Strong and Weak Convergence of the Sequence of Successive Approximations for Quasi-Nonexpansive Mappings

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INTRODUCTION

Let $X$ be a real Banach space, $D$ a closed subset of $X$, and $T$ a continuous mapping of $D$ into $X$. Assuming that for a given $x_0$ in $D$ and $\lambda$ in $(0, 1)$ the sequence of iterates $\{x_n\}$ determined by the successive iteration method

(i) $x_n = T(x_{n-1}) = T^n(x_0), \ n = 1, 2, 3, \ldots$,

or the simple iteration method

(ii) $x_n = T_\lambda(x_{n-1}) = T_\lambda^n(x_0), \ T_\lambda = \lambda I + (1 - \lambda) T, \ n = 1, 2, \ldots$,

is well-defined, the purpose of this paper is to obtain conditions, as general as possible, on $T$, $D$, and $X$ which would guarantee the convergence (i.e., the strong convergence) and, under weaker conditions on $T$, the weak convergence of the iterates $\{x_n\}$ to a fixed point of $T$ in $D$. It will be seen from the survey below that our results (which, for the case of convergence of (i) and (ii), are the best possible) unify and extend to a larger class of mappings and, in some cases, to more general Banach spaces many of the known results concerning the convergence and the weak convergence of (i) and (ii) for nonlinear mappings.

Concerning the convergence of (i) and (ii). To describe our problem more precisely and to put our discussion and results in proper perspective, we first outline in chronological order the main known results concerning the convergence of the methods (i) and (ii).

The first basic result is the classical Picard–Banach–Caccioppoli principle which essentially states that if $T$ is a strict contraction of $D$ into $D$ (i.e.,

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\[ \| T(x) - T(y) \| \leq q \| x - y \| \text{ for all } x, y \text{ in } D \text{ and some } q < 1, \]
given by (i) converges to a unique fixed point of \( T \). It is known (e.g., a rotation of the unit disc) that if \( T \) is assumed to be only nonexpansive on \( D \) (i.e., \( \| T(x) - T(y) \| \leq \| x - y \| \forall x, y \in D \)), then \( T^n(x_0) \) need no longer converge; in fact, in general, \( T \) need not have a fixed point (see [11]). However, it was shown by Krasnoselsky [24] that if \( X \) is uniformly convex, \( D \) a closed bounded convex subset of \( X \), and \( T \) a compact (i.e., \( T \) is continuous and \( T(D) \) is relatively compact) map of \( D \) into \( D \), then \( \{ T^n(x_0) \} \) converges to a fixed point of \( T \). Schaefer [40] extended the result of [24] to the case when \( \{ x_n \} \) is given by (ii), while Edelstein [16] extended it to the case when \( X \) is strictly convex. In the case that \( X \) is a Hilbert space and \( D \) a closed ball \( B(0, r) \), Petryshyn [31] extended the results of [24, 40] to demicompact (see Section 2) nonexpansive mappings \( T \) of \( B \) into \( X \) which satisfied the Leray-Schauder conditions on the boundary \( \partial B \) of \( B \). The method used in [31] is the so-called iteration-retraction method which, in view of the results of [12], can only work for Hilbert spaces and which reduces to (ii) in case \( T \) maps \( B \) into itself. Browder and Petryshyn [8, 9] carried further the results of [24, 40, 31], investigating the convergence of \( \{ x_n \} \) given by (i) and/or by (ii) for nonexpansive maps \( T \) of \( X \) into \( X \) which are asymptotically regular (see Section 2) and for which \( I - T \) maps bounded closed sets into closed sets. See also [34] where similar results are obtained for maps from a closed bounded convex subset \( D \) of \( X \) into \( D \). Further extensions concerning the convergence of (i) and (ii) have been obtained by Diaz and Metcalf [13, 14] and by Dotson [15] for quasi-nonexpansive maps (i.e., \( T \) is such that \( \| T(x) - p \| \leq \| x - p \| \) for \( x \) in \( D \) and \( p \in F(T) \), set of fixed points of \( T \)) and by Outlaw [30] for certain nonexpansive mappings. Petryshyn and Tucker [38] considered the case of nonexpansive and \( \mathcal{P}_1 \)-compact maps while Petryshyn [35] studied the convergence of (ii) when \( T \) is nonexpansive and condensing (see Section 2).

It is interesting to observe that, in order to establish the convergence of \( \{ T^n(x_0) \} \) or \( \{ T^n(x_0) \} \) to a fixed point of \( T \), each of the above authors had to impose certain additional conditions on the nonexpansive or quasi-nonexpansive map \( T \) with \( F(T) \neq \emptyset \). J. Lindenstrauss informed the first author that he had constructed an example of a nonexpansive map \( T \) of the unit ball \( B(0, 1) \) of a Hilbert space into \( B(0, 1) \) for which \( \{ T^n(x_0) \} \) does not converge to a fixed point of \( T \) although \( F(T) \neq F(T) \) and \( T \) is asymptotically regular on \( B \). Thus, even for a nonexpansive map \( T \) of \( B \) into \( B \) with \( F(T) \neq \emptyset \) and with \( T \) asymptotically regular on \( B \), some additional condition has to be imposed on \( T \) for the sequence \( \{ x_n \} \) given by (ii) to be convergent to a fixed point of \( T \).

In Section 1 we show in Theorem 1.1 that if \( D \) is a closed subset of a Banach space \( X \) and \( T \) is a continuous map of \( D \) into \( X \) such that \( F(T) \neq \emptyset \), \( T \) is
quasi-nonexpansive, and \( T^n(x_0) \in D \) for \( n \geq 1 \) and some \( x_0 \) in \( D \), then \( \{T^n(x_0)\} \) converges to a fixed point of \( T \) in \( D \) if and only if

\[(iii) \quad d(T^n(x_0), F(T)) \to 0 \text{ as } n \to \infty,\]

where \( d(A, C) \) denotes the distance between the sets \( A \) and \( C \). In case \( D \) is also convex and \( T_\lambda^n(x_0) \in D \) for \( n \geq 1 \) for some \( x_0 \) in \( D \) and \( \lambda \) in \( (0, 1) \), then under the same conditions on \( T \), \( \{T_\lambda^n(x_0)\} \) converges to a fixed point of \( T \) if and only if

\[(iv) \quad d(T_\lambda^n(x_0), F(T)) \to 0 \text{ as } n \to \infty.\]

The characterization Theorem 1.1 is then used to obtain two other new theorems in Section 1 which exhibit sufficient conditions for the convergence of (i) and similarly of (ii) which are more practical.

In Section 2 we use Theorems 1.1, 1.1′, 1.2, and 1.3 to obtain new as well as some known results concerning the convergence of \( \{x_n\} \) given by (ii) for various special classes of mappings of quasi-nonexpansive and 1-set and/or 1-ball contractive mappings. In particular we deduce from our theorems and corollaries the known convergence results obtained in [8, 14, 15, 24, 30, 34, 35, 38, 40]. Some new results are also contained in Section 2.

In Section 3, Part 1, we study the convergence of (i) and (ii) under the assumption that \( T \) is strictly quasi-nonexpansive and that \( T \) satisfies the so-called Frum–Ketkov condition [18]. All these convergence results appear to be new. In Part 2 of Section 3 we study the convergence of (i) and (ii) for the case when \( X \) is a strictly convex Banach \( \pi \)-space and \( T \) is a quasi-nonexpansive mapping which is \( P_1 \)-compact in the sense of Petryshyn [32, 34]. As special cases we deduce the corresponding convergence results of Petryshyn and Tucker [38].

Concerning the weak convergence of (i) and (ii). We have noted that, even for a nonexpansive map \( T \) of a unit ball in a Hilbert space into itself, for \( \{x_n\} \) given by (i) or by (ii) to converge to a fixed point of \( T \) the additional condition (iii) or (iv) has to be imposed. The question arises whether \( \{x_n\} \) converges weakly to a fixed point of \( T \) without any additional conditions.

In this direction the first result is due to Schaefer [40] which says that if \( X \) is a Hilbert space, \( D \) a closed bounded convex subset of \( X \), and \( T \) a nonexpansive weakly continuous map of \( D \) into \( D \), then \( \{x_n\} \) given by (ii) converges weakly to a fixed point of \( T \). Browder and Petryshyn [8] have shown that if \( X \) is a reflexive Banach space and \( T \) a nonexpansive and asymptotically regular map of \( D = X \) into \( X \) such that \( I - T \) is demiclosed (see Section 4) and \( F(T) \neq \varnothing \), then a weak limit of a weakly convergent subsequence of \( \{T^n(x_0)\} \) is a fixed point of \( T \), and moreover, \( \{x_n\} \) converges weakly to a fixed point of \( T \) if \( F(T) \) contains just one point. In particular, if \( X \) is a Hilbert space or a Banach space with a weakly continuous duality mapping and \( \{x_n\} \) is
determined by (ii), then $T_\lambda$ is asymptotically regular and $I - T_\lambda$ is demiclosed and so weak limit points of \{\(T_\lambda^n(x_0)\}\} are fixed points of $T$. Opial [29] extended the results of [40, 8] by showing that if $X$ is a uniformly convex Banach space with a weakly continuous duality map (and, in particular, a Hilbert space) and if $T$ is a nonexpansive asymptotically regular map of a closed convex subset $D$ of $X$ into $D$ with $F(T) \neq \emptyset$, then \{\(x_n\)\} given by (i) is weakly convergent to a fixed point of $T$. In the same setting, if \{\(x_n\)\} is given by (ii), then \(\{x_n\}\) converges weakly to a fixed point of $T$ without the condition that $T$ be asymptotically regular since for $T_\lambda$ the latter follows from the results in [8]. Results related to [40, 8, 29] have been also obtained by Dotson [15] for the so-called Mann method. In case of Hilbert spaces, further extensions have been obtained by Browder and Petryshyn [9].

The purpose of Section 4 is to unify and extend the results of [8, 29, 40] and of [3, 15] to various classes of quasi-nonexpansive mappings (for which the map $I - T$ or $I - T_\lambda$ satisfies a condition (see Condition 3.1 in Theorem 3.1) which is weaker than the demiclosedness condition used in [8, 15, 29]) and to Banach spaces which are slightly more general than those used in [15, 29].

1. CONVERGENCE OF ITERATES OF QUASI-NONEXPANSIVE MAPPINGS

In this section the investigation of the convergence of iterates of quasi-nonexpansive mappings is carried out usually under the assumption that the set of fixed point is already known to be nonempty. A simple yet central approach is developed from which varyingly-derived known results of a number of authors as well as some new ones will be deduced.

Let $X$ be a real Banach space with norm $\| \cdot \|$. If $A$ and $B$ are two sets in $X$, denote the distance between $A$ and $B$ by

\[ d(A, B) = \inf \{ \| a - b \| : a \in A, b \in B \}, \]

and the distance between a point $p$ and a set $A$ by $d(p, A)$. If $T$ maps $D \subset X$ into $X$, then denote the set of fixed points of $T$ in $D$ by $F_D(T)$, or simply $F(T)$, whenever the underlying set is clear.

The first basic result of this section is the following new theorem characterizing the convergence of iterates.

**Theorem 1.1.** Let $D$ be a closed subset of a Banach space $X$ and let $T$ map $D$ continuously into $X$ such that

1. $F(T) \neq \emptyset$
2. For each $x \in D$ and every $p \in F(T)$,

\[ \| Tx - p \| \leq \| x - p \|. \]
(1.3) There exist an $x_0 \in D$ such that
\[ x_n = T^n(x_0) \in D \quad \text{for each } n \geq 1. \]

Then $\{x_n\}$ converges to a fixed point of $T$ in $D$ if and only if
\[ \lim_{n \to \infty} d(x_n, F(T)) = 0. \]

Proof. Clearly the condition $\lim_{n \to \infty} d(x_n, F(T)) = 0$ is necessary. For the sufficiency, assume $\lim_{n \to \infty} d(x_n, F(T)) = 0$. Show now that $\{x_n\}$ is a Cauchy sequence. Given $\epsilon > 0$, then there exists an $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $d(x_n, F(T)) < \epsilon/2$. Since for all $l, k \geq n_1$,
\[ \| x_l - x_k \| \leq \| x_l - p \| + \| x_k - p \|, \]
where $p \in F(T)$, (1.2) implies that
\[ \| x_l - x_k \| \leq 2 \| x_{n_1} - p \|, \quad p \in F(T). \]

Taking the infimum over $p \in F(T)$, we get the relation,
\[ \| x_l - x_0 \| \leq 2d(x_{n_1}, F(T)) < \epsilon. \]

So $\{x_n\}$ is Cauchy and hence converges to some $x^* \in D$, since $D$ is closed. Furthermore, since $T$ is continuous $F(T)$ is closed and therefore
\[ \lim_{n \to \infty} d(x_n, F(T)) = 0 \]
implies that $x^* \in F(T)$.

Q.E.D.

Condition (1.2), which will be referred to as "$T$ is quasi-nonexpansive", was introduced by Tricomi [41] for real functions, and later studied by Diaz and Metcalf [13, 14] and by Dotson [15] for mappings in Banach spaces. That this class of mappings properly includes nonexpansive mappings is seen by the following example [15].

Example 1.1. Let $X$ be the real line and let $T$ be defined as follows:
\[ T(0) = 0 \]
\[ Tx = \frac{x}{2} \sin \left( \frac{1}{x} \right), \quad \text{for } x \neq 0. \]
The only fixed point of $T$ is 0, since if $x \neq 0$ and $Tx = x$, then

$$x = \frac{x}{2} \sin \left( \frac{1}{x} \right), \quad \text{or} \quad 2 = \sin \left( \frac{1}{x} \right)$$

which is impossible.

$T$ is quasi-nonexpansive since if $y \in X, \ p = 0$, then

$$\| Ty - p \| = \| Ty - 0 \| = \left| \frac{y}{2} \right| \| \sin \left( \frac{1}{y} \right) \| \leq \frac{|y|}{2} < |y| = \| y - p \| .$$

However $T$ is not a nonexpansive mapping. This is seen by choosing $x = 2/\pi$ and $y = 2/3\pi$. For then,

$$\| Tx - Ty \| = \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{2}{3\pi} \sin \frac{3\pi}{2} = \frac{2}{\pi} \cdot \frac{4}{3} = \frac{8}{3\pi},$$

whereas,

$$\| x - y \| = \frac{4}{3\pi} .$$

See note added in proof.

