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Maximal Subgroups of the Classical Groups Associated with Non-isotropic Subspaces of a Vector Space

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INTRODUCTION

In [6] we showed that if V is a finite-dimensional vector space and if H is a symplectic, special orthogonal, orthogonal, special unitary or unitary group acting on V, then with a few exceptions, the stabilizer in H of a totally singular subspace is maximal. We further indicated that if the stabilizer in H of an arbitrary subspace is maximal, then that subspace will usually be totally singular, non-isotropic, or isotropic but non-singular of dimension 1 (this only occurs in the case of an orthogonal group over a field of characteristic two). In this paper we consider the stabilizer of a non-isotropic subspace with the restriction that either the subspace or its conjugate will have a singular 1-dimensional subspace. There is one general exception: when H contains elements that interchange the subspace and its conjugate. We show that in this case the subgroup of H consisting of the elements that either stabilize the subspace or interchange it with its conjugate is in most cases maximal. There are a number of more specific exceptions listed in the next section.

As in [6], our approach is geometric in nature. We show that any subgroup of H properly containing the given stabilizer contains every transvection or every semi-transvection in H, and deduce that it must therefore be the whole of H.

1. NOTATION

Let V be an n-dimensional vector space over a field K. When n is even, let A be a non-degenerate alternating form on V and let $Sp_n(K)$ be the

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symplectic group of A. Let Q be a quadratic form on V whose associated symmetric bilinear form, given by

$$B(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}), \qquad \forall \mathbf{x}, \mathbf{y} \in V,$$

is non-degenerate, and let $O_n(K)$ and $SO_n(K)$ be respectively the orthogonal and special orthogonal groups of Q. If K is a field with a non-trivial involutory automorphism J, then let K_0 be the fixed subfield of J. It can be shown that K is a normal separable extension of K_0 of degree 2. Given $\lambda \in K$, we shall often write $\bar{\lambda}$ in place of $J(\lambda)$. Let C be a non-degenerate hermitian form on V, thus

$$C(\mathbf{x}, \lambda\mathbf{y} + \mu\mathbf{z}) = \lambda C(\mathbf{x}, \mathbf{y}) + \mu C(\mathbf{x}, \mathbf{z}),$$
$$C(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \bar{\lambda} C(\mathbf{x}, \mathbf{z}) + \bar{\mu} C(\mathbf{y}, \mathbf{z}),$$

and

$$C(\mathbf{y}, \mathbf{x}) = \overline{C(\mathbf{x}, \mathbf{y})}, \qquad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V,\ \forall \lambda, \mu \in K.$$

Let $U_n(K)$ and $SU_n(K)$ be respectively the unitary and special unitary groups of C. We denote the indices of Q and C by $\nu(Q)$ and $\nu(C)$, respectively, or by ν where no confusion arises.

When we wish to describe a property which relates to more than one of A, B and C, we shall often use $(\ ,\ )$ in place of $A(\ ,\ )$, $B(\ ,\ )$ or $C(\ ,\ )$. We will use H to refer to one of $Sp_n(K)$, $O_n(K)$ and $U_n(K)$, and H_1 to refer to $SO_n(K)$ or $SU_n(K)$. For any subspace U of V, we shall denote its conjugate with respect to the appropriate form by U'; it will be evident from the context which form is being considered. When H is $Sp_n(K)$ or $U_n(K)$, we shall use the terms "singular" and "totally singular" in place of the usual terms "isotropic" and "totally isotropic"; this is solely for convenience. We note that an element of H stabilizes U if and only if it stabilizes U'; so the stabilizer in H of U is also the stabilizer of U'. We also note that if U is non-isotropic, then we can write $V = U \oplus U'$. Throughout this paper we shall say that two subspaces are isomorphic only if they are isomorphic with respect to the appropriate form. Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be isomorphic if there exists $h \in H$ such that $h(\mathbf{x}) = \mathbf{y}$ (whence $A(\mathbf{y}, \mathbf{y}) = A(\mathbf{x}, \mathbf{x})$, $Q(\mathbf{y}) = Q(\mathbf{x})$ or $C(\mathbf{y}, \mathbf{y}) = C(\mathbf{x}, \mathbf{x})$).

Let U be a non-isotropic subspace of V of dimension $r \geqslant 1$ such that U' has a singular 1-dimensional subspace; this imposes the requirement that $n - r \geqslant 2$ and $n \geqslant 3$. Let $G = \mathrm{Stab}_H U$, let $E = \mathrm{Stab}_H\{U, U'\}$, let $G_1 = \mathrm{Stab}_{H_1} U\ (= H_1 \cap G)$ and let $E_1 = \mathrm{Stab}_{H_1}\{U, U'\}\ (= H_1 \cap E)$; if U is isomorphic to U', then $G < E$ and $G_1 < E_1$, but otherwise $G = E$ and $G_1 = E_1$. We show that E_1 and E are maximal in H_1 and H, respectively, except in the cases listed below.

We denote the dimensions of the maximal totally singular subspaces of U and U' by ν_1 and ν_2, respectively; by definition, $\nu_2 > 0$. As we are interested in the maximality of certain subgroups, and as $G = \mathrm{Stab}_H U'$, we may assume that if $\nu_1 > 0$, then $r \leqslant n - r$. Note that this implies that when K is finite, $r \leqslant n - r$ whatever the value of ν_1, because if $\nu_1 = 0$, then $r \leqslant 2$.

Throughout the remainder of this paper, we except (unless stated otherwise) the following cases where E_1 is not maximal in H_1 and E is not maximal in H.

When $H = O_n(K)$:

(i) $K = GF(5)$, $n = 3$ and $r = 1$;

(ii) $K = GF(3)$ and $n - r = 2$;

(iii) $K = GF(3)$, $r = 2$ and $\nu_1 = 1$;

(iv) $K = GF(3)$, $n = 4$, $r = 1$ and $\nu = 2$;

(v) $K = GF(3)$, $n = 5$, $r = 2$ and $\nu_1 = 0$;

(vi) $K = GF(2)$, $n \geqslant 6$, $r = 2$ and $\nu_1 = 1$;

(vii) $K = GF(2)$, $n = 6$, $r = 2$, $\nu_1 = 0$ and $\nu_2 = 2$;

(viii) $K = GF(5)$, $n = 4$, $r = 2$ and $\nu_1 = \nu_2 = 1$;

(ix) $K = GF(3)$, $n = 6$ and U is isomorphic to U';

(x) $K = GF(2)$, $n = 8$, $r = 4$ and $\nu_1 = \nu_2 = 2$.

When $H = U_n(K)$

(xi) $K = GF(4)$, $n = 3$ and $r = 1$;

(xii) $K = GF(4)$, $n - r = r = 2$ and $\nu_1 = \nu_2 = 1$.

We also except (unless stated otherwise) the case:

(xiii) $H = O_4(K)$, $K \neq GF(3)$, $GF(5)$, $r = 2$ and $\nu_1 = \nu_2 = 1$,

where E is maximal in H, but E_1 is not maximal in H_1, and the cases:

(xiv) $H = O_4(GF(2))$, $r = 2$, $\nu_1 = 0$ and $\nu_2 = 1$;

(xv) $H = U_4(K), K \neq GF(4)$, $r = 2$ and $\nu_1 = \nu_2 = 1$,

for which we require a separate proof.

2. Preliminary Results and Definitions

This section has three parts. The first consists of definitions and elementary results, including a definition of a semi-transvection. The second part consists of vector space properties when H is one of $O_n(K)$, $U_n(K)$, and in the third part we give some field properties of K when it has a non-trivial involutory automorphism. We do not exclude here cases (i)–(xv) listed above.

In the first part, H will be any one of $Sp_n(K)$, $O_n(K)$ or $U_n(K)$ unless stated otherwise.

PROPOSITION 2.1. *Let U be a non-isotropic subspace of V.*

(i) *If $H(U)$ and $H(U')$ are the groups corresponding to H of U and U', then* $\mathrm{Stab}_H U$ *is isomorphic to the direct product $H(U) \times H(U')$.*

(ii) *If S_1 and S_2 are isomorphisms, $U_1 \to W_1$ and: $U_2 \to W_2$, respectively where U_1, $W_1 \subseteq U$ and U_2, $W_2 \subseteq U'$, then there is an element of* $\mathrm{Stab}_H U$ *extending both S_1 and S_2.*

Proof. (i) See Dieudonné [2].

(ii) Use (i) together with Witt's Theorem (cf. [1, p. 71]). ∎

DEFINITIONS. In $Sp_n(K)$, a transvection centered on a non-zero vector $\mathbf{x}$ is given by

$$: \mathbf{v} \mapsto \mathbf{v} + \lambda A(\mathbf{x}, \mathbf{v})\, \mathbf{x}$$

for some $\lambda \in K \backslash \{0\}$.

In $U_n(K)$, a transvection centered on a non-zero singular vector $\mathbf{x}$ is given by

$$: \mathbf{v} \mapsto \mathbf{v} + \lambda C(\mathbf{x}, \mathbf{v})\, \mathbf{x}$$

for some $\lambda \in K \backslash \{0\}$ such that $\bar{\lambda} = -\lambda$. Such maps lie in $SU_n(K)$ (cf. [2, p. 49]).

In $O_n(K)$, a symmetry or -1 —quasi-symmetry centered on a non-singular vector $\mathbf{y}$ is given by

$$: \mathbf{v} \mapsto \mathbf{v} - [B(\mathbf{y}, \mathbf{v})/Q(\mathbf{y})]\, \mathbf{y}.$$

In $U_n(K)$, if $\lambda \in K \backslash \{1\}$ such that $\lambda \cdot \bar{\lambda} = 1$, then the λ-quasi-symmetry centered on a non-singular vector $\mathbf{y}$ is given by

$$: \mathbf{v} \mapsto \mathbf{v} + (\lambda - 1)[C(\mathbf{y}, \mathbf{v})/C(\mathbf{y}, \mathbf{y})]\, \mathbf{y}.$$

Remarks 2.2. A transvection [respectively, quasi-symmetry] centered on a singular [non-singular] vector $\mathbf{z} \in V$ stabilizes a subspace Z if and only if $\mathbf{z} \in Z \cup Z'$. This is because if $\mathbf{z} \notin Z \cup Z'$ and if $\mathbf{w} \in Z$ is not orthogonal to $\mathbf{z}$, then the transvection [quasi-symmetry] moves $\mathbf{w}$ out of Z. If $\mathbf{z} \in Z \cup Z'$, then the transvection [quasi-symmetry] fixes one of, and therefore both of, Z, Z'.

Every quasi-symmetry in H lies outside H_1. Let $H = U_n(K)$ and let $\mathbf{y}$ be a non-singular vector in V, then $V = \langle \mathbf{y} \rangle \oplus \langle \mathbf{y} \rangle'$. The λ-quasi-symmetry centered on $\mathbf{y}$ takes $\mathbf{y}$ to $\lambda\mathbf{y}$ and fixes each vector in $\langle \mathbf{y} \rangle'$, so it has determinant λ. If $R \neq H_1$ is a coset of H_1 in H, then there exists $\mu \in K$ with $\mu \cdot \bar{\mu} = 1$ such that $R = \{h \in H : \det h = \mu\}$ (cf. [2, p. 56]), so for any given non-singular vector $\mathbf{y}$, there is a quasi-symmetry centered on $\mathbf{y}$ lying in R.

DEFINITION. Let H be one of $O_n(K)$, $U_n(K)$. Let $\mathbf{x}$ be a non-zero singular vector in V, let $\mathbf{w} \in \langle \mathbf{x} \rangle'$ and let $\rho_{\mathbf{x},\mathbf{w}}$ be the isomorphism of $\langle \mathbf{x} \rangle'$ defined by

$$: \mathbf{v} \mapsto \mathbf{v} + (\mathbf{w}, \mathbf{v})\, \mathbf{x}.$$

We denote the set of elements of H that extend $\rho_{\mathbf{x},\mathbf{w}}$ by $P_{\mathbf{x},\mathbf{w}}$ (non-empty by Witt's theorem) and call those elements semi-transvections centered on $\mathbf{x}$.

Certain properties of semi-transvections in $O_n(K)$ have been given by Tamagawa in [7]; we refer to these results (altering the notation) and give the corresponding results for unitary semi-transvections.

