REGULAR AND SELF-SIMILAR SOLUTIONS
OF NONLINEAR SCHröDINGER EQUATIONS

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ABSTRACT. We study the Cauchy problem for the nonlinear Schrödinger equations with nonlinear term $|u|^a u$. For some admissible $a$, we show the existence of global solutions and we calculate the regularity of those solutions. Also, we give some necessary conditions and some sufficient conditions on initial data for the existence of self-similar solutions.

1. Introduction

The aim of this paper consists in a detailed investigation for the problem of global solutions of the nonlinear Schrödinger equations:

\begin{equation}
\begin{cases}
  i\partial_t u + Au = \gamma |u|^a u, \\
  u(0, x) = f(x).
\end{cases}
\end{equation}

Here $\gamma \in \mathbb{R}$, $a > 0$, $u = u(t, x)$ is a complex-valued function defined on $[0, +\infty] \times \mathbb{R}^n$ and the initial data $f$ is a complex valued function on $\mathbb{R}^n$.

Our goal is to prove the existence and the regularity of global self-similar solutions for some admissible parameters $a$. Recall that $u(t, x)$ is a self-similar solution of (NLS) if $u$ is a solution of (NLS) and $u(t, x) = \lambda^{\frac{2}{n}} u(\lambda^2 t, \lambda x)$ for all $\lambda > 0$. For such solution it is well known that $u$ is of the form

$$u(t, x) = t^{-\frac{1}{a}} W \left( \frac{x}{\sqrt{t}} \right)$$

for some $W : \mathbb{R}^n \rightarrow \mathbb{C}$ called the profile of the solution, and the initial data $f$ is of the form

\begin{equation}
f(x) = \frac{\Omega(x')}{|x|^\frac{n}{2}},
\end{equation}

where $x' = \frac{x}{|x|}$ and $\Omega$ is defined on the unit sphere $S^{n-1}$ of $\mathbb{R}^n$. Therefore, equation (NLS) of $u$ can be studied through a semilinear elliptic equation on $W$. However, this semilinear elliptic equation is usually very complicated.

To our knowledge, the case of radially symmetric solutions is the only one which has been studied (see [KW], [RX]) and this impose that the initial data is of the form:

$$f(x) = \frac{C}{|x|^\frac{n}{2}}, \quad C \in \mathbb{C}.$$
On the other hand, to prove by standard methods existence of solutions with initial data of type (1) (see [CW1]), additional difficulties occur because those functions never belong to Lebesgue or Sobolev spaces. This is why the two above methods cannot be used.

Using recent technics, we prove the existence of self-similar solutions of (NLS) not necessarily radial-symmetric. Roughly speaking, for some admissible $\alpha$, we solve (NLS) in a family of non standard functional spaces which contain homogeneous functions like in (1). We obtain new global existence results for the (NLS) equation with small initial data which allows us to prove that there exists a large class of self-similar solutions. More precisely, we give necessary condition and sufficient condition on $\alpha$ to have such solutions. Also, a local in time version of our results give a simple proof of the locally well posedness of (NLS) in $H^s(\mathbb{R}^n)$ for super-critical values of $s$ (i.e. $s > s_c = \frac{n}{2} - \frac{2}{p}$). Our results complete and improve the recent results of [CW2] and [CW3] where self-similar solutions have been built with additional restrictions on $\alpha$ and $\Omega$.

Let us noted that such a method was recently used for Navier-Stokes equation [CP], for nonlinear heat equations [BCG], [R] and for the nonlinear wave equations [RY].

This paper is organized as follows. First we define the admissibility of the parameter $\alpha$ with its range of regularity $I_\alpha$ and we introduce the spaces of resolution $E_q(s \in I_\alpha)$ Then in section 3 we state the main results. Section 4 is devoted to the proof of some nonlinear estimates. In Section 5 we prove the global existence Theorem and related results.

In the sequel, $C$ will denote a constant which may differ at each appearance, possibly depending on the dimension or other parameters. As usual $S = S(\mathbb{R}^n)$ denotes the Schwartz’s space of test functions, $S'(\mathbb{R}^n)$ is its dual. For $f \in S'$. $\hat{f}$ denotes the Fourier transform of $f$ and for $p \geq 1$ we set $p' = \frac{p}{p-1}$.

2. Preliminaries

2.1. Function spaces

The spaces of solutions of (NLS) that will be considered here are based on the homogeneous Sobolev spaces. The usual way to define the homogeneous Sobolev space $\tilde{H}^s_p(\mathbb{R}^n)$ is the following. Let $p$ and $s$ such that $1 < p < +\infty$, $0 \leq s < \frac{n}{p}$. Define $q$ by $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$ and let $(-\Delta)^{\frac{s}{2}}$ the operator with symbol $|\xi|^s$. Then $\tilde{H}^s_p$ is the set of all $f \in L^q(\mathbb{R}^n)$ such that $(-\Delta)^{\frac{s}{2}} f \in L^p$. Note that $\tilde{H}^s_p$ is a Banach space of tempered distributions equipped by the norm

$$\|f\|_{\tilde{H}^s_p} = \|(-\Delta)^{\frac{s}{2}} f\|_{L^p}.$$ (2)

Moreover, for all $\lambda > 0$.

$$\|f_\lambda\|_{\tilde{H}^s_p} = \lambda^{s - \frac{n}{p}} \|f\|_{\tilde{H}^s_p},$$ (3)

where $f_\lambda(x) = f(\lambda x)$. The second way to define $\tilde{H}^s_p$ makes use of the Fourier-analytic approach which will be of great interest for us.