The above characterization theorem yields the validity of the following practically useful theorem. The following definition will be needed first.

**Definition 1.1** (Browder and Petryshyn [8]). If $T$ is a mapping of $D \subseteq X$ into $D$ such that for every $x \in D$,

$$\lim_{n \to \infty} \| T^n(x) - T^{n+1}(x) \| = 0,$$

then $T$ is said to be asymptotically regular on $D$. $T : D \to X$ is asymptotically regular at $x_0 \in D$ if $T^n(x_0) - T^{n+1}(x_0) \to 0$ as $n \to \infty$ whenever $T^n(x_0)$ is defined for all $n$.

**Theorem 1.2.** Let $D$ be a closed subset of a Banach space $X$, and let $T$ map $D$ continuously into $X$. Assume that

1. $F(T) \neq \emptyset$.
2. $T$ is quasi-nonexpansive.
3. There exist an $x_0 \in D$ such that $x_n = T^n(x_0) \in D$ for all $n \geq 1$.
4. $T$ is asymptotically regular at $x_0$.
5. If $\{ y_n \} \subseteq D, \ n \geq 1, \ and \ \|(I - T)y_n\| \to 0$ as $n \to \infty$, then

$$\lim_n d(y_n, F(T)) = 0.$$

Then $\{ x_n \}$ converges to a fixed point of $T$ in $D$. 
Proof. Since $T^n(x_0) \in D$ for $n \geq 1$, $T$ is asymptotically regular at $x_0$, and

$$(I - T) T^n(x_0) = T^n(x_0) - T^{n+1}(x_0),$$

we see that $\lim_{n \to \infty} \| (I - T) x_n \| = 0$. Hence, by (1.5),

$$\lim_{n \to \infty} \inf d(x_n, F(T)) = 0.$$

By (1.2), the sequence $\{d(x_n, F(T))\}_{n \geq 1}$ is monotonically decreasing and hence $\lim_{n \to \infty} d(x_n, F(T)) = 0$. Therefore, by Theorem 1.1, $\{x_n\}$ converges strongly to a fixed point of $T$ in $D$. Q.E.D.

The following compact nonexpansive operator defined on a unit ball $B$ in a Hilbert space provides an example of a mapping which is asymptotically regular at some points in $B$ but not at others.

Example 1.2. Let $B = B(0, 1)$ be the unit ball in $\mathbb{R}^2$ with the usual norm. Define $T: B \to B$ by

$$T: (x, y) \mapsto \left(-\frac{x}{2}, -y\right),$$

where $(x, y)$ denotes the usual coordinates for $\mathbb{R}^2$.

(a) $T$ is nonexpansive. If $(x_1, y_1)$ and $(x_2, y_2) \in B$, then

$$\| T(x_1, y_1) - T(x_2, y_2) \|^2 = \left\| \left(-\frac{x_1}{2}, -y_1\right) - \left(-\frac{x_2}{2}, -y_2\right) \right\|^2$$

$$= \frac{1}{4} (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\leq (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= \|(x_1, y_1) - (x_2, y_2)\|^2.$$

Hence $T$ is a nonexpansive mapping of $B$ into $B$, and since $B$ is compact, then $T$ is also a compact mapping.

(b) $F(T) \neq \emptyset$; in fact $(x, y) = (0, 0)$ is the only fixed point of $T$ in $B$.

(c) If $(x, y) \in B$, then, by a simple calculation,

$$\| T^n(x, y) - T^{n+1}(x, y) \|^2 = \left(\frac{3}{2^{n+1}}\right)^2 (2y)^2,$$

for any $n$. \hfill (')

Hence, at all points $x$ in $B$ on the line $y = 0$, $T$ is asymptotically regular at $x$, and $T$ is not asymptotically regular at any other points in $B$.

As another consequence of Theorem 1.1, the following practically useful theorem provides general sufficient conditions for the convergence of iterates.
Theorem 1.3. Let $D$ be a closed subset of a Banach space $X$. Let $T$ map $D$ continuously into $X$ such that

(1.1) $F(T) \neq \emptyset$.

(1.2) $T$ is quasi-nonexpansive.

(1.6) For every $x \in D - F$, there exists $p_x \in F(T)$ such that

$$\|Tx - p_x\| < \|x - p_x\|.$$ 

(1.7) There exists $x_0 \in D$ such that $T^n(x_0) \in D$, for all $n \geq 1$, and

$$\{x_n\} \equiv \{T^n(x_0)\}_{n \geq 0}$$

contains a convergent subsequence $\{x_{n_j}\}_{j \geq 1}$ converging to some $x^* \in D$.

Then $x^* \in F(T)$ and $\{x_n\}$ converges to $x^*$.

Proof. Condition (1.2) implies that $\lim_{n \to \infty} d(x_n, F(T)) = d > 0$ exists. Hence it suffices to show that $d = 0$, for then, Theorem 1.1 may be applied. If $x^* \in F(T)$, then $d = 0$. If $x^* \notin F(T)$, then by (1.6), there exists $p = p_x \in F(T)$ such that $\|Tx^* - p\| < \|x^* - p\|$. But also by the continuity of $T$ and the condition (1.7) we have the relation

$$\|Tx^* - p\| = \|T(\lim_{j \to \infty} x_{n_j}) - p\| = \lim_{j \to \infty} \|T^{n+1}(x_0) - p\|$$

$$= \lim_{n \to \infty} \|T^n(x_0) - p\| = \lim_{j \to \infty} \|T^n(x_0) - p\|$$

$$= \lim_{j \to \infty} \|x_{n_j} - p\| = \lim_{j \to \infty} \|x_{n_j} - p\| = \|x^* - p\|,$$

where the middle equalities hold since (1.2) implies that $\lim_{n \to \infty} \|T^n(x_0) - p\|$ exists. This is a contradiction, hence $x^* \in F(T)$ and the theorem is proven.

Q.E.D.

In the presence of (1.2), (1.6) is implied by: (1.8) For every $x \in D$, where $x \notin F(T)$, $d(Tx, F(T)) < d(x, F(T))$. In condition (1.7) a convergent subsequence must be assumed in view of the Hilbert space example of Diaz and Metcalf [14, p. 471]. The following corollary of Theorem 1.3 is due to Diaz and Metcalf [14].

Corollary 1.1. Let $D$ be a closed subset of a Banach space $X$, and let $T$ map $D$ continuously into $D$ such that

(1.1) $F(T) \neq \emptyset$.

(1.9) For every $x \in D$, where $x \notin F(T)$, and every $p \in F(T)$,

$$\|Tx - p\| < \|x - p\|.$$
Let \( x_0 \) be an arbitrary element of \( D \), and define \( x_n = T^n x_0 \), \( n \geq 1 \). If \( \{ x_n \} \) contains a convergent subsequence, then the whole sequence converges to a fixed point of \( T \) in \( D \).

The proof of the above corollary is immediate since (1.9) implies both (1.2) and (1.6). Although Theorem 1.3 and the above theorem of Diaz and Metcalf were proven in the same manner, the following example satisfies the hypotheses of Theorem 1.3 but not those of Corollary 1.1, i.e., Theorem 1.3 is a proper generalization of Corollary 1.1.

**Example 1.3.** Let \( H \) be a separable real Hilbert space with orthonormal basis \( \{ e_i \}_{i=0}^\infty \). If \( x \in H \), denote \( x = (x_0, x_1, \ldots) \), where \( x_i \) is the \( i \)th coefficient in the representation of \( x \) in the basis \( \{ e_i \} \). Let \( H^+ = \{ x \in H : x_1 \geq 0 \} \), and let \( \hat{a} = (1, 0, 0, \ldots) \). Take as a domain the set \( D = H \cap B(\hat{a}, 1) \), where \( B(\hat{a}, 1) \) is the ball of radius 1 with center \( \hat{a} \). Note that if \( x \in D \), then \( x_0 \geq 0 \) and \( x_1 \geq 0 \). Define the mapping \( T : D \to H \) as follows: for every \( x \in D \), \( x = (x_0, x_1, x_2, \ldots) \), define

\[
T(x) = ((x_0^2 + x_1^2)^{1/2}, 0, x_2, x_3, \ldots).
\]

(a) Now, \( T \) is nonexpansive, for if \( \hat{x}, \hat{y} \in D \), where \( \hat{x} = (x_0, x_1, \ldots) \) and \( \hat{y} = (y_0, y_1, \ldots) \), then

\[
\| \hat{x} - \hat{y} \|^2 = \sum_{i \geq 0} |x_i - y_i|^2
\]

and

\[
\| T\hat{x} - T\hat{y} \|^2 = [(x_0^2 + x_1^2)^{1/2} - (y_0^2 + y_1^2)^{1/2}]^2 + \sum_{i \geq 2} |x_i - y_i|^2.
\]

So it suffices to show

\[
[(x_0^2 + x_1^2)^{1/2} - (y_0^2 + y_1^2)^{1/2}] \leq (x_0 - y_0)^2 + (x_1 - y_1)^2,
\]

or equivalently, to show

\[
(x_0^2 + x_1^2)^{1/2} (y_0^2 + y_1^2)^{1/2} \geq x_0 y_0 + x_1 y_1.
\]

Since \( x_1, y_1, x_0, y_0 \) are all nonnegative, then squaring both sides, it suffices to show

\[
2x_0 y_0 x_1 y_1 \leq x_1^2 y_0^2 + x_0^2 y_1^2,
\]

which is always the case since

\[
(x_1 y_0 - y_1 x_0)^2 \geq 0.
\]

(b) Now \( T(D) \subseteq D \). Indeed, by the form of \( T \), \( T(D) \subseteq H^+ \). Since \( \hat{a} \) is a fixed point of \( T \) and \( T \) is nonexpansive, then \( T(D) \subseteq B(\hat{a}, 1) \). Therefore \( T(D) \subseteq H^+ \cap B(\hat{a}, 1) = D \).
(c) \( F(T) = \{ \dot{x} \in D \mid x_1 = 0 \} \), clearly, and note that \( T(D) = F(T) \).

(d) Condition (1.6) is satisfied, for if \( x \in D - F(T) \), then let \( p_x = T x \in F(T) \). So then

\[
0 = \| T x - p_x \| < \| x - p_x \|. 
\]

(e) If \( x_0 \in D \), arbitrarily, then the sequence of iterates becomes constant after one step, hence (1.7) is satisfied for every \( x_0 \in D \).

(f) Condition (1.9) of Corollary 1.1 is not satisfied, since \( \hat{0} \in F(T) \) and for every \( \dot{x} \in D - F \)

\[
\| \dot{x} - \hat{0} \| = \| \dot{x} \| = \| T \dot{x} \| = \| T \dot{x} - \hat{0} \|. 
\]

So this is an example of a nonexpansive, noncompact mapping \( T \) defined on a closed bounded convex set \( D \) in a Hilbert space, \( T: D \to D \), and the iterates of any point converge to a fixed point of \( T \). In addition, convergence is guaranteed by Theorem 1.3 but not by Corollary 1.1, whose hypotheses are not satisfied.

In the following characterization and later sections we shall say that a mapping \( T: D \to X \) is conditionally quasi-nonexpansive if \( T \) is quasi-nonexpansive whenever \( F(T) \neq \emptyset \). The following theorem is proven without the prior assumption of knowledge about \( F(T) \).

**Theorem 1.4.** Let \( D \) be a closed subset of a Banach space \( X \). Let \( T \) be a conditionally quasi-nonexpansive mapping of \( D \) into \( X \). Suppose \( \{T^n(x_0)\}_{n \geq 1} \subseteq D \) for some \( x_0 \in D \). Then the sequence \( \{T^n(x_0)\}_{n \geq 1} \) converges strongly to a fixed point of \( T \) in \( D \) if and only if

\[
(1.4) \quad T \text{ is asymptotically regular at } x_0. \\
(1.10) \quad \text{There exists a compact set } K \text{ such that } \lim_{n \to \infty} d(T^n(x_0), K) = 0. 
\]

**Proof.** The forward implication is immediate. For the reverse implication, assume (1.4) and (1.10). Since (1.10) holds, with \( K \) compact, then there exists \( y_0 \in K \cap D \) and a subsequence \( \{T^{n_i}(x_0)\}_{i \geq 1} \) of \( \{T^n(x_0)\}_{n \geq 1} \) such that

\[
T^{n_i}(x_0) \to y_0. 
\]

By the continuity of \( T \), \( T^{n_{i+1}}(x_0) \to Ty_0 \). Since \( T \) is asymptotically regular at \( x_0 \), the inequality

\[
\| y_0 - Ty_0 \| \leq \| y_0 - T^{n_i}(x_0) \| + \| T^{n_{i+1}}(x_0) - T(y_0) \| \\
+ \| T^{n_i}(x_0) - T^{n_{i+1}}(x_0) \| 
\]

implies \( Ty_0 = y_0 \). Hence \( y_0 \in F(T) \), and therefore the conditional quasi nonexpansiveness of \( T \) implies that the whole sequence \( \{T^n(x_0)\}_{n \geq 1} \) converges strongly to \( y_0 \). Q.E.D.
This characterization of the strong convergence of iterates differs from
Theorem 1.1 in that Theorem 1.4 does not require the assumption that
$F(T) \neq \emptyset$ or some knowledge about $F(T)$ (such as $d(T^n(x_0),F(T)) \to 0$ as
$n \to \infty$) but, instead, it requires $T$ to satisfy (1.4) and (1.10).

Remark 1.1. Although we have formulated Theorems 1.1 to 1.4 in terms
of Banach spaces, a careful examination shows that only the distance function
between points and sets has been used. Hence Theorems 1.1 to 1.4 are also
valid for general complete metric spaces.

For Banach spaces, Theorems 1.1 to 1.4 can also be formulated for the
sequence $\{x_n\}$ given by the iteration method (ii). Thus, for example, the
characterization Theorem 1.1 yields the following result for $T_\lambda$.

**Theorem 1.1'**. Let $D$ be a closed convex subset of a Banach space $X$ and
let $T$ be a continuous mapping of $D$ into $X$ such that

1. $F(T) \neq \emptyset$.
2. $T$ is quasi-nonexpansive.
3. There exists an $x_0$ in $D$ such that $x_n = T_\lambda^n(x_0) \in D$ for each $n \geq 1$
   and some $\lambda$ in $(0, 1)$.

