Partitioning the edge set of a bipartite graph into chain packings: complexity of some variations

D. de Werra*

Institut de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

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Abstract

An extension of the decomposition theorem of Birkhoff–von Neumann theorem is given for the case where entries can be positive or negative. The special case of an integral matrix is discussed and some balancing properties which are trivial for the classical case (non-negative entries only in the matrix) are shown to be NP-complete. Some relaxations are presented for solving the balancing problem.

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1. Introduction

Bistochastic matrices (i.e., matrices with non-negative entries where the sums of all rows and of all columns are equal (to one)) have been extensively studied in various types of applications (timetabling, telecommunications, etc.) or simply from a theoretical point of view (see [1,3,4,9]).

The case where the entries of A are simply real has been examined in [10]; other applications to scheduling based on totally unimodular matrices are given in [6]. Here we shall introduce some balancing requirements in the problem by analogy to what is done in classical timetabling applications, and we will study the complexity of the problem (for a concise presentation of notions of complexity, see for instance [2]).

* Fax: +41-21-6935840.
E-mail address: dewerra.imae@epfl.ch (D. de Werra).

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Although the problems considered here will have an immediate interpretation in terms of alternating chain packings in graphs, we shall use a matrix formulation; a graph theoretical interpretation will be sketched whenever it will be necessary.

For an \((m \times n)\) matrix \(A = (a_{ij})\) with real entries, we will introduce the following:

\[
\begin{align*}
r(A, i) &= \sum_{j=1}^{n} a_{ij} \quad (i = 1, \ldots, m), \\
c(A, j) &= \sum_{i=1}^{m} a_{ij} \quad (j = 1, \ldots, n), \\
T(A) &= \max \left( \max_i |r(A, i)|, \max_j |c(A, j)| \right).
\end{align*}
\]

Matrix \(A\) will be called \textit{regular} if each row \(i\) and each column \(j\) satisfy:

\[ |r(A, i)| = |c(A, j)| = T(A). \]

We shall say that an \((m \times n)\) matrix \(P\) is a \textit{subpermutation matrix} of \(A\) if it satisfies the following:

\[
\begin{align*}
p_{ij} &\in \mathbb{Z} \quad \text{for all } i, j, & (1.1) \\
p_{ij} \neq 0 \Rightarrow \text{sgn}(p_{ij}) = \text{sgn}(a_{ij}), & (1.2) \\
r(P, i) &= \text{sgn}(r(A, i)) \quad \text{for all rows } i, & (1.3) \\
c(P, j) &= \text{sgn}(c(A, j)) \quad \text{for all columns } j.
\end{align*}
\]

It was shown in [10] that for any regular matrix \(A\) with real entries there exists a finite \(r\), a collection \(P_1, \ldots, P_r\) of subpermutation matrices of \(A\) and coefficients \(\lambda_1, \ldots, \lambda_r > 0\) such that

\[
\sum_{s=1}^{r} \lambda_s P_s = A, & (1.4) \\
\sum_{s=1}^{r} \lambda_s = T(A). & (1.5)
\]

\textbf{Remark 1.1.} In the case where \(A\) is not necessarily regular, we may define a \textit{generalized subpermutation matrix} of \(A\) as an \((m \times n)\) matrix \(P\) satisfying (1.1), (1.2) and

\[
\begin{align*}
0 &\leq r(P, i) \leq 1 \quad \text{for each row } i \text{ with } r(A, i) \geq 0, \\
-1 &\leq r(P, i) \leq 0 \quad \text{for each row } i \text{ with } r(A, i) < 0, \\
0 &\leq c(P, j) \leq 1 \quad \text{for each column } j \text{ with } c(A, j) \geq 0, \\
1 &\leq c(P, j) \leq 0 \quad \text{for each column } j \text{ with } c(A, j) < 0.
\end{align*}
\]

Similar matrices will be used in the relaxed problem presented in Section 3.
The above result can then be formulated as in [10] for arbitrary real matrices; in the decomposition satisfying (1.4) and (1.5) the subpermutation matrices have to be replaced by generalized subpermutation matrices.

In our complexity study we shall restrict our attention to integral matrices; this will allow us to apply graph theoretical concepts in a more direct way.

