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On Asian option pricing for NIG Lévy processes☆

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Abstract

In this paper, we derive approximations and bounds for the Esscher price of European-style arithmetic and geometric average options. The asset price process is assumed to be of exponential Lévy type with normal inverse Gaussian (NIG) distributed log-returns. Numerical illustrations of the accuracy of these bounds as well as approximations and comparisons of the NIG average option prices with the corresponding Black–Scholes prices are given.

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1. Introduction

In recent years it has been realized that distributions of logarithmic asset returns can often be fitted extremely well by normal inverse Gaussian (NIG) distributions (see e.g. [3,4,23,24]). It is therefore of particular interest to study stochastic process models for stock prices and asset returns that capture this property. For a general survey on stochastic processes of normal inverse Gaussian type, we refer to Barndorff-Nielsen [5].

As a member of the family of generalized hyperbolic (GH) distributions, the normal inverse Gaussian distribution is infinitely divisible and thus generates a Lévy process $(Z_t)_{t\geqslant 0}$, which gives rise to the following exponential Lévy model (see e.g. [11]). By setting

$$S_t = S_0 \exp(Z_t),$$

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where $(S_t)_{t\geqslant 0}$ denotes the asset price process over time, the log-returns of this model produce exactly a normal inverse Gaussian distribution. $(S_t)_{t\geqslant 0}$ is again a Lévy process. The applicability of such a Lévy model in practice is e.g. discussed in Geman [12].

Since this market model is incomplete (cf. [7]), there are many candidates of equivalent martingale measures for risk-neutral valuation of derivative securities. One mathematically tractable choice is the so-called Esscher equivalent measure, a concept which was introduced to mathematical finance by Madan and Milne [20]; see also Gerber and Shiu [13]. This particular choice of the pricing measure can be justified both within utility and equilibrium theory (cf. [6,14,15]).

In this paper we observe that the Esscher equivalent measure in the NIG model has a particularly simple structure and we exploit this property in various ways to obtain easy computable approximations and bounds for the Esscher price of arithmetic and geometric average options. For that purpose we adapt several techniques developed for the Black–Scholes setting to our situation.

In Section 2 we introduce various properties of the normal inverse Gaussian distribution needed for the development of the NIG asset price model and the derivation of the Esscher equivalent measure in Section 3. Section 4 uses the notion of stop-loss transforms, which is well-known to actuaries, to obtain upper bounds for arithmetic average option Esscher prices. Two approximation techniques for arithmetic average option prices are developed in Section 5. Section 6 contains approximation methods for geometric average rate options and gives bounds for the arithmetic average option prices in terms of the geometric price. Finally, in Section 7 we give numerical illustrations of these bounds and approximations and also compare the Asian Esscher option prices in the NIG model with the corresponding prices in the Black–Scholes world showing significant price differences, which indicates that a careful choice of the asset price model is an important issue in practice.

2. The normal inverse Gaussian distribution

The normal inverse Gaussian distribution is defined by the density

$$f_{\mathrm{NIG}(\alpha,\beta,\delta,\mu)}(x) = f_{\mathrm{NIG}}(x) = c\exp(\beta(x-\mu))\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \tag{1}$$

with $0 \leqslant |\beta| \leqslant \alpha$, $\delta \geqslant 0$, $\mu \in \mathbb{R}$ and

$$c = \frac{\alpha\delta}{\pi}\exp(\delta\sqrt{\alpha^2-\beta^2}).$$

Here $K_1(x)$ denotes the modified Bessel function of the third kind of order 1. In general, for a real number v, the function $K_v(x)$ satisfies the differential equation

$$x^2y'' + xy' - (x^2+v^2)y = 0.$$

(cf. [1]). The moment generating function of (1) is given by

$$M_{\mathrm{NIG}}(u) = \exp(\delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+u)^2}) + \mu u)$$

from which one can deduce the following important convolution property:

$$f_{\mathrm{NIG}(\alpha,\beta,\delta_1,\mu_1)} * f_{\mathrm{NIG}(\alpha,\beta,\delta_2,\mu_2)} = f_{\mathrm{NIG}(\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)}. \tag{2}$$

The NIG distribution is the special case $\lambda = -\frac{1}{2}$ of the generalized hyperbolic distribution given by the density

$$f_{\mathrm{GH}}(x) = \frac{\zeta^{\lambda}}{\sqrt{2\pi}\alpha^{2\lambda-1}\delta^{2\lambda}K_{\lambda}(\zeta)} \mathrm{e}^{\beta(x-\mu)}\eta(x)^{\lambda-1/2}K_{\lambda-1/2}(\eta(x)),$$

with $\zeta = \delta\sqrt{\alpha^2 - \beta^2}$ and $\eta(x) = \alpha\sqrt{\delta^2 + (x-\mu)^2}$, which was introduced by Barndorff–Nielsen [2].

3. The NIG Lévy asset price model

As already mentioned in the introduction, the NIG distribution is infinitely divisible and hence generates a Lévy process $(Z_t)_{t \geqslant 0}$ (i.e. a stochastic process with stationary and independent increments, $Z_0 = 0$ a.s. and Z_1 is NIG-distributed; from the convolution property (2) it follows that the increments are NIG-distributed for arbitrary time intervals).

Let now S_t for $t \geqslant 0$ denote the price of a nondividend-paying stock at time t and consider the following dynamics for the stock price process (see [9])

$$\mathrm{d}S_t = S_{t^-}(\mathrm{d}Z_t + \mathrm{e}^{\Delta Z_t} - 1 - \Delta Z_t), \tag{3}$$

where $(Z_t)_{t \geqslant 0}$ denotes the NIG Lévy motion, Z_{t^-} the left hand limit of the path at time t and $\Delta Z_t = Z_t - Z_{t^-}$ the jump at time t. Then, the solution of the stochastic differential equation (3) is given by

$$S_t = S_0 \exp(Z_t)$$

and it follows that the log-returns $\ln(S_t/S_{t-1})$ are indeed NIG-distributed.