Then $\{x_n\}$ converges to a fixed point of $T$ in $D$ if and only if

$$d(T_\lambda^n(x_0),F(T)) \to 0 \text{ as } n \to \infty.$$ 

Proof. To prove Theorem 1.1', it suffices to show that the operator $T_\lambda$
satisfies conditions (1.1), (1.2), and (1.3) of Theorem 1.1. Now, since $D$
is also convex, $T_\lambda$ is well-defined on $D$ and $F(T) = F(T_\lambda)$. Since, for each $\lambda$
in $(0, 1)$, $x$ in $D$, and $p$ in $F(T)$, the condition (1.1) implies that

$$\| T_\lambda(x) - p \| = \| \lambda x + (1 - \lambda) T(x) - \lambda p - (1 - \lambda) p \|$$

$$\leq \lambda \| x - p \| + (1 - \lambda) \| T(x) - p \| \leq \| x - p \|,$$

we see that $T_\lambda$ is also quasi-nonexpansive. Now, by hypothesis, there exists an
$x_0$ in $D$ such that $T_\lambda^n(x_0) \in D$ for each $n \geq 1$. Hence Theorem 1.1' follows
from Theorem 1.1, that is, it is in fact a restatement of Theorem 1.1 for the
mapping $T_\lambda$. Q.E.D

2. Applications to Nonexpansive Mappings and to
1-set and 1-ball Contractions

In this section we use Theorem 1.1 and its consequences, Theorems 1.2
and 1.3, to obtain new as well as some known results concerning the con-
vergence of the iterates $\{T_\lambda^n(x_0)\}$ for various special classes of quasinon-
expansive mappings.
Before we state certain corollaries for nonexpansive mappings defined on closed bounded convex subsets of \( X \) we need the following definition.

Following Petryshyn [31] we say that a map \( T \) of \( D \subseteq X \) into \( X \) is demicompact at \( f \) if, for any bounded sequence \( \{ x_n \} \) in \( D \) such that \( x_n - T(x_n) \to f \) as \( n \to \infty \), there exists a subsequence \( \{ x_{n_j} \} \) and an \( x \) in \( D \) such that \( x_{n_j} \to x \) as \( j \to \infty \) and \( x - T(x) = f \). \( T: D \to X \) is demicompact on \( D \) if \( T \) is demicompact for each such \( f \).

Clearly, when \( T \) is demicompact on \( D \), then it is demicompact at \( 0 \) but the converse is not true. It is also obvious that if \( T: D \to X \) is compact, then \( T \) is demicompact on \( D \). Also, if \( S: D \to X \) is a strict contraction and \( C: D \to X \) is compact, then \( T = S + C: D \to X \) is demicompact on \( D \). For the discussion of demicompact mappings see [31].

As a first consequence of Theorem 1.2 we have the following corollary.

**Corollary 2.1.** Let \( X \) be a uniformly convex Banach space, \( D \) a closed bounded convex set in \( X \), and \( T \) a nonexpansive mapping of \( D \) into \( D \) such that \( T \) satisfies any one of the following two conditions:

1. \((I - T)\) maps closed sets in \( D \) into closed sets in \( X \).
2. \( T \) is demicompact at \( 0 \).

For any \( \lambda, 0 < \lambda < 1 \), define \( T_\lambda = \lambda I + (1 - \lambda) T \). Then for any \( x_0 \in D \), the iterates \( x_n = T_\lambda^n(x_0), n \geq 1 \), converge strongly to a fixed point of \( T \) in \( D \).

**Proof.** It suffices to show that \( T_\lambda \) satisfies conditions (1.1)–(1.5) of Theorem 1.2. By a result due to Browder [5], Göhde [21], and Kirk [23], a nonexpansive self-mapping of a closed bounded convex set in a uniformly convex space has a fixed point, i.e., \( F(T) \neq \emptyset \). Clearly \( F(T) = F(T_\lambda) \neq \emptyset \) and \( T_\lambda \) maps \( D \) into \( D \) since \( D \) is convex. The nonexpansiveness of \( T \) (and hence of \( T_\lambda \)) implies (1.2). The condition (1.3) holds for every \( x_0 \) in \( D \) and Browder and Petryshyn [8] showed that \( T_\lambda \) is asymptotically regular on \( D \) and hence (1.4) also holds for every \( x_0 \) in \( D \). Suppose \( \{ y_n \} \subseteq D, n \geq 1 \), and \( \| (I - T_\lambda) y_n \| \to 0 \) as \( n \to \infty \). Assume first that (2.1) holds and let \( G \) be the strong closure of the set \( \{ y_n \} \). \( G \) is a subset of \( D \), and by (2.1) and (\( I - T_\lambda \)) \( G = (1 - \lambda) (I - T) (G) \) we see that (\( I - T_\lambda \))(\( G \)) is closed; hence \( 0 \in (I - T_\lambda)(G) \). So then there exists \( y^* \in G \) such that (\( I - T_\lambda \)) \( y^* = 0 \), and there exists \( \{ y_{n_j} \}_{j \geq 1} \), a subsequence of \( \{ y_n \} \), such that \( y_{n_j} \to y^* \) as \( j \to \infty \) with \( y^* \in F(T) \). Hence \( d(y_{n_j}, F(T)) \to 0 \) as \( j \to \infty \) and therefore

\[
\lim \inf_n d(y_n, F(T)) = 0,
\]

which yields the validity of condition (1.5). If \( T \) satisfies condition (2.2), then (1.5) follows from the demicompactness of \( T \) at \( 0 \).
Remark 2.1. Joran Lindenstrauss informed the first author that he had constructed an example of a nonexpansive mapping $T$ of a unit ball $B(0, 1)$ of a Hilbert space into itself with $F(T) \neq \emptyset$ for which the sequence $\{T_1^n(x_0)\}$ does not converge to a fixed point of $T$. Consequently, for the sequence $\{x_n\}$ of iterates constructed by the simple method $x_n = T_1^n(x_0)$ to be convergent to a fixed point of a nonexpansive mapping $T: D \rightarrow D$ (with $F(T) \neq \emptyset$) some additional condition on $T$ has to be imposed. It appears that our hypothesis 

$d(T_1^n(x_0), F(T_1)) \rightarrow 0$ as $n \rightarrow \infty$'' is the weakest (since it is also a necessary) condition which insures the convergence of $\{T_1^n(x_0)\}$ to a fixed point of $T$ in $D$.

Remark 2.2. Corollary 2.1 has first been obtained by Krasnoselsky [24] for the case when $T$ is compact and $\lambda = \frac{1}{2}$ and later by Schaefer [40] for $T$ compact and $\lambda \in (0, 1)$. In case $T$ is demicompact on $D$, Corollary 2.1 was proved by Petryshyn [31], and by Browder and Petryshyn [8] for $T$ satisfying condition (2.1).

Remark 2.3. The condition (2.1) and (2.2) are not related. There are mappings (e.g. $T = I$) for which (2.1) holds but not (2.2) and there are mappings (e.g. generalized contractions in the sense of Belluce and Kirk [3]) for which (2.2) holds but not necessarily (2.1).

Corollary 2.2. Let $X$ be a uniformly convex space, $D$ a closed bounded convex subset of $X$, and $T$ a nonexpansive mapping of $D$ into $D$. Assume

(2.3) There exists a number $c > 0$ such that for each $x \in D$,

$$
\|(I - T)x\| \geq cd(x, F(T)).
$$

Let $x_0$ be an arbitrary element of $D$ and define $x_n = T_1^n(x_0)$, $n \geq 1$, for any fixed $\lambda$, $0 < \lambda < 1$. Then $\{x_n\}_{n \geq 0}$ converges strongly to a fixed point of $T$ in $D$.

Proof. Again, it suffices to verify the assumptions of Theorem 1.2. Conditions (1.1), (1.2), (1.3) and (1.4) are satisfied as in the previous Corollary. If $\{y_n\} \subseteq D$ and $\|(I - T_\lambda)y_n\| \rightarrow 0$ as $n \rightarrow \infty$, then (2.3) implies that $\lim_n d(y_n, F(T)) = 0$, since $I - T_\lambda = (1 - \lambda)(I - T)$ and $F(T) = F(T_\lambda)$ which in turn, gives (1.5) for $T_\lambda$ and $F(T_\lambda)$.

Q.E.D.

Apparently, unaware of the result of Browder and Petryshyn on the asymptotic regularity of $T_\lambda$, Outlaw [30] established directly that $(I - T)x_n \rightarrow 0$ as $n \rightarrow \infty$, and with this proved the preceding Corollary under the restriction that $c < 1$ and $\lambda = \frac{1}{2}$.

To obtain further special cases we need the following definition.

Definition 2.1 (Kuratowski [25]). Let $X$ be a real Banach space and $D$
a bounded subset of $X$. The set-measure of noncompactness of $D$, $\gamma(D)$, is defined to be

$$\gamma(D) = \inf\{d > 0 \mid D \text{ can be covered by a finite number of sets of diameter less than or equal to } d\}.$$ 

It follows immediately that $\gamma(D) = \gamma(D)$, $\gamma(\lambda D) = |\lambda| \gamma(D)$, $\gamma(D) \leq \gamma(Q)$ whenever $D \subseteq Q$ and $Q$ is a bounded subset of $X$, $\gamma(D) = 0$ if and only if $D$ is compact; furthermore, if $\overline{co}(D)$ denotes the convex closure of $D$ and $D + Q = \{x + y \mid x \in D, y \in Q\}$, then it was shown by Darbo [10] that

$$\gamma(D) = \gamma(\overline{co}(D)) \quad \text{and} \quad \gamma(D + Q) \leq \gamma(D) + \gamma(Q).$$

Closely associated with $\gamma$ is the concept of $k$-set-contraction defined in [25] to be a bounded continuous mapping of a subset $G$ of $X$ into $X$ such that $\gamma(T(D)) < k\gamma(D)$ for each bounded subset $D$ of $G$ and some constant $k \geq 0$. It follows from this definition that $C: G \to X$ is compact if and only if $C$ is $0$-set-contractive and that every Lipschitzian mapping $S: G \to X$ with Lipschitz constant $l > 0$ is $k$-set-contractive with $k = l$. Clearly the mapping $T = S + C: G \to X$ is also $k$-set-contractive with $k = l$. In what follows we shall also need the concept of a condensing mapping introduced first by Sadovsky [39] for the ball-measure of noncompactness (see the definition below) and later by Furi and Vignoli [19] for the set-measure $\gamma$. A bounded continuous mapping $T$ of $G$ into $X$ is set-condensing (or densifying by [19]) if $\gamma(T(D)) < \gamma(D)$ for each bounded subset $D$ of $G$ with $\gamma(D) > 0$. It follows that every $k$-set-contractive mapping with $k < 1$ is set-condensing and that every set-condensing mapping is $1$-set-contractive but the reverse implications do not hold (see, for example, [26, 27]).

A ball-measure of noncompactness of $D$ with respect to $X$, $\chi_x(D)$, has been introduced in [20] by defining

$$\chi_x(D) = \inf\{r > 0 \mid D \text{ can be covered by a finite number of balls with centers in } X \text{ and radius } r\}.$$ 

The measures $\gamma$ and $\chi_x$ are different although they have a good deal in common (see [27, 19]). In this paper we consider the ball-measure of noncompactness only with respect to $X$ and therefore for notational simplicity we shall write $\chi$ instead of $\chi_x$.

As in the case of $\gamma$, corresponding to $\chi$ we have $k$-ball-contractions and ball-condensing mappings. It is obvious that for $\chi$ one also proves that $T: G \to X$ is compact if and only if $T$ is $0$-ball contractive. However, if, for example, $T: G \to X$ is contractive (i.e. Lipschitzian of constant $l < 1$), then it is unknown whether the map $T$ is $k$-ball-contractive with $k = l$. On the other hand, as has been shown in [37], there are $1$-ball-contractive maps $T$ of $X$
into $X$ which need not be 1-set-contractive. The reason for introducing here $k$-ball-contractions and ball-condensing mappings is that for fixed point theory (see [37]) and (as we shall see) for iteration methods the same argument works for mappings $T: D \to X$ defined either in terms of $\gamma$ or in terms of $\chi$.

Recall that $T: G \subseteq X \to X$ is said to be strictly nonexpansive if

$$\|Tx - Ty\| < \|x - y\|$$

for $x$ and $y$ in $G$.

The following corollary of Theorem 1.3 is due to Petryshyn [35], who generalized the results of [24, 40, 16] to strictly convex Banach spaces and to set-condensing mappings.

**Corollary 2.3.** Let $X$ be a Banach space, and $D$ a closed bounded convex subset of $X$. Let $T$ be either a set-condensing or a ball-condensing nonexpansive mapping of $D$ into $D$. Suppose further that either $X$ is strictly convex or $T$ is strictly nonexpansive. For any $\lambda, 0 < \lambda < 1$, let $T_\lambda = \lambda I + (1 - \lambda)T$. Then for every $x_0 \in D$, the sequence $\{T_\lambda^n(x_0)\}_{n \geq 0}$ converges strongly to a fixed point of $T$ in $D$.

**Proof.** We prove Corollary 2.3 for the case when $T$ is set-condensing since the ball-condensing case is handled similarly. It suffices to show that $T_\lambda$ satisfies the conditions of Theorem 1.3. By a result of [19, 27, 39], since $T$ is set-condensing, $F(T) \neq \emptyset$; hence $F(T_\lambda) \neq \emptyset$. Since $T$ is nonexpansive, (1.2) is satisfied. If $T$ is strictly nonexpansive, then (1.6) is satisfied. If it is the case that $X$ is strictly convex, then for $x \in D - F(T)$ and $p \in F(T)$

$$\|\lambda(x - p) + (1 - \lambda)(Tx - p)\| = \|T_\lambda x - p\| \leq \|x - p\|.$$  

The strict inequality must hold since $X$ is strictly convex; hence (1.6) holds also in this case. Since $T$ is a set-condensing mapping, $T_\lambda$ is also set-condensing. Then, it remains to show that given some $x_0 \in D$, $\{T_\lambda^n(x_0)\}$ contains a convergent subsequence, i.e., that (1.7) holds. Consider the set $C = \{T_\lambda^n(x_0) \mid n \geq 0\}$. If $\gamma(C) > 0$, let $T_\lambda(C) = \{T_\lambda^n(x_0) \mid n \geq 1\}$. Since $T_\lambda$ is condensing, $\gamma(T_\lambda(C)) < \gamma(C)$. But $C = T_\lambda(C) \cup \{x_0\}$, which implies that

$$\gamma(C) \leq \max\{\gamma(T_\lambda(C)), \gamma(\{x_0\})\} = \gamma(T_\lambda(C)) < \gamma(C),$$

a contradiction. Therefore $\gamma(C) = 0$ and $C$ is precompact. Q.E.D.