For an integral \((m \times n)\) matrix \(B\) we define the cardinality \(|B|\) of \(B\) by

\[
|B| = \sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij}|.
\]

Notice that for an integral matrix \(A\), the coefficients \(\lambda_1, \ldots, \lambda_r\) in the decomposition into (generalized) subpermutation matrices \(P_1, \ldots, P_r\) can be integers. So by repeating \(\lambda_s\) times matrix \(P_s\) we may assume that we have a decomposition

\[
A = P_1 + \cdots + P_r.
\]

We will study in Section 2 the complexity of finding a decomposition (1.7) such that

\[
\max_{p,q \leq r} |P_p| - |P_q| \leq h,
\]

where \(h \geq 1\) is a given integer.

In some of the applications of decompositions of real matrices \(A\) (see [7]), it is sometimes desirable to balance in some way some additional parameters of the matrices \(P_1, \ldots, P_r\) of a decomposition; this may correspond to balancing some load; in timetabling applications for instance, balancing the row sums or column sums corresponds to balancing the load of a class (or of a teacher) on the various days of the week.

Here we shall require that some balancing properties hold for positive entries as well as for negative entries of each row or column.

Let us define for each row \(i\) and for each column \(j\) of \(A\)

\[
\begin{align*}
  r^+(A, i) &= \sum_j (a_{ij} \mid a_{ij} > 0), \\
  r^-(A, i) &= \sum_j (|a_{ij}| \mid a_{ij} < 0), \\
  c^+(A, j) &= \sum_i (a_{ij} \mid a_{ij} > 0), \\
  c^-(A, j) &= \sum_i (|a_{ij}| \mid a_{ij} < 0).
\end{align*}
\]

A decomposition (1.7) of a regular integral matrix \(A\) into subpermutation matrices will be called canonical if it satisfies for each row \(i\) and for each column \(j\) and for any two subpermutation matrices \(P_p, P_q\) (\(p, q \leq r\)):

\[
\begin{align*}
|r^+(P_p, i) - r^+(P_q, i)| &\leq 1, \\
|r^-(P_p, i) - r^-(P_q, i)| &\leq 1, \\
|c^+(P_p, j) - c^+(P_q, j)| &\leq 1, \\
|c^-(P_p, j) - c^-(P_q, j)| &\leq 1.
\end{align*}
\]

(1.8)
All decompositions (1.7) considered in the remainder of the paper will be canonical; we will in fact mention shortly decompositions (instead of canonical decompositions) when no confusion is possible.

2. The cardinality constrained decomposition problem

We shall in fact examine the general case of constraints bearing on the cardinalities of the matrices $P_1, \ldots, P_r$ in a decomposition (1.7).

Given an $(m \times n)$ regular matrix $A$ with integral entries, we may ask whether given $r = T(A)$ positive integers $p_1, \ldots, p_r$ with $\sum_{s=1}^r p_s = |A|$ there exists a (canonical) decomposition (1.7) into subpermutation matrices $P_1, \ldots, P_r$ satisfying

$$|P_i| = p_i \quad (i = 1, \ldots, r).$$

(2.1)

Let us call this problem CCDP (cardinality constrained decomposition problem); the main result of this section is the following:

**Proposition 2.1.** CCDP is NP-complete even for $r = T(A) = 3$.

Before proving Proposition 2.1, let us show that a (canonical) decomposition (1.7) of $A$ can be found by a simple edge coloring argument.

Given an $(m \times n)$ regular integral matrix $A$ (with $r = T(A)$) we construct a bipartite multigraph $B = (V', V'', E)$ as follows: we introduce in $V''$ (respectively, in $V'$) a node $r''_i$ (respectively, $r'_i$) for each row $i$ of $A$ and similarly a node $c''_j$ (respectively, $c'_j$) for each column $j$ of $A$.