Since our aim is the risk-neutral valuation of derivative securities in this model and since the model is incomplete, we have to choose an equivalent martingale measure. In this paper we choose the method of Esscher transforms to find such a measure. This approach is applicable, whenever the stochastic process $(Z_t)_{t \geqslant 0}$ has stationary and independent increments (see [10,13]). Apart from its mathematical simplicity, this particular choice can also be economically justified (see [21] for a survey on this issue).

From (2) it follows that the density of Z_t is given by

$$f^{*t}_{\mathrm{NIG}}(x) = f_{\mathrm{NIG}(\alpha,\beta,t\delta,t\mu)}(x).$$

For a real number θ let us consider the Esscher transform

$$f^{*t}_{\mathrm{NIG}}(x;\theta) = \frac{\mathrm{e}^{\theta x} f^{*t}_{\mathrm{NIG}}(x)}{\int_{-\infty}^{\infty} \mathrm{e}^{\theta y} f^{*t}_{\mathrm{NIG}}(y)\,\mathrm{d}y} = \frac{\mathrm{e}^{\theta x}}{M_{\mathrm{NIG}}(\theta)^t} f^{*t}_{\mathrm{NIG}}(x), \tag{4}$$

of the one-dimensional marginal distributions $f^{*t}_{\mathrm{NIG}}(x)$ of $(Z_t)_{t \geqslant 0}$. For any Lévy process $(Z_t)_{t \geqslant 0}$ (on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$) it is now possible to define a locally equivalent probability measure $\mathbb{P}^{\theta}$ through

$$\mathrm{d}\mathbb{P}^{\theta} = \exp(\theta Z_t - t \log M_{\mathrm{NIG}}(\theta))\,\mathrm{d}\mathbb{P},$$

such that $(Z^{\theta}_t)_{t \geqslant 0}$ defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P}^{\theta})$ is again a Lévy process and the one-dimensional marginal distributions of $(Z^{\theta}_t)_{t \geqslant 0}$ are the Esscher transforms of the corresponding marginals of $(Z_t)_{t \geqslant 0}$ (see e.g. [22]). $\mathbb{P}^{\theta}$ is called the Esscher equivalent measure.

The parameter θ can now be chosen in such a way, that the discounted stock price process $(e^{-rt}S_t)_{t\geqslant 0}$ is a $\mathbb{P}^\theta$-martingale, namely if θ is the (unique) solution of

$$r = \mu + \delta(\sqrt{\alpha^2 - (\beta+\theta)^2} - \sqrt{\alpha^2 - (\beta+\theta+1)^2}), \tag{5}$$

which can be derived explicitly. Here r is the constant daily interest rate. For further details concerning the construction of the Esscher equivalent martingale measure we refer to Gerber and Shiu [13].

The following observation will substantially simplify the calculation of Esscher prices in the NIG model:

Lemma 1. *The Esscher transform of a* NIG-*distributed random variable is again* NIG-*distributed. In particular*,

$$f_{\mathrm{NIG}(\alpha,\beta,\delta,\mu)}(x;\theta) = f_{\mathrm{NIG}(\alpha,\beta+\theta,\delta,\mu)}(x). \tag{6}$$

Proof. From

$$\begin{aligned} f_{\mathrm{NIG}(\alpha,\beta,\delta,\mu)}(x;\theta) &= \frac{e^{\theta x}}{M_{\mathrm{NIG}}(\theta)} \frac{\alpha\delta}{\pi} \exp(\delta\sqrt{\alpha^2-\beta^2} + \beta(x-\mu)) \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \\ &= \frac{\alpha\delta}{\pi} \frac{e^{\theta\mu+\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+\theta)^2}}}{M_{\mathrm{NIG}}(\theta)} e^{\delta\sqrt{\alpha^2-(\beta+\theta)^2}+(\beta+\theta)(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \end{aligned}$$

it follows that we have to prove

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\,dx = e^{\theta\mu+\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+\theta)^2}},$$

which is equivalent to

$$\int_{-\infty}^{\infty} \frac{\alpha\delta}{\pi} \exp(\delta\sqrt{\alpha^2-(\beta+\theta)^2} + (\beta+\theta)(x-\mu)) \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} = 1. \tag{7}$$

But Eq. (7) holds for $|\beta+\theta| < \alpha$ and the latter condition is always satisfied due to the choice of the Esscher optimal θ as the solution of (5). □

As a first example, the value at time t of a European call option with exercise price K and maturity T can be represented by a simple analytical expression: From $\mathrm{EC}_t = \mathbb{E}^\theta[e^{-r(T-t)}(S_T - K)^+ \mid \mathscr{F}_t]$ and Lemma 1 it follows that

$$\mathrm{EC}_t = S_t \int_k^\infty f_{\mathrm{NIG}(\alpha,\beta+\theta+1,(T-t)\delta,(T-t)\mu)}(x)\,dx - e^{-r(T-t)}K \int_k^\infty f_{\mathrm{NIG}(\alpha,\beta+\theta,(T-t)\delta,(T-t)\mu)}(x)\,dx \tag{8}$$

with $k = \ln(K/S_t)$. This value can be computed numerically.

4. Stop-loss transforms and upper bounds for arithmetic average options

We will now focus on the evaluation of the Esscher price of a European-style arithmetic average call option at time t given by

$$\mathrm{AA}_t = \frac{\mathrm{e}^{-r(T-t)}}{n}\, \mathbb{E}^\theta \left[\left(\sum_{k=0}^{n-1} S_{T-k} - nK \right)^+ \middle| \mathscr{F}_t \right], \tag{9}$$

where n is the number of averaging days, K the strike price, T the time to expiration and r the risk-free interest rate.

The main difficulty here is to find the distribution of $\sum S_i$, which is a sum of dependent random variables. Simon et al. [25] recently derived upper bounds for the price of an arithmetic average option in an arbitrage-free and complete market by means of stop-loss transforms and the theory of comonotone risks. In the sequel we will adapt their technique to the NIG model.

