will substantially simplify the calculation of Esscher prices in the NIG model:

**Lemma 1.** The Esscher transform of a NIG-distributed random variable is again NIG-distributed. In particular,
\begin{equation}
    f_{\text{NIG}(x,\beta,\delta,\mu)}(x;\theta) = f_{\text{NIG}(x,\beta+\delta,\mu)}(x). \tag{6}
\end{equation}

**Proof.** From
\begin{align*}
    f_{\text{NIG}(x,\beta,\delta,\mu)}(x;\theta) &= \frac{e^{\theta x}}{M_{\text{NIG}}(\theta)} \frac{\alpha \delta}{\pi} \exp(\delta \sqrt{x^2 - \beta^2} + \beta(x - \mu)) \frac{K_1(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}} \\
    &= \frac{\alpha \delta}{\pi} \frac{e^{\theta x + \delta \sqrt{x^2 - \beta^2} - \delta \sqrt{x^2 - (\beta + \theta)^2}}}{M_{\text{NIG}}(\theta)} \frac{e^{\delta \sqrt{x^2 - (\beta + \theta)^2} + (\beta + \theta)(x - \mu)} K_1(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}}
\end{align*}
it follows that we have to prove
\begin{equation}
    \int_{-\infty}^{\infty} e^{\theta x + \delta \sqrt{x^2 - \beta^2} + \beta(x - \mu)} \frac{K_1(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}} \frac{d x}{\pi} = e^{\theta x + \delta \sqrt{x^2 - \beta^2} - \delta \sqrt{x^2 - (\beta + \theta)^2}}, \tag{7}
\end{equation}
which is equivalent to
\begin{equation}
    \int_{-\infty}^{\infty} \frac{\alpha \delta}{\pi} \exp(\delta \sqrt{x^2 - (\beta + \theta)^2} + (\beta + \theta)(x - \mu)) \frac{K_1(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}} = 1. \tag{7}
\end{equation}
But Eq. (7) holds for $|\beta + \theta| < \alpha$ and the latter condition is always satisfied due to the choice of the Esscher optimal $\theta$ as the solution of (5). □

As a first example, the value at time $t$ of a European call option with exercise price $K$ and maturity $T$ can be represented by a simple analytical expression: From $EC_t = \mathbb{E}^{\theta}[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t]$ and Lemma 1 it follows that
\begin{equation}
    EC_t = S_t \int_{k}^{\infty} f_{\text{NIG}(x,\beta+\delta,\mu)}(x;\theta) \, dx - e^{-r(T-t)}K \int_{0}^{\infty} f_{\text{NIG}(x,\beta+\delta,\mu)}(x;\theta) \, dx \tag{8}
\end{equation}
with $k = \ln(K/S_t)$. This value can be computed numerically.
4. Stop-loss transforms and upper bounds for arithmetic average options

We will now focus on the evaluation of the Esscher price of a European-style arithmetic average call option at time \( t \) given by

\[
AA_t = \frac{e^{-r(T-t)}}{n} \mathbb{E}^0 \left[ \left( \sum_{k=0}^{n-1} S_{T-k} - nK \right) \bigg| \mathcal{F}_t \right],
\]

(9)

where \( n \) is the number of averaging days, \( K \) the strike price, \( T \) the time to expiration and \( r \) the risk-free interest rate.

The main difficulty here is to find the distribution of \( \sum S_i \), which is a sum of dependent random variables. Simon et al. [25] recently derived upper bounds for the price of an arithmetic average option in an arbitrage-free and complete market by means of stop-loss transforms and the theory of comonotone risks. In the sequel we will adapt their technique to the NIG model.

Let \( F(x) \) be a distribution function with support \( D \subseteq \mathbb{R}^+ \), then its stop-loss transform \( \Psi_F(r) \) is defined by

\[
\Psi_F(r) = \int_{[r,\infty) \cap D} (x - r) \, dF(x).
\]

A stop-loss ordering of distribution functions \( F(x) \) and \( G(x) \) with support in \( \mathbb{R}^+ \) can now be defined in the following way: \( F(x) \) is said to precede \( G(x) \) in stop-loss order (\( F \leq_{sl} G \)), if

\[
\Psi_F(r) \leq \Psi_G(r) \quad \text{for all } r \in \mathbb{R}^+.
\]

Next, we can rewrite the price of the arithmetic average option given by (9) to

\[
AA_t = \frac{e^{-r(T-t)}}{n} \Psi_{F_{A_n(T)}}(nK)
\]

(10)

for a given value \( S_t = s \) with \( F_{A_n(T)} = \mathbb{P}^0(A_n(T) \leq x \mid S_t = s) \), where

\[
A_n(T) = \sum_{k=0}^{n-1} S_{T-k}.
\]

In this way we have transformed the problem of pricing an arithmetic average option to calculating the stop-loss transform of a sum of dependent risks. Hence we can apply results on bounds for stop-loss transforms to our option pricing problem.