Let $\psi \in S$ be such that $\hat{\psi}$ is supported by the set $\{|\xi| \leq 2\}$ and

$$\sum_{i \in \mathbb{Z}} \psi(2^i \xi) = 1 \text{ for } \xi \neq 0.$$ (4)
Define \( \varphi \) by
\[
\varphi(\xi) = 1 - \sum_{j \geq 1} \psi(2^{-j} \xi).
\]

We denote by \( \Delta_j \) and \( S_j \) the convolution operators whose symbols are respectively \( \hat{\psi}(2^{-j} \xi) \) and \( \hat{\varphi}(2^{-j} \xi) \). Then the homogeneous Sobolev space \( H_p^s \) is also the space of all \( f \in L^q(\mathbb{R}^n) \) such that
\[
||\sum_{j \in \mathbb{Z}} 4^{sj} |\Delta_j f|^2|^{\frac{1}{2}}||_{L^p} < +\infty
\]
and (6) defines a norm on \( \dot{H}_p^s \) which is equivalent to the norm (2). Such definition is the so-called realization of homogeneous Sobolev spaces [B]. Recall that for \( s > 0, 1 < p < +\infty \), the nonhomogeneous Sobolev space \( H_p^s \) (without dot) is given by
\[
H_p^s \simeq L^p \cap \dot{H}_p^s.
\]

It remains clear that this definition of homogeneous spaces guarantees (3). Also we have the following Littlewood-Paley decomposition ([B], [SY]), for \( f \in \dot{H}_p^s \) and \( k \in \mathbb{Z} \),
\[
f = \lim_{m \to +\infty} \sum_{j = -m}^{m} \Delta_j(f) = \sum_{j \in \mathbb{Z}} \Delta_j(f),
\]
and
\[
S_k(f) = \lim_{m \to +\infty} \sum_{j = -m}^{k} \Delta_j(f) = \sum_{j \leq k} \Delta_j(f),
\]
(convergence in \( S' \)). Such decompositions will play an important role in the investigation of nonlinear estimates.

We return now to the equation (NLS). As mentioned above, we shall say that \( u \) is a self-similar solution of (NLS) if \( u \) satisfies (NLS) and
\[
\forall \lambda > 0, \ u(t, x) = u_\lambda(t, x) = \lambda^z u(\lambda^2 t, \lambda x).
\]
Therefore the space \( E \) of the solutions must satisfies
\[
||u_\lambda||_E = ||u||_E,
\]
for all \( u \in E \) and for all \( \lambda > 0 \).

On the other hand, it will be convenient that \( E \) allows to measure the Sobolev-regularity and the time decay of the solutions. To do so, let \( s, p, \) and \( \theta > 0 \) such that \( 0 \leq s < \frac{n}{p} \), \( 2 \leq p < +\infty \). Let \( E_s \) be the space of all Bochner measurable functions \( u : ]0, +\infty[ \to H_p^s \) such that
\[
||u||_{E_s} = \sup_{t > 0} t^\theta ||u(t, x)||_{H_p^s} < +\infty.
\]
By means of (3) and (10) we obtain the relation:
\[
\theta = \frac{1}{2} \left( \frac{2}{\alpha} - \frac{n}{p} + s \right).
\]
Other relations will be given in the next subsection and it will be explained that \( E_s \) depends only on the parameters \( \alpha \) and \( s \).
2.2. Admissibility of \( \alpha \) and range of regularity

In practice we solve \((NLS)\) through the integral equation

\[
\mathcal{I}L = S(t)f - i\gamma \int_0^1 S(t - \tau)(|u(\tau)|^\alpha u(\tau))d\tau,
\]

where \( S(t) \) is the unitary group \( e^{it\Delta} \) given by the linear Schrodinger equation.

The proofs of our results are based on the fixed point Theorem applied to \((IN)\) in the space \( E_s \) for suitable values of \( s \). Concerning the Schrodinger group, we will only use the well known \( L^p \to L^p \) estimate:

\[
\forall p \geq 2, \forall t \neq 0, \|S(t)f\|_{L^p} \leq C|t|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)}\|f\|_{L^p}.
\]

In view of \((13)\), it is then natural to study the mapping properties of the nonlinear operator \( u \to |u|^\alpha u \) from \( \dot{H}^s_p \) into \( \dot{H}^s_p \). In Section 4 below we prove the following nonlinear estimate: for \( \alpha \geq 1, \)

\[
\| |f|^\alpha f - |g|^\alpha g\|_{\dot{H}^s_p} \leq C\|f - g\|_{\dot{H}^s_p}\||f|_{\dot{H}^s_p}^\alpha + \|g|_{\dot{H}^s_p}^\alpha,\]

where \( p \) is given by

\[
p = \frac{n(\alpha + 2)}{\alpha n + \alpha}.
\]

and \( s \) satisfies

\[
0 \leq s < Min\left(\alpha, \frac{n}\alpha \right) \text{ if } \alpha \notin 2\mathbb{N}.
\]

\[
0 \leq s < \frac{n}{p} \text{ if } \alpha \in 2\mathbb{N}.
\]

This special value of \( p \) appears already in [CW1] for the study of \((NLS)\) in \( H^s(\mathbb{R}^n) = \dot{H}_x^s \). By virtue of \((3)\) such value is the unique one for which \((14)\) holds. When \( \alpha \notin 2\mathbb{N} \), the restriction \( s < \alpha \) comes from the lack of smoothness for the function \( x \to |x|^\alpha x \) at 0.