Then for each entry $a_{ij} > 0$ (respectively, $< 0$) we introduce $|a_{ij}|$ edges $[r''_i, c'_j]$ (respectively, $[c''_j, r'_i]$) into $E$. The degrees $d_B(v)$ of nodes $v$ in $B$ satisfy

$$d_B(r''_i) = r(A,i) ; \quad d_B(r''_i) = r^+(A,i) \quad (i = 1, \ldots, m),$$

$$d_B(c''_j) = c(A,j) ; \quad d_B(c''_j) = c^+(A,j) \quad (j = 1, \ldots, n).$$

Observe that:

$$d_B(r''_i) - d_B(r'_i) = r \sgn(r(A,i)),$$

$$d_B(c''_j) - d_B(c'_j) = r \sgn(c(A,j)).$$

Now consider each row: if $r^+(A,i) \mod r = r^-(A,i) \mod r = \alpha_i$ we introduce $r - \alpha_i$ edges $[r''_i, r'_i]$ in $B$; similarly for each column $j$, if $c^+(A,j) \mod r = c^-(A,j) \mod r = \beta_j$ we introduce $r - \beta_j$ edges $[c''_j, c'_j]$ in $B$. Notice that now all degrees in $B$ are multiples of $r$.

Now given a subset $H$ of edges in $B$, we denote by $h(v)$ the number of edges of $H$ which are adjacent to $v$. We recall the definition of an equitable $r$-coloring of the edges of a multigraph $B$: it is a partition of the edge set of $B$ into subsets (color
classes) $H_1, \ldots, H_r$ such that for each node $v$ of $B$ and for any two colors $p, q$ we have

$$-1 \leq h_p(v) - h_q(v) \leq 1.$$ 

It is known [8] that a bipartite multigraph has an equitable $r$-coloring of its edges for any $r \geq 1$.

Lemma 2.1. There is a one-to-one correspondence between the (canonical) decompositions $(P_1, \ldots, P_r)$ of $A$ and the equitable $r$-colorings $(H_1, \ldots, H_r)$ of the edges of $B$.

Proof. (A) Assume first that the associated multigraph $B$ of $A$ has an equitable $r$-coloring $(H_1, \ldots, H_r)$.

Let $H_p$ be the $p$th color class; it defines a subpermutation matrix $P_p$ of $A$ as follows: initially set $(P_p)_{ij} = 0$ for all $i, j$. Then for each edge $[r''_i, c'_j]$ (respectively, $[c''_j, r'_i]$) in $H_p$ we increase (respectively, decrease) $(P_p)_{ij}$ by one. We do nothing for the edges $[c''_j, c'_j]$ and $[r''_i, r'_i]$.

Since all degrees are multiples of $r$, we have for each $r(i)$ $h_p(r''_i) - h_p(r'_i) = \sgn r(A, i)$ and for each $c_j$ $h_p(c'_j) - h_p(c''_j) = \sgn c(A, j)$. We may remove the edges $[c''_j, c'_j]$ and $[r''_i, r'_i]$ from $B$ and adjust the values of all $h_p(v)$ ($p = 1, \ldots, r$); the above equalities will still hold. By the above construction of $P_p$ we have $r^+(P, i) = h_p(r''_i)$, $r^-(P, i) = h_p(r'_i)$, $c^+(P, j) = h_p(c'_j)$ and $c^-(P, j) = h_p(c''_j)$. So $P_p$ is a subpermutation matrix of $A$; since the coloring is equitable, the decomposition $P_1, \ldots, P_r$ is canonical ($|r^+(P, i) - r^-(P, i)| = |h_p(r''_i) - h_p(r'_i)| \leq 1$ and similarly for the other nodes of $B$).

(B) Conversely given a (canonical) decomposition $(P_1, \ldots, P_r)$ of $A$ we may associate with it an equitable $r$-coloring of the edges of $B$.

Consider a matrix $P_p$: for each entry $(P_p)_{ij} > 0$ (respectively, $<0$) introduce $[(P_p)_{ij}]$ edges $[r''_i, c'_j]$ (respectively, $[c''_j, r'_i]$) into $H_p$.

Now examine row $i$ of $A$: We have $r^+(P_q, i) - r^-(P_q, i) = \sgn(r(A, i))$ for each $q \leq r$.