Let $\mathbf{y}$ be a singular vector in V such that $(\mathbf{x}, \mathbf{y}) = 1$; then for a set of semi-transvections $P_{\mathbf{x},w}$, we may assume that $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle'$ (because otherwise we could replace $\mathbf{w}$ by $\mathbf{w} - (\mathbf{y}, \mathbf{w})\, \mathbf{x}$ without altering $P_{\mathbf{x},\mathbf{w}}$). If $H = O_n(K)$ and if $\rho \in P_{\mathbf{x},\mathbf{w}}$ where $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle'$, then Tamagawa shows that

$$\rho(\mathbf{y}) = \mathbf{y} - Q(\mathbf{w}) \cdot \mathbf{w} - \mathbf{w},$$

whence $P_{\mathbf{x},\mathbf{w}} = \{\rho\}$. If $H = U_n(K)$ and $\rho \in P_{\mathbf{x},\mathbf{w}}$ where $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle'$, then consideration of the equations

$$C(\rho(\mathbf{y}), \rho(\mathbf{v})) = 0, \qquad \forall \mathbf{v} \in (\langle \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle')$$

and

$$C(\rho(\mathbf{y}), \rho(\mathbf{x})) = 1$$

shows that

$$\rho(\mathbf{y}) = \mathbf{y} + \beta\mathbf{x} - \mathbf{w},$$

where $\beta + \bar{\beta} = -C(\mathbf{w}, \mathbf{w})$. Indeed, for any such $\beta \in K$ there is an element of $P_{\mathbf{x},\mathbf{w}}$ taking $\mathbf{y}$ to $\mathbf{y} + \beta\mathbf{x} - \mathbf{w}$.

The elements of $P_{\mathbf{x},\mathbf{x}} (= P_{\mathbf{x},0})$ are the elements of H that fix every vector in $\langle \mathbf{x} \rangle'$, i.e., $P_{\mathbf{x},\mathbf{x}}$ consists of the transvections centered on $\mathbf{x}$, together with the identity element. For any $P_{\mathbf{x},\mathbf{w}}$, if $\rho \in P_{\mathbf{x},\mathbf{w}}$, then we can write $P_{\mathbf{x},\mathbf{w}} = \rho \cdot P_{\mathbf{x},\mathbf{x}} = P_{\mathbf{x},\mathbf{x}} \cdot \rho$.

If we define the product $P_{\mathbf{x},\mathbf{u}} \cdot P_{\mathbf{x},\mathbf{w}}$ to be $\{\sigma \cdot \rho : \sigma \in P_{\mathbf{x},\mathbf{u}}, \rho \in P_{\mathbf{x},\mathbf{w}}\}$, then we can deduce:

PROPOSITION 2.3. *If* $\mathbf{x}$ *is a non-zero singular vector in* V, then

$$\begin{aligned}
P_{\mathbf{x},\mathbf{u}} \cdot P_{\mathbf{x},\mathbf{w}} &= P_{\mathbf{x},\mathbf{u}+\mathbf{w}}, && \forall \mathbf{u}, \mathbf{w} \in \langle \mathbf{x} \rangle', \\
P_{\lambda\mathbf{x},\mathbf{w}} &= P_{\mathbf{x},\lambda\mathbf{w}}, && \forall \mathbf{w} \in \langle \mathbf{x} \rangle', \lambda \in K \textit{ when } H = O_n(K), \\
P_{\lambda\mathbf{x},\mathbf{w}} &= P_{\mathbf{x},\bar{\lambda}\mathbf{w}}, && \forall \mathbf{w} \in \langle \mathbf{x} \rangle', \lambda \in K \textit{ when } H = U_n(K), \\
hP_{\mathbf{x},\mathbf{w}}h^{-1} &= P_{h\mathbf{x},h\mathbf{w}}, && \forall \mathbf{w} \in \langle \mathbf{x} \rangle', h \in H, \\
P_{\mathbf{x},\mathbf{x}} &= P_{\mathbf{x},\mathbf{w}} && \textit{if and only if } \mathbf{w} \in \langle \mathbf{x} \rangle.
\end{aligned}$$

If $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle'$ and if we extend $\{\mathbf{x}, \mathbf{w}, \mathbf{y}\}$ to an ordered base for V, then with respect to that base, the matrix of $\rho \in P_{\mathbf{x},\mathbf{w}}$ is upper triangular with all the diagonal entries being 1. Hence every semi-transvection in $U_n(K)$ lies in $SU_n(K)$. Tamagawa showed that every semi-transvection in $O_n(K)$ lies in $SO_n(K)$.

If $H = O_n(K)$ and if P is a hyperbolic 2-dimensional subspace of V, then we define $SO(P)$ to be the subgroup of $SO_n(K)$ consisting of the elements that fix every vector in P'.

Result 2.4. (Tamagawa [7, Lemmas 11 and 12]). If $H = O_n(K)$, if P is a hyperbolic 2-dimensional subspace of V and if T is the subgroup of H_1 generated by the semi-transvections in H_1, then $H_1 = T \cdot SO(P)$, except when $n = 4$, $\nu = 2$ and $K = GF(2)$.

PROPOSITION 2.5. *If $H = U_n(K)$, then H_1 is generated by its semi-transvections, except perhaps when $n = 3$ and $K = GF(4)$.*

Proof. Every transvection is a semi-transvection and it is known that H_1 is generated by its transvections, except when $H = U_3(GF(4))$ (cf. [3, p. 49]), so the result follows. ∎

The following result is evident from the definition of a semi-transvection.

PROPOSITION 2.6. *If Z is a non-isotropic subspace of V and if $\rho \in P_{\mathbf{x},\mathbf{w}}$ where $\mathbf{x} \in Z$, then ρ stabilizes Z if and only if $\mathbf{w} \in Z$.*

Remark 2.7. If $\mathbf{x}$ is a non-zero singular vector and if $\mathbf{w} \in \langle \mathbf{x} \rangle'$, then it can be shown that for $\rho \in P_{\mathbf{x},\mathbf{w}}$ and $\mathbf{v} \in V$, we can write

$$\rho(\mathbf{v}) = \mathbf{v} + [\beta \cdot (\mathbf{x}, \mathbf{v}) + (\mathbf{w}, \mathbf{v})]\, \mathbf{x} - (\mathbf{x}, \mathbf{v})\, \mathbf{w},$$

where $\beta = -Q(\mathbf{w})$ when $H = O_n(K)$ and $\beta + \bar{\beta} = -C(\mathbf{w}, \mathbf{w})$ when $H = U_n(K)$.

For the second part of Section 2 we shall assume that H is one of $O_n(K)$, $U_n(K)$.

PROPOSITION 2.8. *A non-isotropic subspace of V that has a singular 1-dimensional subspace has a base of singular vectors.*

Proof. See [3, pp. 21 and 34]. ∎

PROPOSITION 2.9. *Any subspace of V that contains a non-singular vector has a base of non-singular vectors, except when $H = O_n(GF(2))$.*

Proof. Suppose that the proposition is false and let $Z \subseteq V$ be a counterexample; we show that a contradiction results. Let $\mathbf{z}$ be a non-singular vector in Z and let $\mathbf{w}$ be a non-zero vector in Z that cannot be expressed as the sum of one or more non-singular vectors in Z, then $\mathbf{w}$ must be singular and $\mathbf{z} + \lambda\mathbf{w}$ must be singular for every $\lambda \in K\backslash\{0\}$.

If $H = O_n(K)$, then

$$\begin{aligned} 0 &= Q(\mathbf{z} + \lambda\mathbf{w}) \\ &= Q(\mathbf{z}) + \lambda B(\mathbf{z}, \mathbf{w}) \qquad \text{for every } \lambda \in K\backslash\{0\}. \end{aligned}$$

As $H \neq O_n(GF(2))$, we conclude that $Q(\mathbf{z}) = 0$. But we chose $\mathbf{z}$ to be non-singular, so we have arrived at a contradiction.

If $H = U_n(K)$, then

$$\begin{aligned} 0 &= C(\mathbf{z} + \lambda\mathbf{w}, \mathbf{z} + \lambda\mathbf{w}) \\ &= C(\mathbf{z}, \mathbf{z}) + \lambda C(\mathbf{z}, \mathbf{w}) + \bar{\lambda}\overline{C(\mathbf{z}, \mathbf{w})} \qquad \text{for every } \lambda \in K\backslash\{0\}. \end{aligned}$$

Thus the non-singularity of $\mathbf{z}$ implies that $C(\mathbf{z}, \mathbf{w}) \neq 0$. But if $\lambda = -C(\mathbf{z}, \mathbf{z})/C(\mathbf{z}, \mathbf{w})$, then $\lambda \in K\backslash\{0\}$ and $C(\mathbf{z} + \lambda\mathbf{w}, \mathbf{z} + \lambda\mathbf{w})$ $(= -C(\mathbf{z}, \mathbf{z}))$ is non-zero, giving a contradiction. The proposition is therefore proved. ∎

Remark. Suppose that $H = O_n(GF(2))$. A hyperbolic 2-dimensional subspace W of V has three 1-dimensional subspaces, only one of which is non-singular; moreover, if W_0 is a totally singular subspace of W', then $W + W_0$ contains a non-singular vector, but fails to have a base of non-singular vectors. If Z is a non-isotropic subspace, then the standard canonical form of Q (restricted to Z) indicates a base of non-singular vectors for Z, unless Z is a hyperbolic 2-dimensional subspace.

This remark, together with Proposition 2.9, yields the following result.

COROLLARY TO PROPOSITION 2.9. *A non-isotropic subspace Z of V has a base of non-singular vectors unless $H = O_n(GF(2))$ and Z is a hyperbolic 2-dimensional subspace.*

PROPOSITION 2.10. *If Z is a non-isotropic subspace of V of dimension $m \geqslant 2$ and if $\mathbf{z}$ is a non-singular vector in Z, then Z has a base of vectors* *isomorphic to* $\mathbf{z}$, *except when* $H = O_n(GF(2))$ *or* $O_n(GF(3))$, *and* Z *is a hyperbolic* 2-*dimensional subspace.*

Proof. If H is one of $O_n(GF(2))$, $U_n(GF(4))$, then V has only one isomorphism class of non-singular vectors, so the result follows immediately from the corollary above. We suppose now that $H \neq O_n(GF(2))$ or $U_n(GF(4))$.

Suppose that the proposition is false; we show that a contradiction results. Let Z_0 be the subspace of Z spanned by the vectors isomorphic to $\mathbf{z}$, then $\mathbf{z} \in Z_0 \subsetneqq Z$, and any isomorphism of Z fixes Z_0; in particular, any symmetry or quasi-symmetry centered on a vector in Z stabilizes Z_0. Thus by Remark 2.2, any non-singular vector in Z lies in Z_0 or $Z_0' \cap Z$. By the corollary above, Z has a base of non-singular vectors, so $Z = Z_0 + (Z_0' \cap Z)$; consideration of dimensions show that this sum is direct and hence that Z_0 and $Z_0' \cap Z$ are non-isotropic. Moreover, for any non-singular vectors $\mathbf{u} \in Z_0$, $\mathbf{v} \in Z_0' \cap Z$, the (non-zero) vector $\mathbf{u} + \mathbf{v}$ must be singular. Thus if $H = O_n(K)$, then $Q(\mathbf{v}) = -Q(\mathbf{u})$, and if $H = U_n(K)$, then $C(\mathbf{v}, \mathbf{v}) = -C(\mathbf{u}, \mathbf{u})$. It follows that each of Z_0 and $Z_0' \cap Z$ have one isomorphism class of non-singular vectors, and in particular that $\lambda\mathbf{z}$ is isomorphic to $\mathbf{z}$ for each $\lambda \in K \backslash \{0\}$.

If $H = O_n(K)$ with $K \neq GF(3)$, then there exists $\lambda \in K \backslash \{0\}$ such that $\lambda^2 \neq 1$, i.e., such that $\lambda\mathbf{z}$ is not isomorphic to $\mathbf{z}$, giving a contradiction as required. If $H = U_n(K)$, then there exists $\lambda \in K \backslash \{0\}$ such that $\lambda \cdot \bar{\lambda} \neq 1$ (if K has characteristic 2, then we can take $\lambda \in K \backslash \{0, 1\}$; otherwise we can choose $\mu \in K \backslash \{0\}$ such that $\bar{\mu} = -\mu$ and take λ to be one of μ, $\mu + 1$), i.e., such that $\lambda\mathbf{z}$ is not isomorphic to $\mathbf{z}$, giving a contradiction.

We have one case left to consider, when $H = O_n(GF(3))$. We have already established that Z contains a (non-zero) singular vector, and we have excluded the case where Z is hyperbolic of dimension 2, so Z must have dimension $\geqslant 3$. Thus one of Z_0, $Z_0' \cap Z$ has dimension $\geqslant 2$. However, consider a non-isotropic subspace W of dimension $\geqslant 2$ having only one isomorphism class of non-singular vectors. Let $\mathbf{w}$ be a non-singular vector in W and let $\mathbf{w}^*$ be a non-singular vector in $\langle \mathbf{w} \rangle' \cap W$, then $\mathbf{w}^*$ is isomorphic to $\mathbf{w}$, but $Q(\mathbf{w} + \mathbf{w}^*) = -Q(\mathbf{w})$; so $\mathbf{w} + \mathbf{w}^*$ is not isomorphic to $\mathbf{w}$, giving a contradiction. Hence we have the required contradiction, even when $H = O_n(GF(3))$. ∎

Remark. Note that in the last part of the proof of Proposition 2.10, we have actually shown that if $H = O_n(GF(3))$, then every non-isotropic subspace of dimension $\geqslant 2$ contains elements of each isomorphism class of non-singular vectors in V. In particular, a hyperbolic 2-dimensional subspace has two non-singular 1-dimensional subspaces; they are orthogonal but not isomorphic.