Such a restriction seems to be natural when one tries to establish nonlinear estimates of type \((14)\) ([CW1], [RS], [R], [RY]).

On the other hand, in order to apply the contraction principle to \((IN)\) in \( E_s \) (see the proof of Theorem 1) we need to impose that

\[
\theta(\alpha + 1) < 1 \text{ and } \frac{n}{2} \left(1 - \frac{2}{p}\right) < 1.
\]

Regarding \((18)\), in view of \((12)\) and \((15)\), we obtain then a second restriction on \( s \),

\[
s_{\min} = \frac{n}{2} \frac{\alpha + 2}{\alpha} < s < s_{\max} = \frac{n}{2} - \frac{\alpha + 2}{\alpha(\alpha + 1)}.
\]

We are now able to define the admissible \( \alpha \) and its corresponding range \( I_\alpha \) of regularity.
DEFINITION.
1) If $\alpha < 1$, $I_\alpha = \{0\} \cup [s_{\min}, s_{\max}].$
2) If $\alpha \geq 1$ and $\alpha \notin 2\mathbb{N}$, $I_\alpha = [s_{\min}, s_{\max}] \cap [0, \alpha[.$
3) If $\alpha \geq 1$ and $\alpha \in 2\mathbb{N}$, $I_\alpha = [s_{\min}, s_{\max}] \cap [0, +\infty[.$

DEFINITION. We will say that $\alpha$ is admissible if $I_\alpha$ is not empty.
In the following proposition, we characterize the admissible parameters $\alpha$.

PROPOSITION 1. Let $\alpha_0$ be the non negative root of the equation

$$(20) \quad Q(x) = nx^2 + x(n - 2) - 4 = 0$$

and let $a^+$ be the largest non negative root of the equation

$$(21) \quad P(x) = 2x^2 + x(2 - n) + 4 = 0.$$ 

1) If $\alpha \in 2\mathbb{N}$ or $n \leq 6$, $\alpha$ is admissible if and only if $\alpha > \alpha_0$.
2) For the case $n = 7$ and $\alpha \notin 2\mathbb{N}$, $\alpha$ is admissible if and only if

$$\alpha \in \left( \alpha_0, \frac{4}{n - 2} \right] \cap [1, +\infty[.$$ 

3) For a $\notin 2\mathbb{N}$ and $n \geq 8$, $\alpha$ is admissible if and only if

$$\alpha \in \left( \alpha_0, \frac{4}{n - 2} \right] \cap [1, +\infty[.$$ 

Proof. First we assume that $\alpha \geq 1$. In view of our definitions, it is then necessary that $s_{\max} > 0$ which is equivalent to $\alpha_0 < \alpha$ where $\alpha_0$ is the non negative root of the equation $Q(x) = 0$. Also, as mentioned above, for a $\notin 2\mathbb{N}$ nonlinear inequalities of type (14) required that $s_{\min} < a$ which is equivalent to $P(\alpha) > 0$. When $n \leq 7$, one can check that $P(\alpha) > 0$ for all $\alpha$. However, for $n \geq 8$, $P(\alpha)$ has two non negative roots $\alpha_-$ and $\alpha_+$ such that

$$\alpha_0 < \frac{4}{n - 2} < a_- < 1 < \alpha_+ < \frac{n}{2}.$$ 

In particular, if $n \geq 8$ we are led to suppose that $\alpha > \alpha_+.$

Next we consider the case $\alpha < 1$. In the same manner than previously, we must have $s_{\max} > 0$ which is equivalent to $\alpha > \alpha_0.$ Note that for $n = 1, 2$, and 3 we have $\alpha_0 \geq 1.$ Hence it is enough to consider the case $n \geq 4.$ But for $a < 1$ and $s > 0$, nonlinear estimates (14) fail. However, (14) remains true for $s = 0.$ So, we solve $(NLs)$ in $E_s$ for the special value $s = 0$ (i.e. in weighted in time Lebesgue spaces). In particular we need to impose that $0 \in I_\alpha$ which is equivalent to $\alpha < \frac{4}{n-2}$. 

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
3. Statements of the main results

Throughout this section, for admissible $\alpha$ and $s \in I_\alpha$, we consider the space $E_s$ defined by (11) where $p$ and $\theta$ are fixed as above by

$$p = \frac{n(\alpha + 2)}{\alpha s + n} \quad \text{and} \quad \theta = \frac{1}{2} \left( \frac{2}{\alpha} - \frac{n}{p} + s \right).$$

Our first result states the existence of global solutions for all initial data $f$ such that $u_0(t,x) = (S(t)f)(x)$ belongs to $E_s$ with small norm.