If $r^+(A, i) (\mod r) \equiv r^-(A, i) (\mod r) = \alpha_i$ with $0 < \alpha_i < r$, then there are $r - \alpha_i$ matrices $P_q$ with

$$r^+(P_q, i) = \lfloor r^+(A, i)/r \rfloor \quad \text{and} \quad r^-(P_q, i) = \lfloor r^-(A, i)/r \rfloor.$$ 

Let $Z$ be the set consisting of these $\alpha_i$ indices. For the remaining $r - \alpha_i$ matrices $P_q$ we have

$$r^+(P_q, i) = \lceil r^+(A, i)/r \rceil \quad \text{and} \quad r^-(P_q, i) = \lceil r^-(A, i)/r \rceil.$$

In other words we have constructed a (partial) coloring of the edges of $B$ such that for $q \in Z$

$$h_q(r''_i) = \lfloor r^+(A, i)/r \rfloor, \quad h_q(r'_i) = \lfloor r^-(A, i)/r \rfloor.$$
We have precisely \( \alpha_i \) edges \([r''_i, r'_i]\) in \( B \) which are still uncolored; we color these \( \alpha_i \) edges with \( \alpha_i \) colors \( q \) with \( q \in \mathbb{Z} \) (giving a different color to each edge). After that we will have \( h_p(r''_i) = d_B(r''_i)/r \) and \( h_p(r'_i) = d_B(r'_i)/r \) for each color \( p \leq r \) and for each row \( i \) of \( A \).

The same construction is repeated for all indices \( i \leq m \) and for each column \( j \) of \( A \). This will give us an equitable \( r \)-coloring of the edges of \( B \).

Consider now the bipartite multigraph \( B = (V'', V', E) \); according to our construction we have \(|E| \equiv 0 \pmod{r} \), so let \( h = |E|/r \); in any equitable \( r \)-coloring \((H_1, \ldots, H_r)\) of the edges of \( B \) we have \(|H_p| = h \) for \( p = 1, \ldots, r \). For each row \( i \) of \( A \) let \( \mathcal{F}(r_i) \) be the family of \( \alpha_i \geq 0 \) parallel edges \([r''_i, r'_i]\) introduced into \( B \) and similarly for each column \( j \) of \( A \) let \( \mathcal{F}(c_j) \) be the family of \( \beta_j \geq 0 \) parallel edges \([c''_j, c'_j]\) introduced into \( B \). Finally let

\[ \mathcal{F} = \mathcal{F}(r_1) \cup \cdots \cup \mathcal{F}(r_m) \cup \mathcal{F}(c_1) \cup \cdots \cup \mathcal{F}(c_n). \]

Now given positive integers \( p_1, \ldots, p_r \) satisfying \( \sum_{i=1}^r p_i = |A| \), there exists a (canonical) decomposition \( P_1, \ldots, P_r \) of \( A \) satisfying (2.1) iff \( B \) has an equitable \( r \)-coloring \((H_1, \ldots, H_r)\) satisfying

\[ |H_s \cap \mathcal{F}| = p_s \quad \text{for} \quad s = 1, \ldots, r. \]

This follows from Lemma 2.1 and from the fact that the edges in \( \mathcal{F} \) are “artificial”, i.e., they do not correspond to entries in \( A \) but they have been introduced for making all degrees multiples of \( r \).

In an equivalent way, we may state:

**Lemma 2.2.** There exists a (canonical) decomposition of \( A \) satisfying (2.1) iff the associated multigraph \( B \) has an equitable \( r \)-coloring \((H_1, \ldots, H_r)\) satisfying

\[ |H_s \cap \mathcal{F}| = \frac{|A| + |\mathcal{F}|}{r} - p_s \quad \text{for} \quad s = 1, \ldots, r. \]

**Lemma 2.3.** An \((n + m) \times (n + m)\) integral matrix \( C \) is the node–node incidence matrix of a multigraph \( B \) associated to a regular integral \((m \times n)\) matrix \( A \) with \( r = \text{T}(A) \) iff it satisfies (possibly after permutation of rows and of columns):

(i) the diagonal terms \( c_{ii} \) are non-negative integers,
(ii) \( c_{ij} = 0 \) \( (i \neq j) \) for \( i, j \leq m \) and \( i, j > m \),
(iii) \( r(C, i) \equiv 0 \pmod{r} \), \( c(C, i) \equiv 0 \pmod{r} \) for \( i = 1, \ldots, m + n \),
(iv) \( |r(C, i) - c(C, i)| = r \) for \( i = 1, \ldots, m + n \),
(v) \( c_{ij} \cdot c_{ji} = 0 \) for all \( i, j \) \( (i \neq j) \).

**Proof.** It is easy to verify that starting from \( A \), the construction of the associated multigraph \( B \) gives a node–node incidence matrix \( C \) satisfying (i)-(v).