**Remark 2.2.** Corollary 2.3 contains, as a special case, a result due to Edelstein [16] obtained by him for the case when $T$ is a compact nonexpansive map of $D$ into $D$ and $X$ is strictly convex.
Remark 2.3. In case $S: D \to X$ is strictly contractive ($l < 1$) and $C: D \to X$ is compact, and $T = S + C: D \to D$, then Corollary 2.3 is applicable to the mapping $T = S + C$ since it is $k$-set-contractive with $k = l < 1$ and, thus, set-condensing.

Corollary 2.4. Let $X$ be a Banach space and suppose $T$ is a nonexpansive set-condensing or ball-condensing mapping of a closed ball $B = B(0, r) \subseteq X$ into $X$ satisfying:

$$(\Pi_1) \quad \text{If } Tx = \alpha x \text{ and } x \in \partial B, \text{ then } \alpha < 1.$$ 

Suppose further that either $X$ is strictly convex or that $T$ is strictly nonexpansive. Then $F(T)$ is convex and compact, and there exists a convex open set $Q \subseteq B$ which properly contains $F(T)$ such that for every $x_0 \in Q$ and for any fixed $\lambda$, $0 < \lambda < 1$, the sequence of iterates of $T_\lambda$ at $x_0$ is well-defined and converge strongly to a fixed point of $T$ in $B$.

Proof. Since $T$ is set-condensing or ball-condensing and since it satisfies the boundary condition $(\Pi_1)$ on $\partial B$, Theorem 1 of Petryshyn [35] implies that $F(T)$ is nonempty and compact. Moreover $(\Pi_1)$ forces $F(T)$ to be contained in the interior of $B$. In fact, there exists $d_0 > 0$ such that if $x \in X$ and $d(x, F(T)) \leq d_0$, then $x \in B$. For, if not, then for every $n \geq 1$ there would exist $x_n \notin B$ such that $d(x_n, F(T)) < 1/n$. Since $F(T)$ is compact, there exists a subsequence $\{x_{n_j}\}_{j=1}^\infty$ of $\{x_n\}$ and $x^* \in X$ such that $x_{n_j} \to x^*$ as $j \to \infty$, with $x^* \in F(T)$ because $F(T)$ is closed. Since $x_{n_j} \notin B$, $x^* \in \text{cl}(X - B)$, i.e., $x^* \notin \text{interior } B$, which contradicts the fact that $F(T) \subseteq \text{interior } B$. So then there exists $d_0 > 0$ such that $d(x, F(T)) \leq d_0$ implies $x \in B$. Let

$$Q = \{x \in X \mid d(x, F(T)) < d_0\},$$

and note that $Q$ is open and $F(T) \subseteq Q \subseteq B$. Since $T$ is nonexpansive, for every $y \in Q$ the inequality

$$d(Ty, F(T)) \leq d(y, F(T)) < d_0$$

implies that $Ty \in Q$, i.e., $T(Q) \subseteq Q$. Now, to show $T_\lambda(Q) \subseteq Q$ for $\lambda$ fixed $\lambda \in (0, 1)$, it suffices to show that $Q$ is a convex set for then $I(Q) \subseteq Q$ and $T(Q) \subseteq Q$ imply $T_\lambda(Q) \subseteq Q$. To verify that $Q$ is a convex set, it suffices to show that $F(T)$ is a convex set, since a $d_0$-neighborhood of a convex set is convex. In the case that $T$ is strictly nonexpansive, $F(T)$ consists of a single point, hence is convex. In the case that $X$ is strictly convex, then by an argument of Schaefer [40], $F(T)$ is convex since $T$ is nonexpansive. Therefore $T_\lambda(Q) \subseteq Q$, and by the continuity of $T$, $T_\lambda(Q) \subseteq Q$. So then, since (1.6) holds, to prove Corollary 2.4, it now suffices to show that (1.7) holds for $x_0 \in Q$, for
then the conditions of Theorem 1.3 will be verified. Let \( x_0 \in Q \), and consider \( \{T^n(x_0)\}_{n \geq 0} \). The sequence is contained in \( Q \) and, since \( T_\lambda \) is condensing, \( \{T^n_\lambda(x_0)\} \) contains a convergent subsequence by the same argument as in the proof of Corollary 2.3. Q.E.D.

We add in passing that the boundary condition \((\Pi, \langle \cdot, \cdot \rangle)\) is equivalent to the Leray-Schauder condition, i.e.,

\[
T_\lambda x \neq \eta x \quad \text{for } x \in \partial B \quad \text{and} \quad \eta \geq 1.
\]

It is satisfied in any of the following cases:

(a) \( T(B) \subseteq \text{interior}(B) \)

(b) \( T(\partial B) \subseteq \text{interior}(B) \)

(c) \( \| x \|^2 + \| Tx - x \|^2 > \| Tx \|^2 \) for \( x \in \partial B \) (see Altman [1])

(d) \( \langle Tx, Jx \rangle < \langle x, Jx \rangle \) for \( x \in \partial B \), where \( J : X \to 2^* \) is a duality mapping, i.e., \( J(0) = \{0\} \) and

\[
J(x) = \{w \in X^* \mid \langle w, x \rangle = \| w \| \| x \| \text{ and } \| x \| = \| w \|, \ x \in X, \ x \neq 0\}.
\]

For \( X = \text{Hilbert space} \) and \( J = I \), this reduces to \( \langle Tx, x \rangle \leq \| x \|^2 \) for \( x \in \partial B \).

Before we state our next consequence of Theorem 1.1 we need the following slight generalization of Theorem 5 of Browder and Petryshyn [8]. Our proof of Lemma 2.1 follows the arguments of [8].

**Lemma 2.1.** Let \( X \) be a uniformly convex Banach space, \( D \) a subset of \( X \), and \( T \) a mapping of \( D \) into \( X \) such that \( F(T) \neq \emptyset \) and \( T \) is quasi-nonexpansive. If there exists an \( x_0 \in D \) and a \( \lambda \) in \( (0, 1) \) such that \( T^n_\lambda(x_0) \) is defined and lies in \( D \) for each \( n \geq 1 \) where \( T_\lambda = \lambda I + (1 - \lambda) \) \( T \), then \( T^n_\lambda(x_0) \to T^{n+1}(x_0) \to 0 \) as \( n \to \infty \), i.e., \( T_\lambda \) is asymptotically regular at \( x_0 \).

**Proof.** Let \( p \) be any element in \( F(T) \) and let \( x_0 \) be an element in \( D \) and \( \lambda \) a number in \( (0, 1) \) such that \( x_n = T^n_\lambda(x_0) \in D \) for \( n \geq 1 \). Note that \( T_\lambda \) is also quasi-nonexpansive since \( F(T_\lambda) = F(T) \neq \emptyset \) and for all \( x \) in \( D \)

\[
\| T_\lambda(x) - p \| = \| \lambda x - \lambda p + (1 - \lambda)(Tx - p) \| \\
\leq \lambda \| x - p \| + (1 - \lambda) \| x - p \| \\
= \| x - p \|
\]

by the quasi-nonexpansiveness of \( T \). Hence

\[
\| x_{n+1} - p \| = \| T_\lambda(x_n) - p \| \leq \| x_n - p \| \quad \text{for each } n \geq 1
\]

and therefore \( \| x_n - p \| \to d_0 \) for some \( d_0 \geq 0 \). If \( d_0 = 0 \), then \( x_n \to p \) as \( n \to \infty \) and so in this case \( x_n - x_{n+1} = T^n_\lambda(x_0) - T^{n+1}_\lambda(x_0) \to 0 \) as \( n \to \infty \),
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i.e., $T_\lambda$ is asymptotically regular at $x_0$. Suppose now that $d_0 > 0$. Since

$$\| x_n - p \| \to d_0, \| T_\lambda(x_n) - p \| \leq \| x_n - p \|$$

for each $n$, and

$$\| T_\lambda(x_n) - p \| = \| x_{n+1} - p \| \to d_0$$

as $n \to \infty$, it follows from the uniform convexity of $X$ that

$$\|(x_n - p) - (T_\lambda x_n - p)\| \to 0$$

as $n \to \infty$, i.e.,

$$\| x_n - T_\lambda(x_n) \| = \| T_\lambda^n(x_0) - T_\lambda^{n+1}(x_0) \| \to 0$$

as $n \to \infty$.

In the sequel we shall also need the following generalization of the result of Schaefer [40].

**Lemma 2.2.** Let $X$ be a strictly convex Banach space and $D$ a closed convex subset of $X$. If $T$ is a continuous mapping of $D$ into $X$ such that $F(T) \neq \emptyset$ and

$$\| T(x) - p \| \leq \| x - p \| \quad \text{for } x \in D - F \text{ and } p \in F,$$

then $F(T)$ is a convex set.

**Proof.** Let $x$ and $y$ be any two distinct points of $F(T)$ and, for $t \in (0, 1)$, let $x_t = tx + (1 - t)y$. Now, since $D$ is convex, $x_t \in D$ for each $t$ in $(0, 1)$. Suppose, contrary to our assertion, that $x_t \notin F(T)$ for some $t \in (0, 1)$, i.e., $x_t \in D - F$. Hence it follows from (1.2) that

$$\| x - y \| \leq \| x - T(z_t) \| + \| T(z_t) - y \| \leq \| x - x_t \| + \| x_t - y \|$$

Since $X$ is strictly convex, it follows that

$$x - T(z_t) = a(T(z_t) - y), \quad a > 0.$$

But this implies that

$$T(z_t) = \frac{1}{1 + a} x + \frac{a}{1 + a} y,$$

i.e., $T(z_t)$ lies on the line segment determined by $x$ and $y$. On the other hand, $\| T(z_t) - x \| \leq \| z_t - x \|$ and $\| T(z_t) - y \| \leq \| z_t - y \|$. Thus $T(z_t)$ must coincide with $z_t$ and Lemma 2.2 is proved.

Our next result is the following new theorem for 1-set-contractive and 1-ball-contractive mappings.

**Theorem 2.1.** Let $X$ be a uniformly convex Banach space, $D$ a bounded
open subset of \( X \), and let \( T \) be either a 1-set-contractive or a 1-ball-contractive mapping of \( D \) into \( X \) such that

\begin{equation}
\text{(2.4) There exists a } y \text{ in } D \text{ such that}
\end{equation}

\[
T x - y \neq \lambda (x - y) \quad \text{for all } x \text{ in } \partial D \text{ and } \lambda > 1.
\]

\begin{equation}
\text{(2.5) } T \text{ is conditionally quasi-nonexpansive.}
\end{equation}

\begin{equation}
\text{(2.6) There exists an } x_0 \text{ in } D \text{ and a } \lambda \text{ in } (0, 1) \text{ such that } x_n = T_{\lambda}^n(x_0) \text{ is}
\end{equation}

defined and lies in \( D \) for each \( n \geq 1 \).

\begin{equation}
\text{(2.7) } T \text{ is either demicompact at } 0 \text{ or } I - T \text{ maps closed sets in } D \text{ into}
\end{equation}
closed sets in \( X \).

Then the sequence \( \{ x_n \} \) converges strongly to a fixed point of \( T \) in \( D \).

**Proof.** To prove Theorem 2.1, it suffices to show that the mapping \( T_{\lambda} \) satisfies conditions (1.1)–(1.5) of Theorem 1.2.

Now, since \( T : D \to X \) is either a 1-set-contractive or 1-ball-contractive and since \( T \) satisfies the conditions (2.4) and (2.7), the fixed point theorem of Petryshyn [37] implies that \( F(T) \neq \emptyset \); hence \( F(T_{\lambda}) = F(T) \neq \emptyset \) and so condition (1.1) of Theorem 1.2 holds. Since \( T \) (and hence \( T_{\lambda} \)) is conditionally quasi-nonexpansive, it follows that (1.2) also holds. Condition (1.3) is true by assumption, while (1.4) follows from Lemma 2.1. The proof that (1.5) also holds follows from either one of the assumptions (2.7) just as in the proof of Corollary 2.1. Q.E.D.

**Remark 2.4.** If \( D \) in Corollaries 2.1 and 2.3 is the closure of an open bounded convex subset in \( X \), then it is easy to see that Corollaries 2.1 and 2.3 are special cases of Theorem 2.1.

### 3. Further Applications

In the first part of this section we apply Theorem 1.3 to the study of the convergence of iterates \( \{ T^n(x) \} \) and \( \{ T_{\lambda}^n(x) \} \) under the assumptions that \( T \) or \( T_{\lambda} \) is strictly quasi-nonexpansive and that \( T \) satisfies the so-called Frum-Ketkov [18] condition (see condition 3.1 below). In the second part of this section we use Theorems 1.1 and 1.2 to obtain a slight generalization of Theorem 6.1 of Petryshyn and Tucker [38] concerning the convergence of iterates \( \{ T_{\lambda}^n(x_0) \} \) with \( T \) a nonexpansive generalized projectionally-compact \((P\text{-}\text{compact})\) mapping. The latter class of mappings has been introduced by Petryshyn [34, 32] for the constructive approach (via finite dimensional approximations) in the study of fixed point and solvability problems for various classes of nonlinear mappings.
3.1. Our first result in this section is the following new result.

**Theorem 3.1.** Let $D$ be a closed convex subset of a real Banach space $X$ and let $T$ be a conditionally quasi-nonexpansive mapping of $D$ into itself. Suppose further that $T$ satisfies the following conditions:

1. There exists a compact set $K \subset X$ and a constant $k < 1$ such that $d(T(x), K) \leq kd(x, K)$ for each $x \in D$.
2. $T$ is conditionally strictly quasi-nonexpansive.

Then the sequence $\{T^n(x_0)\}$ converges to a fixed point of $T$ for each $x_0 \in D$.