A positive random vector \( (X_1, \ldots, X_n) \) with marginal distributions \( F_1(x_1), \ldots, F_n(x_n) \) is called comonotone, if \( F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = \min\{F_1(x_1), \ldots, F_n(x_n)\} \) holds for every \( x_1, \ldots, x_n \geq 0 \). It immediately follows that a comonotone random vector \( (X_1, \ldots, X_n) \) with given marginal distributions \( F_1(x_1), \ldots, F_n(x_n) \) is uniquely determined. It can easily be shown (see e.g. [8]) that an upper bound for the stop-loss transform of a sum of dependent random variables \( \sum_{k=1}^n X_k \) with marginal distributions \( F_1(x_1), \ldots, F_n(x_n) \) is now given by the stop-loss transform of the sum \( \sum_{k=1}^n Y_k \), where \( (Y_1, \ldots, Y_n) \) is the comonotone random vector with marginal distributions \( F_1(x_1), \ldots, F_n(x_n) \), i.e. \( \sum_{k=1}^n X_k \leq_{sl} \sum_{k=1}^n Y_k \). Let us define \( F_{R}(x) := \mathbb{P}^0 \left( \sum_{k=1}^n Y_k \leq x \right) \), then we have ([8])

\[
F_R^{-1}(x) = \sum_{k=1}^n F_k^{-1}(x) \quad \text{for each } x \in \mathbb{R}^+
\]
and
\[ \Psi_{F_R}(x) = \sum_{k=1}^{n} \Psi_{F_k}(F_{R_k}^{-1}(F_R(x))) \quad \text{for each } x \in \mathbb{R}^+. \]

Hence this upper bound of the arithmetic average option price can be viewed as a sum of prices of European call options with strike prices \( F_{R_k}^{-1}(F_R(x)) \). The following proposition is an adaption of the result of Simon et al. [25] to our situation. Let in the sequel \( F(x_2, t_2, x_1, t_1) \) denote the conditional distribution function of \( S_{t_2} \) under the equivalent Esscher martingale measure \( \mathbb{P}^\theta \) given \( S_{t_1} = x_1 \), i.e.

\[ F(x_2, t_2, x_1, t_1) = \mathbb{P}^\theta(S_2 \leq x_2 | S_{t_1} = x_1) \]

\[ = \mathbb{P}^\theta \left( Z_{t_2-t_1} \leq \ln \frac{x_2}{S_{t_1}} \right) \]

\[ = \int_{-\infty}^{\ln(x_2/S_{t_1})} \mathcal{NIG}(\mu, \beta, \nu, T_2-t_1) dz, \quad t_2 \geq t_1. \]

**Proposition 1** (Simon et al. [25]). Let \( k^* \) be such that \( T - k^* \leq t < T - k^* + 1 \) and \( K_j = nK - \sum_{k=j}^{n-1} S_{T-k} \) for \( j < n \), \( K_n = nK \). Let \( AA_t \) be the price of an arithmetic average option at time \( t \) as given in (9) and let furthermore \( EC_t(K_k, T - k) \) be the price of a European option with strike price \( K_k \) and time to expiration \( T - k \). Then we have for \( K_{k^*} > 0 \)

\[ AA_t \leq \frac{e^{-r(T-t)}}{n} \sum_{k=0}^{k^*-1} \Psi_{F,(T-k,s,t)}(K_k) = \frac{1}{n} \sum_{k=0}^{k^*-1} e^{-kr} EC_t(K_k, T - k), \quad (11) \]

where

\[ K_k = F^{-1}(F_R(K_{k^*}), T - k, s, t), \quad k = 0, \ldots, k^*. \quad (12) \]

Moreover, this choice of the strike prices \( K_k \) is best possible.

In case \( K_{k^*} \leq 0 \), we have

\[ AA_t = \frac{S_t}{n} \sum_{k=0}^{k^*-1} e^{-kr} + \frac{e^{-r(T-t)}}{n} \sum_{k=k^*}^{n-1} S_{T-k} - e^{-r(T-t)} K. \]

Proposition 1 shows (for \( K_{k^*} > 0 \)) that the price of an arithmetic average option is bounded from above by the price of a portfolio of time-delayed European-style call options with exercise prices \( K_k \), and this bound is optimized by the above choice of \( K_k \).

In order to obtain a bound for the arithmetic average option price we thus have to calculate \( k^* \) strike prices \( K_k \) using (12) and then evaluate (11) using (8). A numerical illustration of the accuracy of these bounds is given in Section 7.