Let $F(x)$ be a distribution function with support $D \subseteq \mathbb{R}^+$, then its stop-loss transform $\Psi_F(r)$ is defined by

$$\Psi_F(r) = \int_{[r,\infty)\cap D} (x-r)\,\mathrm{d}F(x).$$

A stop-loss ordering of distribution functions $F(x)$ and $G(x)$ with support in $\mathbb{R}^+$ can now be defined in the following way: $F(x)$ is said to precede $G(x)$ in stop-loss order ($F \leqslant_{\mathrm{sl}} G$), if

$$\Psi_F(r) \leqslant \Psi_G(r) \qquad \text{for all } r \in \mathbb{R}^+.$$

Next, we can rewrite the price of the arithmetic average option given by (9) to

$$\mathrm{AA}_t = \frac{\mathrm{e}^{-r(T-t)}}{n}\, \Psi_{F^s_{A_n(T)}}(nK) \tag{10}$$

for a given value $S_t = s$ with $F^s_{A_n(T)} = \mathbb{P}^\theta(A_n(T) \leqslant x | S_t = s)$, where

$$A_n(T) = \sum_{k=0}^{n-1} S_{T-k}.$$

In this way we have transformed the problem of pricing an arithmetic average option to calculating the stop-loss transform of a sum of dependent risks. Hence we can apply results on bounds for stop-loss transforms to our option pricing problem.

A positive random vector $(X_1,\dots,X_n)$ with marginal distributions $F_1(x_1),\dots,F_n(x_n)$ is called comonotone, if $F_{X_1,\dots,X_n}(x_1,\dots,x_n) = \min\{F_1(x_1),\dots,F_n(x_n)\}$ holds for every $x_1,\dots,x_n \geqslant 0$. It immediately follows that a comonotone random vector $(X_1,\dots,X_n)$ with given marginal distributions $F_1(x_1),\dots,F_n(x_n)$ is uniquely determined. It can easily be shown (see e.g. [8]) that an upper bound for the stop-loss transform of a sum of dependent random variables $\sum_{k=1}^n X_k$ with marginal distributions $F_1(x_1),\dots,F_n(x_n)$ is now given by the stop-loss transform of the sum $\sum_{k=1}^n Y_k$, where $(Y_1,\dots,Y_n)$ is the comonotone random vector with marginal distributions $F_1(x_1),\dots,F_n(x_n)$, i.e. $\sum_{k=1}^n X_k \leqslant_{\mathrm{sl}} \sum_{k=1}^n Y_k$. Let us define $F_R(x) := \mathbb{P}^\theta\left(\sum_{k=1}^n Y_k \leqslant x\right)$, then we have ([8])

$$F_R^{-1}(x) = \sum_{k=1}^n F_k^{-1}(x) \quad \text{for each } x \in \mathbb{R}^+$$

and

$$\Psi_{F_R}(x) = \sum_{k=1}^{n} \Psi_{F_k}(F_k^{-1}(F_R(x))) \quad \text{for each } x \in \mathbb{R}^+.$$

Hence this upper bound of the arithmetic average option price can be viewed as a sum of prices of European call options with strike prices $F_k^{-1}(F_R(x))$. The following proposition is an adaption of the result of Simon et al. [25] to our situation. Let in the sequel $F(x_2, t_2, x_1, t_1)$ denote the conditional distribution function of S_{t_2} under the equivalent Esscher martingale measure $\mathbb{P}^\theta$ given $S_{t_1} = x_1$, i.e.

$$\begin{aligned} F(x_2, t_2, x_1, t_1) &= \mathbb{P}^\theta(S_{t_2} \leqslant x_2 | S_{t_1} = x_1) \\ &= \mathbb{P}^\theta\left(Z_{t_2 - t_1} \leqslant \ln \frac{x_2}{S_{t_1}}\right) \\ &= \int_{-\infty}^{\ln(x_2/S_{t_1})} f_{\mathrm{NIG}(\alpha, \beta+\theta, (t_2-t_1)\delta, (t_2-t_1)\mu)}(z)\,\mathrm{d}z, \quad t_2 \geqslant t_1. \end{aligned}$$

Proposition 1 (Simon et al. [25]). *Let k^* be such that $T - k^* \leqslant t < T - k^* + 1$ and $K_j = nK - \sum_{k=j}^{n-1} S_{T-k}$ for $j < n$, $K_n = nK$. Let AA_t be the price of an arithmetic average option at time t as given in (9) and let furthermore $\mathrm{EC}_t(\kappa_k, T-k)$ be the price of a European option with strike price κ_k and time to expiration $T-k$. Then we have for $K_{k^*} > 0$*

$$\mathrm{AA}_t \leqslant \frac{\mathrm{e}^{-r(T-t)}}{n} \sum_{k=0}^{k^*-1} \Psi_{F(\cdot, T-k, s, t)}(\kappa_k) = \frac{1}{n} \sum_{k=0}^{k^*-1} \mathrm{e}^{-kr} \mathrm{EC}_t(\kappa_k, T-k), \tag{11}$$

where

$$\kappa_k = F^{-1}(F_R(K_{k^*}), T-k, s, t), \quad k = 0, \ldots, k^*. \tag{12}$$

Moreover, this choice of the strike prices κ_k is best possible.

In case $K_{k^*} \leqslant 0$, we have

$$\mathrm{AA}_t = \frac{S_t}{n} \sum_{k=0}^{k^*-1} \mathrm{e}^{-kr} + \frac{\mathrm{e}^{-r(T-t)}}{n} \sum_{k=k^*}^{n-1} S_{T-k} - \mathrm{e}^{-r(T-t)} K.$$

Proposition 1 shows (for $K_{k^*} > 0$) that the price of an arithmetic average option is bounded from above by the price of a portfolio of time-delayed European-style call options with exercise prices κ_k, and this bound is optimized by the above choice of κ_k.

In order to obtain a bound for the arithmetic average option price we thus have to calculate k^* strike prices κ_k using (12) and then evaluate (11) using (8). A numerical illustration of the accuracy of these bounds is given in Section 7.