PROPOSITION 2.11. *If Z is a non-isotropic 2-dimensional subspace of V and if $\mathbf{z} \in Z$ is non-singular, then $Z \backslash (\langle \mathbf{z} \rangle \cup \langle \mathbf{z} \rangle' \cap Z)$ contains a vector $\mathbf{w}$ isomorphic to $\mathbf{z}$, except in the following cases:* (a) $H = O_n(GF(3))$; (b) $H = U_n(GF(4))$; *and* (c) *Z is hyperbolic and $H = O_n(GF(2))$ or $O_n(GF(5))$.*

Proof. We suppose the proposition to be false and arrive at a contradiction. By Proposition 2.10, there is a vector $\mathbf{v} \in Z$ isomorphic to $\mathbf{z}$ such that $\{\mathbf{z}, \mathbf{v}\}$ is a base for Z; by our supposition, $\mathbf{v} \in \langle \mathbf{z} \rangle'$. Since Z is non-isotropic, the characteristic of K must be other than two when $H = O_n(K)$. We may assume that $H \neq O_n(GF(5))$, because otherwise $\mathbf{z} + 2\mathbf{v}$ is singular, i.e., Z is hyperbolic, an excepted case.

If $H = O_n(K)$, then $K \neq GF(2)$, $GF(3)$ or $GF(5)$, so there exists $\lambda \in K$ such that $\lambda^2 \notin \{0, 1, -1\}$. Let

$$\mathbf{w} = (\lambda^2 - 1)\,\mathbf{z}/(\lambda^2 + 1) + 2\lambda \mathbf{v}/(\lambda^2 + 1),$$

then $\mathbf{w}$ is isomorphic to $\mathbf{z}$, but does not lie in $\langle \mathbf{z} \rangle \cup \langle \mathbf{z} \rangle'$, giving a contradiction.

If $H = U_n(K)$, then $K \neq GF(4)$; so there exists $\lambda \in K \backslash \{0, 1, -1\}$ such that $\bar{\lambda} = -\lambda$. Let

$$\mathbf{w} = \mathbf{z}/(1 + \lambda) + \lambda \mathbf{v}/(1 + \lambda),$$

then $\mathbf{w}$ is isomorphic to $\mathbf{z}$, but does not lie in $\langle \mathbf{z} \rangle \cup \langle \mathbf{z} \rangle'$, giving a contradiction, and thereby completing the proof of the proposition. ∎

PROPOSITION 2.12. *Any complement of a totally isotropic subspace of V in its conjugate is non-isotropic.*

Proof. Let W be a totally isotropic subspace of V and let X be a complement of W in W'; then the following are equivalent expressions for $X \cap X'$: $(W' \cap X) \cap X'$; $X \cap (W' \cap X')$; $X \cap (W + X)'$; $X \cap (W')'$; and $X \cap W$, but $X \cap W = \{0\}$ so $X \cap X' = \{0\}$. Thus X is non-isotropic. ∎

PROPOSITION 2.13. *If $H = O_n(GF(2))$, if Z is a non-isotropic subspace of V of dimension $m \geqslant 4$ and if $\mathbf{z} \in Z$ is non-singular, then $Z \cap \langle \mathbf{z} \rangle'$ has a base of non-singular vectors. If $m = 4$, then $\langle \mathbf{z} \rangle$ has a non-hyperbolic 2-dimensional complement in $Z \cap \langle \mathbf{z} \rangle'$.*

Proof. If $m \geqslant 6$, then any base of non-singular vectors for a complement of $\langle \mathbf{z} \rangle$ in $\langle \mathbf{z} \rangle' \cap Z$ (see Proposition 2.10), together with $\mathbf{z}$ forms a base for $Z \cap \langle \mathbf{z} \rangle'$.

Suppose that $m = 4$ and let X be a complement (necessarily 2-dimensional) of $\langle \mathbf{z} \rangle$ in $Z \cap \langle \mathbf{z} \rangle'$. If X is hyperbolic, then it contains non-zero singular vectors $\mathbf{x}$ and $\mathbf{y}$ such that $X = \{0, \mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}\}$. The subspace

$\{0, \mathbf{x}+\mathbf{y}, \mathbf{y}+\mathbf{z}, \mathbf{x}+\mathbf{z}\}$ is then a non-hyperbolic 2-dimensional complement of $\langle \mathbf{z} \rangle$ in $Z \cap \langle \mathbf{z} \rangle'$. Thus $\langle \mathbf{z} \rangle$ has a non-hyperbolic 2-dimensional complement in $Z \cap \langle \mathbf{z} \rangle'$ and a base of non-singular vectors for $Z \cap \langle \mathbf{z} \rangle'$ may be constructed as above. ∎

For the third part of Section 2, K will be a field with a non-trivial involutory automorphism J (implying $|K| \geqslant 4$) whose fixed subfield is K_0.

PROPOSITION 2.14. *If $\lambda \in K$ and if $\lambda \cdot \bar{\lambda} = 1$, then there exists $\mu \in K$ such that $\mu \cdot \bar{\mu}^{-1} = \lambda$.*

Proof. This follows from Hilbert's "Theorem 90" (cf. [5]) and the fact that K is a normal separable extension of K_0 of degree 2. ∎

PROPOSITION 2.15. *There exists $\lambda \in K$ such that $\lambda \cdot \bar{\lambda} = 1$ and $\lambda^n \neq 1$ in the following cases*: (a) *$n = 3$ and $K \neq GF(4)$*; (b) *n is a positive integer and K is infinite of characteristic two*; *and* (c) *$n = 4$ and K is finite of characteristic two.*

Proof. If the characteristic of K is other than two, then we can take $\lambda = -1$ for part (a).

Suppose that K has characteristic two and that $n < |K_0|$. Let $\beta \in K \backslash K_0$, let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ be distinct elements of K_0 and let $\gamma_i = (\alpha_i \bar{\beta} + \beta)/(\alpha_i \beta + \bar{\beta})$ for $i = 1, 2, \ldots, n+1$, then the γ_i's are distinct and $\gamma_i \cdot \bar{\gamma}_i = 1$ for each i. Since K has at most n nth roots of 1, it follows that one of the γ_i's is not an nth root. This proves (b) and completes the proof of (a).

If K is finite of characteristic two, then the multiplicative group of K has odd order (>1). Thus if $\beta \in K \backslash K_0$ and if $\lambda = \beta/\bar{\beta}$, then $\lambda \cdot \bar{\lambda} = 1$ and $\lambda^4 \neq 1$, proving (c). ∎

COROLLARY TO PROPOSITION 2.15. *There exists $\mu \in K$ such that $\bar{\mu}^2/\mu \in K \backslash K_0$ except when $K = GF(4)$.*

Proof. By Propositions 2.15(a) and 2.14, there exists $\mu \in K \backslash \{0\}$ such that $(\mu \cdot \bar{\mu}^{-1})^3 \neq 1$. It follows that $\bar{\mu}^2/\mu \neq \overline{(\bar{\mu}^2/\mu)}$, i.e., that $(\bar{\mu}^2/\mu) \in K \backslash K_0$. ∎

3. THE SYMPLECTIC GROUP

Let $H = Sp_n(K)$. As U is non-isotropic, r must be even; $\nu_1 = r/2$ and $\nu_2 = (n-r)/2$, so $r \leqslant n - r$. Let $F \leqslant H$ such that $E < F$. We show that F contains every transvection in H, whence $F = H$ and G is maximal in H.

PROPOSITION 3.1. *F acts transitively on the non-zero vectors of V.*

Proof. Let $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ be the sets of non-zero vectors of U, U' and $V\backslash(U \cup U')$, respectively; then any element of $\mathscr{C}_3$ can be written as the sum of an element of $\mathscr{C}_1$ and an element of $\mathscr{C}_2$. Hence, by Proposition 2.1, $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ are orbits of G.

Let $f \in F\backslash E$, then $fU' \nsubseteq U, U'$, and so there exist non-zero vectors $\mathbf{u}$, $\mathbf{v} \in U'$ such that $f(\mathbf{u}) \notin U'$ and $f(\mathbf{v}) \notin U$, i.e., $\mathbf{u}$, $\mathbf{v} \in \mathscr{C}_2$, $f(\mathbf{u}) \notin \mathscr{C}_2$ and $f(\mathbf{v}) \notin \mathscr{C}_1$. We have three possibilities: (a) $f(\mathbf{u}) \in \mathscr{C}_3$; (b) $f(\mathbf{v}) \in \mathscr{C}_3$; and (c) $f(\mathbf{u}) \in \mathscr{C}_1$, $f(\mathbf{v}) \in \mathscr{C}_2$ in which case $\mathbf{u}+\mathbf{v} \in \mathscr{C}_2$, but $f(\mathbf{u}+\mathbf{v}) \in \mathscr{C}_3$. In each case $\mathscr{C}_2$ and $\mathscr{C}_3$ lie in the same orbit of F. Since $G < F$, it follows that $\mathscr{C}_1$ is not an orbit of F, so F can have only one orbit. ∎

THEOREM 3.2. *E is maximal in H.*

Proof. Let t be any transvection in H, centered on a vector $\mathbf{w}$ say. By Proposition 3.1, there exists $f \in F$ such that $f(\mathbf{w}) \in U$. The element ftf^{-1} is a transvection centered on a vector in U which (by Remark 2.2) therefore lies in G. Hence $t \in F$ and so F contains every transvection in H. It is known that H is generated by its transvections; so $F = H$ and E is maximal in H. ∎

Remark. Let $N = GSp_n(K)$, let $L = \mathrm{Stab}_N\{U, U'\}$ and let $M \leqslant N$ such that $L < M$. A similar argument to that used above would show that M contains H. Let k be any element of N, then since H acts transitively on the non-isotropic subspaces of V (by Witt's theorem), there exists $h \in H$ such that $hkU = U$, i.e., such that $hk \in L$. But this implies that $k \in M$, so $M = N$ and therefore L is maximal in N.

4. THE ORTHOGONAL AND UNITARY GROUPS

Let H be one of $O_n(K)$, $U_n(K)$, and let $F \leqslant H$ and $F_1 \leqslant H_1$ such that $E < F$ and $E_1 < F_1$. We show that F_1 contains every semi-transvection in H_1, and deduce that $F_1 = H_1$, whence E_1 is maximal in H_1. We then deduce that $F = H$, whence E is maximal in H.

Our first objective is to show that there exists $f \in F_1\backslash G_1$ such that $U' \cap fU'$ has a singular 1-dimensional subspace; we refer to this property as condition IV. Our approach is to suppose that condition IV is not satisfied and to reach a contradiction to this supposition. It will simplify our notation if we define the following:

Condition I. There exists $f_1 \in F_1\backslash G_1$ and a singular vector $\mathbf{z} \in U'$ such that if we write $f_1(\mathbf{z}) = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1 \in U$ and $\mathbf{z}_2 \in U'$, then $\mathbf{z}_1$ and $\mathbf{z}_2$ are non-singular.

Condition II(a). There exists $f_2 \in F_1\backslash G_1$ and a singular vector $\mathbf{y} \in U'$

such that if we write $f_2(\mathbf{y}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ are non-singular and $f_2 U \not\subseteq U'$.

Condition II(b). There exists $f_2 \in F_1 \backslash G_1$ and a singular vector $\mathbf{y} \in U'$ such that if we write $f_2(\mathbf{y}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ are non-singular and $f_2 U' \neq \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$.

Condition III. There exists $f_3 \in F_1 \backslash G_1$ and a singular vector $\mathbf{x} \in U'$ such that if we write $f_3(\mathbf{x}) = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 \in U$ and $\mathbf{x}_2 \in U'$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are non-singular, and $(U \cap \langle \mathbf{x}_1 \rangle') \cup (U' \cap \langle \mathbf{x}_2 \rangle')$ contains a non-singular vector that does not lie in $f_3 U \cup f_3 U'$.

We first consider the action of G_1 and F_1 on the singular 1-dimensional subspaces of V.

PROPOSITION 4.1. *If $v_1 > 0$, then there exist singular vectors $\mathbf{a}_1, \mathbf{b}_1 \in U$, $\mathbf{a}_2, \mathbf{b}_2 \in U'$ and non-singular vectors $\mathbf{c}_1 \in U$, $\mathbf{c}_2 \in U'$ such that $\mathbf{a}_1 + \mathbf{b}_1$ and $\mathbf{a}_2 + \mathbf{b}_2$ are non-singular, but $\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{a}_2 + \mathbf{b}_2$, $\mathbf{c}_1 + \mathbf{c}_2$, $\mathbf{c}_1 + \mathbf{a}_1 + \mathbf{b}_1$ and $\mathbf{c}_2 + \mathbf{a}_2 + \mathbf{b}_2$ are non-zero and singular.*

Proof. Let P_1 and P_2 be hyperbolic 2-dimensional subspaces of U and U', respectively, and let θ be an isomorphism: $P_1 \to P_2$. We consider separately the cases: $H = U_n(K)$, or $H = O_n(K)$ and K does not have characteristic two; $H = O_n(K)$ and K has characteristic two, but $K \neq GF(2)$; and $H = O_n(GF(2))$.

