Some Algebraic Identities Concerning Determinants and Permanents

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ABSTRACT

Some algebraic identities are presented which give expansions for determinants of square matrices in terms of permanents of principal submatrices, and vice versa. Particular cases of these reciprocal formulae yield the recurrence of Muir.

Let $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ be a diagonal matrix, and $A = (a_{ij})_{1 \leqslant i, j \leqslant n}$ a $n \times n$ matrix. For $\lambda \subseteq [n] = \{1, 2, \ldots, n\}$, $A[\lambda]$ denotes the principal submatrix of A with row and column labels in λ, and X_λ denotes the product of elements of X with indices in λ. For $\sigma \in D(\lambda)$, the set of divisions (ordered partitions with no empty parts) of λ, $\iota(\sigma)$ is defined to be the number of nonempty parts of σ. In particular, we set $D(\varnothing) = \varnothing$ and $X_\varnothing = \det(A[\varnothing]) = \operatorname{per}(A[\varnothing]) = 1$ for the empty set $\varnothing$. For brevity, the following notation will be used throughout the paper:

$$\det(A_\sigma) = \prod_{i=1}^{\iota(\sigma)} \det(A[\sigma_i]),$$

$$\operatorname{per}(A_\sigma) = \prod_{i=1}^{\iota(\sigma)} \operatorname{per}(A[\sigma_i]),$$

where $\sigma = \dot{\bigcup} \sigma_i \in D(\lambda)$. On the basis of these preliminaries, we are ready to state our main results.

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THEOREM 1. *Let λ^c denote the complement of λ in $[n]$, and $|\lambda|$ the order of λ. There holds the identity*

$$\det(X + A) = \sum_{\lambda \subseteq [n]} \sum_{\sigma \in D(\lambda)} (-1)^{|\lambda| + \iota(\sigma)} X_{\lambda^c} \operatorname{per}(A_\sigma). \tag{1}$$

Before proving the theorem we state an interesting consequence as follows.

COROLLARY 2.

$$\det(A) = \sum_{\sigma \in D([n])} (-1)^{n + \iota(\sigma)} \operatorname{per}(A_\sigma). \tag{2}$$

Proof of theorem. It is obvious that

$$\det(X + A) = \sum_{\lambda \subseteq [n]} X_{\lambda^c} \det(A[\lambda]).$$

Hence it suffices to show that

$$\det(A[\lambda]) = \sum_{\sigma \in D(\lambda)} (-1)^{|\lambda| + \iota(\sigma)} \operatorname{per}(A_\sigma)$$

or equivalently to show the truth of (2).

For each permutation w of $[n]$, denote by $[w]P(A)$ the coefficient of $\prod a_{iw(i)}$ in the polynomial function $P(A)$. Suppose the cyclic type of w is $(1^{c_1}2^{c_2}\dots n^{c_n})$, i.e., the decomposition of w into disjoint cycles has c_i cycles of length i ($1 \leqslant i \leqslant n$). Then $[w]\operatorname{per}(A_\sigma) = 1$ if and only if the factors of cyclic decomposition for w constitute the parts of σ; otherwise $[w]\operatorname{per}(A) = 0$. Let $d(n,k)$ denote the number of ways distributing n distinct balls into k distinct boxes with no boxes being empty. Then we have a known expression

$$d(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$

With $c = c_1 + c_2 + \cdots + c_n$, we have

$$[w] \sum_{\sigma \in D([n])} (-1)^{n+\iota(\sigma)} \mathrm{per}(A_\sigma) = \sum_{\sigma \in D([n])} (-1)^{n+\iota(\sigma)} [w] \mathrm{per}(A_\sigma)$$

$$= \sum_{k=0}^{c} (-1)^{n+k} d(c,k)$$

$$= \sum_{i=0}^{c} (-1)^{n+i} \binom{c+1}{i+1} i^c$$

$$= (-1)^{n+c} = [w] \det(A).$$

The last step follows from the fact that $(c+1)$th difference of x^c with respect to x is equal to zero. Hence for each permutation w of $[n]$, the coefficients of $[w]$ on both sides of (2) are identical, i.e., (2) holds. This completes the proof of Theorem 1. ■

At the end of last century, Muir [2] established a recurrence relation between permanents and determinants (cf. [1]) which can be restated as follows:

Proposition 3.

$$\sum_{\lambda \subseteq [n]} (-1)^{|\lambda|} \mathrm{per}(A[\lambda^c]) \det(A[\lambda]) = 0. \tag{3}$$

This can be verified directly by means of Corollary 2. In fact, substitution of (2) into the left-hand side of (3) gives

$$\sum_{\lambda \subseteq [n]} \sum_{\sigma \in D(\lambda)} (-1)^{\iota(\sigma)} \mathrm{per}(A[\lambda^c]) \mathrm{per}(A_\sigma)$$

$$= \sum_{\sigma \in D([n])} (-1)^{\iota(\sigma)} \mathrm{per}(A_\sigma) + \sum_{\lambda \subset [n]} \sum_{\sigma \in D(\lambda)} (-1)^{\iota(\sigma)} \mathrm{per}(A[\lambda^c]) \mathrm{per}(A_\sigma)$$

$$= \sum_{\sigma \in D([n])} (-1)^{\iota(\sigma)} \mathrm{per}(A_\sigma) + \sum_{\sigma \in D([n])} (-1)^{\iota(\sigma)-1} \mathrm{per}(A_\sigma) = 0.$$

Hence (2) yields Muir's recurrence (3).