Conversely given a matrix \( C \) satisfying the above properties, we obtain the matrix \( A \) as follows:
for \( i = 1, \ldots, m \) and for \( j = 1, \ldots, n \) we set \( a_{ij} = c_{i,m+j}; \)

for \( i = m+1, \ldots, m+n \) and for \( j = 1, \ldots, m \) we examine \( c_{ij}; \)

if \( c_{ij} > 0 \), we set \( a_{j,i-m} = -c_{ij}. \)

The given matrix \( C \) does correspond to \( B \) since every entry \( c_{m+i,j} (1 \leq i \leq m, 1 \leq j \leq n) \) corresponds to edges \([c^r_j, r^c_i]\) in \( B \) (which are associated to negative entries \( a_{ij} \) of \( A \)) and every entry \( c_{i,m+j} (1 \leq i \leq m, 1 \leq j \leq n) \) corresponds to edges \([r^c_i, c^r_j]\) in \( B \) (which are associated to positive entries \( a_{ij} \) of \( A \)). □

Let us now consider the problem COLFEAS which will be used in the proof of Proposition 2.1.

Given a bipartite multigraph \( G = (V, W, E) \) with maximum degree 3 and given three positive integers \( h_1, h_2, h_3 \) with \( h_1 + h_2 + h_3 = |E| \), is there an (equitable) 3-coloring \((F_1, F_2, F_3)\) of the edges of \( G \) with \(|F_i| = h_i (i = 1, 2, 3)\)?

It is known that COLFEAS is NP-complete [5].

**Proof of Proposition 2.1.** We shall transform COLFEAS into CCDP, i.e., into a problem of construction of an equitable 3-coloring \((H_1', H_2', H_3')\) of edges with some cardinality constraints in a multigraph \( \mathcal{G} \) for which the node–node incidence matrix satisfies the conditions (i)--(v) of Lemma 2.3.

Let us start from graph \( G = (V, W, E) \); we replace each edge \([v, w]\) in \( E \) by the chain of length 9 represented in Fig. 1(a); let \( G_1 = (R_1, C_1, E_1) \) be the resulting graph. In the same way, we start from \( G \) again and replace each edge \([v, w]\) by the chain of length 9 represented in Fig. 1(b). Let \( G_2 = (R_2, C_2, E_2) \) be the resulting graph.

It is clear that \( G_1, G_2 \) are bipartite and that in any 3-coloring of the edges the first and last edges of a chain replacing an edge of \( G \) have the same color.

Furthermore in each such chain, the coloring is unique (up to a permutation of the colors on the parallel edges) as soon as the color of the first edge is fixed). In the intermediate edges and on the last edge, all colors occur exactly 4 times. Notice that \(|E_i| = |E| + 12|E| \) for \( i = 1, 2.\)

![Fig. 1. Expansion of edges in G1 and G2. Replacement of [v, w] with v ∈ V, w ∈ W to get G1 and (b) G2.](image-url)
So we can state:

**Fact 2.1.** The following statements are equivalent:

(a) $G_1$ has a 3-coloring $(H^1_1, H^2_1, H^3_1)$ of its edges with
\[ |H^i_j| = h_i + 4|E| \quad (i = 1, 2, 3), \]

(b) $G_2$ has a 3-coloring $(H^1_2, H^2_2, H^3_2)$ of its edges with
\[ |H^i_j| = h_i + 4|E| \quad (i = 1, 2, 3) \]

(c) $G$ has a 3-coloring $(H_1, H_2, H_3)$ of its edges with
\[ |H^i| = h_i \quad (i = 1, 2, 3). \]

Notice that there is a one-to-one correspondence between the nodes of $G_1$ and those of $G_2$; in general for each node $u$ of $G$, we will denote by $u'$ its copy in $G_i$ ($i = 1, 2$).

From the construction of $G_1, G_2$ we can also state the following:

**Fact 2.2.** For any choice of two nodes $x^i, y^i$ in $G_i$, if $[x^i, y^i] \in E_i$, then $[x^i, y^j] \notin E_j$ for $i, j \in \{1, 2\}, i \neq j$. In other words if two nodes are linked in $G_1$ (respectively, $G_2$), the corresponding nodes are not linked in $G_2$ (respectively, $G_1$).

This fact will be needed to make sure that the node–node incidence matrix that we will construct will satisfy property (v) of Lemma 2.3.