**Theorem 1.** Let $\alpha$ admissible, $s \in I_\alpha$, and set $u_0(t,x) = [S(t)f](x)$. There exists $\varepsilon > 0$ such that if

$$\|u_0\|_{E_s} \leq \varepsilon,$$

then there exists a unique solution $u(t,x) \in E_s$ of (NLS) with $\|u\|_{E_s} \leq 2\varepsilon$.

**Remark 1.** Theorem 1 contained the results of the recent works of T. Cazenave and F. Weissler ([CW2] and [CW3]) for which only the cases $s = 0$ and $s = 1$ are considered.

**Remark 2.** One can easily obtain a local version of Theorem 1 as follows. Let $T > 0$ and consider $f$ such that

$$\sup_{t \in [0,T]} t^\theta \|S(t)f\|_{\dot{H}_p^s} \leq \varepsilon.$$

Then there exists $u(t,x)$ a unique local solution to (NLS) which belongs to $E_{s,T}$ the set of tempered distributions $\nu(t,x)$ such that

$$\sup_{t \in [0,T]} t^\theta \|\nu(t,x)\|_{\dot{H}_p^s} \leq 2\varepsilon.$$

Theorem 1 raises the problem of the characterization of initial data $f$ for which $u_0(t,x) = (S(t)f)(x)$ belongs to $E_s$. Our considerations will be restricted to two cases. First we study the homogeneous initial data $\hat{f}$ (Theorem 2) and next we will consider the case when $\hat{f} \in H^s$ (see Remark 6 below).

The following Theorem gives sufficient condition and necessary condition on $f$ for which $u_0 \in E_s$. This conditions will be based on the operator $\Delta_0$ (see subsection 2.1) defined by

$$\Delta_0(f) = \psi * f.$$

**Theorem 2.** Let $\hat{f}$ be a homogeneous distribution such that (1) holds.

1) If $[S(t)f](x) \in E_s$ then $\Delta_0 \hat{f} \in L^p$.

2) If $\Delta_0 \hat{f} \in L^p$ then $[S(t)f](x) \in E_s$.

From Theorems 1 and 2, we deduce the following result about the existence of self-similar solutions for (NLS) equations.

**Theorem 3.** Let $\alpha$ admissible and $\Omega \in C^\infty(S^{n-1})$. Set

$$f(x) = \frac{\Omega(\frac{x}{|x|^2})}{|x|^\frac{n}{2}}.$$
and \( u_0(t,x) = (S(t)f)(x) \). Then

\[
\|u_0\|_{E,} \leq C\|\Omega\|_{C^*}.
\]

In particular, there exists \( \varepsilon > 0 \) such that if

\[
\|\Omega\|_{C^*} \leq \varepsilon,
\]

then there exists a unique global self-similar solution of (NLS) with \( f \) as initial data.

**Remark 3.** - Related to the study of blow-up phenomenon, there exists previous works on self-similar solutions for the pseudoconformal (NLS) equations (see [KW] and [RX]). Those works are based on the study of the semilinear elliptic equation satisfied by the profile \( W \) but only for the case of radially symmetric solutions. In Theorem 3 there is no any radial symmetric assumptions on the solutions.

**Remark 4.** - Theorem 3 proves the existence of no necessarily radial-symmetric self-similar solutions for a larger range of nonlinearity than previously known. Recently in [CW2] and [CW3] self-similar solutions of (NLS) have been built in the following cases:

- for \( a > \alpha_0 \) when \( n = 1,2,3,4,5,6 \).
- for \( \alpha \in ]\alpha_0, \frac{4}{n-4} [ \) if \( n = 5,6 \).
- for \(\alpha \in ]\alpha_0, \frac{4}{n-4} [\cup ]1, \frac{4}{3} [ \) if \( n = 7 \).
- for \(\alpha \in ]\alpha_0, \frac{4}{n-2} [ \) if \( n \geq 8 \).

This above results are all included in Theorem 3. More precisely we obtain existence of self-similar solutions of (NLS):

- for \( a > \alpha_0 \) when \( n = 1,2,3,4,5,6 \).
- for \( \alpha \in ]\alpha_0, \frac{4}{3} [\cup ]1, +\infty [ \) in the special case \( n = 7 \).
- for \(\alpha \in ]\alpha_0, \frac{4}{n-2} [\cup ]a^+, +\infty [ \) if \( n \geq 8 \).

When \( \alpha \geq 1 \), Theorem 3 also prove that, for \( t > 0 \), the self-similar solutions of (NLS) are in \( \dot{H}^s \) for all \( s \in I_\alpha \).

**Remark 5.** - Also, in [CW2] and [CW3] the existence of self-similar solutions have been proved for homogeneous initial data \( f \) of the form

\[
f(x) = \frac{P_k(x)}{|x|^{k+\frac{2}{n}}},
\]

where \( P_k \) is an homogeneous polynomial of degree \( k \). Again Theorem 3 allows to consider a large scale of homogeneous functions than in the above cases.