### 5. Two approximations for the distribution of the arithmetic mean

In this section we study approximations of the arithmetic option price (9) by means of Edgeworth series expansions. For notational simplicity we will assume \( t = 0 \) and \( n = T \) in (9), i.e. the averaging
starts at time $t = 1$ and we determine the price at time $t = 0$

$$AA_0 = \frac{e^{-rT}}{n} \mathbb{E}^0 \left[ \left( \sum_{k=1}^{n} S_k - nK \right) \right]_{\mathcal{F}_0}. \quad (13)$$

The extension to the general case is straightforward. Recall that the cumulants of a random variable $X$ with distribution function $F$ are defined by

$$\chi_i(F) = \left[ \frac{\partial^i \ln \mathbb{E}[e^{tX}]}{\partial t^i} \right]_{t=0}, \quad i = 1, 2, \ldots$$

and can also be expressed in terms of moments. For the first four cumulants we have

- $\chi_1(F) = \mathbb{E}[X]$,
- $\chi_2(F) = \mathbb{E}[(X - \mathbb{E}[X])^2]$,
- $\chi_3(F) = \mathbb{E}[(X - \mathbb{E}[X])^3]$,
- $\chi_4(F) = \mathbb{E}[(X - \mathbb{E}[X])^4] - 3\chi_2^2(F)$.

In the sequel we will make use of the following classical result:

**Lemma 2** (Jarrow and Rudd [17]). Let $F$ and $G$ be two continuous distribution functions with $G \in \mathcal{C}^5$ and $\chi_1(F) = \chi_1(G)$, and assume that the first five moments of both distributions exist. Then we can expand the density $f(x)$ in terms of the density $g(x)$ as follows

$$f(x) = g(x) + \frac{\chi_2(F) - \chi_2(G)}{2} \frac{\partial^2 g}{\partial x^2}(x) - \frac{\chi_3(F) - \chi_3(G)}{3!} \frac{\partial^3 g}{\partial x^3}(x)$$

$$+ \frac{\chi_4(F) - \chi_4(G) + 3(\chi_2(F) - \chi_2(G))^2}{4!} \frac{\partial^4 g}{\partial x^4}(x) + \varepsilon(x),$$

where $\varepsilon(x)$ is a residual error term.

We will now approximate the distribution function of $\sum_{k=1}^{n} S_k$ (which we denote by $F$) by a lognormal distribution $G$ (see [19,26] for a similar procedure in the Black–Scholes case). The density of the lognormal distribution is given by

$$g(x) = f_{\text{LN}}(x) = \frac{1}{\sigma \sqrt{2\pi \sigma x}} \exp \left( - \frac{(\ln x - \mu)^2}{2\sigma^2} \right) \chi_{(0, \infty)}(x).$$

Let us define

$$R_i = \frac{S_i}{S_{i-1}}, \quad i = 1, \ldots, n.$$ 

and

$$L_n = 1,$$

$$L_{i-1} = 1 + R_i L_i, \quad i = 2, \ldots, n.$$
Then we have

\[ \sum_{k=1}^{n} S_k = S_0(R_1 + R_1R_2 + \cdots + R_1R_2\ldots R_n) = S_0R_1L_1. \]

Since we can rewrite equation Eq. (13) to

\[ A A_0 = \frac{e^{-rT}S_0}{n} E^\theta \left[ \left( R_1L_1 - \frac{nK}{S_0} \right) \mid \mathcal{F}_0 \right], \]

it remains to determine \( E^\theta[(R_1L_1)^m] \) for \( m = 1, 2, 3, 4 \). Because of the independent increments property of a Lévy process, we have \( E^\theta[(R_1L_1)^m] = E^\theta[R_1^m]E^\theta[L_1^m] \) and

\[ E^\theta[L_{m-1}^m] = E^\theta[(1 + L_iR_i)^m] = \sum_{k=0}^{m} \binom{m}{k} E^\theta[L_i^k]E^\theta[R_i^k]. \] (14)

In order to apply recursion (14), we need to determine the moments \( E^\theta[R_i^k] \):

**Lemma 3.** For all \( k \in \mathbb{N} \) we have

\[ E^\theta[R_i^k] = \exp(\delta(\sqrt{\bar{\alpha}^2 - (\beta + \theta)^2} - \sqrt{\bar{\alpha}^2 - (\beta + \theta + k)^2} + k\mu). \] (15)