Now for each node $u^i$ of $G_1$ we will introduce a node $\tilde{u}^1$ with 3 parallel edges linking $u^i$ and $\tilde{u}^1$. Similarly for each node $\tilde{u}^2$ of $G_2$ we introduce no additional edges (so all nodes $\tilde{u}^2$ are isolated).

Let us now call $\hat{G}_1$ and $\hat{G}_2$ the current graphs. Notice that they are still bipartite; if we denote $\hat{G}_i$ by $(\tilde{R}_i, \tilde{C}_i, \tilde{E}_i)$ for $i = 1, 2$ and if we let $m = |R_i|, n = |C_i|$ then for $i = 1, 2$ we have
\[ |\tilde{R}_i| = |\tilde{C}_i| = m + n \quad \text{and} \quad |\tilde{E}_i| = |E_i| + 3(m + n). \]

Observe that we still have a one-to-one correspondence between the nodes of $\hat{G}_1$ and $\hat{G}_2$: this was true for the nodes of $G_1$ and $G_2$. For the nodes $\tilde{u}^1, \tilde{u}^2$, we simply associate each $\tilde{u}^1$ in $G_1$ (associated to the node $u^1$ of $G_1$ corresponding to $u$ in $G$) to the node $\tilde{u}^2$ of $G_2$ (associated to the node $u^2$ of $G^2$ corresponding to $u$ in $G$).

We can now state:

**Fact 2.3.** The following statements are equivalent:

(a) $\hat{G}_1$ has an equitable 3-coloring $(H^1_1, H^2_1, H^3_1)$ of its edges with
\[ |H^i_1| = h_i + 4|E| + (m + n) \quad (i = 1, 2, 3), \]
Fig. 2. Construction of $\mathcal{G}$ from given graphs $G_1$ and $G_2$.

(b) $\hat{G}_2$ has an equitable 3-coloring $(H^1_2, H^2_2, H^3_2)$ of its edges with

$$|H^i_2| = h_i + 4|E| \quad (i = 1, 2, 3),$$

(c) $G$ has a 3-coloring $(H^1_1, H^2_1, H^3_1)$ of its edges with

$$|H^i_1| = h_i \quad (i = 1, 2, 3).$$

Now we shall introduce a collection $\mathcal{F}$ of families of parallel edges between some pairs $u^1, u^2$ of nodes with $u^i \in R_i \cup C_i$ $(i = 1, 2)$.

We consider each node $u$ of $G$ (i.e., $u \in V \cup W$): if $d_G(u) \equiv \alpha$ with $0 < \alpha < 2$ then we introduce $\alpha$ edges between nodes $u^1$ of $G_1$ and $u^2$ of $G_2$. The construction is illustrated in Fig. 2.

**Fact 2.4.** The resulting graph $\mathcal{G} = (\mathcal{U}, \mathcal{V}, \mathcal{E})$ is bipartite and $d_{\mathcal{G}}(x) \equiv 0 \pmod{3}$ for each node $x$.

Observe that by taking $\mathcal{U} = \hat{R}_1 \cup \hat{C}_2$ and $\mathcal{V} = \hat{R}_2 \cup \hat{C}_1$ we may verify that each edge of $\mathcal{G}$ has one endpoint in $\mathcal{U}$ and the other in $\mathcal{V}$.

**Fact 2.5.** The node–node incidence matrix of $\mathcal{G}$ satisfies properties (i)–(v) of Lemma 2.3.

This is verified easily if we construct the matrix by assigning one row to each node of $R_1, \hat{R}_1 - R_1, C_2, \hat{C}_2 - C_2$ in this order and one column to each node of $R_2, \hat{R}_2 - R_2, C_1, \hat{C}_1 - C_1$ in this order (assuming that in each of these sets we use the same order for the nodes of one graph $\hat{G}_i$ and for the corresponding nodes of the other graph $\hat{G}_j$ $(j \neq i)$).

Observe now that in any equitable 3-coloring $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ of $\mathcal{G}$ we will have

$$|\mathcal{H}_i| = |\mathcal{E}|/3.$$
where

\[ |\mathcal{E}| = |\hat{E}_1| + |\hat{E}_2| + |\mathcal{F}| = 2|E| + 3(m + n) + 24|E| + |\mathcal{F}| = 2|E| + |\mathcal{F}| + 3(m + n + 8|E|). \]

It follows from this that \((2|E| + |\mathcal{F}|) \equiv 0 \pmod{3}\).