5. Two approximations for the distribution of the arithmetic mean

In this section we study approximations of the arithmetic option price (9) by means of Edgeworth series expansions. For notational simplicity we will assume $t = 0$ and $n = T$ in (9), i.e. the averaging starts at time $t=1$ and we determine the price at time $t=0$

$$\mathrm{AA}_0 = \frac{\mathrm{e}^{-rT}}{n}\,\mathbb{E}^{\theta}\left[\left(\sum_{k=1}^{n} S_k - nK\right)^{+}\middle|\,\mathscr{F}_0\right]. \tag{13}$$

The extension to the general case is straightforward. Recall that the cumulants of a random variable X with distribution function F are defined by

$$\chi_i(F) = \left[\frac{\partial^i \ln \mathbb{E}[\mathrm{e}^{tX}]}{\partial t^i}\right]_{t=0}, \quad i=1,2,\ldots$$

and can also be expressed in terms of moments. For the first four cumulants we have

$$\chi_1(F) = \mathbb{E}[X],$$

$$\chi_2(F) = \mathbb{E}[(X-\mathbb{E}[X])^2],$$

$$\chi_3(F) = \mathbb{E}[(X-\mathbb{E}[X])^3],$$

$$\chi_4(F) = \mathbb{E}[(X-\mathbb{E}[X])^4] - 3\chi_2^2(F).$$

In the sequel we will make use of the following classical result:

Lemma 2 (Jarrow and Rudd [17]). *Let F and G be two continuous distribution functions with $G\in\mathscr{C}^5$ and $\chi_1(F)=\chi_1(G)$, and assume that the first five moments of both distributions exist. Then we can expand the density $f(x)$ in terms of the density $g(x)$ as follows*

$$f(x) = g(x) + \frac{\chi_2(F)-\chi_2(G)}{2}\frac{\partial^2 g}{\partial x^2}(x) - \frac{\chi_3(F)-\chi_3(G)}{3!}\frac{\partial^3 g}{\partial x^3}(x)$$
$$+\frac{\chi_4(F)-\chi_4(G)+3(\chi_2(F)-\chi_2(G))^2}{4!}\frac{\partial^4 g}{\partial x^4}(x) + \varepsilon(x),$$

where $\varepsilon(x)$ is a residual error term.

We will now approximate the distribution function of $\sum_{k=1}^{n} S_k$ (which we denote by F) by a lognormal distribution G (see [19,26] for a similar procedure in the Black–Scholes case). The density of the lognormal distribution is given by

$$g(x) = f_{\mathrm{LN}}(x) = \frac{1}{\sigma\sqrt{2\pi}x}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\chi_{(0,\infty)}(x).$$

Let us define

$$R_i = \frac{S_i}{S_{i-1}}, \quad i=1,\ldots,n.$$

and

$$L_n = 1$$

$$L_{i-1} = 1 + R_i L_i, \quad i=2,\ldots,n.$$

Then we have

$$\sum_{k=1}^{n} S_k = S_0(R_1 + R_1R_2 + \cdots + R_1R_2\ldots R_n) = S_0R_1L_1.$$

Since we can rewrite equation Eq. (13) to

$$\mathrm{AA}_0 = \frac{\mathrm{e}^{-rT}S_0}{n}\,\mathbb{E}^{\theta}\left[\left(R_1L_1 - \frac{nK}{S_0}\right)^{+}\middle|\,\mathscr{F}_0\right],$$

it remains to determine $\mathbb{E}^{\theta}[(R_1L_1)^m]$ for $m=1,2,3,4$. Because of the independent increments property of a Lévy process, we have $\mathbb{E}^{\theta}[(R_1L_1)^m] = \mathbb{E}^{\theta}[R_1^m]\mathbb{E}^{\theta}[L_1^m]$ and

$$\mathbb{E}^{\theta}[L_{i-1}^m] = \mathbb{E}^{\theta}[(1 + L_iR_i)^m] = \sum_{k=0}^{m}\binom{m}{k}\mathbb{E}^{\theta}[L_i^k]\mathbb{E}^{\theta}[R_i^k]. \tag{14}$$

In order to apply recursion (14), we need to determine the moments $\mathbb{E}^{\theta}[R_i^k]$:

Lemma 3. *For all $k \in \mathbb{N}$ we have*

$$\mathbb{E}^{\theta}[R_i^k] = \exp(\delta(\sqrt{\alpha^2 - (\beta+\theta)^2} - \sqrt{\alpha^2 - (\beta+\theta+k)^2}) + k\mu). \tag{15}$$

Proof. Since R_i is log-NIG distributed (namely $R_i \overset{\mathrm{d}}{\sim} \mathrm{LNIG}(\alpha, \beta+\theta, \delta, \mu)$), the result follows from

$$\begin{aligned}\mathbb{E}^{\theta}[R_i^k] &= \int_0^{\infty} x^k\,\frac{\alpha\delta}{\pi x}\exp(\delta\sqrt{\alpha^2 - (\beta+\theta)^2} + (\beta+\theta)(\ln x - \mu))\frac{K_1(\alpha\sqrt{\delta^2 + (\ln x - \mu)^2})}{\sqrt{\delta^2 + (\ln x - \mu)^2}}\,\mathrm{d}x \\ &= \int_0^{\infty} \mathrm{e}^{uk}\,\frac{\alpha\delta}{\pi}\exp(\delta\sqrt{\alpha^2 - (\beta+\theta)^2} + (\beta+\theta)(u - \mu))\frac{K_1(\alpha\sqrt{\delta^2 + (u - \mu)^2})}{\sqrt{\delta^2 + (u - \mu)^2}}\,\mathrm{d}u \\ &= M_{\mathrm{NIG}(\alpha,\beta+\theta,\delta,\mu)}(k). \qquad \square\end{aligned}$$

The moments $\mathbb{E}^{\theta}[L_1^m]$ $(m=1,2,3,4)$ and subsequently the cumulants $\chi_i(F)$ can now be calculated recursively using (14), (15) and the fact that $\mathbb{E}^{\theta}[L_n^k] = 1$ for all $k \in \{0,\ldots,m\}$. The parameters of the approximating lognormal distribution are chosen in such a way that the first two moments of the approximating log-normal and the original distribution coincide (a so-called Wilkinson approximation):

$$\begin{aligned}\tilde{\mu} &= 2\ln(\chi_1(F)) - \tfrac{1}{2}\ln(\chi_1^2(F) + \chi_2(F)) \\ \tilde{\sigma}^2 &= \ln(\chi_1^2(F) + \chi_2(F)) - 2\ln(\chi_1(F)).\end{aligned}$$

In this way we have derived a lognormal approximation pricing formula for an arithmetic average option in the NIG model, which we call the Turnbull–Wakeman price $\mathrm{AA}_0^{\mathrm{TW}}$ at time $t=0$.