Suppose that $H = U_n(K)$, or that $H = O_n(K)$ and K has characteristic other than two. Let $\mathbf{d}_1$ be a non-singular vector in P_1, then $\langle \mathbf{d}_1 \rangle$ is non-isotropic, so $\langle \mathbf{d}_1 \rangle' \cap P_1$ contains a non-singular vector, $\mathbf{c}_1$ say, such that $\mathbf{c}_1 + \mathbf{d}_1$ is singular (and necessarily non-zero). Let $\mathbf{d}_2 = \theta(\mathbf{c}_1)$, $\mathbf{c}_2 = \theta(d_1)$, then $\mathbf{c}_2 + \mathbf{d}_2$ is singular, and as P_1 and P_2 are orthogonal, $\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{d}_1 + \mathbf{d}_2$ are singular. Let $\{\mathbf{a}_1, \mathbf{b}_1\}$ and $\{\mathbf{a}_2, \mathbf{b}_2\}$ be bases of singular vectors for P_1 and P_2, respectively (cf. Proposition 2.8). Replacing $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{a}_2$ and $\mathbf{b}_2$ by scalar multiples if necessary, we may assume that $\mathbf{a}_1 + \mathbf{b}_1 = \mathbf{d}_1$ and $\mathbf{a}_2 + \mathbf{b}_2 = \mathbf{d}_2$. The vectors $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{a}_2, \mathbf{b}_2$ and $\mathbf{c}_2$ now have the required properties.

Suppose that $H = O_n(K)$ and that K has characteristic two, but that $K \neq GF(2)$. By the corollary to Proposition 2.9, P_1 has a base of non-singular vectors $\{\mathbf{d}_1, \mathbf{c}_1\}$. Replacing $\mathbf{c}_1$ by a scalar multiple if necessary, we may assume that $\mathbf{c}_1 + \mathbf{d}_1$ is singular. Let $\mathbf{c}_2 = \theta(\mathbf{c}_1)$, $\mathbf{d}_2 = \theta(\mathbf{d}_1)$, then $\mathbf{c}_2 + \mathbf{d}_2$, $\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{d}_1 + \mathbf{d}_2$ are singular. As above, there are singular vectors $\mathbf{a}_1$, $\mathbf{b}_1 \in P_1$ and $\mathbf{a}_2, \mathbf{b}_2 \in P_2$ such that $\mathbf{a}_1 + \mathbf{b}_1 = \mathbf{d}_1$ and $\mathbf{a}_2 + \mathbf{b}_2 = \mathbf{d}_2$. The vectors $\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{a}_2, \mathbf{b}_2$ and $\mathbf{c}_2$ have the required properties.

Suppose that $H = O_n(GF(2))$. Since $v_1 > 0$ and since we have excepted the case $r = 2$ and $v_1 = 1$, it follows that $n - r \geqslant r \geqslant 4$. Thus there are non-singular vectors $\mathbf{c}_1 \in P_1' \cap U$ and $\mathbf{c}_2 \in P_2' \cap U'$. For $i = 1, 2$, P_i contains a non-singular vector $\mathbf{d}_i$ and two non-zero singular vectors $\mathbf{a}_i$, $\mathbf{b}_i$ whose sum is

$\mathbf{d}_i$. The vectors $\mathbf{c}_1+\mathbf{d}_1$, $\mathbf{c}_2+\mathbf{d}_2$, $\mathbf{c}_1+\mathbf{c}_2$ and $\mathbf{d}_1+\mathbf{d}_2$ are singular, so $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, $\mathbf{a}_2$, $\mathbf{b}_2$ and $\mathbf{c}_2$ have the required properties. ∎

PROPOSITION 4.2. *Suppose that* $\nu_1>0$. *Let* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *be the sets of non-zero singular vectors of* U *and* U', *respectively, and let* $\mathscr{C}_3=\{\mathbf{u}+\mathbf{v}:\mathbf{u}\in\mathscr{C}_1, \mathbf{v}\in\mathscr{C}_2\}$, *then* G_1 *acts transitively on each of* $\mathscr{C}_1$, $\mathscr{C}_2$ *and* $\mathscr{C}_3$.

Proof. Clearly G and therefore G_1 acts on each of $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$, and by Proposition 2.1 the action of G is transitive in each case. As $\nu_l>0$, it follows that $n-r\geqslant r\geqslant 2$; we have excluded the case $n-r=r=2$ and $\nu_1=\nu_2=1$, so $n-r\geqslant 3$. For $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ to be orbits of G_1, we need only show that given $\mathbf{w}\in\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$, each coset of G_1 in G contains an element fixing $\mathbf{w}$. But $\langle\mathbf{w}\rangle'\cap U'$ has dimension $\geqslant n-r-1>(n-r)/2$ and so cannot be totally singular, i.e., $\langle\mathbf{w}\rangle'\cap U'$ contains a non-singular vector $\mathbf{z}$. By Remark 2.2, each coset of H_1 in H (other than H_1 itself) contains a quasi-symmetry centered on $\mathbf{z}$; such an element lies in G and fixes $\mathbf{w}$. As $G_1=H_1\cap G$, it follows that each coset of G_1 in G contains an element fixing $\mathbf{w}$, as required. Hence G_1 acts transitively on each of $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$. ∎

Remark. Notice that $\mathscr{C}_2$ is still an orbit of G_1 if $\nu_1=0$. To adapt the proof of Proposition 4.2, we would need the non-singular vector $\mathbf{z}$ to lie in U.

PROPOSITION 4.3. *If Condition* IV *is not satisfied, then Condition* I *is satisfied.*

Proof. First suppose that $\nu_1=0$. By Proposition 2.8, U' has a base of singular vectors, so if $f_1\in F_1\backslash G_1$, then there exists a singular vector $\mathbf{z}\in U'$ such that $f_1(\mathbf{z})\notin U'$. If we write $f_1(\mathbf{z})=\mathbf{z}_1+\mathbf{z}_2$, where $\mathbf{z}_1\in U$ and $\mathbf{z}_2\in U'$, then $\mathbf{z}_1$ must be non-zero and therefore non-singular. Thus $\mathbf{z}_2$ is non-singular and Condition I is satisfied.

Suppose now that $\nu_1>0$, then $r\leqslant n-r$. Let $h\in F_1\backslash E_1$, then $hU'\nsubseteq U$, so there exists a singular vector $\mathbf{v}\in U'$ such that $h(\mathbf{v})\notin U$. Since Condition IV is not satisfied, $h(\mathbf{v})\notin U'$, so if we write $h(\mathbf{v})=\mathbf{v}_1+\mathbf{v}_2$, where $\mathbf{v}_1\in U$ and $\mathbf{v}_2\in U'$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-zero and are either both non-singular or both singular. If they are non-singular, then Condition I is satisfied.

Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both singular. Then in the notation of Proposition 4.2, $\mathbf{v}\in\mathscr{C}_2$ and $h(\mathbf{v})\in\mathscr{C}_3$. Thus $\mathscr{C}_2$ and $\mathscr{C}_3$ lie in the same F_1-orbit; Condition I will be satisfied if this orbit does not lie inside $\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$. Let $\mathbf{a}_1$, $\mathbf{b}_1$, $\mathbf{c}_1$, $\mathbf{a}_2$, $\mathbf{b}_2$ and $\mathbf{c}_2$ be as in Proposition 4.1; then $\mathbf{a}_1+\mathbf{a}_2$, $\mathbf{b}_1+\mathbf{b}_2$ and $\mathbf{a}_1+\mathbf{b}_1+\mathbf{c}_1+\mathbf{a}_2+\mathbf{b}_2+\mathbf{c}_2$ lie in $\mathscr{C}_3$, but $\mathbf{c}_1+\mathbf{c}_2$ and $\mathbf{a}_1+\mathbf{b}_1+\mathbf{a}_2+\mathbf{b}_2$ lie outside $\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$.

Let us suppose that Condition I is not satisfied. Let $k\in F_1$ such that $k(\mathbf{a}_1+\mathbf{a}_2)\in\mathscr{C}_2$; then as the F_1-orbit containing $\mathscr{C}_2$ and $\mathscr{C}_3$ lies inside $\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$, it follows that $k(\mathbf{b}_1+\mathbf{b}_2)\in\mathscr{C}_1\cup\mathscr{C}_2\cup\mathscr{C}_3$ and $k(\mathbf{a}_1+\mathbf{a}_2)+$

$k(\mathbf{b}_1+\mathbf{b}_2)=k(\mathbf{a}_1+\mathbf{b}_1+\mathbf{a}_2+\mathbf{b}_2)\notin\mathscr{C}_2\cup\mathscr{C}_3$. But $k(\mathbf{b}_1+\mathbf{b}_2)\in\mathscr{C}_1\cup\mathscr{C}_2$ leads to a contradiction of the latter statement, so $k(\mathbf{b}_1+\mathbf{b}_2)\in\mathscr{C}_3$, implying that $k(\mathbf{a}_1+\mathbf{a}_2)+k(\mathbf{b}_1+\mathbf{b}_2)\in\mathscr{C}_1$. Thus the F_1-orbit containing $\mathscr{C}_2$ and $\mathscr{C}_3$ cannot contain $\mathscr{C}_1$, i.e., $\mathscr{C}_2\cup\mathscr{C}_3$ is an orbit of F_1 and in particular $k(\mathbf{a}_1+\mathbf{b}_1+\mathbf{c}_1+\mathbf{a}_2+\mathbf{b}_2+\mathbf{c}_2)\in\mathscr{C}_2\cup\mathscr{C}_3$. We can write

$$k(\mathbf{c}_1+\mathbf{c}_2)=-k(\mathbf{a}_1+\mathbf{a}_2+\mathbf{b}_1+\mathbf{b}_2)+k(\mathbf{a}_1+\mathbf{b}_1+\mathbf{c}_1+\mathbf{a}_2+\mathbf{b}_2+\mathbf{c}_2)$$

which must lie in $\mathscr{C}_2\cup\mathscr{C}_3$, so $\mathbf{c}_1+\mathbf{c}_2\in\mathscr{C}_2\cup\mathscr{C}_3$, a contradiction. Hence Condition I is satisfied. ∎

PROPOSITION 4.4. *Suppose that $n-r=r+1$. If Condition* IV *is not satisfied, then Condition* II(a) *is satisfied.*

Proof. As n is odd, K does not have characteristic two when $H=O_n(K)$. By Proposition 4.3, there exists $f_1\in F_1\backslash G_1$ and a singular vector $\mathbf{z}\in U'$ such that if we write $f_1(\mathbf{z})=\mathbf{z}_1+\mathbf{z}_2$, where $\mathbf{z}_1\in U$ and $\mathbf{z}_2\in U'$, then $\mathbf{z}_1$ and $\mathbf{z}_2$ are non-singular. Condition II(a) is satisfied unless $f_1U\subseteq U'$.

Suppose that $f_1U\subseteq U'$, or equivalently $U\subseteq f_1U'$. Thus $\mathbf{z}_1\in f_1U'$; as $\mathbf{z}_1+\mathbf{z}_2\in f_1U'$, it follows that $\mathbf{z}_2\in f_1U'$ and therefore $f_1U'=\langle\mathbf{z}_2\rangle\oplus U$ (by consideration of dimensions). By Proposition 2.10, there is a base for U' of vectors isomorphic to $\mathbf{z}_2$; let $\mathbf{v}$ be an element of that base not lying in $\langle\mathbf{z}_2\rangle$. By Proposition 2.1, there exists $g\in G$ such that $g(\mathbf{z}_2)=\mathbf{v}$; we may assume that $g\in G_1$, because otherwise we could replace g by g_0g where $g_0\in G\cap g^{-1}H_1$ is a quasi-symmetry centered on a non-singular vector in $\langle\mathbf{v}\rangle'\cap U'$ (cf. Remark 2.2). Clearly $gf_1U'\neq f_1U'$, so $f_1^{-1}gf_1\in F_1\backslash G_1$ and $f_1^{-1}gf_1U'\cap U'=f_1^{-1}U$. As Condition IV is not satisfied it follows that $\nu_1=0$.

Let $f_2=f_1^{-1}gf_1$, then $f_2(\mathbf{z})\notin U'$. If we write $f_2(\mathbf{z})=\mathbf{y}_1+\mathbf{y}_2$, where $\mathbf{y}_1\in U$ and $\mathbf{y}_2\in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ must be non-singular. Since $f_2U'\cap U'$ has dimension r, Condition II(a) is satisfied unless $r=1$.