**Proof.** Since \( R_i \) is log-NIG distributed (namely \( R_i \sim \text{LNIG}(\bar{\alpha}, \beta + \theta, \delta, \mu) \)), the result follows from

\[ E^\theta[R_i^k] = \int_0^{\infty} x^k \frac{2\delta}{\pi x} \exp(\delta(\sqrt{\bar{\alpha}^2 - (\beta + \theta)^2} + (\beta + \theta)(\ln x - \mu)) \frac{K_1(\bar{\alpha}\sqrt{\delta^2 + (\ln x - \mu)^2})}{\delta^2 + (\ln x - \mu)^2} dx \]

\[ = \int_0^{\infty} e^{uk} \frac{2\delta}{\pi} \exp(\delta(\sqrt{\bar{\alpha}^2 - (\beta + \theta)^2} + (\beta + \theta)(u - \mu)) \frac{K_1(\bar{\alpha}\sqrt{\delta^2 + (u - \mu)^2})}{\delta^2 + (u - \mu)^2} du \]

\[ = \text{MNIG}(\bar{\alpha}, \beta + \theta, \delta, \mu)(k). \]

The moments \( E^\theta[L_i^m] \) \( (m = 1, 2, 3, 4) \) and subsequently the cumulants \( \chi_i(F) \) can now be calculated recursively using (14), (15) and the fact that \( E^\theta[L_i^k] = 1 \) for all \( k \in \{0, \ldots, m\} \). The parameters of the approximating lognormal distribution are chosen in such a way that the first two moments of the approximating log-normal and the original distribution coincide (a so-called Wilkinson approximation):

\[ \bar{\mu} = 2 \ln(\chi_1(F)) - \frac{1}{2} \ln(\chi_1^2(F) + \chi_2(F)) \]

\[ \bar{\sigma}^2 = \ln(\chi_1^2(F) + \chi_2(F)) - 2 \ln(\chi_1(F)). \]

In this way we have derived a lognormal approximation pricing formula for an arithmetic average option in the NIG model, which we call the Turnbull–Wakeman price \( AA_0^{TW} \) at time \( t = 0 \).
Proposition 2. The price at time $0$ of a European-style arithmetic average option in the NIG model with maturity $T$ and strike price $K$ is

$$
A_{0}^{TW} = e^{-rT} \frac{S_{0}}{n} \left( e^{\beta + \frac{\sigma^{2}}{2}} \Phi \left( \frac{\bar{\mu} + \bar{\sigma}^{2} - \ln(nK/S_{0})}{\bar{\sigma}} \right) - \frac{nK}{S_{0}} \varphi \left( \frac{\bar{\mu} - \ln(nK/S_{0})}{\bar{\sigma}} \right) \right) - e^{-rT} \frac{S_{0}}{n} \left( \frac{\chi_3(F) - \chi_3(G) \cdot \tilde{c}g(nK/S_{0})}{3!} + \frac{\chi_4(F) - \chi_4(G) \cdot \tilde{c}^{2}g(nK/S_{0})}{4!} \right).
$$

If only the first two cumulants are considered, we call the corresponding approximation the Levy price $A_{0}^{L}$ given by

$$
A_{0}^{L} = e^{-rT} \frac{S_{0}}{n} \left( e^{\beta + \frac{\sigma^{2}}{2}} \Phi \left( \frac{\bar{\mu} + \bar{\sigma}^{2} - \ln nK/S_{0}}{\bar{\sigma}} \right) - \frac{nK}{S_{0}} \varphi \left( \frac{\bar{\mu} - \ln nK/S_{0}}{\bar{\sigma}} \right) \right).
$$

Another possibility is to approximate the arithmetic average $\sum_{k=1}^{n} S_{k}$ by a NIG distribution by matching the first four moments. Since the cumulants of a NIG(\(\chi, \beta, \delta, \mu\))-distributed random variable $H$ are given by

$$
\chi_1(H) = \mu + \frac{\beta \delta}{(\sigma^2 - \beta^2)^{1/2}},
$$

$$
\chi_2(H) = \frac{\sigma^2 \delta}{(\sigma^2 - \beta^2)^{3/2}},
$$

$$
\chi_3(H) = \frac{3 \sigma^2 \beta \delta}{(\sigma^2 - \beta^2)^{5/2}},
$$

$$
\chi_4(H) = \frac{3 \sigma^2 \delta (\sigma^2 + 4 \beta^2)}{(\sigma^2 - \beta^2)^{7/2}},
$$

the parameters $\tilde{\sigma}, \tilde{\beta}, \tilde{\delta}$ and $\tilde{\mu}$ of the approximating NIG distribution are the solution of the equation system $\chi_k(H) = \chi_k(F)$ for $k = 1, \ldots, 4$, where $F$ again denotes the distribution function of $\sum_{k=1}^{n} S_{k}$. This leads to

$$
\tilde{\beta} = \frac{3 \chi_2(F) \chi_3(F)}{3 \chi_2(F) \chi_4(F) - 5(\chi_3(F))^{2}},
$$