**Lemma 2.4.** \(G\) has a 3-coloring \((H_1, H_2, H_3)\) of its edges with \(|H_i| = h_i (i = 1, 2, 3)\) iff \(G\) has an equitable 3-coloring \((\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)\) of its edges with:

\[ |\mathcal{F} \cap \mathcal{H}_i| = \left| \mathcal{F} \right| + 2|E| - 2h_i \quad (i = 1, 2, 3). \]

**Proof.** (A) Assume first that \(G\) has a 3-coloring \((H_1, H_2, H_3)\) of its edges satisfying \(\left| H_i \right| = h_i (i = 1, 2, 3)\); we construct the corresponding colorings in \(G_1\) and in \(G_2\). Then by giving colors 1, 2, 3 to each family of 3 parallel edges introduced, we get equitable 3-colorings of \(\hat{G}_1\) and \(\hat{G}_2\) satisfying (a) and (b) of Fact 2.3.

At this stage we have an equitable 3-coloring of \(G - F = \hat{G}_1 \cup \hat{G}_2\) with exactly \(2h_1 + 8|E| + (m + n)\) edges of color \(i \quad (i = 1, 2, 3)\).

Since we started from the same coloring of \(G\) to color \(G_1\) and \(G_2\), we can extend the coloring to an equitable 3-coloring of \(G\); for each node \(u\) of \(G\), the number of occurrences of color \(p \leq 3\) on edges adjacent to \(u\) is \(h_p(u)\); by construction we have \(h_p(u^1) = h_p(u) + 1 = h_p(u^2) + 1\) since we have introduced 3 parallel edges between \(u^1\) and \(u^2\). So we can color the edges of \(\mathcal{F}\) between \(u^1\) and \(u^2\) (if any) with the colors occurring the smallest number of times around \(u^1\) (and \(u^2\)), by construction they are the same.

We finally get an equitable edge 3-coloring \((\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)\) of \(G\) (where each family of parallel edges is colored with different colors). It satisfies \(\left| \mathcal{H}_i \right| = \left| \mathcal{E} \right| / 3\).

So we have

\[ |\mathcal{H}_i| = 2h_i + 8|E| + (m + n) + |\mathcal{H}_i \cap \mathcal{F}| = \frac{2|E| + |\mathcal{F}|}{3} + (m + n) + 8|E| \]

which gives

\[ |\mathcal{F} \cap \mathcal{H}_i| = \frac{|\mathcal{F}| + 2|E|}{3} - 2h_i \quad \text{for all} \quad i. \]

(B) Conversely assume that we have an equitable 3-coloring \((\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)\) of the edges of \(G\) satisfying the above equalities. It is well known [7] that there exists one such that in each family of parallel edges (there are at most 3 edges in such families in \(G\)) all colors are different.

Such a coloring satisfies \(\left| \mathcal{H}_i \right| = (2|E| + |\mathcal{F}|)/3 + (m + n) + 8|E|\). We may remove all families of 3 parallel edges introduced between nodes \(u^1\) and \(u^2\); so we have \(\left| \mathcal{H}_i \right| = (2|E| + |\mathcal{F}|)/3 + 8|E|\) edges of color \(i\); we may also replace the chains of length 9 introduced earlier in \(G_1\) and in \(G_2\) by single edges (having the same
color as the first edge and the last edge in the chain). So we are left with $|\mathcal{H}_i| = (2|E| + |\mathcal{F}|)/3$ edges of each color $i$.

Now we notice that all degrees in the remaining graph are all equal to three. The graph consists of two copies of $G$ where the corresponding nodes $u^1, u^2$ are linked in parallel edges of $F$. This means that for each color $i \leq 3$ and for each pair of corresponding nodes $u^1, u^2$, we have $h_i(u^1) = h_i(u^2)$; this holds also after removal of all edges in $\mathcal{F}$. For each color $i$, we are left with $(2|E| + |\mathcal{F}|)/3 - |\mathcal{F} \cap \mathcal{H}_i|$ edges of this color. By hypothesis, this number is equal to $2h_i$. Now since the number of occurrences $h_i(u^1)$ of each color $i$ at node $u^1$ is the same as $h_i(u^2)$, we have exactly the same number of edges of color $i$ in each one of the two copies of $G$. So each copy has exactly $h_i$ edges of color $i$ and $G$ has the required coloring. □

Now we have reduced COLFEAS to a CCDP problem; the reduction is polynomial as shown in the construction. So we have proved Proposition 2.1. □

Remark 2.1. The construction used in the above proof shows that the problem remains NP-complete even if all row sums and all column sums of $A$ are equal to $T(A)$ (i.e., we have no negative lines, but only possibly negative entries in $A$).