Suppose that $r=1$, then $n-r=2$. By Proposition 2.11, we may assume that the vector $\mathbf{v}\in U'$ (as above) is isomorphic to $\mathbf{z}_2$, but does not lie in $\langle\mathbf{z}_2\rangle\cup\langle\mathbf{z}_2\rangle'$. Thus $gf_1U\,(=\langle\mathbf{v}\rangle'\cap U')$ cannot lie in f_1U' $(=\langle\mathbf{z}_2\rangle\oplus U)$. We conclude that $f_2U\nsubseteq U'$ and so Condition II(a) is satisfied. ∎

PROPOSITION 4.5. *Suppose that $n-r=2\leqslant r$ and that K does not have characteristic two when $H=O_n(K)$. If Condition* IV *is not satisfied, then Condition* II(b) *is satisfied.*

Proof. We note that $H\neq O_n(GF(3))$ or $U_n(GF(4))$ and that if $H=O_n(GF(5))$, then $r=2$ and $\nu_1=0$.

By Proposition 4.3, there exists $f_1\in F_1\backslash G_1$ and a singular vector $\mathbf{z}\in U'$ such that if we write $f_1(\mathbf{z})=\mathbf{z}_1+\mathbf{z}_2$, where $\mathbf{z}_1\in U$ and $\mathbf{z}_2\in U'$, then $\mathbf{z}_1$ and $\mathbf{z}_2$ are non-singular. Condition II(b) is satisfied unless $f_1U'=\langle\mathbf{z}_1,\mathbf{z}_2\rangle$.

Suppose that $f_1 U' = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. Let $\mathbf{v} \in U$ be isomorphic to $\mathbf{z}_1$ but not contained in $\langle \mathbf{z}_1 \rangle \cup \langle \mathbf{z}_1 \rangle'$, constructed by applying Proposition 2.11 to a non-isotropic subspace of U containing $\mathbf{z}_1$, then we can write $\mathbf{v} = \alpha \mathbf{z}_1 + \beta \mathbf{u}$ for some non-singular $\mathbf{u} \in U \cap \langle \mathbf{z}_1 \rangle' \subseteq f_1 U$ and some $\alpha, \beta \in K \backslash \{0\}$. By Proposition 2.1, there is an element $g \in G$ that fixes $\mathbf{z}_2$ and takes $\mathbf{z}_1$ to $\mathbf{v}$; any quasi-symmetry centered on a non-singular vector in $U' \cap \langle \mathbf{z}_2 \rangle'$ fixes $\mathbf{z}_2$ and $\mathbf{v}$, and lies in G, so $G \cap g^{-1} H_1$ contains an element fixing $\mathbf{z}_2$ and $\mathbf{v}$ (cf. Remark 2.2), and we may therefore assume that $g \in G_1$. Let $f_2 = f_1^{-1} g f_1$, then

$$f_2(\mathbf{z}) = f_1^{-1}(\mathbf{z}_2) + \alpha f_1^{-1}(\mathbf{z}_1) + \beta f_1^{-1}(\mathbf{u}),$$

with $f_1^{-1}(\mathbf{z}_2)$, $f_1^{-1}(\mathbf{z}_1) \in U'$ and $f_1^{-1}(\mathbf{u}) \in U$, so $f_2 \in F_1 \backslash G_1$. We know that $f_1^{-1}(\mathbf{z}_2) = f_2(f_1^{-1}(\mathbf{z}_2)) \in f_2 U'$ and that $f_1^{-1}(\alpha \mathbf{z}_1 + \beta \mathbf{u}) = f_2(f_1^{-1}(\mathbf{z}_1)) \in f_2 U'$ so $f_1^{-1}(\mathbf{u}) \notin f_2 U'$. Hence if we write $f_2(\mathbf{z}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ are non-singular and $f_2 U' \neq \langle \mathbf{y}_1, \mathbf{y}_2 \rangle$. Thus Condition II(b) is satisfied. ∎

PROPOSITON 4.6. *Suppose that $H \neq U_n(GF(4))$. If Condition* IV *is not satisfied, then Condition* III *is satisfied.*

Proof. By Propositions 4.3, 4.4 and 4.5, there exists $f_2 \in F_1 \backslash G_1$ and a singular vector $\mathbf{y} \in U'$ such that if we write $f_2(\mathbf{y}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ are non-singular, $f_2 U \nsubseteq U'$ when $r + 1 = n - r$, and $f_2 U' \neq \langle y_1, y_2 \rangle$ when $n - r = 2 \leqslant r$. We give separate proofs for each of the cases: (i) $H \neq O_n(GF(3))$ and K does not have characteristic two when $H = O_n(K)$; (ii) $H = O_n(K)$ and K has characteristic two, but $K \neq GF(2)$; (iii) $H = O_n(GF(3))$; and (iv) $H = O_n(GF(2))$.

(i) Suppose that $H \neq O_n(GF(3))$ and that K does not have characteristic two when $H = O_n(K)$. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the sets of non-singular vectors of $\langle \mathbf{y}_1 \rangle' \cap U$ and $\langle \mathbf{y}_2 \rangle' \cap U'$, respectively. We show that $\mathscr{B}_1 \cup \mathscr{B}_2 \nsubseteq f_2 U \cup f_2 U'$.

Suppose that $\mathscr{B}_2 \subseteq f_2 U \cup f_2 U'$ and that $\mathscr{B}_2 \cap f_2 U$ and $\mathscr{B}_2 \cap f_2 U'$ are non-empty. For any $\mathbf{v}_1 \in \mathscr{B}_2 \cap f_2 U$ and any $\mathbf{v}_2 \in \mathscr{B}_2 \cap f_2 U'$, the vector $\mathbf{v}_1 + \mathbf{v}_2$ lies in $\langle \mathbf{y}_2 \rangle' \cap U'$ but not in $f_2 U \cup f_2 U'$, and must therefore be singular. Thus $\lambda \mathbf{v}_1$ must be isomorphic to $\mathbf{v}_1$ for every $\lambda \in K \backslash \{0\}$ and as in the proof of Proposition 2.10, this contradicts $H \neq O_n(GF(3))$. Hence $\mathscr{B}_2 \subseteq f_2 U'$ or $\mathscr{B}_2 \subseteq f_2 U$. If $\mathscr{B}_2 \subseteq f_2 U'$, then Proposition 2.9 implies that $\langle \mathbf{y}_2 \rangle' \cap U' \subseteq f_2 U'$. But $f_2 U'$ would then contain the isotropic $n - r$-dimensional subspace $\langle f_2(\mathbf{y}) \rangle + (\langle \mathbf{y}_2 \rangle' \cap U')$ which is impossible, so $\mathscr{B}_2 \subseteq f_2 U$. Proposition 2.9 implies that $\langle \mathbf{y}_2 \rangle' \cap U' \subseteq f_2 U$, whence $r \geqslant n - r - 1$. As $f_2 U \nsubseteq U'$ when $r + 1 = n - r$, it follows that $r \geqslant n - r \geqslant 2$.

Suppose that $\mathscr{B}_2 \subseteq f_2 U$. If $\mathscr{B}_1 \cap f_2 U$ and $\mathscr{B}_1 \cap f_2 U'$ are non-empty, then we arrive at a contradiction, as with $\mathscr{B}_2$. If $\mathscr{B}_1 \subseteq f_2 U$, then $\langle \mathbf{y}_1 \rangle' \cap U \subseteq f_2 U$

(by Proposition 2.9); so $f_2 U$ contains $(\langle \mathbf{y}_1\rangle' \cap U) + (\langle \mathbf{y}_2\rangle' \cap U')$. But then $f_2 U' = \langle \mathbf{y}_1, \mathbf{y}_2\rangle$, contrary to our choice of f_2. Thus $\mathscr{B}_1 \subseteq f_2 U'$, whence $\langle \mathbf{y}_1\rangle' \cap U \subseteq f_2 U'$. However, this implies that $f_2 U'$ contains the isotropic r-dimensional subspace $\langle f_2(\mathbf{y})\rangle + (\langle \mathbf{y}_1\rangle' \cap U)$, contrary to $r \geqslant n-r$. Hence if $\mathscr{B}_2 \subseteq f_2 U \cup f_2 U'$, then $\mathscr{B}_1 \nsubseteq f_2 U \cup f_2 U'$, so $\mathscr{B}_1 \cup \mathscr{B}_2 \nsubseteq f_2 U \cup f_2 U'$ and Condition III is satisfied with $\mathbf{x} = \mathbf{y}$ and $f_3 = f_2$.

(ii) Suppose that $H = O_n(K)$ and that K has characteristic two but that $K \neq GF(2)$. If $\{\mathbf{y}_1, \mathbf{y}_2\} \nsubseteq f_2 U \cup f_2 U'$, then Condition III is satisfied with $\mathbf{x} = \mathbf{y}$ and $f_3 = f_2$, because $\mathbf{y}_1, \mathbf{y}_2 \in (\langle \mathbf{y}_1\rangle' \cap U) \cup (\langle \mathbf{y}_2\rangle' \cap U')$. Suppose that $\mathbf{y}_1, \mathbf{y}_2 \in f_2 U \cup f_2 U'$, then $\mathbf{y}_1, \mathbf{y}_2 \in f_2 U'$ and $f_2 U'$ contains a totally isotropic 2-dimensional subspace, whence $n - r \geqslant 4$ and $\langle \mathbf{y}_2\rangle' \cap U'$ has dimension $\geqslant 3$. By Proposition 2.9, the subspace $\langle \mathbf{y}_2\rangle' \cap U'$ has a base of non-singular vectors. These cannot all lie in $f_2 U'$, because otherwise $f_2 U'$ would contain the isotropic $n-r$-dimensional subspace $\langle \mathbf{y}_1\rangle + (\langle \mathbf{y}_2\rangle' \cap U')$, which would be absurd. Let $\mathbf{v}$ be a base element not lying in $f_2 U'$. If $\mathbf{v} \notin f_2 U$, then Condition III is satisfied. If $\mathbf{v} \in f_2 U$, then for any $\lambda \in K \backslash \{0\}$ such that $\lambda^2 \neq Q(\mathbf{y}_2)/Q\mathbf{v}$), the vector $\mathbf{y}_2 + \lambda \mathbf{v}$ is non-singular and lies in $\langle \mathbf{y}_2\rangle'$ but does not lie in $f_2 U \cup f_2 U'$; so Condition III is satisfied, with $\mathbf{x} = \mathbf{y}$ and $f_3 = f_2$.

(iii) Suppose that $H = O_n(GF(3))$. Notice that there are two isomorphism classes of non-singular vectors (corresponding to the values $+1$ and -1 taken by Q) and two isomorphism classes of non-isotropic subspaces of any given dimension. As will be shown in the proof of Lemma 4.8, the failure of Condition IV to be satisfied implies that $\mathbf{y}_1$ and $\mathbf{y}_2$ lie in $f_2 U'$.

Suppose that $n - r \geqslant 4$, then $n - r \geqslant r$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-r-1}\}$ be a base for $\langle \mathbf{y}_2\rangle' \cap U'$ of vectors isomorphic to $\mathbf{y}_1$ (cf. Proposition 2.10 and the remark that follows it), then none of the $\mathbf{v}_i$'s can lie in $f_2 U'$ (otherwise Condition IV would be satisfied by $\mathbf{y}_2 + \mathbf{v}_i$ for some i). Condition III is then satisfied unless $\mathbf{v}_i \in f_2 U$ for each i. If $\mathbf{v}_i \in f_2 U$ for each i, then $\langle \mathbf{y}_2\rangle' \cap U' \subseteq f_2 U$ and $r = n - r$. Thus $f_2 U \cap U$ has dimension $\leqslant 1$, and so $\langle \mathbf{y}_1\rangle' \cap U$ has a base of non-singular vectors that cannot lie in $f_2 U$; at least one of these vectors must lie outside $f_2 U'$ (by consideration of dimensions) so that Condition III is satisfied.

If $n - r = 3$, then either $r = 1 = \nu$ or $r = 3$ and U is not isomorphic to U'.

Suppose that $n - r = 3$ and $r = 1 = \nu$. Then $\langle \mathbf{y}_2\rangle' \cap U'$ is non-hyperbolic and therefore has a base $\{\mathbf{v}_1, \mathbf{v}_2\}$ of vectors isomorphic to $\mathbf{y}_1$. As above $\mathbf{v}_1$, $\mathbf{v}_2 \notin f_2 U'$; at most one of $\mathbf{v}_1$, $\mathbf{v}_2$ can lie in $f_2 U$, so Condition III must be satisfied.