**Proof.** To prove Theorem 3.1, it suffices to show that $T$ satisfies the conditions of Theorem 1.3. In view of (3.1), the fixed point theorem of Frum–Ketkov [18] with a correct proof by Nussbaum [28] shows that $F(T) \neq \emptyset$, i.e., (1.1) of Theorem 1.3 holds and so does (1.2) since $T$ is conditionally quasi-nonexpansive. In view of (3.2), condition (1.6) of Theorem 1.3 is also verified. Now, since for any $x_0 \in D$ the relation $d(T^n(x_0), K) \leq k^n d(x_0, K)$ implies that $\lim_n d(T^n(x_0), K) = 0$, the compactness of $K$ forces $\{T^n(x_0)\}$ to contain a convergent subsequence, i.e., (1.7) also holds. Hence Theorem 3.1 follows from Theorem 1.3. Q.E.D.

Note that condition (3.2) is implied by the assumption that $T$ is strictly nonexpansive.

If $X$ is assumed to be a Banach $\Pi_1$-space, then the following result also holds. Recall first that $X$ is said to be a $\Pi_1$-space for some $\alpha \geq 1$ if there exists a monotonically increasing sequence $\{X_n\}$ of finite dimensional subspaces of $X$ and a sequence of bounded linear projections $\{P_n\}$ such that $P_n(X) = X_n$, $\|P_n\| \leq \alpha$ for each $n$, and $P_n(x) \to x$ for each $x \in X$.

**Theorem 3.2.** Let $X$ be a Banach $\Pi_1$-space and let $T$ be a conditionally strictly quasi-nonexpansive mapping of $B(0, r)$ into $X$ such that $T(\partial B) \subset \text{Int}(B)$. Suppose there exists a compact set $K$ in $X$ and a number $k < 1$ such that (3.1) holds for each $x$ in $B$.

Then there exists a convex open set $Q \subset B$ such that for each $x_0 \in Q$ the sequence $\{T^n(x_0)\}$ converges to a fixed point of $T$ in $B$.

**Proof.** Again, it suffices to verify the assumptions of Theorem 1.3. The theorem of Frum–Ketkov with proof by Nussbaum [27] shows that $F(T) \neq \emptyset$, i.e., (1.1) of Theorem 1.3 holds and hence (1.2) and (1.6) also hold since $T$ is conditionally strictly quasi-nonexpansive. Note that $F(T)$ is compact and lies in $\text{Int}(B)$ and hence, as in the proof of Corollary 2.4, there exists a number $d_0 > 0$ such that if $x \in X$ and $d(x, F(T)) < d_0$, then $x \in B$. Let

$$Q = \{x \in X \mid d(x, F(T)) < d_0\},$$
and note that $Q$ is open and $F(T) \subseteq Q \subseteq B$. Since $T$ is strictly quasi-non-expansive, for each $y$ in $B_0$ and $p$ in $F(T)$, the inequality $\|Ty - p\| \leq \|y - p\|$ implies that

$$d(Ty, F(T)) \leq d(y, F(T)) < d_0,$$

i.e., $Ty \in Q$. Hence $T(Q) \subseteq Q$ and so $T(\overline{Q}) \subseteq \overline{Q} \subseteq B$. Consequently, for each $x_0$ in $Q$ the sequence $\{T^n(x_0)\} \subseteq Q \subseteq B$ and, by (3.1), $d(T^n(x_0), K) \to 0$ as $n \to \infty$. This and the compactness of $K$ implies that $\{T^n(x_0)\}$ has a convergent subsequence, i.e., (1.7) of Theorem 1.3 holds. Thus, Theorem 3.2 follows from Theorem 1.3. Q.E.D.

If in Theorem 3.1 we assume that $K$ is also convex, then the assertion remains valid for $T_\lambda$ without the additional assumption (3.2). To establish this claim we need the following lemma.

**Lemma 3.1.** Let $D$ be a closed convex subset of $X$ and $T$ a mapping of $D$ into $D$ such that

$$d(T(x), K) \leq kd(x, K) \quad \text{for all } x \text{ in } D,$$

for some convex compact set $K$ in $X$ and constant $k < 1$.

If $\lambda$ is any number in $(0, 1)$ and $T_\lambda = \lambda I + (1 - \lambda) T$, then

$$d(T_\lambda(x), K) \leq k_\lambda d(x, K) \quad \text{for each } x \in D,$$

where $k_\lambda = \lambda + (1 - \lambda) k < 1$.

**Proof.** Clearly $0 < k_\lambda < 1$. Let $\lambda$ be fixed, $0 < \lambda < 1$, and $x \in D$, fixed. Now it suffices to show

$$d(T_\lambda(x), K) \leq k_\lambda d(x, K).$$

Given $\delta > 0$, there exist $y_\delta \in K$ and $z_\delta \in K$ such that

$$\|x - y_\delta\| \leq d(x, K) + \frac{\delta}{2\lambda},$$

$$\|Tx - z_\delta\| \leq d(Tx, K) + \frac{\delta}{2(1 - \lambda)}.$$

Let $w_\delta = \lambda y_\delta + (1 - \lambda) z_\delta$, and note that $w_\delta \in K$ since $K$ is convex. So then,

$$d(T_\lambda x, K) \leq \|T_\lambda x - w_\delta\| = \|\lambda(x - y_\delta) + (1 - \lambda)(Tx - z_\delta)\|$$

$$\leq \lambda \|x - y_\delta\| + (1 - \lambda) \|Tx - z_\delta\|$$

$$\leq \lambda \left[d(x, K) + \frac{\delta}{2\lambda}\right] + (1 - \lambda) \left[d(Tx, K) + \frac{\delta}{2(1 - \lambda)}\right]$$

$$\leq \left[\lambda + (1 - \lambda) k\right] d(x, K) + \delta$$

$$\leq k_\lambda d(x, K) + \delta.$$

Since $\delta > 0$ was chosen arbitrarily, the lemma is proven. Q.E.D.
Theorem 3.3. Let $D$ be a closed convex set in $X$ and let $T$ be a conditionally quasi-nonexpansive mapping of $D$ into $D$. Suppose further that $T$ satisfies the following conditions:

(3.3) There exists a convex compact set $K$ in $X$ and a number $k < 1$ such that $d(Tx, K) \leq kd(x, K)$ for each $x$ in $D$.

(3.4) $X$ is strictly convex.

Then the sequence $\{T^n(x_0)\}$ converges to a fixed point of $T$ in $D$ for any $x_0$ in $D$ and $\lambda$ in $(0, 1)$.

Proof. To prove Theorem 3.3, it suffices to show that $T_\lambda$ satisfies the conditions of Theorem 3.1. Now, $T_\lambda$ maps $D$ into itself since $D$ is convex and, since $K$ is also convex, it follows from Lemma 3.1 that for each $x$ in $D$

$$d(T_\lambda(x), K) \leq k_\lambda d(x, K), \quad k_\lambda = \lambda + (1 - \lambda) k < 1.$$ 

Furthermore, it is not hard to see that $T_\lambda$ is also conditionally strictly quasi-nonexpansive since $X$ is strictly convex. Consequently, $T_\lambda$ satisfies all conditions of Theorem 3.1 and, therefore, the conclusion of Theorem 3.3 holds.

Q.E.D.

It may be of interest to note that under the assumption that $K$ is also convex in (3.1) the fact that $F(T) \neq \emptyset$ is deducible from the following simple theorem without recourse to the work of [28].

Theorem 3.4. Let $D$ be a closed convex subset of $X$ and $T$ a continuous mapping of $D$ into $D$ such that

(3.5) There exists a compact convex set $K \subset X$ such that

$$d(T(x), K) \leq d(x, K) \quad \text{for } x \text{ in } D.$$ 

(3.6) If $x \in D - K$, then $d(T(x), K) < d(x, K)$.

(3.7) There exists an $x_0$ in $D$ such that $\{T^n(x_0)\}$ contains a convergent subsequence, say, $\{T^{n_j}(x_0)\}$.

Then $T$ has a fixed point in $D$, i.e., $F(T) \neq \emptyset$.

Proof. Let $\{T^{n_j}(x_0)\}$ be a convergent subsequence of $\{T^n(x_0)\}_{n \geq 1}$ and let $x^* \in D$ be its limit. The condition (3.5) implies that $\lim_{n \to \infty} d(T^n(x_0), K)$ exists and equals some $d_0 > 0$. In fact, $d_0 = 0$. For if it were true that $d_0 > 0$, then $x^* \notin K$ and (3.6) would imply that $d(Tx^*, K) < d(x^*, K)$.
On the other hand, 
\[ d(Tx^*, K) = d(T(\lim_{j \to \infty} T^{n_j}(x_0)), K) \]
\[ = \lim_{j \to \infty} d(T^{n_j+1}(x_0), K) \]
\[ = \lim_{n \to \infty} d(T^n(x_0), K) \]
\[ = d(\lim_{j \to \infty} T^{n_j}(x_0), K) \]
\[ = d(x^*, K), \]
which gives a contradiction. Hence \( d_0 = 0 \) and \( x^* \in K \cap D \). Since \( K \) and \( D \) are convex and \( K \) is compact, \( K \cap D \) is also compact and convex, and \( T(K \cap D) \subseteq K \cap D \), by condition (3.5). Hence by Tikhonoff's fixed point theorem, \( F_{K \cap D}(T) \neq \emptyset \) i.e. \( F(T) \neq \emptyset \). Q.E.D.

Remark 3.1. If \( D \) is a closed bounded convex subset of \( X \) and \( T \) is a compact nonexpansive mapping of \( D \) into itself and if we set \( K = \overline{co}(T(D)) \), then, by Mazur's theorem, \( K \) is a convex compact set in \( X \) and \( d(T(x), K) \leq kd(x, K) \) for all \( x \) in \( D \) and any \( k \) in \( (0, 1) \). Thus, the convergence theorems of Krasnoselsky [24], Schaefer [40], and Edelstein [16] can also be deduced from Theorem 3.3.

The following example of a nonexpansive mapping shows that Theorem 3.3 is a proper generalization of the results mentioned above.

Example 3.1. Let \( X = l_p, 1 < p < \infty \), that is, infinite sequences of real numbers \( \hat{x} = (x_1, x_2, \ldots) \) whose norm \( \| \hat{x} \| = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \) is finite. The space \( l_p \) is uniformly convex. Let \( \hat{e}_i \) be the unit vectors in \( l_p \) of the form \( \hat{e}_i = \delta_{ij} \hat{e}_i \), where \( \delta_{ij} \) is the Kronecker delta. The collection \( \{ \hat{e}_i \mid i \geq 1 \} \) forms a Schauder basis for \( l_p \), that is, each \( \hat{x} \in l_p \) has a unique representation in terms of this collection, i.e., \( \hat{x} = \sum_{i=1}^{\infty} x_i \hat{e}_i \). Let \( B \) be the unit ball in \( l_p \) with center \( \hat{0} \) and let \( \{ f_i \}_{i=1}^{\infty} \) be a collection of nonexpansive self-mappings of the interval \([-1, 1]\) with \( f_i(0) = 0, i \geq 1 \). Define \( T \) for \( \hat{x} = (x_1, x_2, \ldots) \in B \) by
\[ T(\hat{x}) = f_1(x_1) \hat{e}_1 + \frac{1}{2} \sum_{i=1}^{\infty} f_i(x_i) \hat{e}_i. \]
To show that \( T(\hat{x}) \in B \) for any \( \hat{x} \in B \), it suffices to show that \( T \) is nonexpansive and to note that \( T(\hat{0}) = \hat{0} \in l_p \). If
\[ \hat{x} = (x_1, x_2, \ldots) \in B \quad \text{and} \quad \hat{y} = (y_1, y_2, \ldots) \in B, \]
then
\[ T\hat{x} - T\hat{y} = \frac{1}{2} \sum_{i=1}^{\infty} [f_i(x_i) - f_i(y_i)] \hat{e}_i + [f_1(x_1) - f_1(y_1)] \hat{e}_1. \]
So,
\[ \| T\xi - T\eta \|_p^p \leq 2^{-p} \sum_{i>1} |f_i(x_i) - f_i(y_i)|^p + |f_1(x_1) - f_1(y_1)|^p \]
\[ \leq 2^{-p} \sum_{i>1} |x_i - y_i|^p + |x_1 - y_1|^p \]
\[ \leq \sum_{i>1} |x_i - y_i|^p = \| \xi - \eta \|_p^p. \]

Therefore \( T \) is well-defined, nonexpansive, and \( T(B) \subseteq B \). Let
\[ K = \{ \xi \in l_p : x_i = 0, i > 1; |x_1| \leq 1 \}. \]

Then \( K \) is convex and compact, and for any \( \xi \in B \),
\[ d(T\xi, K) \leq \frac{1}{2} d(\xi, K). \]

To see this, note that \( \hat{k} \in K \) implies
\[ \| \xi - \hat{k} \| = \left( \sum_{i>1} |x_i|^p + |x_1 - k_1|^p \right)^{1/p}. \]

Hence
\[ d(\xi, K) = \inf_{\hat{k} \in K} \| \xi - \hat{k} \| = \left( \sum_{i>1} |x_i|^p \right)^{1/p}. \]

Similarly
\[ d(T\xi, K) = \frac{1}{2} \left( \sum_{i>1} |f_i(x_i)|^p \right)^{1/p}, \]
which gives
\[ d(T\xi, K) = \frac{1}{2} \left( \sum_{i>1} |f_i(x_i)|^p \right)^{1/p} \leq \frac{1}{2} \left( \sum_{i>1} |x_i|^p \right)^{1/p} = \frac{1}{2} d(\xi, K). \]

Therefore \( T \) is, in general, a nonlinear noncompact nonexpansive mapping of \( B \) into \( B \) in a uniformly convex Banach space satisfying (3.3). Hence, by Theorem 3.3, for any \( x_0 \) in \( B \) and \( \lambda \in (0, 1) \), \( \{ T_\lambda^n(x_0) \} \) converges to a fixed point of \( T \) in \( B \).

3.2. In this section we outline briefly the applicability of Theorems 1.1 and 1.2 to the study of iterates involving a class of continuous \( P_1 \)-compact maps which forms a subclass of a class of generalized \( P \)-compact (i.e., \( P_\gamma \)-compact) mappings.