**Remark 6.** - Now, we consider initial data \( f \) in \( H^\gamma \). We assume again that \( \alpha \) is admissible and satisfies the following addiional condition:

\[
\alpha \geq \alpha_c = \frac{4}{n-2},
\]

(\( \alpha_c \) is the so called \( H^1 \) critical value). Let \( f \in \dot{H}^s(\mathbb{R}^n) \) where:

\[
\hat{s} > s_c = \frac{n}{2} - \frac{2}{\alpha},
\]

\[
 (22) \]
Thanks to Sobolev embedding, one can check that there always exists $s \in I$ such that $H^s \subset E$. So, for $0 < t < T$ we have

$$t^\theta \|S(t)f\|_{H^s} \leq t^\theta \|S(t)f\|_{H^i} \leq t^\theta \|f\|_{H^s} \leq T^\theta \|f\|_{H^s}.$$  

First, choosing $T$ small enough, it follows that $\|S(t)f\|_{E,s} \leq \varepsilon$ and, from the local version of Theorem 1 there exists a unique solution $u \in E_s$ of (NLS) with $f$ as initial data. Next using the results of [CW1], there exists $T' > 0$ and a solution $w \in C([0, T'], H^s)$ of (NLS) with $f$ as initial data. Again by the embedding $H^s \subset E$, it remains clear that $w \in E_{s,T'}(T'' > 0)$. Now, uniqueness now prove that $u = w$ on $[0, \min(T, T', T'')]$. Thus our “weak” solutions are in fact some “classical” solutions of (NLS).

4. Nonlinear inequalities

Our purpose in this section is to prove nonlinear inequalities of type (14).

**Proposition 2.** Let $\alpha > 0$, $s \in I$, and let

$$p = \frac{n(\alpha + 2)}{\alpha s + n}$$

It holds

$$\| |f|^\alpha f\|_{H^s} \leq C \|f\|^{\alpha + 1}_{H^s},$$

and

$$\| |f|^\alpha f - |g|^\alpha g\|_{H^s} \leq C \|f - g\|_{H^s} \|f\|^\alpha_{H^s} + \|g\|^\alpha_{H^s}.$$  

To prove Proposition 2 we will use the two following Lemmas. The first Lemma is expressed by the following result of Runst and Sickel [RS] (Theorem 5-4-3/1 (ii), p. 363-364) for the nonhomogeneous Sobolev spaces. There are no difficulties to overtake this result to the homogeneous situation.

**Lemma 1.** Let $\alpha > 0$, $s > 0$ and $1 < r < \infty$ such that

$$s < \min\left(\frac{n}{r}, \alpha + 1\right) \text{ and } (\alpha + 1)\left(\frac{n}{r} - s\right) \leq n.$$  

Define $t$ by

$$t = s + \left(\frac{n}{\alpha + 1}\left(\frac{n}{r} - s\right)\right).$$

a) Then, for all $f \in \dot{H}^s_r(\mathbb{R}^n)$ we have:

$$\| |f|^\alpha f\|_{H^i} \leq C \|f\|^{\alpha + 1}_{H^i},$$

and

$$\| |f|^\alpha f\|_{H^i} \leq C \|f\|^{\alpha + 1}_{H^i}.$$  

b) Furthermore, if $\alpha$ is an even integer (respectively an odd integer) then (26) (respectively (27)) holds without the restriction $s < \alpha + 1$.  

TOME 77 - 1998 - N°10
As mentioned above the restriction \( s < \alpha + 1 \) comes from the lack of smoothness of \( x \to |x|^\alpha x \) at \( 0 \). The restriction \( (\alpha + 1) \left( \frac{p}{p} - s \right) \leq n \) guarantee that \( |u|^\alpha u \) belongs to the space of distributions \( \mathcal{S}' \). Indeed, Sobolev embedding guarantees that \( u \in L^z \) for some \( z \geq \alpha + 1 \).

The second Lemma deals with product of functions in Sobolev spaces. It will be used in the proof of (25).

**Lemma 2.** Let \( 1 < p_0, p_1, p_2 < +\infty \) and let \( 0 \leq s < \frac{n}{p_1} \) such that

\[
s + \frac{n}{p_0} = \frac{n}{p_1} + \frac{n}{p_2}.
\]

Then

\[
\|fg\|_{\dot{H}^s_{p_0}} \leq C\|f\|_{\dot{H}^s_{p_1}}\|g\|_{\dot{H}^s_{p_2}}.
\]

Proof. First note that for \( s = 0 \), (29) is just the Holder’s inequality. Hence we only consider the case \( s > 0 \). Before proving this result let us recall a supplement about Sobolev spaces (see for example [Ma] about nonhomogeneous situation). For \( s > 0 \) and \( R > 0 \), there exists \( C > 0 \) such that

\[
\left\| \sum_{j \in \mathbb{Z}} f_j \right\|_{\dot{H}^s_{p_0}} \leq C\left\| \left\{ \sum_{j \in \mathbb{Z}} 4^{sj} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{p_0},
\]

holds for any sequence \( \{f_j\}_{j \in \mathbb{Z}} \) of functions such that \( \hat{f}_j \) is supported in the ball \( B_j = \{ |\xi| \leq R 2^j \} \).

On the other hand, using (7) and (8) there are no difficulties to decompose the product \( fg \ ( f \in \dot{H}^{s_1}_{p_1}, g \in \dot{H}^{s_2}_{p_2} \) as follows

\[
fg = \Pi_1(f, g) + \Pi_2(g, f),
\]

where

\[
\Pi_1(f, g) = \sum_{j \in \mathbb{Z}} \Delta_j(f)S_j(g)
\]

and

\[
\Pi_2(f, g) = \sum_{j \in \mathbb{Z}} \Delta_j(f)S_{j-1}(g)
\]

We only prove the estimate for \( \Pi_1 (f, g) \). The estimate for \( \Pi_2 (g, f) \) is similar.