Suppose that $n - r = r = 3$ and that U is not isomorphic to U'. Then the subspaces $\langle \mathbf{y}_2\rangle' \cap U'$ and $\langle \mathbf{y}_1\rangle' \cap U$ are isomorphic. If $\langle \mathbf{y}_2\rangle' \cap U'$ is non-hyperbolic, then we can use the argument given for the case $n - r \geqslant 4$ to deduce that Condition III is satisfied. Suppose that $\langle \mathbf{y}_2\rangle' \cap U'$ is hyperbolic. If Condition III is not satisfied, then either $f_2 U' \cap U$ is hyperbolic of dimension 2 of $f_2U' \cap U'$ is non-hyperbolic of dimension 2. In the first case we can find an element $g_1 \in G_1$ that fixes $f_2U' \cap U$ but moves $\langle \mathbf{y}_2 \rangle$, so that $f_2^{-1}g_1f_2 \in F_1 \backslash G_1$, but then Condition IV is satisfied. In the second case we can find an element $g_2 \in G_1$ that moves $\mathbf{y}_1$ into f_2U and moves $\mathbf{y}_2$ out of f_2U'; the element $f_2^{-1}g_2f_2 \in F_1 \backslash G_1$ takes $\mathbf{y}$ to a vector whose U' component $\mathbf{y}_2^*$ is non-singular but not isomorphic to $\mathbf{y}_2$. Hence $\langle \mathbf{y}_2^* \rangle' \cap U'$ is non-hyperbolic and Condition III is satisfied, with $\mathbf{x} = \mathbf{y}$ and $f_3 = f_2^{-1}g_2f_2$.

(iv) Suppose that $H = O_n(GF(2))$. As in (ii), Condition III is satisfied unless $\mathbf{y}_1, \mathbf{y}_2 \in f_2U'$. We therefore assume that $\mathbf{y}_1, \mathbf{y}_2 \in f_2U'$, whence $n - r \geqslant 4$.

Suppose that $n - r \geqslant 6$. Proposition 2.9 may be readily extended to give a base of non-singular vectors for $\langle \mathbf{y}_2 \rangle' \cap U'$ that does not contain $\mathbf{y}_2$. None of these vectors can lie in f_2U' (otherwise Condition IV would be satisfied) and they cannot all lie in f_2U (otherwise $\mathbf{y}_2 \in f_2U$, a contradiction); so Condition III is satisfied.

If $n - r = 4$, then either $\nu_2 = 1$, $r = 2$ and $\nu_1 = 0$, or $r = 4$ and one of ν_1, $\nu_2 < 2$.

Suppose that $n - r = 4$, $\nu_2 = 1$, $r = 2$ and $\nu_1 = 0$. By Proposition 2.1, there is non-hyperbolic complement of $\langle \mathbf{y}_2 \rangle$ in $\langle \mathbf{y}_2 \rangle' \cap U'$. The non-zero vectors in this complement cannot all lie in f_2U and none can lie in f_2U' (otherwise Condition IV would be satisfied), so Condition III is satisfied.

Suppose that $n - r = r = 4$ and that one of ν_1, $\nu_2 < 2$. By Proposition 2.1, there are non-hyperbolic complements of $\langle \mathbf{y}_1 \rangle$ in $\langle \mathbf{y}_1 \rangle' \cap U$ and $\langle \mathbf{y}_2 \rangle$ in $\langle \mathbf{y}_2 \rangle' \cap U'$. None of the non-zero vectors in these complements can lie in f_2U' (otherwise f_2U' has a 3-dimensional totally isotropic subspace), so Condition III is satisfied unless f_2U is the sum of these components, i.e., $\nu_1 = 2$. But in this latter circumstance $\nu_2 = 1$ and $f_2U' \cap U'$ is then hyperbolic so that Condition IV is satisfied, a contradiction. ∎

PROPOSITION 4.7. *Suppose that $H = U_n(K)$, that K is finite and that $n \geqslant 4$, then Condition* IV *is satisfied.*

Proof. It is well known that every non-isotropic subspace of dimension $\geqslant 2$ has a singular 1-dimensional subspace, so $n - r \geqslant r$. Moreover, the case $n - r = r = 2$ and $\nu_1 = \nu_2 = 1$ is excluded so $n - r \geqslant 3$.

We suppose the proposition to be false and arrive at a contradiction. By Propositions 4.3 and 4.4, there exists $f_2 \in F_1 \backslash G_1$ and a singular vector $\mathbf{y} \in U'$ such that $f_2U \nsubseteq U'$ when $r + 1 = n - r$ and such that if we write $f_2(\mathbf{y}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$, then $\mathbf{y}_1$ and $\mathbf{y}_2$ are non-singular, whence $\langle \mathbf{y}_2 \rangle' \cap U'$ and (when $r \geqslant 2$) $\langle \mathbf{y}_1 \rangle' \cap U$ are non-isotropic. We claim that there is a non-zero singular vector $\mathbf{v} \in ((\langle \mathbf{y}_2 \rangle' \cap U') \cup (\langle \mathbf{y}_1 \rangle' \cap U)$ that does not lie in $f_2U \cup f_2U'$. If not, then the fallacy of the proposition implies that $\langle \mathbf{y}_2 \rangle' \cap U' \subseteq f_2U$ (cf. Proposition 2.8) which in turn implies that

$r = n - r$. Thus $\langle \mathbf{y}_1 \rangle' \cap U$ has a base of singular vectors, none of which can lie in $f_2 U$ (otherwise $f_2 U$ would be isotropic of dimension $\geqslant r$, an absurdity) and not all of which can lie in $f_2 U'$ (otherwise $f_2 U'$ would contain the isotropic r-dimensional subspace $\langle f_2(\mathbf{y}) \rangle + (\langle \mathbf{y}_1 \rangle' \cap U)$), a contradiction. Thus there is a vector $\mathbf{v}$ as required. Let t be a transvection centered on $\mathbf{v}$, then $t \in G_1$ and $tf_2 U \neq f_2 U$ (cf. Remark 2.2) so $f_2^{-1} t f_2 \in F_1 \backslash G_1$. But $f_2^{-1} t f_2 U' \cap U'$ contains the singular 1-dimensional subspace $\langle \mathbf{y} \rangle$, contradicting the fallacy of the proposition. Hence Condition IV must be satisfied. ∎

LEMMA 4.8. *There exists $f \in F_1 \backslash G_1$ such that $U' \cap fU'$ has a singular 1-dimensional subspace.*

Proof. We have proved the lemma for the case: $H = U_n(K)$, K finite and $n \geqslant 4$, in Proposition 4.7, so we may except that case in this proof. We suppose the lemma to be false and arrive at a contradiction. By Proposition 4.6, there exist $f_3 \in F_1 \backslash G_1$ and a singular vector $\mathbf{x} \in U'$ such that if we write $f_3(\mathbf{x}) = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 \in U$ and $\mathbf{x}_2 \in U'$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are non-singular and such that there exists a non-singular vector $\mathbf{u} \in (U \cap \langle \mathbf{x}_1 \rangle') \cup (U' \cap \langle \mathbf{x}_2 \rangle')$ that does not lie in $f_2 U \cup f_2 U'$.

Suppose that K does not have characteristic two when $H = U_n(K)$. Let s_0, s_1 and s_2 be the -1-quasi-symmetries centered on $\mathbf{u}$, $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively; then $s_1 s_2$, $s_0 s_1 \in G_1$ and $f_3^{-1} s_1 s_2 f_3 U' \cap U'$ contains the singular 1-dimensional subspace $\langle \mathbf{x} \rangle$, so the fallacy of the lemma implies that $s_1 s_2 f_3 U = f_3 U$. Thus $s_1 s_2(\mathbf{v}) - \mathbf{v} \in f_3 U$ for every $\mathbf{v} \in f_3 U$. Let $\mathbf{v} \in f_3 U$; then $(\mathbf{x}_2, \mathbf{v}) = -(\mathbf{x}_1, \mathbf{v})$. If $H = O_n(K)$, then

$$\begin{aligned} s_1 s_2(\mathbf{v}) - \mathbf{v} &= -[B(\mathbf{x}_1, \mathbf{v})/Q(\mathbf{x}_1)]\, \mathbf{x}_1 - [B(\mathbf{x}_2, \mathbf{v})/Q(\mathbf{x}_2)]\, \mathbf{x}_2 \\ &= -[B(\mathbf{x}_1, \mathbf{v})/Q(\mathbf{x}_1)] f_3(\mathbf{x}). \end{aligned}$$

This implies that $B(\mathbf{x}_1, \mathbf{v}) = 0$ for every $\mathbf{v} \in f_3 U$, so $\mathbf{x}_1, \mathbf{x}_2 \in f_3 U'$. Similarly, if $H = U_n(K)$, then

$$s_1 s_2(\mathbf{v}) - \mathbf{v} = [-2C(\mathbf{x}_1, \mathbf{v})/C(\mathbf{x}_1, \mathbf{x}_1)] f_3(\mathbf{x}),$$

so $\mathbf{x}_1, \mathbf{x}_2 \in f_3 U'$. Hence $s_0 s_1 f_3 U = s_0 f_3 U \neq f_3 U$, i.e., $f_3^{-1} s_0 s_1 f_3 \in F_1 \backslash G_1$. But $f_3^{-1} s_0 s_1 f_3 U' \cap U'$ contains the hyperbolic 2-dimensional subspace $\langle f_3^{-1}(\mathbf{x}_1), f_3^{-1}(\mathbf{x}_2) \rangle$, contrary to the fallacy of the lemma; so Condition IV must be satisfied.

Suppose that $H = U_n(K)$, that K has characteristic two and that K is infinite when $n \geqslant 4$. By Proposition 2.1, there exists $\lambda \in K$ such that $\lambda \cdot \bar{\lambda} = 1$ and $\lambda^n \neq 1$. Let q be the λ^n-quasi-symmetry centered on $\mathbf{u}$ and let k be the element of G taking $\mathbf{v}$ to $\lambda^{-1}\mathbf{v}$ for each $\mathbf{v} \in V$, then $qk \in G_1$ and $qkf_3 U = qf_3 U \neq f_3 U$, so $f_3^{-1} qkf_3 \in F_1 \backslash G_1$. But $f_3^{-1} qkf_3 U' \cap U'$ contains the singular 1-

dimensional subspace $\langle \mathbf{x} \rangle$, contradicting the fallacy of the lemma, so Condition IV must be satisfied. ∎

We now prove a series of results that will establish that F_1 contains every semi-transvection in H_1.

PROPOSITION 4.9. *There exists a non-zero singular vector* $\mathbf{x} \in U'$ *and a non-zero vector* $\mathbf{z} \in U$ *such that* $P_{\mathbf{x},\mathbf{z}} \subseteq F_1$.

Proof. By Lemma 4.8, there exists $f \in F_1 \backslash G_1$ such that $fU' \cap U'$ has a singular 1-dimensional subspace. Let $\mathbf{x}$ be a non-zero vector in such a subspace. By Proposition 2.1, there exists $g \in G$ such that $gf(\mathbf{x}) = \mathbf{x}$; by premultiplying g by a quasi-symmetry centered on a non-singular vector in U if necessary, we may assume that $g \in G_1$. Thus $gf \in F_1 \backslash G_1$ and $gf(\mathbf{x}) = \mathbf{x}$. Hence we may assume that $f(\mathbf{x}) = \mathbf{x}$.

Suppose that f does not fix $U' \cap \langle \mathbf{x} \rangle'$ and let $\mathbf{v} \in U' \cap \langle \mathbf{x} \rangle'$ such that $f(\mathbf{v}) \notin U' \cap \langle \mathbf{x} \rangle'$. If we write $f(\mathbf{v}) = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in U$ and $\mathbf{v}_2 \in U'$, then $f(\mathbf{v}) \in \langle \mathbf{x} \rangle'$, so $\mathbf{v}_2 \in U' \cap \langle \mathbf{x} \rangle'$ and therefore $\mathbf{v}_1 \neq 0$. By Proposition 2.6, the sets $P_{\mathbf{x},\mathbf{v}}$ and $P_{\mathbf{x},-\mathbf{v}_2}$ lie in G_1, so F_1 contains $fP_{\mathbf{x},\mathbf{v}}f^{-1} \cdot P_{\mathbf{x},-\mathbf{v}_2}$. But $fP_{\mathbf{x},\mathbf{v}}f^{-1} \cdot P_{\mathbf{x},-\mathbf{v}_2} = P_{\mathbf{x},\mathbf{v}_1}$ (by Proposition 2.3), so if $\mathbf{z} = \mathbf{v}_1$, then $\mathbf{z}$ is a non-zero vector in U such that F_1 contains $P_{\mathbf{x},\mathbf{z}}$.