Proposition 2. *The price at time* 0 *of a European-style arithmetic average option in the* NIG *model with maturity* T *and strike price* K *is*

$$\mathrm{AA}_0^{\mathrm{TW}} = \mathrm{e}^{-rT}\frac{S_0}{n}\left(\mathrm{e}^{\tilde{\mu}+(\tilde{\sigma}^2/2)}\Phi\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln(nK/S_0)}{\tilde{\sigma}}\right)-\frac{nK}{S_0}\Phi\left(\frac{\tilde{\mu}-\ln(nK/S_0)}{\tilde{\sigma}}\right)\right)$$
$$-\mathrm{e}^{-rT}\frac{S_0}{n}\left(\frac{\chi_3(F)-\chi_3(G)}{3!}\frac{\partial g(nK/S_0)}{\partial x}+\frac{\chi_4(F)-\chi_4(G)}{4!}\frac{\partial^2 g(nK/S_0)}{\partial x^2}\right)$$

If only the first two cumulants are considered, we call the corresponding approximation the Levy price $\mathrm{AA}_0^{\mathrm{L}}$ given by

$$\mathrm{AA}_0^{\mathrm{L}} = \mathrm{e}^{-rT}\frac{S_0}{n}\left(\mathrm{e}^{\tilde{\mu}+\frac{\tilde{\sigma}^2}{2}}\Phi\left(\frac{\tilde{\mu}+\tilde{\sigma}^2-\ln nK/S_0}{\tilde{\sigma}}\right)-\frac{nK}{S_0}\Phi\left(\frac{\tilde{\mu}-\ln nK/S_0}{\tilde{\sigma}}\right)\right).$$

Another possibility is to approximate the arithmetic average $\sum_{k=1}^n S_k$ by a NIG distribution by matching the first four moments. Since the cumulants of a NIG(α,β,δ,μ)-distributed random variable H are given by

$$\chi_1(H)=\mu+\frac{\beta\delta}{(\alpha^2-\beta^2)^{1/2}},$$

$$\chi_2(H)=\frac{\alpha^2\delta}{(\alpha^2-\beta^2)^{3/2}},$$

$$\chi_3(H)=\frac{3\alpha^2\beta\delta}{(\alpha^2-\beta^2)^{5/2}},$$

$$\chi_4(H)=\frac{3\alpha^2\delta(\alpha^2+4\beta^2)}{(\alpha^2-\beta^2)^{7/2}},$$

the parameters $\tilde{\alpha},\tilde{\beta},\tilde{\delta}$ and $\tilde{\mu}$ of the approximating NIG distribution are the solution of the equation system $\chi_k(H)=\chi_k(F)$ for $k=1,\dots,4$, where F again denotes the distribution function of $\sum_{k=1}^n S_k$. This leads to

$$\tilde{\beta}=\frac{3\chi_2(F)\chi_3(F)}{3\chi_2(F)\chi_4(F)-5(\chi_3(F))^2},$$

$$\tilde{\alpha}^2=\frac{3\chi_2(F)\tilde{\beta}}{\chi_3(F)}+\tilde{\beta}^2,$$

$$\tilde{\delta}=\frac{\chi_2(F)(\tilde{\alpha}^2-\tilde{\beta}^2)^{3/2}}{\tilde{\alpha}^2},$$

$$\tilde{\mu}=\chi_1(F)-\frac{\tilde{\beta}\tilde{\delta}}{(\tilde{\alpha}^2-\tilde{\beta}^2)^{1/2}}.$$

Finally, the option price is given by

$$\mathrm{AA}_0^{\mathrm{NIG}} = \mathrm{e}^{-rT}\,\frac{S_0}{n}\int_d^{\infty}(x-d)\,f_{\mathrm{NIG}(\tilde{\alpha},\tilde{\beta},\tilde{\delta},\tilde{\mu})}(x)\,\mathrm{d}x \quad \text{with} \quad d = nK/S_0.$$

This value can be calculated numerically (see Section 7 for a numerical example).

6. Arithmetic and geometric average options

Let us consider a geometric average option with Esscher price given by

$$\mathrm{GA}_0 = \mathrm{e}^{-rT}\ \mathbb{E}^{\theta}[(G_T - K)^+|\mathscr{F}_0], \tag{16}$$

where $G_T = (\prod_{k=1}^n S_k)^{1/n}$ and K denotes the strike price. Again, we have chosen $t=0$ and $n=T$ (the generalization to arbitrary $t \geqslant 0$ and arbitrary starting times of the averaging period is straightforward). Taking the logarithm of G_T, we get

$$\ln G_T = \frac{1}{n}\sum_{k=1}^{n}\ln S_k = \ln S_0 + \frac{1}{n}\sum_{k=1}^{n}\ln\frac{S_k}{S_0},$$

which can also be written as

$$\ln G_T = \ln S_0 + \frac{1}{n}\left(n\ln\frac{S_1}{S_0} + (n-1)\ln\frac{S_2}{S_1} + \cdots + \ln\frac{S_n}{S_{n-1}}\right)$$

and hence we have

$$\ln G_T = \ln S_0 + X_1 + \frac{n-1}{n}X_2 + \ldots + \frac{1}{n}X_n$$

with $X_k \overset{\mathrm{iid}}{\sim} \mathrm{NIG}$. Unfortunately, the distribution of $\ln G_T$ is not NIG anymore. However, we will approximate it by a NIG distribution. For that purpose, we determine the cumulants:

$$\chi_1(\ln G_T) = \ln S_0 + \frac{(n+1)}{2}\chi_1(X_1),$$

$$\chi_2(\ln G_T) = \frac{1}{n^2}\chi_2(X_1)\left(\frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)\right),$$

$$\chi_3(\ln G_T) = \frac{1}{n^3}\chi_3(X_1)\left(\frac{1}{4}(n+1)^4 - \frac{1}{2}(n+1)^3 + \frac{1}{4}(n+1)^2\right),$$

$$\chi_4(\ln G_T) = \frac{1}{n^4}\chi_4(X_1)\left(\frac{1}{5}(n+1)^5 - \frac{1}{2}(n+1)^4 + \frac{1}{3}(n+1)^3 - \frac{1}{30}(n+1)\right).$$