3. Extended subpermutation matrices

Since balancing cardinalities of the (generalized) subpermutation matrices $P_i$ in a decomposition $P_1, \ldots, P_r$ of an integral matrix $A$ is not always possible, we may try to relax the requirements on the generalized subpermutation matrices in order to improve the balancing properties.

Given an integral and regular $(m \times n)$ matrix $A$ we shall say that an $(m \times n)$ matrix $P$ is a $t$-extended subpermutation matrix of $A$ if in addition to (1.1) and (1.2) the following holds:

\begin{align}
0 \leq r(P, i) &\leq t \quad \text{for each row } i \text{ with } r(A, i) \geq 0, \\
-t \leq r(P, i) &\leq 0 \quad \text{for each row } i \text{ with } r(A, i) < 0, \\
0 \leq c(P, j) &\leq t \quad \text{for each column } j \text{ with } c(A, j) \geq 0, \\
-t \leq c(P, j) &\leq 0 \quad \text{for each column } j \text{ with } c(A, j) < 0,
\end{align}

where $t \geq 1$ is a given positive integer.

We may ask what is the smallest $t$ such that for any integer matrix $A$ with $r = T(A)$ there exists a decomposition (1.7) into $t$-extended subpermutation matrices with

$$\max_{P, q} |P_p| - |P_q| \leq 1.$$  \tag{3.2}

Proposition 3.1. For any integral regular matrix $A$ there exists a finite $r$ and 2-extended subpermutation matrices $P_1, \ldots, P_r$ such that (3.1) and (3.2) are satisfied.
Proof. We construct the bipartite multigraph $B$ associated to $A$ as in Section 2; let $r = T(A)$. $B$ has an equitable $r$-coloring $(H_1, \ldots, H_r)$ of its edges with
\[
\max_{p,q}(|H_p| - |H_q|) \leq 1.
\]
(3.3)
Consider the set $H_p$ of edges of color $p$ and a row $i$ of $A$. Let
\[
e = \text{sgn}(r(A,i)), \quad a = \lfloor dB(r''_i)/r \rfloor, \quad b = \lceil dB(r''_i)/r \rceil.
\]
Since $A$ is regular, we have
\[
d_B(r''_i) - d_B(r'_i) = re.
\]
Since the $r$-coloring is equitable
\[
a \leq h_p(r''_i) \leq b,
\]
\[
a - e \leq h_p(r'_i) \leq b - e.
\]
(3.4)
So for the matrix $P_p$ defined by the edges of $H_p$ we have
\[
r(P_p,i) = h_p(r''_i) - h_p(r'_i),
\]
and (3.4) implies:
\[
e - 1 \leq a - b + e \leq r(P_p,i) \leq b - a + e \leq 1 + e.
\]
(3.5)
Hence for a row $i$ with $r(A,i) = +r$ (3.5) gives $0 \leq r(P_p,i) \leq 2$ while for a row $i$ with $r(A,i) = -r$, it gives $-2 \leq r(P_p,i) \leq 0$. It follows from (3.3) that the 2-extended subpermutation matrices $P_1, \ldots, P_r$ constructed satisfy (3.2). □

4. Final remarks

The original theorem of Birkhoff–von Neumann (in its integral version) is in fact the theorem of Koenig (see [1]) on edge colorings of bipartite multigraphs.

Allowing arbitrary signs in the (integral) matrix $A$ amounts to consider an oriented version of the Koenig theorem: instead of having $a_{ij}$ undirected edges between nodes $r_i$ and $c_j$, we may consider that we have $|a_{ij}|$ arcs $(r_i, c_j)$ if $a_{ij} > 0$ or $|a_{ij}|$ arcs $(c_j, r_i)$ if $a_{ij} < 0$. As discussed in [10], instead of matchings, we have collections of oriented paths with disjoint end nodes.

References