Notice that (3) is symmetric in det and per. This suggests the following dual results for (1)–(2), which can be proved by the same methods.

THEOREM 4.

$$\operatorname{per}(X+A)=\sum_{\lambda\subseteq[n]}\ \sum_{\sigma\in D(\lambda)}(-1)^{|\lambda|+\iota(\sigma)}X_{\lambda'}\det(A_\sigma). \tag{4}$$

COROLLARY 5.

$$\operatorname{per}(A)=\sum_{\sigma\in D([n])}(-1)^{n+\iota(\sigma)}\det(A_\sigma). \tag{5}$$

Next we shall discuss some applications of the identities just established. Before doing so some definitions should be introduced. Denote by $\langle\overline{m}\rangle$ the multiset $\{1^{m_1}2^{m_2}\dots n^{m_n}\}$. If we define $D(\langle\overline{m}\rangle)$ and $D^0(\langle\overline{m}\rangle)$ as the sets of divisions of $\langle\overline{m}\rangle$ whose parts are subsets of $\langle\overline{m}\rangle$ and $[n]$ respectively [i.e., the distinction between $D(\langle\overline{m}\rangle)$ and $D^0(\langle\overline{m}\rangle)$ is just that the former includes the multisets as possible parts while the latter does not], then we have the following expansions.

THEOREM 6.

$$\operatorname{per}^{-1}(I-XA)=\sum_{\overline{m}}X^{\overline{m}}\sum_{\sigma\in D^0(\langle\overline{m}\rangle)}(-1)^{|\overline{m}|+\iota(\sigma)}\operatorname{per}(A_\sigma), \tag{6}$$

$$\det{}^{-1}(I-XA)=\sum_{\overline{m}}X^{\overline{m}}\sum_{\sigma\in D^0(\langle\overline{m}\rangle)}(-1)^{|\overline{m}|+\iota(\sigma)}\det(A_\sigma), \tag{7}$$

where I denotes the identity matrix, $\overline{m}=(m_1,m_2,\dots,m_n)$ the nonnegative integer vector of n dimensions with coordinate sum $|\overline{m}|$, and $X^{\overline{m}}=\prod_{i=1}^{n}x_i^{m_i}$.

Proof. Note that

$$\operatorname{per}(I-XA)=1+\sum_{\varnothing\subset\lambda\subseteq[n]}X_\lambda\sum_{\sigma\in D(\lambda)}(-1)^{\iota(\sigma)}\det(A_\sigma)$$

by (4). Hence

$$\operatorname{per}^{-1}(I - XA) = \sum_k (-1)^k \Big\{ \sum_{\varnothing \subset \lambda \subseteq [n]} X_\lambda \sum_{\sigma \in D(\lambda)} (-1)^{\iota(\sigma)} \det(A_\sigma) \Big\}^k$$

$$= \sum_{\overline{m}} X^{\overline{m}} \sum_{\lambda \in D^0(\langle \overline{m} \rangle)} (-1)^{\iota(\lambda)} \prod_{j=1}^{\iota(\lambda)} \sum_{\sigma \in D(\lambda_j)} (-1)^{\iota(\sigma)} \det(A_\sigma)$$

$$= \sum_{\overline{m}} X^{\overline{m}} \sum_{\lambda \in D^0(\langle \overline{m} \rangle)} (-1)^{|\overline{m}| + \iota(\lambda)} \operatorname{per}(A_\lambda).$$

The last step follows from (5). This completes the proof of (6). Dually we can give a similar derivation for (7). ■

By comparing (2) and (5) with (6)–(7), we obtain the following interesting consequences, in which $[X^{\overline{m}}]P(X)$ denotes the coefficient of $X^{\overline{m}}$ in the expansion of the formal power series $P(X)$.

PROPOSITION 7.

$$\det(A) = [x_1 x_2 \cdots x_n] \operatorname{per}^{-1}(I - XA), \tag{8}$$

$$\operatorname{per}(A) = [x_1 x_2 \cdots x_n] \det^{-1}(I - XA). \tag{9}$$

The second of these is a special case of MacMahon's master theorem. In fact, the summation region $D^0(\langle \overline{m} \rangle)$ for the right hand side of (7) can be replaced by $D(\langle \overline{m} \rangle)$ because the repeated indices in one part of σ will force $\det(A_\sigma)$ to be zero. It follows again from (5) that

$$\frac{\operatorname{per}(A_{\langle \overline{m} \rangle})}{\prod_{k=1}^n m_k!} = \sum_{\sigma \in D^0(\langle \overline{m} \rangle)} (-1)^{|\overline{m}| + \iota(\sigma)} \det(A_\sigma), \tag{10}$$

where $A_{\langle \overline{m} \rangle}$ denotes the matrix in which the first m_1 rows are identical with row suffix equal 1, the next m_2 rows are identical with row suffix equal to 2, etc., and the columns are indexed similarly. It has been shown by Vere-Jones

[3] that (10) is equivalent to the master theorem. An intriguing open question is whether (10) admits a dual proposition related in a similar way to (6).

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