$$
\tilde{\sigma}^2 = \frac{3 \chi_2(F) \tilde{\beta}}{\chi_3(F)} + \tilde{\beta}^2,
$$

$$
\tilde{\delta} = \frac{\chi_2(F)(\tilde{\sigma}^2 - \tilde{\beta}^2)^{3/2}}{\tilde{\sigma}^2},
$$

$$
\tilde{\mu} = \chi_1(F) - \frac{\tilde{\beta} \delta}{(\tilde{\sigma}^2 - \tilde{\beta}^2)^{1/2}}.
$$
Finally, the option price is given by
\[ A_{0}^{\text{NIG}} = e^{-rT} \frac{S_0}{n} \int_{d}^{\infty} (x - d) f_{\text{NIG}(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\mu})}(x) \, dx \quad \text{with} \quad d = nK/S_0. \]

This value can be calculated numerically (see Section 7 for a numerical example).

6. Arithmetic and geometric average options

Let us consider a geometric average option with Esscher price given by
\[ GA_0 = e^{-rT} \mathbb{E}^d[(G_T - K)^+]|\mathcal{F}_0], \tag{16} \]
where \( G_T = (\prod_{k=1}^{n} S_k)^{1/n} \) and \( K \) denotes the strike price. Again, we have chosen \( t = 0 \) and \( n = T \) (the generalization to arbitrary \( t \geq 0 \) and arbitrary starting times of the averaging period is straightforward). Taking the logarithm of \( G_T \), we get
\[ \ln G_T = \frac{1}{n} \sum_{k=1}^{n} \ln S_k = \ln S_0 + \frac{1}{n} \sum_{k=1}^{n} \ln \frac{S_k}{S_0}, \]
which can also be written as
\[ \ln G_T = \ln S_0 + \frac{1}{n} \left( n \ln \frac{S_1}{S_0} + (n - 1) \ln \frac{S_2}{S_1} + \cdots + \ln \frac{S_n}{S_{n-1}} \right) \]
and hence we have
\[ \ln G_T = \ln S_0 + X_1 + \frac{n - 1}{n} X_2 + \cdots + \frac{1}{n} X_n \]
with \( X_k \text{iid} \sim \text{NIG} \). Unfortunately, the distribution of \( \ln G_T \) is not NIG anymore. However, we will approximate it by a NIG distribution. For that purpose, we determine the cumulants:
\[ \chi_1(\ln G_T) = \ln S_0 + \frac{(n + 1)}{2} \chi_1(X_1), \]
\[ \chi_2(\ln G_T) = \frac{1}{n^2} \chi_2(X_1) \left( \frac{1}{3} (n + 1)^3 - \frac{1}{2} (n + 1)^2 + \frac{1}{6} (n + 1) \right), \]
\[ \chi_3(\ln G_T) = \frac{1}{n^3} \chi_3(X_1) \left( \frac{1}{4} (n + 1)^4 - \frac{1}{2} (n + 1)^3 + \frac{1}{4} (n + 1)^2 \right), \]
\[ \chi_4(\ln G_T) = \frac{1}{n^4} \chi_4(X_1) \left( \frac{1}{5} (n + 1)^5 - \frac{1}{2} (n + 1)^4 + \frac{1}{3} (n + 1)^3 - \frac{1}{30} (n + 1) \right). \]

As above, the parameters \( \alpha^*, \beta^*, \delta^*, \mu^* \) of the approximating NIG distribution \( X \) can be obtained by solving the system of equations \( \chi_k(X) = \chi_k(\ln G_T) \) \( (k = 1, \ldots, 4) \). Then, the price of a European-style geometric average option at time 0, where the distribution of \( \ln G_T \) is approximated by the NIG
Table 1
Comparison of simulated Asian option prices and the SL upper bound of Section 4