Definition 3.1 (Petryshyn [34]). Let \( X \) be a Banach \( \Pi_\alpha \)-space. \( T : D \subseteq X \rightarrow X \) is said to be \( P_1 \)-compact at \( f \) if for any \( p \geq 1 \) and any bounded
sequence \( \{x_{n_j} | x_{n_j} \in X_{n_j} \cap D \} \) such that \( P_{n_j}T(x_{n_j}) - p x_{n_j} \to f \) as \( j \to \infty \), there exists an \( x \in D \) and a subsequence \( \{x_{n_{1(k)}}\} \) such that \( x_{n_{1(k)}} \to x \) and \( P_{n_{1(k)}}T(x_{n_{1(k)}}) - px_{n_{1(k)}} \to T(x) - p(x) (= f \) for continuous \( T \)) as \( k \to \infty \). \( T \) is \( P_1 \)-compact if \( T \) is \( P_1 \)-compact at each such \( f \).

Remark 3.2. Note that the class of \( P_1 \)-compact mappings includes the class of \( P \)-compact mappings (see [34, 38]) as its proper subclass. We remark that upon the examination of the proofs of the fixed point theorems in [32, 33, 38] it is clear that the only property of \( T \) which has been used there is contained in the requirement that \( T \) be \( P_1 \)-compact at \( f = 0 \). This remark is of practical usefulness since there are mappings \( T \) which are \( P_1 \)-compact at 0 but for which it is unknown whether they are \( P_1 \)-compact. Thus, for example, it has been shown by Fitzpatrick [17] (see also [43] where a similar result has been obtained earlier for the case when \( T \) is defined on all of \( X \)) that if \( T \) is a generalized contraction in the sense of Belluce and Kirk [2] of \( B(0, r) \) into itself, then \( T \) is \( P_1 \)-compact at 0 but it is unknown whether a generalized contraction (or even a strict contraction) defined only on \( B(0, r) \) is \( P_1 \)-compact. We recall (see [2]) that \( T : D \to X \) is a generalized contraction if for each \( x \) in \( D \) there exists an \( a(x) \) with \( 0 < a(x) < 1 \) such that

\[
\| T(x) - T(y) \| \leq a(x) \| x - y \| \quad \text{for each } y \text{ in } D.
\]

Our first result in this section is the following generalization of Theorem 6.1(c) of Petryshyn-Tucker [38].

**Theorem 3.5.** Let \( X \) be a strictly convex Banach \( \Pi_\alpha \)-space, \( D \) a closed bounded convex subset of \( X \) with an interior \( \text{Int}(D) \neq \emptyset \), and \( T \) a continuous conditionally quasi-nonexpansive mapping of \( D \) into \( D \) which is \( P_1 \)-compact at 0 and which has no fixed points on \( \partial D \). If \( T \) is also asymptotically regular on \( D \), then for each \( x_0 \) in \( D \) the sequence \( \{T^n(x_0)\} \) converges to a fixed point of \( T \).

**Proof.** By the fixed point theorem of Petryshyn-Tucker [38], \( F(T) \neq \emptyset \). Since \( X \) is strictly convex and \( T \) is conditionally quasi-nonexpansive, it follows from Lemma 2.2 that \( F(T) \) is a closed convex set which lies in \( \text{Int}(D) \) because \( T \) has no fixed points on \( \partial D \); moreover, for \( \{x_n\} \) given by \( \{T^n(x_0)\} \) we see that the sequence \( \{d(x_n, F(T))\} \) is a monotonically decreasing sequence with zero as its lower bound. We claim that \( d(x_n, F(T)) \to 0 \) as \( n \to \infty \). Suppose, to the contrary, that \( d(x_n, F(T)) \to \alpha_0 > 0 \) as \( n \to \infty \). By the monotonicity of \( \{d(x_n, F(T))\} \), it follows that \( d(x_n, F) > \alpha_0 \) for each \( n \). Hence, by Lemma 6.2 in [38] (which is also valid for the case when \( T \) is only assumed to be \( P_1 \)-compact at 0), there exists a real number \( \varepsilon_0 - \varepsilon(x_0) > 0 \) such that \( \| x_n - T(x_n) \| \geq \varepsilon_0 \) for each \( n \), in contradiction to the assumed asymptotic regularity of \( T \). Thus, \( \lim_n d(x_n, F) = 0 \) and so the conclusion of Theorem 3.5 follows from Theorem 1.1. Q.E.D.
A consequence of Theorem 3.5 is the following corollary which includes Theorem 6.2(c) in [38] for the case when $T$ is nonexpansive and $T$ is $P_1$-compact on $D$.

**Corollary 3.1.** Let $X$ be a uniformly convex Banach $\Pi_\alpha$-space, $D$ a closed bounded convex set in $X$ with $\text{Int}(D) \neq \emptyset$, and $T$ a continuous conditionally quasi-nonexpansive mapping of $D$ into itself which is $P_1$-compact at 0 and which has no fixed points on $\partial D$. Then for each $x_0$ in $D$ and $\lambda$ in $(0, 1)$ the sequence $\{T_\lambda^n(x_0)\}$ converges to a fixed point of $T$.

**Proof.** To prove Corollary 3.1, it suffices to show that $T_\lambda$ satisfies the conditions of Theorem 3.5 for each fixed $\lambda$ in $(0, 1)$, since $T_\lambda$ maps $D$ into $D$ and $T_\lambda$ is conditionally quasi-nonexpansive with no fixed points on $\partial D$. Furthermore, $T_\lambda$ is $P_1$-compact at 0. Indeed, let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}D\}$ be any bounded sequence and let $p$ be any real number with $p > 1$ such that $P_{n_j}T_\lambda(x_{n_j}) - px_{n_j} \to 0$ as $j \to \infty$. Then, since $P_{n_j}$ are linear,

$$P_{n_j}T_\lambda(x_{n_j}) - px_{n_j} = (1 - \lambda) P_{n_j}T(x_{n_j}) + (\lambda - p)x_{n_j} \to 0 \quad \text{as} \quad j \to \infty.$$

Hence,

$$P_{n_j}T(x_{n_j}) - \left(\frac{\lambda - 1}{1 - \lambda}\right)x_{n_j} \to 0 \quad \text{as} \quad j \to \infty$$

with

$$\left(\frac{\lambda - 1}{1 - \lambda}\right) \geq 1 \quad \text{for each} \quad \lambda \in (0, 1)$$

and any $p \geq 1$. Since $T$ is $P_1$-compact at 0, there exist $x$ in $D$ and $\{x_{n_{j(k)}}\}$ such that $x_{n_{j(k)}} \to x$ and

$$P_{n_{j(k)}}T(x_{n_{j(k)}}) - \left(\frac{\lambda - 1}{1 - \lambda}\right)x_{n_{j(k)}} \to T(x) - \left(\frac{\lambda - 1}{1 - \lambda}\right)x = 0, \quad \text{as} \quad k \to \infty,$$

i.e., $T_\lambda(x) - p(x) = 0$ and so $T_\lambda$ is $P_1$-compact at 0. Since $X$ is uniformly convex and $T$ is conditionally quasi-nonexpansive, to show that $T_\lambda$ is also asymptotically regular on $D$ it suffices, by Lemma 2.1, to show that $F(T_\lambda) \neq \emptyset$. But the latter fact follows from the fixed point theorem in [38]. Q.E.D.

We complete this section by showing that Theorem 6.1(b) in [38] can also be deduced as a special case of the following corollary to Theorem 1.2.

**Corollary 3.2.** Let $X$ be uniformly convex Banach $\Pi_\alpha$-space and $T$ a Lipschitzian conditionally quasi-nonexpansive mapping of $B(0, r)$ into $X$ such that
(3.8) \( T(x) \neq \gamma x \) for all \( x \) in \( \partial B \) and \( \gamma > 1 \).

(3.9) There exists an \( x_0 \) in \( B \) and \( \lambda \in (0, 1) \) such that \( T_\lambda^n(x_0) \subseteq B \) for each \( n \geq 1 \).

(3.10) \( T \) is \( P_1 \)-compact at 0.

Then the sequence \( \{x_n\} = \{T_\lambda^n(x_0)\} \) converges to a fixed point of \( T \).

**Proof.** To prove Corollary 3.2, it suffices to show that \( T_\lambda \) satisfies conditions (1.1) to (1.5) of Theorem 1.2.

Now, since \( T: B \to X \) is \( P_1 \)-compact at 0, the fixed point Theorem in [32] and Remark 3.2 imply that \( F(T) = F(T_\lambda) \neq 0 \), i.e., (1.1) of Theorem 1.2 holds. This and the conditional quasi-nonexpansiveness of \( T \) implies the validity of (1.2). Condition (1.3) for \( T_\lambda \) is valid by assumption while (1.4) follows from Lemma 2.1. Now, to show that (1.5) of Theorem 1.2 also holds, it suffices to show that \( T_\lambda \) is demicompact at 0. This will follow from Lemma 3.2 whose proof follows the arguments of [34].

**Lemma 3.2.** Let \( X \) be a Banach \( \Pi_1 \)-space and let \( T \) be a Lipschitzian mapping of \( B(0, r) \) into \( X \) which is \( P_1 \)-compact at 0. Then \( T \) is demicompact at 0.

**Proof.** Let \( \{u_k\} \) be a sequence in \( B(0, r) \) such that \( u_k - T(u_k) \to 0 \) as \( k \to \infty \). Since \( X \) is a \( \Pi_1 \)-space, for each integer \( k \geq 1 \) and \( \varepsilon_k = 1/k \), there exists an integer \( n(k) \geq k \) such that \( \|u_k - P_{n(k)}u_k\| \leq \varepsilon_k \) with \( w_{n(k)} = P_{n(k)}(u_k) \in X_{n(k)} \cap B(0, r) \) for each \( k \) since \( \|P_{n(k)}\| = 1 \) for all \( k \). Since \( T \) is Lipschitzian, say with constant \( L > 0 \), and \( u_k - T(u_k) \to 0 \) as \( k \to \infty \), it follows that

\[
\|P_{n(k)}T(w_{n(k)}) - w_{n(k)}\| \\
\leq \|P_{n(k)}T(w_{n(k)}) - P_{n(k)}T(u_k)\| + \|P_{n(k)}T(u_k) - w_{n(k)}\| \\
\leq L\|w_{n(k)} - u_k\| + \|T(u_k) - u_k\| \to 0 \quad \text{as} \quad k \to \infty.
\]

Thus, because \( T \) is \( P_1 \)-compact at 0, there exists a subsequence \( \{w_{n(j)}\} \) of \( \{w_{n(k)}\} \) and an \( u \) in \( B(0, r) \) such that \( w_{n(j)} \to u \) and

\[ P_{n(j)}T(w_{n(j)}) - w_{n(j)} \to T(u) - u = 0 \quad \text{as} \quad j \to \infty, \]

and moreover,

\[ \|u_j - u\| \leq \|u_j - w_{n(j)}\| + \|w_{n(j)} - u\| \to 0 \quad \text{as} \quad j \to \infty, \]

i.e., \( T \) is demicompact at 0. Q.E.D.
4. WEAK CONVERGENCE OF ITERATES OF QUASI-NONEXPANSIVE MAPPINGS

It has been observed in Section 2 (see Remark 2.1) that even when \( T \) is a nonexpansive mapping of a unit ball \( B(0, 1) \) in a Hilbert space \( X \) into \( B(0, 1) \), the sequence \( \{x_n\} \) of iterates obtained by the simple method

\[
x_n = \lambda x_{n-1} + (1 - \lambda) T(x_{n-1}), \quad n = 1, 2, ..., x_0 \in D, \lambda \in (0, 1), \tag{4.0a}
\]

need not converge (strongly) to a fixed point of \( T \). However, Theorem 1.1 shows that \( \{x_n\} \) converges to a fixed point of \( T \) if and only if the following additional condition (4.0b) holds:

\[
d(x_n, F(T)) \to 0 \quad \text{as} \quad n \to \infty. \tag{4.0b}
\]

In Sections 1 to 3 we studied the iterants for various classes of nonexpansive and quasi-nonexpansive mappings for which the condition (4.0b) is shown to hold.

Generalizing certain results of Browder and Petryshyn [8] and of Schaefer [40], Opial [29] has shown that for certain uniformly convex Banach spaces (including Hilbert spaces and \( l_p \) spaces for \( 1 < p < \infty \)) the sequence \( \{x_n\} \) determined by (4.0a) converges weakly to a fixed point of \( T \) even for more general convex closed domains \( D \). The results in [40, 8, 29] were obtained for nonexpansive mappings \( T \) of \( D \) into \( D \) for which \( I - T \) is demiclosed (see definition below).

The purpose of this section is to unify and extend further the results of [40, 8, 29] as well as those of [15, 31] to various classes of quasi-nonexpansive mappings \( T: D \to X \) for which \( I - T \) satisfies a condition (see condition 4.3 in Theorem 4.1 below) which is weaker than the demiclosedness condition and for Banach spaces which are slightly more general than those used in [8, 29, 15].

**Theorem 4.1.** Let \( X \) be a Banach space, \( D \) a closed convex subset of \( X \), and \( T \) a mapping of \( D \) into \( X \) such that

1. \( \quad \) There exists an \( x_0 \) in \( D \) such that \( x_n = T^n(x_0) \in D \) for \( n \geq 1 \) and \( \{x_n\} \) is weakly sequentially compact.

2. \( \) \( T \) is asymptotically regular at \( x_0 \).

3. \( \) If \( \{x_{n_j}\} \) is any subsequence of \( \{x_n\} \) such that \( x_{n_j} \to \bar{x} \in D \) and \( (I - T)(x_{n_j}) \to 0 \) as \( j \to \infty \), then \( \bar{x} = T(\bar{x}) = 0 \).

Then \( T \) has a fixed point in \( D \) which is obtainable as a (weak) limit of a weakly convergent subsequence of \( \{x_n\} \); moreover, every weakly convergent subsequence of \( \{x_n\} \) has a fixed point of \( T \) for its limit. If additionally we assume that \( T \) has at most one fixed point, then \( \{x_n\} \) is weakly convergent and its weak limit is the unique fixed point of \( T \).
Proof. Since \( \{x_n\} \subset D \) is sequentially weakly compact and \( D \) is a closed convex set and thus weakly closed, there exists a subsequence \( \{x_{n_j}\} \) and an \( \bar{x} \) in \( D \) such that \( x_{n_j} \rightharpoonup \bar{x} \) as \( j \to \infty \). This and the asymptotic regularity of \( T \) at \( x_0 \) imply that \( x_{n_j} - T(x_{n_j}) \to 0 \) as \( j \to \infty \). From this and condition (4.3) it follows that \( \bar{x} = T(\bar{x}) = 0 \), i.e., \( F(T) \neq \emptyset \).