As above, the parameters $\alpha^*, \beta^*, \delta^*, \mu^*$ of the approximating NIG distribution X can be obtained by solving the system of equations $\chi_k(X) = \chi_k(\ln G_T)$ $(k=1,\ldots,4)$. Then, the price of a European-style geometric average option at time 0, where the distribution of $\ln G_T$ is approximated by the NIG

Table 1
Comparison of simulated Asian option prices and the SL upper bound of Section 4

Stock	T	r	K	EC_0	$\mathrm{AA}_0^{\mathrm{MC}}$	SL upper bound	Rel. error (%)
Bayer	10	0.1	34	2.1574	2.0729 ($\pm$0.0016)	2.0759	0.14
($S_0=36$)			35	1.2622	1.1059 ($\pm$0.0014)	1.1246	1.69
			36	0.5849	0.3505 ($\pm$0.0010)	0.3924	11.95
			37	0.1980	0.0539 ($\pm$0.0004)	0.0796	47.68
		0.05	34	2.0946	2.0376 ($\pm$0.0015)	2.0409	0.16
			35	1.2061	1.0718 ($\pm$0.0014)	1.0921	1.89
			36	0.5396	0.3298 ($\pm$0.0010)	0.3723	12.89
			37	0.1795	0.0487 ($\pm$0.0004)	0.0728	49.49
	20	0.1	34	2.3508	2.1435 ($\pm$0.0021)	2.1532	0.45
			35	1.5210	1.2174 ($\pm$0.0019)	1.2549	3.08
			36	0.8771	0.5047 ($\pm$0.0014)	0.5731	13.55
			37	0.4284	0.1399 ($\pm$0.0008)	0.1891	35.17
		0.05	34	2.2306	2.0787 ($\pm$0.0021)	2.0909	0.59
			35	1.4173	1.1577 ($\pm$0.0019)	1.1976	3.45
			36	0.7894	0.4648 ($\pm$0.0013)	0.5270	13.38
			37	0.3796	0.1237 ($\pm$0.0007)	0.1705	37.83
NYSE C.I.	10	0.1	47	3.1885	3.0983 ($\pm$0.0013)	3.0984	0.00
($S_0=50$)			48.5	1.7191	1.6061 ($\pm$0.0013)	1.6107	0.29
			50	0.5181	0.3114 ($\pm$0.0008)	0.3455	10.95
	20		47	3.3807	3.1841 ($\pm$0.0017)	3.1852	0.04
			48.5	1.9558	1.7086 ($\pm$0.0017)	1.7149	0.37
			50	0.8044	0.4610 ($\pm$0.0011)	0.5179	12.34

distribution, is given by

$$\mathrm{GA}_0^{\mathrm{NIG}} = \mathrm{e}^{-rT} \int_K^\infty (x-K) f_{\mathrm{LNIG}(\alpha^*,\beta^*,\delta^*,\mu^*)}(x)\,\mathrm{d}x. \tag{17}$$

This value can be calculated numerically.

Next, we derive bounds for AA_0 using approximation (17) for the geometric average option price. Since the geometric average is always less or equal the arithmetic average, we have

$$\mathrm{GA}_0 = \mathrm{e}^{-rT}\mathbb{E}^\theta[(G_T-K)^+|\mathscr{F}_0] \leqslant \mathrm{e}^{-rT}\mathbb{E}^\theta[(A_T-K)^+|\mathscr{F}_0] = \mathrm{AA}_0,$$

where $A_T = 1/n\sum_{k=1}^n S_k$. Following Vorst [27], a straightforward upper bound for AA_0 can be derived using

$$(A_T-K)^+ = \max\{G_T-K, G_T-A_T\} + A_T - G_T \leqslant (G_T-K)^+ + A_T - G_T,$$

which implies $\mathrm{AA}_0 \leqslant \mathrm{e}^{-rT}\mathbb{E}^\theta[(G_T-K)^+ + A_T - G_T|\mathscr{F}_0]$ and thus the upper bound

$$\mathrm{AA}_0 \leqslant \mathrm{GA}_0 + \mathrm{e}^{-rT}(\mathbb{E}^\theta[A_T|\mathscr{F}_0] - \mathbb{E}^\theta[G_T|\mathscr{F}_0]) =: \mathrm{AA}_0^{\mathrm{U}}. \tag{18}$$

The expected value $\mathbb{E}^\theta[A_T|\mathscr{F}_0] = S_0/n\mathbb{E}^\theta[R_1L_1]$ in (18) can be calculated by recursion (14), and $\mathbb{E}^\theta[G_T|\mathscr{F}_0] \approx M_{\mathrm{NIG}(\alpha^*,\beta^*,\delta^*,\mu^*)}(1)$, since $\ln G_T$ is approximated by a NIG distribution (cf. Lemma 3).