Suppose that f fixes $U' \cap \langle \mathbf{x} \rangle'$. Let $\mathbf{y}$ be a singular vector in U' such that $(\mathbf{x}, \mathbf{y}) = 1$, and write $f(\mathbf{y}) = \mathbf{y}_1 + \mathbf{y}_2$, where $\mathbf{y}_1 \in U$ and $\mathbf{y}_2 \in U'$. We can write

$$U' = \langle \mathbf{y} \rangle \oplus (\langle \mathbf{x} \rangle' \cap U'),$$

so $\mathbf{y}_1 \neq 0$ (otherwise $f \in G_1$). Let $\rho \in P_{\mathbf{x},\mathbf{y}_1}$; then using the general form of a semi-transvection (see Remark 2.7)

$$\begin{aligned} \rho f(\mathbf{y}) &= \rho(\mathbf{y}_1) + \rho(\mathbf{y}_2) \\ &= \mathbf{y}_1 + (\mathbf{y}_1, \mathbf{y}_1)\,\mathbf{x} + \mathbf{y}_2 + \beta \cdot (\mathbf{x}, \mathbf{y}_2)\,\mathbf{x} - (\mathbf{x}, \mathbf{y}_2)\,\mathbf{y}_1, \end{aligned}$$

where $\beta \in K$. As $(\mathbf{x}, \mathbf{y}_2) = (f(\mathbf{x}), f(\mathbf{y})) = 1$, the vector $\rho f(\mathbf{y})$ lies in U'. Moreover ρ fixes $\langle \mathbf{x} \rangle' \cap U'$, so ρf fixes U', i.e., $\rho f \in G_1$. Hence if $\mathbf{z} = \mathbf{y}_1$, then $\mathbf{z}$ is a non-zero vector in U such that $P_{\mathbf{x},\mathbf{z}} \subseteq F_1$. ∎

PROPOSITION 4.10. *If* $\mathbf{x}$ *is a non-zero singular vector in* U' *with a non–zero vector* $\mathbf{z} \in U$ *such that* $P_{\mathbf{x},\mathbf{z}} \subseteq F_1$, *then* $P_{\mathbf{x},\lambda\mathbf{z}} \subseteq F_1$ for every $\lambda \in K$.

Proof. The proposition is trivial if $\lambda = 0$; so we assume that $\lambda \neq 0$. Let $\mathbf{y}$ be a singular vector in U' such that $(\mathbf{x}, \mathbf{y}) = 1$. If $H = O_n(K)$, or if $H = U_n(K)$ and $\lambda \in K_0$, then the map defined by

$$\begin{aligned} :\ &\mathbf{x} \mapsto \lambda\mathbf{x} \\ &\mathbf{y} \mapsto \lambda^{-1}\mathbf{y} \\ &\mathbf{v} \mapsto \mathbf{v}, \qquad \forall \mathbf{v} \in \langle \mathbf{x}, \mathbf{y} \rangle' \end{aligned}$$

lies in G_1; so F_1 contains $gP_{\mathbf{x},\mathbf{z}}g^{-1}$. But $gP_{\mathbf{x},\mathbf{z}}g^{-1} = P_{\mathbf{x},\lambda\mathbf{z}}$ (by Proposition 2.3); so F_1 contains $P_{\mathbf{x},\lambda\mathbf{z}}$.

Suppose that $H = U_n(K)$, that $\lambda \in K \backslash K_0$ and that $\mathbf{z}$ is non-singular. If $K = GF(4)$, then $\lambda \cdot \bar{\lambda} = 1$ and there is a non-singular vector $\mathbf{u} \in (U \cap \langle \mathbf{z} \rangle') \cup (U' \cap \langle \mathbf{x}, \mathbf{y} \rangle')$. Let s_1 and s_2 be respectively the λ-quasi-symmetry centered on $\mathbf{z}$ and the λ^{-1}-quasi-symmetry centered on $\mathbf{u}$, then $s_1 s_2 \in G_1$; so F_1 contains $s_1 s_2 P_{\mathbf{x},\mathbf{z}}(s_1 s_2)^{-1}$, i.e., contains $P_{\mathbf{x},\lambda\mathbf{z}}$. If $K \neq GF(4)$, then by the corollary to Proposition 2.15, there exists $\mu \in K$ such that $\bar{\mu}^2 \cdot \mu^{-1} \notin K_0$. Since K is an extension of K_0 of degree 2, there exist λ_1, $\lambda_2 \in K_0$ such that $\lambda = \lambda_1 + \lambda_2 \bar{\mu}^2 \cdot \mu^{-1}$. Let h be the map defined by

$$
\begin{aligned}
:\ \mathbf{x} &\mapsto \mu \mathbf{x} \\
\mathbf{y} &\mapsto \bar{\mu}^{-1} \mathbf{y} \\
\mathbf{z} &\mapsto \bar{\mu} \cdot \mu^{-1} \mathbf{z} \\
\mathbf{v} &\mapsto \mathbf{v}, \qquad \forall \mathbf{v} \in \langle \mathbf{x}, \mathbf{y}, \mathbf{x} \rangle';
\end{aligned}
$$

then $h \in G_1$; so F_1 contains $hP_{\mathbf{x},\lambda_2\mathbf{z}}h^{-1}$. But $hP_{\mathbf{x},\lambda_2\mathbf{z}}h^{-1} = P_{\mathbf{x},\lambda_2\bar{\mu}^2\mu^{-1}\mathbf{z}}$ (by Proposition 2.3) and F_1 contains $P_{\mathbf{x},\lambda_1\mathbf{z}}$; so F_1 contains $P_{\mathbf{x},\lambda\mathbf{z}}$ $(= P_{\mathbf{x},\lambda_1\mathbf{z}} \cdot hP_{\mathbf{x},\lambda_2\mathbf{z}}h^{-1})$.

Suppose that $H = U_n(K)$, that $\lambda \in K \backslash K_0$ and that $\mathbf{z}$ is singular. Let $\mathbf{z}_0$ be a singular vector in U such that $(\mathbf{z}, \mathbf{z}_0) = 1$ and let $\xi \in K \backslash \{0\}$ such that $\bar{\xi} \cdot \xi^{-1} \notin K_0$ (such exists: take $\zeta \in K \backslash K_0$ and if $\bar{\zeta} \cdot \zeta^{-1} \in K_0$, then $(\overline{\zeta + 1}) \cdot (\zeta + 1)^{-1} \notin K_0$); then there exists λ_3, $\lambda_4 \in K_0$ such that $\lambda = \lambda_3 + \lambda_4 \bar{\xi} \cdot \xi^{-1}$. Let k be the map defined by

$$
\begin{aligned}
:\ \mathbf{x} &\mapsto \xi \mathbf{x} \\
\mathbf{y} &\mapsto \bar{\xi}^{-1} \mathbf{y} \\
\mathbf{z} &\mapsto \xi^{-1} \mathbf{z} \\
\mathbf{z}_0 &\mapsto \bar{\xi} \mathbf{z}_0 \\
\mathbf{v} &\mapsto \mathbf{v}, \qquad \forall \mathbf{v} \in \langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}_0 \rangle';
\end{aligned}
$$

then $k \in G_1$; so F_1 contains $kP_{\mathbf{x},\lambda_4\mathbf{z}}k^{-1}$. Now arguing as above, F_1 contains $P_{\mathbf{x},\lambda\mathbf{z}}$.

Hence F_1 contains $P_{\mathbf{x},\lambda\mathbf{z}}$ for every $\lambda \in K$. ∎

Proposition 4.11. *If $\mathbf{x}$ is a non-zero singular vector in U' with a non-zero vector $\mathbf{z} \in U$ such that $P_{\mathbf{x},\mathbf{z}} \subseteq F_1$ and if $\mathbf{u} \in U$ is isomorphic to $\mathbf{z}$, then $P_{\mathbf{x},\mathbf{u}} \subseteq F_1$.*

Proof. If $\mathbf{z}$ is singular, then by Proposition 4.2 there exists $g \in G_1$ such that $g(\mathbf{x} + \mathbf{z}) = \mathbf{x} + \mathbf{u}$, i.e., such that $g(\mathbf{x}) = \mathbf{x}$ and $g(\mathbf{z}) = \mathbf{u}$. By

Proposition 2.3, $P_{\mathbf{x},\mathbf{u}} = gP_{\mathbf{x},\mathbf{z}}g^{-1}$ and therefore lies in F_1. If $\mathbf{z}$ is non-singular, then by Proposition 2.1, there exists $h \in G$ such that $h(\mathbf{x}) = \mathbf{x}$ and $h(\mathbf{z}) = \mathbf{u}$. If $h \in G_1$, then let q be the identity element; otherwise let q be the quasi-symmetry centered on $\mathbf{u}$ that lies in $h^{-1}H_1$. Thus $qh \in G_1$ and F_1 contains $qhP_{\mathbf{x},\mathbf{z}}(qh)^{-1}$, i.e., F_1 contains $P_{\mathbf{x},\lambda\mathbf{u}}$ for some $\lambda \neq 0$ (by Proposition 2.3). By Proposition 4.10, it follows that F_1 contains $P_{\mathbf{x},\mathbf{u}}$.

PROPOSITION 4.12. *There exists a non-zero singular vector* $\mathbf{x} \in U'$ *such that* $P_{\mathbf{x},\mathbf{w}} \subseteq F_1$ *for every* $\mathbf{w} \in U'$.

Proof. By Proposition 4.9 there exists a non-zero singular vector $\mathbf{x} \in U'$ and a non-zero vector $\mathbf{z} \in U$ such that $P_{\mathbf{x},\mathbf{z}} \subseteq F_1$. By Propositions 2.8 and 2.10, there is a base $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_r\}$ for U of vectors isomorphic to $\mathbf{z}$. Let $\mathbf{w} \in \langle \mathbf{x} \rangle'$, then we can write $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in U$ and $\mathbf{v} \in U' \cap \langle x \rangle'$, and we can write

$$\mathbf{u} = \sum_{i=1}^{r} \lambda_i \mathbf{u}_i,$$

for some $\lambda_i \in K$. Thus by Proposition 2.3,

$$P_{\mathbf{x},\mathbf{w}} = P_{\mathbf{x},\mathbf{v}} \cdot \prod_{i=1}^{r} P_{\mathbf{x},\lambda_i \mathbf{u}_i}.$$

By Propositions 2.6, 4.10 and 4.11, the sets $P_{\mathbf{x},\mathbf{v}}$ and $P_{\mathbf{x},\lambda_i\mathbf{u}_i}$ lie in F_1; so $P_{\mathbf{x},\mathbf{w}} \subseteq F_1$. Hence $P_{\mathbf{x},\mathbf{w}} \subseteq F_1$, for every $\mathbf{w} \in \langle \mathbf{x} \rangle'$. ∎

LEMMA 4.13. F_1 *contains every semi-transvection in* H_1.

Proof. If $\mathbf{x}$ is as in Proposition 4.12 and if we can show that F_1 acts transitively on the non-zero singular vectors of V, then by Proposition 2.3, any semi-transvection is conjugate under F_1 to a semi-transvection centered on $\mathbf{x}$, and is therefore contained in F_1. We know that the set $\mathscr{C}_1$ of non-zero singular vectors of U is a G_1-orbit when $\nu_1 > 0$, and that the set $\mathscr{C}_2$ of non-zero singular vectors of U' is a G_1-orbit (cf. Proposition 4.2 and the remark following it). As F_1 does not stabilize U or U', and as the elements of $\mathscr{C}_2$ and $\mathscr{C}_1$ (when $\nu_1 > 0$) span U' and U, respectively (cf. Proposition 2.8), it follows that $\mathscr{C}_1$ and $\mathscr{C}_2$ cannot be F_1-orbits. Thus to prove that F_1 acts transitively on the non-zero singular vectors of V, we need only show that any singular vector $\mathbf{w} \in V \backslash (U \cup U')$ lies in the F_1-orbit containing $\mathscr{C}_2$.

Let us write $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in U$ and $\mathbf{v} \in U'$; then $\mathbf{v}$ is non-zero, so there is a singular vector $\mathbf{y} \in U'$ such that $(\mathbf{y}, \mathbf{v}) = 1$. Let $\rho \in P_{\mathbf{y},\mathbf{u}}$, then ρ is conjugate under F_1 to a semi-transvection centered on $\mathbf{x}$ and therefore lies in F_1. Using the general form of a semi-transvection (cf. Remark 2.7),

$$\begin{aligned}\rho(\mathbf{w}) &= \rho(\mathbf{u}) + \rho(\mathbf{v}) \\ &= \mathbf{u} + [\beta \cdot (\mathbf{y}, \mathbf{u}) + (\mathbf{u}, \mathbf{u})]\, \mathbf{x} - (\mathbf{y}, \mathbf{u})\, \mathbf{u} + \mathbf{v} \\ &\quad + [\beta \cdot (\mathbf{y}, \mathbf{v}) + (\mathbf{u}, \mathbf{v})]\, \mathbf{x} - (\mathbf{y}, \mathbf{v})\, \mathbf{u},\end{aligned}$$

so $\rho(\mathbf{w}) \in \mathscr{C}_2$. Hence F_1 acts transitively on the non-zero singular vectors of V, whence F_1 contains every semi-transvection in H_1. ∎

LEMMA 4.14. *$F_1 = H_1$ and $F = H$.*

Proof. If $H = U_n(K)$, then by Lemma 4.28, every semi-transvection in H_1 lies in F_1, so, by Proposition 2.5, $F_1 = H_1$.

Suppose that $H = O_n(K)$. Let P be a hyperbolic 2-dimensional subspace of U' and let $SO(P)$ be the subgroup of H_1 consisting of those elements that fix every vector in P'; then $SO(P) \leqslant G_1$. Let T be the subgroup of H_1 generated by the semi-transvections of H_1; then by Lemma 4.13, $T \leqslant F_1$. By Result 2.4, $H_1 = T \cdot SO(P)$, so $F_1 = H_1$ (notice that the excepted case of Result 2.4 is the case $n - r = r = 2$ and $v_1 = v_2 = 1$ which we have excepted).