Since the Fourier transform of \( \Delta_j(f)S_j(g) \) is supported in the ball \( B_j = B(0,2^{j+2}) \), it follows by (30) that:

\[
\|\Pi_1(f, g)\|_{\dot{H}^s_{p_0}} \leq C\left\| \left\{ \sum_{j \in \mathbb{Z}} 4^{sj} |\Delta_j(f)S_j(g)|^2 \right\}^{\frac{1}{2}} \right\|_{p_0}.
\]

Now recall that \( S_j(g) = \varphi_j * g \) where \( \varphi_j(x) = 2^{nj} \varphi(2^j x) \). Then one can prove that

\[
|S_j(g)(x)| \leq C \cdot M(g)(x).
\]

for all \( j \in \mathbb{Z} \) and all \( x \in \mathbb{R}^n \) and where \( M(g) \) is the Hardy-Littlewood maximal function (see [St] p.62-64).
Because of (28) our condition $0 < s < \frac{n}{p_2}$ guarantees that $\frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{r_2}$, where $0 < \frac{1}{r_2} = \frac{1}{p_2} - \frac{s}{n} < 1$. Hence by Holder's inequality we have:

$$||\Pi(f,g)||_{\dot{H}^s_{r_0}} \leq C \left\| M(g) \left[ \sum_{j \in I} 4^{|s_j|} |\Delta_j(f)|^2 \right]^{\frac{1}{2}} \right\|_{p_1},$$

$$\leq C \left\| M(g) \right\|_{L^r_{2}} \left\| \left[ \sum_{j \in I} 4^{|s_j|} |\Delta_j(f)|^2 \right]^{\frac{1}{2}} \right\|_{p_1}.$$ 

To complete the proof we use the boundedness of $g \rightarrow M(g)$ on $L^{r_2}$ and the Sobolev embedding $\dot{H}^s_{p_2} \hookrightarrow L^{r_2}$.

**Proof of Proposition 2.** In fact (24) is a direct application of (26) with $r = p$ and $t = p'$. Indeed for $s \in I_\alpha$ recall that $s < \min\left(\frac{n}{p}, \alpha + 1\right)$ and by (15)

$$p' = \frac{n}{s + (\alpha + 1)\left(\frac{n}{p} - s\right)}.$$ 

Moreover, the restriction

$$(\alpha + 1)\left(\frac{n}{p} - s\right) \leq n$$

is, again by (15), equivalent to

$$\frac{(\alpha + 1)(n - 2s)}{\alpha + 2} \leq n$$

which is always satisfied.

Now we prove (25) and we will only consider the case when $f, g$ are real valued functions. There are no difficulties to generalize this particular case to the complex valued one.

Because of $\alpha > 0$,

$$|a|^{\alpha} a - |b|^{\alpha} b = (\alpha + 1)(a - b) \int_{b}^{1} |t(a - b) + b|^{\alpha} dt$$

holds for all $a, b \in \mathbb{R}$. Hence in our situation we have:

$$|f|^{\alpha} f - |g|^{\alpha} g = (\alpha + 1)(f - g) \int_{0}^{1} |t(f - g) + g|^{\alpha} dt.$$ 

Let

$$p_2 = \frac{n}{s + \alpha\left(\frac{n}{p} - s\right)},$$

it is easy to see that $s + \frac{n}{p'} = \frac{n}{p} + \frac{n}{p_2}$ and $s < \frac{n}{p_2}$. Since $\alpha$ is admissible and $s \in I_\alpha$, we obtain:

$$s < \frac{n}{p} < \frac{n}{p'}.$$
Applying Lemma 2, we obtain that

\[ ||f^n f - g^n g||_{\dot{H}^s} \leq C ||f - g||_{\dot{H}^s} ||G(f, g)||_{\dot{H}^s}, \]

where \( G(f, g) = \int_0^t |t(t - s)| + g^n dt. \)

Now recall that \( 0 < s < \frac{n}{p}, \ s < \alpha \) and

\[ \alpha \left( \frac{n}{p} - s \right) \leq (\alpha + 1) \left( \frac{n}{p} - s \right) \leq n. \]

Lemma 1 applied to \( G(f, g) \) prove that

\[ ||G(f, g)||_{\dot{H}^s} \leq C( ||f||_{\dot{H}^s}^\alpha + ||g||_{\dot{H}^s}^\alpha ). \]

Finally (25) follows from (32) and (33).