Table 2
Comparison of simulated Asian option prices and approximations of Section 5

Stock	T	r	K	AA_0^{MC}	AA_0^{L}	r.e. (%)	AA_0^{TW}	r.e. (%)	AA_0^{NIG}	r.e. (%)
Bayer	10	0.1	34	2.0729	2.0719	0.05	2.0732	0.02	2.0728	0.01
($S_0 = 36$)			35	1.1059	1.1054	0.05	1.1055	0.04	1.1056	0.03
			36	0.3505	0.3577	2.05	0.3487	0.51	0.3498	0.20
			37	0.0539	0.0510	5.38	0.0555	2.97	0.0543	0.74
		0.05	34	2.0376	2.0366	0.05	2.0381	0.02	2.0377	0.01
			35	1.0718	1.0712	0.06	1.0717	0.01	1.0719	0.01
			36	0.3298	0.3367	2.09	0.3282	0.49	0.3292	0.18
			37	0.0487	0.0459	5.75	0.0499	2.46	0.0489	0.41
	20	0.1	34	2.1435	2.1427	0.04	2.1440	0.02	2.1435	0.00
			35	1.2174	1.2197	0.19	1.2164	0.08	1.2172	0.02
			36	0.5047	0.5105	1.15	0.5037	0.20	0.5043	0.08
			37	0.1399	0.1393	0.43	0.1410	0.79	0.1402	0.21
		0.05	34	2.0787	2.0772	0.07	2.0788	0.01	2.0784	0.01
			35	1.1577	1.1596	0.16	1.1566	0.10	1.1574	0.03
			36	0.4648	0.4704	1.21	0.4641	0.15	0.4646	0.04
			37	0.1237	0.1230	0.57	0.1246	0.73	0.1238	0.08
NYSE C.I.	10	0.1	47	3.0963	3.0983	0.07	3.0977	0.05	3.0978	0.05
($S_0 = 50$)			48.5	1.6061	1.6049	0.08	1.6069	0.05	1.6068	0.04
			50	0.3114	0.3162	1.54	0.3101	0.42	0.3109	0.16
	20		47	3.1841	3.1850	0.03	3.1851	0.03	3.1851	0.03
			48.5	1.7086	1.7051	0.21	1.7081	0.03	1.7078	0.05
			50	0.4610	0.4651	0.89	0.4613	0.07	0.4617	0.15

Vorst [27] also proposed the approximation

$$AA_0^V = e^{-rT}\mathbb{E}^\theta[(G_T - K')^+|\mathscr{F}_0] \quad \text{with} \quad K' = K - (\mathbb{E}^\theta[A_T|\mathscr{F}_0] - \mathbb{E}^\theta[G_T|\mathscr{F}_0])$$

for the price of an arithmetic average value option, leading to

$$AA_0^V = e^{-rT}\int_{K'}^{\infty}(x - K')f_{\mathrm{LNIG}(\alpha^*,\beta^*,\delta^*,\mu^*)}(x)\,dx.$$

This integral can be calculated numerically.

Remark. Since our asset price model is arbitrage-free, put-call parity holds and thus the above techniques can also directly be applied to put options of Asian type.

7. Numerical illustrations and comparison with the Black–Scholes model

We now give some numerical illustrations of the accuracy of the bounds and approximations derived in the previous sections for options on the stock of Bayer AG and on the NYSE Composite Index, respectively. For that purpose the parameters of our NIG model are estimated from historical

Table 3
Comparison of simulated Asian option prices and approximations of Section 6

Stock	T	r	K	AA_0^{MC}	AA_0^{V}	r.e.(%)	GA_0^{MC} (± s.e.)	GA_0^{NIG}	r.e.(%)	AA_0^{U}
Bayer	10	0.1	34	2.0729	2.0728	0.01	2.0692 (±0.0015)	2.0691	0.01	2.0728
($S_0 = 36$)			35	1.1059	1.1060	0.01	1.1026 (±0.0014)	1.1025	0.01	1.1063
			36	0.3505	0.3498	0.20	0.3483 (±0.0010)	0.3478	0.14	0.3516
			37	0.0539	0.0537	0.37	0.0530 (±0.0004)	0.0532	0.38	0.0570
			38	0.0052	0.0052	0.00	0.0050 (±0.0001)	0.0052	4.00	0.0089
		0.05	34	2.0376	2.0377	0.01	2.0339 (±0.0015)	2.0340	0.01	2.0377
			35	1.0718	1.0723	0.05	1.0687 (±0.0014)	1.0688	0.01	1.0726
			36	0.3298	0.3291	0.21	0.3278 (±0.0010)	0.3272	0.18	0.3310
			37	0.0487	0.0483	0.82	0.0478 (±0.0004)	0.0479	0.21	0.0517
			38	0.0045	0.0045	0.00	0.0044 (±0.0001)	0.0044	0.00	0.0082
	20	0.1	34	2.1435	2.1437	0.01	2.1363 (±0.0021)	2.1362	0.01	2.1439
			35	1.2174	1.2178	0.03	1.2112 (±0.0019)	1.2113	0.01	1.2189
			36	0.5047	0.5041	0.12	0.5001 (±0.0014)	0.4999	0.04	0.5076
			37	0.1399	0.1389	0.72	0.1372 (±0.0007)	0.1373	0.07	0.1449
			38	0.0273	0.0263	3.66	0.0262 (±0.0003)	0.0259	1.15	0.0336
		0.05	34	2.0787	2.0786	0.01	2.0717 (±0.0021)	2.0713	0.02	2.0788
			35	1.1577	1.1582	0.04	1.1519 (±0.0018)	1.1518	0.01	1.1593
			36	0.4648	0.4644	0.09	0.4606 (±0.0013)	0.4605	0.02	0.4681
			37	0.1237	0.1227	0.81	0.1213 (±0.0007)	0.1212	0.08	0.1288
			38	0.0227	0.0221	2.64	0.0218 (±0.0002)	0.0218	0.00	0.0294
N.C.I.	10	0.1	47	3.0983	3.0978	0.02	3.0965 (±0.0013)	3.0959	0.02	3.0978
($S_0 = 50$)			48.5	1.6061	1.6068	0.04	1.6043 (±0.0013)	1.6050	0.04	1.6068
			50	0.3114	0.3109	0.16	0.3103 (±0.0008)	0.3098	0.16	0.3116
			51.5	0.0055	0.0054	1.82	0.0053 (±0.0001)	0.0054	1.89	0.0072
	20		47	3.1841	3.1851	0.03	3.1803 (±0.0017)	3.1814	0.04	3.1851
			48.5	1.7086	1.7079	0.04	1.7051 (±0.0017)	1.7043	0.05	1.7080
			50	0.4610	0.4615	0.11	0.4587 (±0.0011)	0.4594	0.15	0.4631
			51.5	0.0305	0.0302	0.98	0.0298 (±0.0003)	0.0299	0.34	0.0337