Each coset of H_1 in H (other than H_1) contains a quasi-symmetry centered on a non-singular vector in U, so E contains elements of each coset of H_1 in H. Thus $E_1 < F \cap H_1 \leqslant H_1$. We have already shown that if $E_1 < F_1 \leqslant H_1$, then $F_1 = H_1$, so $F \cap H_1 = H_1$. Hence $F = H$. ∎

We now consider briefly three cases that we have so far excluded.

PROPOSITION 4.15. *Suppose that $H = O_n(K)$, $n - r = r = 2$ and $v_1 = v_2 = 1$, but that $K \neq GF(3)$ or $GF(5)$, then $F = H$.*

Proof. First suppose that $K \neq GF(2)$. Notice that Proposition 4.1 applies to this case and that the proof of the analogues of Propositions 4.2, 4.3, 4.5, and 4.6, Lemma 4.8, Propositions 4.9, 4.10, 4.11, and 4.12 and Lemma 4.13 (in the analogue to Lemma 4.8, we would consider the symmetry s_0 instead of the products $s_1 s_2$ and $s_0 s_1$) would be very similar to the originals. Hence F contains $SO(U')$ and T. It follows that F contains H_1, but F also contains elements of $H \backslash H_1$; so $F = H$.

Now suppose that $K = GF(2)$. In this case H has order 72 and V has only six non-singular 1-dimensional subspaces. These fall into four orbits under G, two orbits under E and just one orbit under F. Thus F contains every symmetry in H. It is well known that the symmetries of H generate a subgroup of order 36; as E has order 8, it follows that $F = H$. ∎

PROPOSITION 4.16. *If $H = O_4(GF(2))$, $n - r = r = 2$, $v_1 = 0$ and $v_2 = 1$, then $F_1 = H_1$ and $F = H$.*

Proof. In this case H_1 is isomorphic to the alternating group A_5, and G_1

has order 6. Thus F_1 must have order 12, 30 or 60. But A_5, being simple, has no subgroup of order 30; moreover, the only subgroups of A_5 of order 12 are those isomorphic to A_4 which has no subgroup of order 6; so F_1 must have order 60. Hence $F_1 = H_1$. As in the proof of Lemma 4.14, it follows that $F = H$. ▮

PROPOSITION 4.17. *If* $H = U_n(K)$, $n - r = r = 2$ *and* $\nu_1 = \nu_2 = 1$, *then* $F_1 = H_1$ *and* $F = H$, *except when* $K = GF(4)$.

Proof. We first show that Condition IV is satisfied. Suppose not and let $h \in F_1 \backslash E_1$. Then $hU \not\subseteq U'$, so there exists a singular vector $\mathbf{x} \in U'$ such that if we write $h(\mathbf{x}) = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1 \in U$ and $\mathbf{x}_2 \in U'$, then $\mathbf{x}_1, \mathbf{x}_2 \neq 0$.

Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are singular. At least one of $\mathbf{x}_1, \mathbf{x}_2$ must lie outside $hU \cup hU'$; let t be a transvection on such a vector; then $t \in G_1$ and $thU \neq hU$ (cf. Remark 2.2); so $h^{-1}th \in F_1 \backslash G_1$. But $h^{-1}thU' \cap U'$ contains the singular 1-dimensional subspace $\langle \mathbf{x} \rangle$, a contradiction to Condition IV not being satisfied.

Now suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are non-singular. As argued in the proof of Proposition 4.5, we may assume that $hU' \neq \langle \mathbf{x}_1, \mathbf{x}_2 \rangle$. Let $\lambda \in K$ such that $\lambda \cdot \bar{\lambda} = 1$ and $\lambda^2 \neq 1$ (such exists: if $\mu \in K \backslash K_0$, then one of $\bar{\mu}/\mu$, $(\bar{\mu} + 1)/(\mu + 1)$ gives the required λ), let s_1 and s_2 be the λ^2 - quasi-symmetries centered on $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, and let k be the map taking $\mathbf{v}$ to $\lambda^{-1}\mathbf{v}$ for each $\mathbf{v} \in V$. Then $s_1 s_2 k \in G_1$ and in the manner of the proof of Lemma 4.8, the failure of Condition IV to be satisfied implies that $\mathbf{x}_1$, $\mathbf{x}_2 \in hU'$, a contradiction.

Hence Condition IV is satisfied. With one amendment we may use the methods of proof of Propositions 4.9, 4.10, 4.11 and 4.12 to show that there exists a non-xero singular vector $\mathbf{x} \in U'$ such that $P_{\mathbf{x},\mathbf{w}} \subseteq F_1$, for every $\mathbf{w} \in \langle \mathbf{x} \rangle'$; the amendment is needed in the analogue to Proposition 4.11 when $\mathbf{z}$ is singular. We need to show that if $\mathbf{z}$ and $\mathbf{u}$ are non-zero singular vectors in U and if $P_{\mathbf{x},\lambda\mathbf{z}} \subseteq F_1$ for every $\lambda \in K$, then $P_{\mathbf{x},\mathbf{u}} \subseteq F_1$. We may assume that $\mathbf{u} \notin \langle \mathbf{z} \rangle$, so $C(\mathbf{z}, \mathbf{v}) \neq 0$. By Proposition 2.14 there exists $\eta \in K$ such that $\eta \cdot \bar{\eta}^{-1} = -C(\mathbf{z}, \mathbf{v})/C(\mathbf{v}, \mathbf{z})$. Let g be the element of G_1 defined by

$$\mathbf{z} \mapsto -\eta^{-1}\mathbf{u},$$

$$\mathbf{u} \mapsto \eta\mathbf{z},$$

$$\mathbf{v} \mapsto \mathbf{v}, \qquad \forall \mathbf{v} \in U';$$

then F_1 contains $gP_{\mathbf{x},-\eta\mathbf{z}}g^{-1}$. Thus by Propositions 2.3, F_1 contains $P_{\mathbf{x},\mathbf{u}}$. We may now use the methods of Lemmas 4.13 and 4.14 (noting that $\mathscr{C}_1$ and $\mathscr{C}_2$ are still orbits of G_1) to deduce that $F_1 = H_1$ and $F = H$. ▮

We have chosen F_1 and F arbitrarily such that $E_1 < F_1 \leqslant H_1$ and $E < F \leqslant H$. We noted at the beginning of this section that $G_1 = E_1$ and

$G = E$ when U is not isomorphic to U' and that $G_1 < E_1$ and $G < E$ when U is isomorphic to U'. Hence by Lemma 4.13 and Propositions 4.14, 4.15 and 4.16, we have proved our main result.

THEOREM. *G_1 and G are maximal in H_1 and H, respectively, except when U is isomorphic to U' and except in cases* (i)–(vii) *and* (xi) (cf. *Section* 1). *If U is isomorphic to U', then E_1 is maximal in H_1 except when $H = O_4(K)$ and except in cases* (ix), (x) *and* (xii), *and E is maximal in H except in cases* (iii), (viii), (ix), (x) *and* (xii).

5. THE ORTHOGONAL AND UNITARY GROUPS: THE EXCEPTIONAL CASES

The cases excluded from the theorem above are all exceptions to the theorem. In this section we explain briefly how these exceptions arise. We define groups $F \leqslant H$ and $F_1 = F \cap H_1$ (unless stated otherwise) and claim that $E_1 < F_1 < H_1$ and $E < F < H$. We omit the proof of this claim for reasons of space, but it is not difficult to construct elements of $H_1 \backslash F_1$, $F_1 \backslash E_1$, $H \backslash F$ and $F \backslash E$ in each case.

(i) Suppose that $H = O_3(GF(5))$ and $r = 1$. Then U' has two subspaces L_1 and L_2 isomorphic to U and these are orthogonal. Let $F = \mathrm{Stab}_H\{U, L_1, L_2\}$; then $G < F < H$.

Suppose that $H = O_4(GF(5))$, $n - r = r = 2$ and $v_1 = v_2 = 1$. Let M_1 be a non-singular 1-dimensional subspace of U, then U has one other subspace isomorphic to M_1, namely, $U \cap M_1'$, and U' has two subspaces, L_1 and L_2, isomorphic to M_1, with $L_2 = U' \cap L_1'$. Let $F = \mathrm{Stab}_H\{L_1, L_2, M_1, M_2\}$; then $E < F < H$.

(ii) and (iii) Suppose that $H = O_n(GF(3))$ and that $n - r = 2$, or $r = 2$ and $v_1 = 1$, but not both, i.e., there exists $W \in \{U, U'\}$ not isomorphic to W' such that W is a hyperbolic 2-dimensional subspace. As we remarked after Proposition 2.10, there are two non-isomorphic subspaces L and M of W. One of these subspaces, M say, must be isomorphic to a subspace of W'. Let $F = \mathrm{Stab}_H L$; then $G = \mathrm{Stab}_H W < F < H$.

Suppose that $H = O_4(GF(3))$, $n - r = r = 2$ and $v_1 = v_2 = 1$. Let L_1 and L_2 be the two non-isomorphic non-singular 1-dimensional subspaces of U and let M_1 and M_2 be the corresponding subspaces of U', with L_1 isomorphic to M_1. Let $F = \mathrm{Stab}_H\{L_1, L_2\}$; then $E < F < H$.

(iv), (v) and (ix) Suppose that $H = O_n(GF(3))$ and that $n - r = 3$. Then U' has thirteen 1-dimensional subspaces, four of which are singular. The nine non-singular 1-dimensional subspaces lie in two isomorphism classes of sizes six and three (consider the projective plane derived from U' and cf. [4]). Let L_1, L_2 and L_3 be the elements of the smaller class; then L_1, L_2 and L_3 are mutually orthogonal. If $n = 4$, $r = 1$ and $\nu = 2$ (case (iv)), then U is isomorphic to L_1; let $F = \mathrm{Stab}_H\{U, L_1, L_2, L_3\}$; then $G < F < H$. If $n = 5$, $r = 2$ and $\nu_1 = 0$ (case (v)), then by Proposition 2.10, there are two subspaces M_1 and M_2 of U orthogonal to L_1, and these are mutually orthogonal. Let $F = \mathrm{Stab}_H\{M_1, M_2, L_1, L_2, L_3\}$; then $G < F < H$. If $n = 6$ and U is isomorphic to U' (case (ix)), then U has three subspaces M_1, M_2 and M_3 isomorphic to L_1, and these are orthogonal. Let $F = \mathrm{Stab}_H\{M_1, M_2, M_3, L_1, L_2, L_3\}$; then $E < F < H$.

(vi) Suppose that $H = O_n(GF(2))$, $n \geqslant 6$, $r = 2$ and $\nu_1 = 1$. There is only one non-singular subspace L of U; let $F = \mathrm{Stab}_H L$; then $G < F < H$.

(vii) and (x) Suppose that $H = O_n(GF(2))$, $n - r = 4$ and $\nu_2 = 2$. Then U' has two non-hyperbolic non-isotropic 2-dimensional subspaces, W_1 and W_2; these are orthogonal. If $r = 2$ and $\nu_1 = 0$ (case (vii)), then U is isomorphic to W_1 and W_2; let $F = \mathrm{Stab}_H\{U, W_1, W_2\}$; then $G < F < H$. If $r = 4$ and $\nu_1 = 2$ (case(x)), then U is isomorphic to U' and therefore has two (orthogonal) subspaces U_1 and U_2 isomorphic to W_1; let $F = \mathrm{Stab}_H\{U_1, U_2, W_1, W_2\}$; then $E < F < H$.

(xi) and (xii) Suppose that $H = U_n(GF(4))$ and that $n - r = 2$. Then U' has five 1-dimensional subspaces of which two, L_1 and L_2, are non-singular; L_1 and L_2 are orthogonal and isomorphic. If $r = 1$ (case (xi)), then U is isomorphic to L_1; let $F = \mathrm{Stab}_H\{U, L_1, L_2\}$, then $G < F < H$. If $r = 2$ and $\nu_1 = \nu_2 = 1$ (case (xii)), then U is isomorphic to U' and therefore has two non-singular orthogonal subspaces M_1 and M_2 isomorphic to L_1. Let $F = \mathrm{Stab}_H\{M_1, M_2, L_1, L_2\}$; then $E < F < H$.

Suppose that $H = O_n(K)$, $n - r = r = 2$ and $\nu_1 = \nu_2 = 1$. Then each of U, U' has two singular 1-dimensional subspaces, say U_1, $U_2 \subseteq U$ and W_1, $W_2 \subseteq U'$. Clearly E acts on the pairs $\{U_1, W_1\}$, $\{U_1, W_2\}$, $\{U_2, W_1\}$ and $\{U_2, W_2\}$; this action is transitive, but the action of E_1 is not, because E_1 stabilizes $\{\{U_1, W_1\}, \{U_2, W_2\}\}$. Let $F_1 = \mathrm{Stab}_{H_1}\{U_1 + W_1, U_2 + W_2\}$; then $E_1 < F_1 < H_1$.

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