<table>
<thead>
<tr>
<th>Stock</th>
<th>( T )</th>
<th>( r )</th>
<th>( K )</th>
<th>( EC_0 )</th>
<th>( AA_0^{MC} )</th>
<th>SL upper bound</th>
<th>Rel. error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayer</td>
<td>10</td>
<td>0.1</td>
<td>34</td>
<td>2.1574</td>
<td>2.0729 (±0.0016)</td>
<td>2.0759</td>
<td>0.14</td>
</tr>
<tr>
<td>(( S_0 = 36 ))</td>
<td>35</td>
<td>1.2622</td>
<td>1.1059 (±0.0014)</td>
<td>1.2146</td>
<td>1.69</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>36</td>
<td>0.5849</td>
<td>0.3505 (±0.0010)</td>
<td>0.3924</td>
<td>11.95</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>37</td>
<td>0.1980</td>
<td>0.0539 (±0.0004)</td>
<td>0.0796</td>
<td>47.68</td>
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<td></td>
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<tr>
<td></td>
<td>0.05</td>
<td>34</td>
<td>2.0946</td>
<td>2.0729 (±0.0016)</td>
<td>2.0759</td>
<td>0.14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>1.2061</td>
<td>1.0718 (±0.0014)</td>
<td>1.0921</td>
<td>1.89</td>
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<td></td>
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<tr>
<td></td>
<td>36</td>
<td>0.5396</td>
<td>0.3298 (±0.0010)</td>
<td>0.3723</td>
<td>12.89</td>
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<tr>
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<td>37</td>
<td>0.1795</td>
<td>0.0487 (±0.0004)</td>
<td>0.0728</td>
<td>49.49</td>
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<td>34</td>
<td>2.3508</td>
<td>2.1435 (±0.0021)</td>
<td>2.1532</td>
<td>0.45</td>
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<tr>
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<td>35</td>
<td>1.5210</td>
<td>1.2174 (±0.0019)</td>
<td>1.2549</td>
<td>3.08</td>
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<tr>
<td></td>
<td>36</td>
<td>0.8771</td>
<td>0.5047 (±0.0014)</td>
<td>0.5731</td>
<td>13.55</td>
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<tr>
<td></td>
<td>37</td>
<td>0.4284</td>
<td>0.2398 (±0.0008)</td>
<td>0.3804</td>
<td>12.89</td>
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</tr>
<tr>
<td>NYSE C.I.</td>
<td>10</td>
<td>0.1</td>
<td>47</td>
<td>3.1885</td>
<td>3.0983 (±0.0013)</td>
<td>3.0984</td>
<td>0.00</td>
</tr>
<tr>
<td>(( S_0 = 50 ))</td>
<td>48.5</td>
<td>1.7191</td>
<td>1.6061 (±0.0013)</td>
<td>1.6107</td>
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</tr>
<tr>
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<td>50</td>
<td>0.5181</td>
<td>0.3114 (±0.0008)</td>
<td>0.3455</td>
<td>10.95</td>
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</tr>
<tr>
<td></td>
<td>20</td>
<td>3.3807</td>
<td>3.1841 (±0.0017)</td>
<td>3.1852</td>
<td>0.04</td>
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<tr>
<td></td>
<td>48.5</td>
<td>1.9558</td>
<td>1.7086 (±0.0017)</td>
<td>1.7149</td>
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<tr>
<td></td>
<td>50</td>
<td>0.8044</td>
<td>0.4610 (±0.0011)</td>
<td>0.5179</td>
<td>12.34</td>
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</tr>
</tbody>
</table>

The expected value \( \mathbb{E}^0[AT|\mathcal{F}_0] = S_0/n\mathbb{E}^0[R_1L_1] \) in (18) can be calculated by recursion (14), and \( \mathbb{E}^0[G_T|\mathcal{F}_0] \approx M_{\text{NIG}}(\alpha^*,\beta^*,\delta^*,\mu^*)(1) \), since \( \ln G_T \) is approximated by a NIG distribution (cf. Lemma 3).
Table 2
Comparison of simulated Asian option prices and approximations of Section 5

<table>
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<tr>
<th>Stock</th>
<th>$T$</th>
<th>$r$</th>
<th>$K$</th>
<th>$A_{0}^{MC}$</th>
<th>$A_{0}^{L}$</th>
<th>r.e. (%)</th>
<th>$A_{0}^{TW}$</th>
<th>r.e. (%)</th>
<th>$A_{0}^{NIG}$</th>
<th>r.e. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayer</td>
<td>10</td>
<td>0.1</td>
<td>34</td>
<td>2.0729</td>
<td>2.0719</td>
<td>0.05</td>
<td>2.0732</td>
<td>0.02</td>
<td>2.0728</td>
<td>0.01</td>
</tr>
<tr>
<td>$(S_0 = 36)$</td>
<td>35</td>
<td>1.1059</td>
<td>1.1054</td>
<td>0.05</td>
<td>1.1055</td>
<td>0.04</td>
<td>1.1056</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>36</td>
<td>0.3505</td>
<td>0.3577</td>
<td>2.05</td>
<td>0.3487</td>
<td>0.51</td>
<td>0.3498</td>
<td>0.20</td>
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<tr>
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<td>37</td>
<td>0.0539</td>
<td>0.0510</td>
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<td>0.0555</td>
<td>2.97</td>
<td>0.0543</td>
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<tr>
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<td>0.05</td>
<td>34</td>
<td>2.0376</td>
<td>2.0366</td>
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<td>2.0381</td>
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<td>2.0377</td>
<td>0.01</td>
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<tr>
<td></td>
<td>35</td>
<td>1.0718</td>
<td>1.0712</td>
<td>0.06</td>
<td>1.0717</td>
<td>0.01</td>
<td>1.0719</td>
<td>0.01</td>
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<tr>
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<td>36</td>
<td>0.3298</td>
<td>0.3367</td>
<td>2.09</td>
<td>0.3382</td>
<td>0.49</td>
<td>0.3392</td>
<td>0.18</td>
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<td>0.0489</td>
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<td>20</td>
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<td>34</td>
<td>2.1435</td>
<td>2.1427</td>
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<tr>
<td></td>
<td>47</td>
<td>3.0963</td>
<td>3.0983</td>
<td>0.07</td>
<td>3.0977</td>
<td>0.05</td>
<td>3.0978</td>
<td>0.05</td>
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<td></td>
<td>48.5</td>
<td>1.6061</td>
<td>1.6049</td>
<td>0.08</td>
<td>1.6069</td>
<td>0.05</td>
<td>1.6068</td>
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<tr>
<td></td>
<td>50</td>
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<tr>
<td></td>
<td>20</td>
<td>3.1841</td>
<td>3.1850</td>
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<td>3.1851</td>
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<td>3.1851</td>
<td>0.03</td>
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<td>48.5</td>
<td>1.7086</td>
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<td>0.03</td>
<td>1.7078</td>
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<tr>
<td></td>
<td>50</td>
<td>0.4610</td>
<td>0.4651</td>
<td>0.89</td>
<td>0.4613</td>
<td>0.07</td>
<td>0.4617</td>
<td>0.15</td>
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</tr>
</tbody>
</table>