Now, if \( \{x_{n_j}\} \) is any weakly convergent subsequence of \( \{x_n\} \) with \( \bar{x} \) as its weak limit, then \( \bar{x} \in D \) and, as before, conditions (4.2) and (4.3) imply that \( \bar{x} \in F(T) \).

If \( T \) has at most one fixed point, then by the preceding result, \( T \) has a unique fixed point, say \( \bar{x} \), in \( D \). But then \( x_n \to \bar{x} \) as \( n \to \infty \) since, by what has been proved above, every weakly convergent subsequence of \( \{x_n\} \) has to have \( \bar{x} \) as its (weak) limit. Q.E.D.

An immediate consequence of Theorem 4.1 is the following corollary.

**Corollary 4.1.** Let \( X \) be a Banach space, \( D \) a convex and weakly compact subset of \( X \), and \( T \) a mapping of \( D \) into \( D \) such that conditions (4.2) and (4.3) of Theorem 4.1 hold for some \( x_0 \) in \( D \). Then \( T \) has a fixed point in \( D \), and moreover, every weakly convergent subsequence of \( \{T^n(x_0)\} \) has a fixed point of \( T \) as its limit.

A special case of Corollary 4.1 is the following theorem due to Belluce and Kirk [3] who proved it under the additional condition that \( T \) is nonexpansive.

**Corollary 4.2.** Let \( X \) be a Banach space and \( D \) a convex and weakly compact subset of \( X \). Let \( T \) be a continuous mapping of \( D \) into itself such that \( T \) is asymptotically regular on \( D \) and \( V = I - T \) is convex on \( D \), i.e.,

\[
\left\| V\left( \frac{x + y}{2} \right) \right\| \leq \frac{1}{2} \left( \| V(x) \| + \| V(y) \| \right) \quad \text{for all } x, y \in D. \tag{4.4}
\]

Then the conclusions of Corollary 4.1 hold.

**Proof.** The validity of Corollary 4.2 follows from Corollary 4.1 and the following lemma.

**Lemma 4.1.** Let \( D \) be a closed convex subset of a Banach space \( X \) and let \( T \) be a continuous mapping of \( D \) into \( X \) such that \( V = I - T \) is convex on \( D \). If \( x_0 \) is an element in \( D \) such that \( T^n(x_0) \in D \) for each \( n \geq 1 \), then \( T \) satisfies the condition (4.3).

**Proof.** Consider the functional \( F(x) = \| x - Tx \| \) for \( x \) in \( D \). It follows from (4.4) that

\[
F\left( \frac{x + y}{2} \right) \leq \frac{F(x) + F(y)}{2} \quad \text{for all } x \text{ and } y \text{ in } D.
\]
i.e., $F$ is weakly convex on $D$. Since $F$ is also continuous, it is not hard to show that $F$ is also strongly convex on $D$, i.e.,

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y) \quad \text{for all } x, y \in D \text{ and } t \in (0, 1).$$

Consequently, $F$ is weakly lower semicontinuous on $D$. Suppose now that \{${x_n}$\} is a subsequence of \{${x_n}$\} = \{${T^n(x_0)}$\} such that $x_{n_j} \rightarrow \bar{x}$ for some $\bar{x}$ in $D$ and $(I - T)(x_{n_j}) \rightarrow 0$ as $j \rightarrow \infty$. In view of this and the weak lower semicontinuity of $F$, we see that

$$0 \leq F(\bar{x}) \leq \liminf_{j} F(x_{n_j}) = \lim_{j} \|x_{n_j} - T(x_{n_j})\| = 0$$

from which (4.3) follows.

**Remark 4.0.** It follows from the proof of Lemma 4.1 that the conclusion of Corollary 4.2 remains valid when the assumption that $I - T$ is convex on $D$ is replaced by the weaker assumption, namely, that $T$ is such that the functional $F(x) = \|x - T(x)\|$ is weakly lower semicontinuous on $D$.

If we are only interested in the problem of finding out when a given sequence of iterates \{${x_n}$\} = \{${T^n(x_0)}$\} $\subseteq D$ is weakly convergent or at least is such that every weakly convergent subsequence of \{${T^n(x_0)}$\} has a fixed point of $T$ as its limit, instead of the assumption that \{${x_n}$\} is weakly compact we may assume that $F(T) \neq \emptyset$. Our first result in this area is the following generalization of the results of Browder and Petryshyn [8].

**Theorem 4.2.** Let $X$ be a reflexive Banach space, $D$ a closed convex subset of $X$, and $T$ a continuous mapping of $D$ into $X$ such that

(4.5) $F(T) \neq \emptyset$.

(4.6) $T$ is quasi-nonexpansive.

(4.7) There exists $x_0$ in $D$ such that $x_n = T^n(x_0) \in D$ for $n \geq 1$.

If $T$ also satisfies conditions (4.2) and (4.3) of Theorem 4.1, then \{${x_n}$\} contains a weakly convergent subsequence with its limit in $F(T)$, and moreover, every weakly convergent subsequence of \{${x_n}$\} has a point in $F(T)$ as its limit. If we also assume that $F(T)$ contains a single point, say $p$, then $x_n \rightarrow p$ as $n \rightarrow \infty$.

**Proof.** Since $T$ is quasi-nonexpansive on $D$ and $F(T) \neq \emptyset$, it follows that for any fixed $p$ in $F(T)$ and any $n$ we have the relation

$$\|x_n - p\| = \|T(x_{n-1}) - p\| \leq \|x_{n-1} - p\|.$$

This implies that \{${x_n}$\} is a bounded sequence which, by (4.7), lies in $D$ and is weakly sequentially compact because $X$ is reflexive. Since, by assumption, $T$ satisfies also conditions (4.2) and (4.3), the conclusions of Theorem 4.2 follow from Theorem 4.1. Q.E.D.
Recall that a mapping \( V: D \rightarrow X \) is said to be \textit{demiclosed} if \( \{x_n\} \) is any sequence in \( D \) such that \( x_n \rightharpoonup \bar{x} \) in \( D \) and \( V(x_n) \rightharpoonup f \) in \( X \), then \( V(\bar{x}) = f \). Note that, for \( D \) closed and convex, every weakly continuous self-mapping of \( D \) is weakly closed, and every weakly closed self-mapping is demiclosed.

\textbf{Remark 4.1.} Since the assumption that \( I - T \) is demiclosed on \( D \) obviously implies the validity of condition (4.3) and since every nonexpansive mappings is quasi-nonexpansive, Theorem 4.2 contains Theorem 3 of Browder and Petryshyn [8] for the case when \( X \) is reflexive. Theorem 4.2 is also related to Theorem 6 in [15]. We add that Lemma 4.1 shows that there are mappings (e.g. convex maps) for which (4.3) holds but for which \( I - T \) need not be demiclosed.

If we assume that \( X \) is uniformly convex and if instead of the iterants \( \{T^n(x)\} \) we consider the iterants \( \{T_\lambda^n(x)\} \) for any \( \lambda \in (0, 1) \), then we may omit the the asymptotic regularity assumption.

\textbf{Theorem 4.3.} Let \( X \) be a uniformly convex Banach space, \( D \) a closed convex subset of \( X \), and \( T \) a continuous mapping of \( D \) into \( X \) such that (4.5) and (4.6) of Theorem 4.2 hold. Suppose there exists an \( x_0 \in D \) such that \( T_\lambda^n(x_0) \in D \) for each \( n \geq 1 \). If \( T_\lambda \) satisfies condition (4.3) of Theorem 4.1, then the conclusions of Theorem 4.2 hold.

\textbf{Proof.} It suffices to show that \( T_\lambda \) satisfies the conditions of Theorem 4.3 for each fixed \( \lambda \in (0, 1) \). Now, it was shown in Section 2, that (4.5) and (4.6) imply the validity of the same for \( T_\lambda \). Furthermore, under present conditions on \( X \) and \( T \), Lemma 2.1 shows that \( T_\lambda \) is asymptotically regular at \( x_0 \), i.e., (4.2) also holds. Hence the conclusions of Theorem 4.3 follow from Theorem 4.2.

\textbf{Corollary 4.3.} Let \( D \) be a closed bounded convex subset of a uniformly convex Banach space. If \( T \) is a continuous mapping of \( D \) into itself such that (4.5) and (4.6) of Theorem 4.2 hold and \( I - T \) is convex on \( D \), then for each \( x \) in \( D \) and \( \lambda \in (0, 1) \) the sequence of iterates \( \{x_n\} \subset D \) determined by \( x_n = T_\lambda^n(x) \) for each \( n \) is such that \( \{x_n\} \) contains a weakly convergent subsequence with its limit in \( F(T) \), and moreover, every weakly convergent subsequence of \( \{x_n\} \) has a point in \( F(T) \) for its limit. If additionally we assume that \( F(T) \) contains only one point, say \( p \), then \( x_n \rightharpoonup p \) as \( n \rightarrow \infty \).

\textbf{Proof.} To prove Corollary 4.3 it suffices to show that, for each fixed \( \lambda \) in \( (0, 1) \), the mapping \( T_\lambda \) satisfies the conditions of Theorem 4.3. Now, since \( D \) is convex and \( T: D \rightarrow D \) is quasi-nonexpansive, it follows that \( T_\lambda : D \rightarrow D, F(T_\lambda) = F(T) \neq \emptyset \), and \( T_\lambda \) is also quasi-nonexpansive. Furthermore, Lemma 2.1 shows that \( T_\lambda \) is asymptotically regular on \( D \). Moreover,
\( V_\lambda = I - T_\lambda \) is also convex since for any given \( \lambda \) in \((0, 1)\) and \( x \) and \( y \) in \( D \) it is easy to see that

\[
\left\| V_\lambda \left( \frac{x + y}{2} \right) \right\| = \left\| \left( 1 - \lambda \right) \left[ \frac{x + y}{2} - T \left( \frac{x + y}{2} \right) \right] \right\|
\leq \frac{1}{2} \left\| \left( 1 - \lambda \right) \left( \| x - T(x) \| + \| y - T(y) \| \right) \right\|
= \frac{1}{2} \left( \| x - T_\lambda(x) \| + \| y - T_\lambda(y) \| \right).
\]

Hence, by Lemma 4.1, \( T_\lambda \) satisfies condition (4.3) for each fixed \( \lambda \) in \((0, 1)\). Thus, \( T_\lambda \) satisfies all the conditions of Theorem 4.3 and so the conclusions of the latter are applicable. Q.E.D.

We add that, under suitable assumptions, Theorems 4.2 and 4.3 are also applicable to 1-set-contractive and 1-ball-contractive mappings defined on bounded sets for which one can show that \( F(T) \neq \emptyset \). Indeed, as an illustration, we state two more corollaries to Theorem 4.3 which, to the best of our knowledge, represents new iteration results for 1-set and 1-ball contractions.

**Corollary 4.4.** Let \( X \) be a uniformly convex Banach space, \( D \) a bounded closed convex subset of \( X \), and \( T \) a conditionally quasi-nonexpansive mapping of \( D \) into \( D \) which is either 1-set-contractive or 1-ball-contractive. If we additionally assume that \( I - T \) is demiclosed at 0 (i.e., if \( \{ y_n \} \) is any sequence in \( D \) such that \( y_n \to y_0 \) in \( D \) and \( y_n - T(y_n) \to 0 \), then \( (I - T)(y_0) = 0 \)), then for any \( x_0 \) in \( D \) and \( \lambda \) in \((0, 1)\) the set \( W(x_0) \subseteq D \) of weak limit points of the sequence \( \{ T_\lambda^n(x_0) \} \) is nonempty and every \( p \) in \( W(x_0) \) is a fixed point of \( T \). If we assume additionally that \( T \) has at most one fixed point, then \( \{ T_\lambda^n(x_0) \} \) converges weakly to that point.

**Proof.** Since \( D \) is convex and \( T(D) \subseteq D \), for any fixed \( w \in D \) and \( k_n \in (0, 1) \), the mapping \( T_\lambda \) defined on \( D \) by \( T_\lambda(x) = (1 - k_n) w + k_n T(x) \) is either a \( k_n \)-set-contractive or a \( k_n \)-ball-contractive mapping with \( k_n < 1 \) of \( D \) into itself. Hence, by the theorem of Darbo [10] or Sadowsky [39], there exists \( y_n \in D \) such that \( y_n = T_\lambda(y_n) \). Taking \( k_n \) in \((0, 1)\) such that \( k_n \to 1 \) as \( n \to \infty \) and noting that \( \{ T(x_n) \} \) is bounded we see that

\[
T(y_n) - y_n = T(y_n) - T_\lambda(y_n) = (1 - k_n) T(x_n) - (1 - k_n) w \to 0
\]
as \( n \to \infty \). Since \( \{ y_n \} \subseteq D \) is bounded, \( D \) convex, and \( X \) reflexive, without loss of generality we may assume that \( y_n \to y_0 \in D \). This and the fact that \( I - T \) is demiclosed at 0 implies that \( y_0 - T(y_0) = 0 \), i.e., \( F(T) \neq \emptyset \).

Since \( T \) is also quasi-nonexpansive and \( X \) is uniformly convex, Lemma 2.1 implies that \( T_\lambda \) is asymptotically regular at each \( x_0 \) in \( D \). Moreover, the demiclosedness of \( I - T \) at 0 implies that \( T_\lambda \) satisfies condition (4.3) of Theorem 4.1. Consequently, the conclusion of Corollary 4.4 follows from Theorem 4.2. Q.E.D.
STRONG AND WEAK CONVERGENCE

We add in passing that in case $T$ is nonexpansive on $D$ it was shown by Browder [7] that $I - T$ is demiclosed. In [26, 27] Nussbaum established the demiclosedness of $I - T$ for a much wider class of continuous mappings $T: D \to X$, the so-called lane mappings, defined as follows: given any $x$ in $D$ and $e > 0$ there exists a weak neighborhood $N_x$ of $x$ in $D$ (depending also on $e$) such that

$$
\| T(u) - T(v) \| \leq \| u - v \| + e \quad \text{for all } u, v \text{ in } N_x.
$$

Since, as has been shown in [26], lane mappings are 1-set-contractive, the iteration results discussed in this section are also applicable to such mappings. In addition to nonexpansive and lane mappings, the class of 1-set and 1-ball contractive mappings contains also a class of semicontractive mappings and mappings of semicontractive type introduced by Browder [6] and further studied by Browder [7], Webb [42], Petryshyn [37], Nussbaum [26], and others.