data of the respective underlying by maximum likelihood methods (the corresponding estimates are $\alpha = 81.6$, $\beta = 3.69$, $\delta = 0.0103$, $\mu = -0.000123$ (Bayer) and $\alpha = 136.29$, $\beta = -8.95$, $\delta = 0.0059$, $\mu = 0.00079$ (NYSE)). Table 1 compares the stop-loss upper bound for the Esscher price of the European-style arithmetic average option of Section 4 with a Monte Carlo simulated price AA_0^{MC} obtained by generating 1 million sample paths (given together with its standard error based on an asymptotic 95% confidence interval). For convenience, the European call option price EC_0 is also given. For this and all the following tables the number of averaging days n equals the number of days T until maturity. The inverse distribution function needed for the calculation of κ_k is interpolated, since there is no analytic expression available. The numerical values indicate that the accuracy of the upper bound is satisfying if the option is in the money.

In Table 2 we compare the approximation techniques for the arithmetic average option developed in Section 5 and give the relative error with respect to AA_0^{MC}. The results show that the NIG approximation outperforms the Turnbull–Wakeman approximation, which itself is superior to the Levy approximation in most cases.

Table 4
Comparison of simulated Asian option prices in the NIG model and the Black–Scholes model

Stock	T	r	K	AA_0^{MC}	$^{(BS)}AA_0$	r.d. (%)	GA_0^{MC}	$^{(BS)}GA_0$	r.d. (%)
Bayer	10	0.1	34	2.0729	2.0718	0.05	2.0692	2.0681	0.05
($S_0 = 36$)			35	1.1059	1.1042	0.15	1.1026	1.1018	0.07
			36	0.3505	0.3577	2.05	0.3483	0.3552	1.98
			37	0.0539	0.0510	5.38	0.0530	0.0503	5.09
			38	0.0052	0.0028	46.15	0.0050	0.0025	50.00
		0.05	34	2.0376	2.0365	0.05	2.0339	2.0328	0.05
			35	1.0718	1.0709	0.08	1.0687	1.0679	0.08
			36	0.3298	0.3369	2.15	0.3278	0.3348	2.14
			37	0.0487	0.0462	5.13	0.0478	0.0455	4.81
			38	0.0045	0.0023	48.89	0.0044	0.0022	100.00
	20	0.1	34	2.1435	2.1416	0.09	2.1363	2.1351	0.06
			35	1.2174	1.2177	0.02	1.2112	1.2127	0.12
			36	0.5047	0.5095	0.95	0.5001	0.5054	1.06
			37	0.1399	0.1385	1.00	0.1372	0.1369	0.22
			38	0.0273	0.0226	17.22	0.0262	0.0222	15.27
		0.05	34	2.0787	2.0761	0.13	2.0717	2.0698	0.09
			35	1.1577	1.1578	0.01	1.1519	1.1531	0.10
			36	0.4648	0.4700	1.12	0.4606	0.4663	1.24
			37	0.1237	0.1226	0.89	0.1213	0.1214	0.08
			38	0.0227	0.0189	16.74	0.0218	0.0188	13.76
NYSE C.I.	10	0.1	47	3.0983	3.0977	0.02	3.0965	3.0959	0.02
($S_0 = 50$)			48.5	1.6061	1.6049	0.08	1.6043	1.6031	0.08
			50	0.3114	0.3166	1.67	0.3103	0.3151	1.55
			51.5	0.0055	0.0042	23.64	0.0053	0.0041	22.64
	20		47	3.1841	3.1850	0.03	3.1803	3.1813	0.03
			48.5	1.7086	1.7053	0.19	1.7051	1.7015	0.21
			50	0.4610	0.4656	1.00	0.4587	0.4627	0.87
			51.5	0.0305	0.0316	3.61	0.0298	0.0300	0.67

The numerical values for the Vorst approximation AA_0^V and its relative error w.r.t. AA_0^{MC} are depicted in Table 3. Moreover, the approximation GA_0^{NIG} is compared with the simulated geometric price GA_0^{MC}. And finally the upper bound AA_0^U is given. Since the expression $\mathbb{E}^\theta[G_T]$ needed for the evaluation of AA_0^U is itself obtained through approximation (cf. Section 6), we always have $AA_0^V \leqslant AA_0^U$, but it may happen that AA_0^U is not an upper bound for the numerical value AA_0^{MC}.

It may be interesting to compare the numerical results for the Esscher option prices obtained in the NIG model to the corresponding values in the Black–Scholes setting. In the Black–Scholes model there is an explicit pricing formula for the geometric average option (see [18])

$$^{(BS)}GA_0 = e^{-rT}(e^{\hat{\mu}+\hat{\sigma}^2/2}\Phi(d_1) - K\Phi(d_2)) \quad \text{with} \quad d_1 = \frac{\hat{\mu}+\hat{\sigma}^2 - \ln K}{\hat{\sigma}}, \; d_2 = d_1 - \hat{\sigma},$$

where

$$\hat{\mu} = \ln S_0 + \frac{T + T/n}{2}\left(r - \frac{\sigma^2}{2}\right), \quad \hat{\sigma}^2 = \sigma^2 T\,\frac{(2n+1)(n+1)}{6n^2}.$$

Here σ^2 denotes the variance of the log-returns, which can again be estimated from historical data. The arithmetic average option price ${}^{(BS)}AA_0$ in the Black–Scholes model can not be obtained by an explicit formula. Thus we use a Quasi-Monte Carlo simulated price (cf. [16]).

In Table 4 the Esscher option prices of the NIG model are compared with the corresponding Black–Scholes prices and the relative difference is given. Note that in the Black–Scholes setting the Esscher pricing principle also yields the correct (unique) option prices. The Black–Scholes prices differ significantly from the NIG Esscher prices; in particular they tend to be lower if the option is in and out of the money and higher than the NIG prices, if the option is at the money. These differences indicate that an appropriate choice of the asset price model is of great importance for the issue of option pricing.

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