Vorst [27] also proposed the approximation

$$A_{0}^{NIG} = e^{-rT}E^{\theta}[(G_T - K')^+ | \mathcal{F}_0] \quad \text{with} \quad K' = K - (E^{\theta}[A_T | \mathcal{F}_0] - E^{\theta}[G_T | \mathcal{F}_0])$$

for the price of an arithmetic average value option, leading to

$$A_{0}^{NIG} = e^{-rT} \int_{K'}^{\infty} (x - K')f_{\text{LNIG}(\sigma^*, \beta^*, \delta^*, \mu^*)}(x) \, dx.$$

This integral can be calculated numerically.

Remark. Since our asset price model is arbitrage-free, put-call parity holds and thus the above techniques can also directly be applied to put options of Asian type.

7. Numerical illustrations and comparison with the Black–Scholes model

We now give some numerical illustrations of the accuracy of the bounds and approximations derived in the previous sections for options on the stock of Bayer AG and on the NYSE Composite Index, respectively. For that purpose the parameters of our NIG model are estimated from historical
Table 3
Comparison of simulated Asian option prices and approximations of Section 6

<table>
<thead>
<tr>
<th>Stock</th>
<th>$T$</th>
<th>$r$</th>
<th>$K$</th>
<th>$A_{0}^{MC}$</th>
<th>$A_{0}^{V}$</th>
<th>r.e.(%)</th>
<th>$G_{0}^{MC}$ (± s.e.)</th>
<th>$G_{0}^{NIG}$</th>
<th>r.e.(%)</th>
<th>$AA_{0}^{U}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bayer (S₀ = 36)</td>
<td>10</td>
<td>0.1</td>
<td>34</td>
<td>2.0729</td>
<td>2.0728</td>
<td>0.01</td>
<td>2.0692 (±0.0015)</td>
<td>2.0691</td>
<td>0.01</td>
<td>2.0728</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td></td>
<td></td>
<td>1.1059</td>
<td>1.1060</td>
<td>0.01</td>
<td>1.1026 (±0.0014)</td>
<td>1.1025</td>
<td>0.01</td>
<td>1.1063</td>
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<td></td>
<td>36</td>
<td></td>
<td></td>
<td>0.3505</td>
<td>0.3498</td>
<td>0.20</td>
<td>0.3483 (±0.0010)</td>
<td>0.3478</td>
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<td>0.3516</td>
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<td>0.0537</td>
<td>0.37</td>
<td>0.0530 (±0.0004)</td>
<td>0.0532</td>
<td>0.38</td>
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<td>38</td>
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<td></td>
<td>0.0052</td>
<td>0.0052</td>
<td>0.00</td>
<td>0.0050 (±0.0001)</td>
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<td>4.00</td>
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<td>2.0339 (±0.0015)</td>
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<td>1.0687 (±0.0014)</td>
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<td>0.3278 (±0.0010)</td>
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<td>0.0478 (±0.0004)</td>
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<td>0.0045</td>
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<td>0.0044 (±0.0001)</td>
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<td>0.01</td>
<td>2.1363 (±0.0021)</td>
<td>2.1362</td>
<td>0.01</td>
<td>2.1439</td>
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<tr>
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<td>1.2178</td>
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<td>1.2112 (±0.0019)</td>
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<td>0.5076</td>
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<td>0.1372 (±0.0007)</td>
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<td>0.0263</td>
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<td>0.0262 (±0.0003)</td>
<td>0.0259</td>
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<td>2.0717 (±0.0021)</td>
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<td>1.1577</td>
<td>1.1582</td>
<td>0.04</td>
<td>1.1519 (±0.0018)</td>
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<td></td>
<td>0.4648</td>
<td>0.4644</td>
<td>0.09</td>
<td>0.4606 (±0.0013)</td>
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<td>0.02</td>
<td>0.4681</td>
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<td>0.1227</td>
<td>0.81</td>
<td>0.1213 (±0.0007)</td>
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<td>0.1288</td>
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<td>0.0218 (±0.0002)</td>
<td>0.0218</td>
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<td>0.0294</td>
</tr>
<tr>
<td>N.C.I. (S₀ = 50)</td>
<td>10</td>
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<td>47</td>
<td>3.0983</td>
<td>3.0978</td>
<td>0.02</td>
<td>3.0965 (±0.0013)</td>
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The data of the respective underlying by maximum likelihood methods (the corresponding estimates are $\alpha = 81.6$, $\beta = 3.69$, $\delta = 0.0103$, $\mu = -0.000123$ (Bayer) and $\alpha = 136.29$, $\beta = -8.95$, $\delta = 0.0059$, $\mu = 0.00079$ (NYSE)). Table 1 compares the stop-loss upper bound for the Esscher price of the European-style arithmetic average option of Section 4 with a Monte Carlo simulated price $A_{0}^{MC}$ obtained by generating 1 million sample paths (given together with its standard error based on an asymptotic 95% confidence interval). For convenience, the European call option price $EC_{0}$ is also given. For this and all the following tables the number of averaging days $n$ equals the number of days $T$ until maturity. The inverse distribution function needed for the calculation of $\kappa_{k}$ is interpolated, since there is no analytic expression available. The numerical values indicate that the accuracy of the upper bound is satisfying if the option is in the money.