If in Corollary 4.4 it is assumed that $D$ is the closure of an open bounded convex subset in $X$, then the following result also hold.

**Corollary 4.5.** Let $X$ be a uniformly convex Banach space, $D$ a bounded open convex subset of $X$, and $T$ a quasi-nonexpansive mapping of the closure $D$ into $X$ such that $T$ is either 1-set-contractive or 1-ball-contractive and there exists an $y_0$ in $D$ for which

$$
T(y) - y_0 \neq \gamma(y - y_0) \quad \text{for all } y \in \partial D, \text{ and all } \gamma > 1.
$$

If additionally we assume that $\{T_\lambda^n(x_0)\} \subseteq D$ for some $x_0$ in $D$ and that $I - T$ is demiclosed at $0$, then the set $W(x_0) \subseteq D$ of weak limit points of $\{T_\lambda^n(x_0)\}$ has the property specified in Corollary 4.4. If we assume additionally that $T$ has at most one fixed point, then $\{T_\lambda^n(x_0)\}$ converges weakly to that point.

**Proof.** As in the proof of Corollary 4.4, it suffices to show that for $T$ satisfying the conditions of Corollary 4.5 the set $F(T) \neq \emptyset$. But the latter fact follows from Petryshyn's fixed point theorem [36].

If we strengthen the conditions on $X$, then we may sharpen the assertions of Theorems 4.2 and 4.3 and their corollaries by eliminating the hypothesis that $F(T)$ consists of a single point. The additional condition on $X$ (see Property (0) below) is the one introduced by Opial in his paper [29] in which he generalized certain results of Schaefer [40] and of Browder and Petryshyn [8] concerning the weak convergence of a sequence of successive iterants for nonexpansive mappings. The theorems obtained here extend those of [40, 8, 29]. They are also related to the results in [15].

Our first result in this area is the following convergence theorem in which, unlike Opial [29], we do not assume that $X$ is uniformly convex.
Theorem 4.4. Let $X$ be a strictly convex and reflexive Banach space, $D$ a closed convex subset of $X$, and $T$ a continuous mapping of $D$ into $X$ such that

(4.5) $F(T) \neq \emptyset$.

(4.6) $T$ is quasi-nonexpansive on $D$.

(4.7) There exists an $x_0$ in $D$ such that $T^n(x_0) \in D$ for $n \geq 1$.

(4.2) $T$ is asymptotically regular at $x_0$.

(4.3) If $\{x_n\}$ is a subsequence of $\{x_n\} = \{T^n(x_0)\}$ such that $x_{n_j} \rightharpoonup \bar{x}$ in $D$ and $(I - T)(x_{n_j}) \to 0$, then $(I - T)(\bar{x}) = 0$.

(4.8) The space $X$ has the Property (0): If $\{y_n\}$ is any sequence in $X$ which converges weakly in $X$ to $y_0$, then

$$\liminf \| y_n - y \| \geq \liminf \| y_n - y_0 \| \quad \text{for all } y \neq y_0.$$

Then the sequence $\{x_n\}$ converges weakly to a fixed point of $T$ in $D$.

Proof. Theorem 4.2 implies that $\{x_n\}$ contains a weakly convergent subsequence with its limit in $F(T)$, and moreover, every weakly convergent subsequence of $\{x_n\}$ has some point in $\bar{p}$ in $F(T)$ for its limit. We will show that our additional condition (4.8) implies that $\bar{p}$ is the same for every weakly convergent subsequence of $\{x_n\}$ and thus that $x_n \to \bar{p}$ as $n \to \infty$.

First, since $X$ is strictly convex and $T: D \to X$ is quasi-nonexpansive, Lemma 2.2 implies that $F(T)$ is a convex subset of $D$ which is obviously closed because $T$ is continuous. Let $B(p_0, r)$ be a closed ball about some point $p_0$ in $F$ which also contains the point $x_0$ from $D$ for which $\{T^n(x_0)\} \subseteq D$. It follows that $\{T^n(x_0)\}$ lies in the bounded closed convex set $D_0 = D \cap B(p_0, r)$. Indeed, (4.6) implies that $\| T(x_k) - p_0 \| \leq \| x_k - p_0 \| \leq r$, i.e., $x_k \in D_0$. If we assume that $x_k$ lies in $D_0$ for $k > 1$, then (4.6) again implies that $x_{k+1}$ lies in $D_0$ since $\| x_{k+1} - p_0 \| = \| T(x_k) - p_0 \| \leq \| x_k - p_0 \| \leq r$. Consequently, we may limit our consideration to the restriction of $T$ to $D_0$. We denote this restriction again by $T$ and note that its set of fixed points $F_0 = F(T | D_0)$ is a nonempty closed bounded convex subset of $D_0$ given by $F_0 = F(T) \cap D_0$. Since for each $x$ in $F_0$ and $n$, in view of (4.6), we have

$$\| x_n - x \| \leq \| T^n(x_0) - x \| = \| T(x_{n-1}) - x \| \leq \| x_{n-1} - x \|,$$

we can define the functional $f$ from $F_0$ to $R^+ = \{ t \in R \mid t \geq 0 \}$ by means of the equation

$$f(x) = \lim_{n} \| x_n - x \|, \quad x \in F_0.$$

It is not hard to show that $f(x)$ thus defined is a continuous real-valued convex functional defined on $F_0$. Hence $f$ is weakly lower semicontinuous and therefore, since $F_0$ is weakly compact (being a bounded closed convex subset of a
reflexive Banach space $X$), $f$ attains its infimum on $F_0$, i.e., there exists an element $\hat{p}$ in $F_0$ such that $f(\hat{p}) = \inf \{ f(x) \mid x \in F_0 \}$. Now we prove that $x_n = T^n(x_0) \to \hat{p}$ as $n \to \infty$. Suppose the contrary. Then, by the reflexivity of $X$ and the boundedness of $\{x_n\}$, there exists a weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$ whose limit $p$, by Theorem 4.2, lies in $F_0$ and which is such that $\hat{p} \neq p$. Now, since $\lim_n \| x_n - x \|$ exists for every $x$ in $F_0$ and since $X$ has Property (0), it follows that

$$\lim_n \| x_n - \hat{p} \| = \lim_j \| x_{n_j} - \hat{p} \| > \lim \| x_{n_j} - p \| = \lim_n \| x_n - p \|,$$

i.e., $f(\hat{p}) > f(p)$, in contradiction to the definition of $\hat{p}$. Hence $\hat{p} = p$ and, therefore, $x_n \to p$ as $n \to \infty$. Q.E.D.

Let $X^*$ be the dual space of $X$ and let $(w, x)$ denote the value of the linear functional $w \in X^*$ at the element $x$ in $X$. Let $\mu$ be a continuous strictly increasing real valued function on $R^+$ with $\mu(0) = 0$. A map $J$, in general multivalued, of $X \to 2X^*$ is called (see [4]) a duality mapping of $X$ into $X^*$ with the gauge function $\mu$ if

$$J(x) = \{ w \in X^* \mid (w, x) = \|w\| \|x\| \quad \text{and} \quad \|w\| = \mu(\|x\|) \}.$$  

We say that $J$ is weakly continuous at a point $x$ if there exists a selection for $J$ which is weakly continuous at $x$. If $J$ is weakly continuous at every point $x$ in $X$, then $J$ is said to be weakly continuous. It has been shown in [22] that if $J$ is a weakly continuous duality mapping of $X$ into $X^*$, then $J$ is necessarily single-valued and the space $X$ has Property (0).

Consequently, as a corollary of Theorem 4.4 we deduce the following new result.

**Theorem 4.5.** Let $D$ be a closed convex subset in a strictly convex and reflexive Banach space having a weakly continuous duality mapping. If $T$ is a continuous asymptotically regular mapping of $D$ into $D$ such that $F(T) \neq \emptyset$, $T$ is quasi-nonexpansive, and $T$ satisfies condition (4.3) for any $x_0$ in $D$, then for any $x_0$ in $D$ the sequence $\{T^n(x_0)\}$ is weakly convergent to a fixed point of $T$.

It was also shown in [22] that if $J$ is a duality mapping of $X$ into $X^*$ which is weakly continuous only at $x = 0$, then

$$\lim \inf \| y_n - y \| \geq \lim \inf \| y_n - y_0 \| \quad \text{for all } y \neq y_0 \quad (4.8_0)$$

whenever $y_n \to y_0$ as $n \to \infty$. Furthermore, it is easy to see that if $X$ is also assumed to be uniformly convex, then the strict inequality holds in $(4.8_0)$ i.e., $X$ has Property (0). Since a uniformly convex Banach space is both strictly convex and reflexive, Theorem 4.5 implies the validity of the following
corollary which generalizes Theorem 2 of Opial [29], established in [29] under the assumption that $X$ is uniformly convex, $J$ is weakly continuous, and $T: D \to D$ is asymptotically regular and nonexpansive with $F(T) \neq \emptyset$.

**Corollary 4.4.** Let $D$ be a closed convex subset in a uniformly convex Banach space having a duality mapping $J$ which is weakly continuous at $x = 0$. If $T$ is a continuous asymptotically regular map of $D$ into $D$ such that $F(T) \neq \emptyset$, $T$ is quasi-nonexpansive, and $T$ satisfies condition (4.3) for any $x_0$ in $D$, then for each $x_0$ in $D$ the sequence $T^n(x_0)$ converges weakly to a fixed point of $T$.

**Remark 4.2.** To see that Theorem 2 in [29] is a special case of Corollary 4.6, it suffices to note that if $X$ is uniformly convex and $T: D \to D$ is nonexpansive with $F(T) \neq \emptyset$, then $I - T$ is demiclosed. Hence $T$ is quasi-nonexpansive and satisfies condition (4.3).

If in Theorem 4.4 we assume that $X$ is a uniformly convex Banach space and if instead of iterates $\{T^n(x_0)\}$ we consider the iterates $\{x_n\}$ given by $x_n = T^n(x_0)$ for $\lambda \in (0, 1)$, then we may omit the hypothesis (4.2) which requires $T$ to be asymptotically regular at $x_0$.

**Theorem 4.6.** Let $D$ be a closed convex set in a uniformly convex Banach space $X$ and let $T$ be a continuous map of $D$ into $X$ such that

\begin{itemize}
  \item[(4.5)] $F(T) \neq \emptyset$.
  \item[(4.6)] $T$ is quasi-nonexpansive.
  \item[(4.7)] There exist $x_0 \in D$ and $\lambda \in (0, 1)$ such that $\{T\lambda^n(x_0)\} \subset D$ for $n \geq 1$.
  \item[(4.3)] If $\{x_{n_j}\}$ is a subsequence of $\{x_n\} = \{T\lambda^n(x_0)\}$ such that $x_{n_j} \to \hat{x}$ in $D$ and $(I - T)(x_{n_j}) \to 0$ as $j \to \infty$, then $(I - T)(\hat{x}) = 0$.
  \item[(4.8)] $X$ has Property (0).
\end{itemize}

Then, under the above conditions, the sequence $\{T\lambda^n(x_0)\}$ converges weakly to a fixed point of $T$.

**Proof.** Theorem 4.6 follows from Theorem 4.4 and Lemma 2.2 since the assumption that $X$ is uniformly convex and $T: D \to X$ is quasi-nonexpansive imply that $X$ and $T\lambda$ satisfy all the conditions of Theorem 4.4.

Our discussion preceding the statement of Corollary 4.4 and Theorem 4.6 imply the validity of the following corollary which generalizes Theorem 3 of Schaefer [40] for weakly continuous nonexpansive mappings in Hilbert spaces as well as Theorem 3 of Opial [29] for nonexpansive mappings and Banach spaces having weakly continuous duality mappings.

**Corollary 4.7.** The assertion of Theorem 4.6 remains valid if instead of the condition that $X$ has Property (0) we assume that $X$ has a duality mapping which is weakly continuous at $x = 0$. 
Remark 4.3. In case $X$ is a Hilbert space and $T$ a continuous quasi-nonexpansive mapping of $D$ into $D$ with $F(T) \neq \emptyset$ and $I - T$ demiclosed, the weak convergence of $\{T^n(x_0)\}$ to a fixed point has also been obtained in [15] from the corresponding result for the normal Mann process.

Remark 4.4. If in Corollaries 4.3, 4.4, and 4.5, it is assumed that $X$ has either Property (0) or a duality mapping which is weakly continuous at $x = 0$, then the sequence of iterates $\{T^n(x_0)\}$ converges weakly to a fixed point of $T$ in $D$ without the additional assumption that $T$ has only one fixed point in $D$.

Note added in proof. In their recent paper, A fixed point theorem in uniformly convex spaces, Bull. U.M.I. 7 (1973), Goebel, Kirk, and Shimi have obtained a fixed point theorem for mappings $T: D \subset X \rightarrow D$ satisfying the condition

$$\|Tx - Ty\| \leq a \|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|)$$

for all $x$ and $y$ in $D$, where $a > 0$, $b > 0$, $c > 0$, and $a + 2b + 2c < 1$. The following simple argument shows that a map $T$ satisfying condition $(\pi)$ and with $F(T) \neq \emptyset$ is quasinonexpansive. Indeed, let $p \in F(T)$ and set $\varepsilon = a + 2b + 2c$. Then, in view of $(\pi),

$$\|Tx - p\| \leq a \|x - p\| + b \|x - Tx\| + c(\|x - p\| + \|p - Tx\|),$$

and, therefore,

$$(1 - c)\|Tx - p\| \leq (a + c)\|x - p\| + b \|x - Tx\|.$$  

On the other hand, $\|x - Tx\| \leq \|x - p\| + \|p - Tx\|$, so that

$$(1 - b - c)\|Tx - p\| \leq (a + b + c)\|x - p\| = (\varepsilon - b - c)\|x - p\|.$$  

This implies that $\|Tx - p\| \leq \|x - p\|$ since $\varepsilon < 1$.

REFERENCES

26. R. D. Nussbaum, The fixed point index and fixed point theorems for k-set-contractions, Ph.D. Dissertation, Univ. of Chicago (1968).