5. Proofs of the main results

5.1. Proof of Theorem 1

We shall make use the fixed point Theorem to solve the integral equation (IN):

\[ u = L(u) = S(t) f - i\gamma \int_0^t S(t - \tau)(|u|^\alpha u(\tau, \cdot)) d\tau. \]

Let \( \alpha \) admissible and \( s \in I_\alpha \). We solve (IN) in a small ball of \( E_s \). First by virtue of the estimate (13) we have

\[ t^\alpha ||L(u)(t, \cdot)||_{\dot{H}^s} \leq t^\alpha ||S(t) f||_{\dot{H}^s} + |\gamma| t^\alpha \int_0^t ||S(t - \tau)|u|^\alpha u||_{\dot{H}^s} d\tau \]

\[ \leq ||u_0||_{E_s} + C t^\alpha \int_0^t |t - \tau|^{-\frac{\alpha}{2}(1 - \frac{s}{p})} |||u|^\alpha u(\tau, \cdot)||_{\dot{H}^s} d\tau. \]

Hence from Proposition 2 it follows

\[ t^\alpha ||L(u)(t, \cdot)||_{\dot{H}^s} \leq \varepsilon + C t^\alpha \int_0^t |t - \tau|^{-\frac{\alpha}{2}(1 - \frac{s}{p})} ||u(\tau, \cdot)||_{\dot{H}^s}^{\alpha + 1} d\tau \]

\[ \leq \varepsilon + C ||u||_{E_s}^{\alpha + 1} t^\alpha \int_0^t |t - \tau|^{-\frac{\alpha}{2}(1 - \frac{s}{p})} \tau^{-\theta(\alpha + 1)} d\tau \]

Because of (12) and (15) we have \( 1 - \frac{n}{2}(1 - \frac{s}{p}) = \theta \alpha = 0 \), it follows that

\[ t^\alpha ||L(u)(t, \cdot)||_{\dot{H}^s} \leq \varepsilon + C ||u||_{E_s}^{\alpha + 1}. \]

Therefore we have

\[ ||L(u)||_{E_s} \leq \varepsilon + C ||u||_{E_s}^{\alpha + 1}. \]

Choose \( \varepsilon \leq \left( \frac{1}{C^{2\alpha + 1}} \right)^\frac{1}{2} \) and let \( B_\varepsilon = \{ u \in E_s, ||u||_{E} \leq 2\varepsilon \} \), then \( L(B_\varepsilon) \subset B_\varepsilon \).
Next as above, by using (25) instead of (24) we derive
\[
|||u^\alpha u(\tau, \cdot) - |v|^\alpha v(\tau, \cdot)|||_{\dot{H}^s_p} \leq C||u(\tau, \cdot) - v(\tau, \cdot)||_{\dot{H}^s_p}|||u(\tau, \cdot)|||_{\dot{H}^s_p}^\alpha + ||v(\tau, \cdot)|||_{\dot{H}^s_p},
\]
so that
\[
|||u^\alpha u(\tau, \cdot) - |v|^\alpha v(\tau, \cdot)|||_{\dot{H}^s_p} \leq C\tau^{-\theta(\alpha+1)}||u - v|||_{E,s}||u|||_{E,s}^\alpha + ||v|||_{E,s}.
\]
Assume that \(||u|||_{E,s} \leq 2\varepsilon\) and \(||v|||_{E,s} \leq 2\varepsilon\). The same reasoning as above gives that:
\[
|||L(u) - L(v)|||_{E,s} \leq C\varepsilon^{\alpha+1}||u - v|||_{E,s} \leq \frac{1}{2}||u - v|||_{E,s},
\]
for \(\varepsilon\) small enough.

This implies that \(L\) is a contraction map from \(B_s\) into \(B_s\). Thus, for all admissible \(\alpha\) and for all \(s \in I_s\), there exists a unique solution \(u \in E_s\) of (NLS) with \(||u|||_{E,s} \leq 2\varepsilon\).

5.2. Proof of Theorem 2

First observe that
\[
u_0(t, x) = t^{-\frac{2}{\alpha}}[S(1)f]\left(\frac{x}{t}\right),
\]
it follows
\[
t^\theta ||u_0(t, \cdot)|||_{\dot{H}^s_p} = t^{\frac{2}{\alpha} - s + \frac{n}{p}}||S(1)f|||_{\dot{H}^s_p}.
\]
Since
\[
\theta = \frac{2}{\alpha} - s + \frac{n}{p} = 0.
\]
we obtain the following Lemma

**Lemma 3.** Suppose that (1) holds. Then \(u_0(t, x) = [S(t)f](x) \in E_s\) if and only if \(S(1)f \in \dot{H}^s_p\).

To prove the first part of Theorem 2, we only use that \(S(1)f \in \dot{H}^s_p\). Let \(h \in S\) be such that \(\hat{h}\) is supported by \(\{\xi : \frac{1}{4} \leq |\xi| \leq 4\}\) and \(\hat{h}(\xi) = 1\) for \(\frac{1}{4} \leq |\xi| \leq 2\). Then we have
\[
\psi = (S(1)\psi) \ast (S(-1)h).
\]
Since \(S(-1)h \in S(\mathbb{R}^n)\), we get
\[
|||\Delta_0(f)|||_{L^p} \leq ||S(-1)h|||_1|||S_0(S(1)f)|||_{L^p} \leq C||S(1)f|||_{\dot{H}^s_p},
\]
which prove part 1) of Theorem 2.