In Table 2 we compare the approximation techniques for the arithmetic average option developed in Section 5 and give the relative error with respect to $A_{0}^{MC}$. The results show that the NIG approximation outperforms the Turnbull–Wakeman approximation, which itself is superior to the Levy approximation in most cases.
Table 4
Comparison of simulated Asian option prices in the NIG model and the Black–Scholes model

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The numerical values for the Vorst approximation $A_{A}^{Y}$ and its relative error w.r.t. $A_{A}^{MC}$ are depicted in Table 3. Moreover, the approximation $G_{A}^{NIG}$ is compared with the simulated geometric price $G_{A}^{MC}$. And finally the upper bound $A_{A}^{U}$ is given. Since the expression $E_{DC2}[G_{T}]$ needed for the evaluation of $A_{A}^{U}$ is itself obtained through approximation (cf. Section 6), we always have $A_{A}^{V} \leq A_{A}^{U}$, but it may happen that $A_{A}^{U}$ is not an upper bound for the numerical value $A_{A}^{MC}$.

It may be interesting to compare the numerical results for the Esscher option prices obtained in the NIG model to the corresponding values in the Black–Scholes setting. In the Black–Scholes model there is an explicit pricing formula for the geometric average option (see [18])

$$
(BS)A_{A}^{0} = e^{-rT}(e^{\hat{\mu} + \hat{\sigma}^{2}/2} \Phi(d_{1}) - K\Phi(d_{2})) \quad \text{with} \quad d_{1} = \frac{\hat{\mu} + \hat{\sigma}^{2} - \ln K}{\hat{\sigma}}, \quad d_{2} = d_{1} - \hat{\sigma},
$$

where

$$\hat{\mu} = \ln S_{0} + \frac{T + T/n}{2} \left( r - \frac{\sigma^{2}}{2} \right), \quad \hat{\sigma}^{2} = \sigma^{2}T \frac{(2n + 1)(n + 1)}{6n^{2}}.$$
Here $\sigma^2$ denotes the variance of the log-returns, which can again be estimated from historical data. The arithmetic average option price $(\text{BS})_{\AA 0}$ in the Black–Scholes model can not be obtained by an explicit formula. Thus we use a Quasi-Monte Carlo simulated price (cf. [16]).

In Table 4 the Esscher option prices of the NIG model are compared with the corresponding Black–Scholes prices and the relative difference is given. Note that in the Black–Scholes setting the Esscher pricing principle also yields the correct (unique) option prices. The Black–Scholes prices differ significantly from the NIG Esscher prices; in particular they tend to be lower if the option is in and out of the money and higher than the NIG prices, if the option is at the money. These differences indicate that an appropriate choice of the asset price model is of great importance for the issue of option pricing.

References