To prove the second part, by means of Lemma 3 it will be sufficient to prove that \(F = S(1)f \in \dot{H}^s_p\). We will prove that \(F \in \dot{B}^{\alpha,1}_p\) (Besov space) which is better than what we want. Indeed,
\[
(34) \quad ||F|||_{\dot{H}^s_p} \leq ||F|||_{\dot{B}^{\alpha,1}_p} = \sum_{j \in \mathbb{Z}} 2^{js}||\Delta_j(f)|||_{L^p}.
\]
To do so, we prove the following estimate
\[
||F|||_{\dot{B}^{\alpha,1}_p} \leq C||\Delta_0(f)|||_{L^p}.
\]
Set \(\varphi = S(1)\varphi\) and \(f_1 = (1 - \varphi) \ast f\) (where \(\varphi\) is given by (5)). We decompose \(F\) as follows
\[
F = F_1 + F_2,
\]
where \(F_1 = S(1)(f_1)\) and \(F_2 = \varphi \ast f\).
(i) Estimate of $F_1$

Since $\hat{F}_1$ is supported in the set $\{\xi/|\xi| \geq 1\}$, it follows that $\Delta_j(F_1) = 0$ for $j \leq -1$. For $j \geq 0$, we observe that:

$$||\Delta_j(F_1)||_{L^p} = ||S(1)\Delta_j(f_1)||_{L^p} \leq C||\Delta_j(f_1)||_{L^{p'}}.$$ 

and

$$||\Delta_j(f_1)||_{L^{p'}} \leq C||\Delta_j(f)||_{L^{p'}}$$

Moreover, condition (1) implies

$$||\Delta_j(f)||_{L^{p'}} = 2^j(\frac{3}{s} - \frac{1}{p'})||\Delta_0(f)||_{L^{p'}}.$$ 

Therefore

$$\sum_{j \geq 0} 2^{sj}||\Delta_j(F_1)||_{L^p} \leq C\sum_{j \geq 0} 2^j(\frac{s+\frac{3}{s} - \frac{1}{p'}}{s})||\Delta_0(f)||_{L^{p'}}.$$ 

Since $s + \frac{2}{\alpha - \frac{n}{p'}} < 0$ (here we use $s < s_{\max}$) it follows

$$||F_1||_{H^s} \leq C||\Delta_0(f)||_{L^{p'}}.$$

(ii) Estimate of $F_2$

In this case $\Delta_j(F_2) = 0$ for $j \geq 2$ because $\hat{F}_2$ is supported in the ball $\{\xi/|\xi| \leq 2\}$. For $j \leq 1$, one can write

$$\Delta_j(F_2) = \Delta_j(f) = \hat{\Delta}_j(\hat{\varphi}),$$

where $\hat{\Delta}_j = \sum_{\ell=-2}^{2} \Delta_{j+\ell}$. Young's inequalities gives

$$\sum_{j \leq 1} 2^{js}||\Delta_j(F_2)||_{L^p} \leq \sum_{j \leq 1} 2^{s_j}||\Delta_j(f)||_{L^p}||\hat{\Delta}_j(\hat{\varphi})||_{L^1}.$$ 

Moreover, observe that $||\hat{\Delta}_j(\hat{\varphi})||_{L^1} = C||\hat{\varphi}||_1$ and from (1) we have

$$\Delta_j(f)(x) = 2^{j\frac{3}{s}}\Delta_0(f)(2^jx).$$

Hence

$$\sum_{j \leq 1} 2^{js}||\Delta_j(F_2)||_{L^p} \leq C||\Delta_0(f)||_{L^p} \sum_{j \leq 1} 2^j(\frac{s+\frac{3}{s} - \frac{n}{p'}}{s}).$$ 

Finally note that $s + \frac{2}{\alpha - \frac{n}{p'}} > 0$ (here we use $s > s_{\min}$) and therefore

$$\sum_{j \leq 1} 2^{js}||\Delta_j(F_2)||_{L^p} \leq C||\Delta_0(f)||_{L^p}.$$ 

Now recall that $p' \leq p$, it follows by Bernstein's inequality that

$$||\Delta_0(f)||_{L^p} \leq C||\Delta_0(f)||_{L^{p'}}$$

which complete the proof.
5.3. Proof of Theorem 3

5.3.1. Part 1)

Thanks to Theorem 2, it is enough to prove that \( \Delta_0(f) \in L^{p'} \). We will need the following Lemma (see our previous paper [RY] on self-similar solutions for the nonlinear wave equation).

**Lemma 4.** Let \( \Omega \in C^k(S^{n-1}) \), \( k \geq 0 \) and let \( f(x) = \frac{\Omega(\frac{x}{|x|^d})}{|x|^d} \)

where \( 0 < d < n \). Then,

\[
|\Delta_0(f)(x)| \leq C||\Omega||_{C^k}(1 + |x|)^{-k-d}.
\]

Consider now \( f \) such as in (1). From Lemma 4, if \( \Omega \) belongs to \( C^k(S^{n-1}) \) where \( k \) satisfies

\[
\alpha \left( \frac{n}{\alpha} \right) > n.
\]

then \( |\Delta_0(f)| \in L^{p'} \). Because \( p' \geq 1 \), it is enough to take \( k = n \).

5.3.2. Part 2)

First by Part 1) we obtain that

\[
||u_0||_{E_\alpha} \leq \varepsilon,
\]

where \( u_0(t,x) = S(t)f(x) \). By virtue of Theorem 1 there exists a unique solution \( u(t,x) \) of (NLS) with \( f \) as initial data. On the other hand, \( u_\lambda \) is also a solution of (NLS) and \( u_\lambda(0,x) = u(0,x) \) by (1). Thus by uniqueness, we obtain that \( u_\lambda = u \) for all \( \lambda > 0 \).

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