A general law of precise asymptotics for the counting process of record times

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Abstract

This paper achieves a general law of precise asymptotics for the counting process of record times of i.i.d. absolutely continuous random variables. It can describe the relations among the boundary function, weighted function, convergence rate and limit value in studies of complete convergence. This extends and generalizes the corresponding results in Stochastic Process. Appl. 101 (2002) 233–239.

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1. Introduction

Since Hsu and Robbins [11] introduced the concept of complete convergence, there have been extensions in two directions. Let \( \{X, X_k: k \geq 1\} \) be a sequence of i.i.d. random variables (r.v.), \( S_n = \sum_{i=1}^{k} X_k, n \geq 1 \), and \( \psi(x) \) and \( f(x) \) be the positive functions defined on \([0, \infty)\). One extension is to discuss the moment conditions, from which it follows that

\[
\sum_{n=1}^{\infty} \psi(n) P(|S_n| \geq \varepsilon f(n)) < \infty, \quad \varepsilon > 0,
\]

\(1.1\)
where $\sum_{n=1}^{\infty} \psi(n) = \infty$. In this direction, one can refer to Hsu and Robbins [11], Erdős [4,5] and Baum and Katz [1], etc. They, respectively, studied the cases in which $\psi(n) \equiv 1$, $f(n) = n$ and $\psi(n) = n^{r/p-2}$, $f(n) = n^{1/p}$, where $0 < p < 2$, $r \geq p$.

Another extension departs from the observation that the convergence rate and limit value of $\sum_{n=1}^{\infty} \psi(n) P(|S_n| \geq \varepsilon f(n))$ as $\varepsilon \downarrow a$, $a \geq 0$. A first result in this direction was Heyde [10], who proved that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n) = E X^2,$$

(1.2)

where $EX = 0$ and $EX^2 < \infty$. For analogous results in the more general case, see [3,7–9, 14], etc. The researches in this field are called the precise asymptotics. Gut [9] studied the precise asymptotics for the counting process of record times of i.i.d. absolutely continuous random variables.

Let $L(1) = 1$ and, recursively

$$L(n) = \min\{k: X_k > X_{L(n-1)}, k > L(n-1)\}, \quad n \geq 2.$$

The associated counting process $\mu(n)$ is defined by

$$\mu(n) = \max\{k: L(k) \leq n\}.$$

Gut [9] achieved the following three results.

**Theorem A1.** Let $1 \leq p < 2$ and $\delta > -1$. Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2p(1+\delta)} \sum_{n \geq 3} \frac{\log n}{n} P(|\mu(n) - \log n| > \varepsilon (\log n)^{1/p}) = \frac{1}{1+\delta} E |N|^{2p(1+\delta)}^{-2p}, \quad (1.3)$$

where r.v. $N \sim N(0, 1)$.

**Theorem A2.** Let $r > 0$. Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{2(\log(n \log n)^{r-1}} \sum_{n \geq 3} \frac{(\log n)^{-1}}{n \log n} P(|\mu(n) - \log n| > \varepsilon (\log n \log \log n)^{1/2})$$

$$= \frac{2}{\sqrt{r}}. \quad (1.4)$$

In the limiting cases corresponding to $\delta = -1$ in Theorem A1 and $r = 0$ in Theorem A2, Gut [9] proved that

**Theorem A3.** The following three limitations hold:

$$\lim_{\varepsilon \downarrow 0} (-\log \varepsilon)^{-1} \sum_{n \geq 3} (n \log n)^{-1} P(|\mu(n) - \log n| > \varepsilon \log n) = 2, \quad (1.5)$$

$$\lim_{\varepsilon \downarrow 0} (-\log \varepsilon)^{-1} \sum_{n \geq 3} (n \log n \log \log n)^{-1} P(|\mu(n) - \log n| > \varepsilon (\log n \log \log n)^{1/2}) = 2, \quad (1.6)$$
\[ \lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n \geq 9} (n \log n \log \log n)^{-1} P\left( |\mu(n) - \log n| > \varepsilon (\log n \log \log n)^{1/2} \right) = 1. \] (1.7)

In this paper, we are concerned on the following three problems.

**Problem 1.** We call \( \varphi(x) \) and \( f(x) \) in (1.1) weighted function and boundary function. In the study of precise asymptotics, we have been extending the scope of weighted functions and boundary functions all the time. Can we get the corresponding results for more general weighted functions and boundary functions?

**Problem 2.** Which kind of relations do there exist among the weighted function, the boundary function, the normalizing constants, the convergence rate and the limit value of the series and \( \varepsilon \)? Does there exist another new kind of convergence rate?

**Problem 3.** Which kind of relations do there exist in Theorems A1–A3 of Gut [9]? Can we use the same form to present the three results?

In this paper we give the general law of precise asymptotics for associated counting process of record times and some corollaries in Section 2, which extend and generalize Theorems A1–A3 of Gut [9]. We point out that the methods and forms of these results are suitable for the studies of precise asymptotics for record times, the point process of jump times, partial sums and so on. The proofs of these results will be given in Section 3.

2. Main results

In the following, we continue to use the notations in Section 1. Without special statements, let \( X, X_1, X_2, \ldots \) be i.i.d. absolutely continuous random variables, and \( \psi(x) = P(|N| > x) = \sqrt{2/\pi} \int_x^\infty e^{-t^2/2} dt, \ x \geq 0 \). Before giving the main results, we first discuss the general form and conditions of precise asymptotics.

Now we decompose boundary function and weighted function so as to make clear of their meanings. Assume there exists some \( n_0 \in \mathbb{Z}^+ \), the following functions are all defined on \([n_0, \infty)\). Denote

\[ f(x) = (\log x)^{1/2} h(x), \quad x \geq n_0, \]

where \((\log x)^{1/2}\) is the normalizing function of \( \mu(n) \), and \( h(x) \) is differentiable. Let \( g(x) \) be differentiable,

\[ \varphi(x) = g'(h(x))h'(x), \quad x \geq n_0. \]

We want to find an appropriate \( a \geq 0 \), and for any \( \varepsilon > a \), to find an appropriate \( G_0(\varepsilon) \) satisfying

\[ G_0(\varepsilon) < \infty, \quad \varepsilon > a \quad \text{and} \quad \lim_{\varepsilon \uparrow a} G_0(\varepsilon) = \infty, \] (2.1)

so that for all \( G(\varepsilon) \sim (G_0(\varepsilon))^{-1}, \ \varepsilon \downarrow a \), we have that
\[
\lim_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n \geq n_0} \varphi(n) P\left( |\mu(n) - E\mu(n)| > \varepsilon (\log n)^{1/2} h(n) \right) = 1. \tag{2.2}
\]

It can be seen that \(G_0(\varepsilon)\) includes the information of the convergence rate, limit value of the series and the limit position of \(\varepsilon\). Choose
\[
G_0(\varepsilon) = \sqrt{\frac{2}{\pi \varepsilon}} \int_{h(n_0)}^\infty g(y) e^{-\varepsilon^2 y^2 / 2} dy, \quad \varepsilon > a, \tag{2.3}
\]
where \(a \geq 0\), such that (2.1) holds.

It is obvious that \(G(\varepsilon)\) is dependent on \(g(x)\) and \(h(x)\). For \(g(x)\) and \(h(x)\), we point out \(h(x)\) cannot be too large, i.e.,
\[
h(x) = O(1)(x - \log x)(\log x)^{-1/2}, \tag{2.4}
\]
otherwise the discussion is trivial. In the following, we will not state this assumption any more. We will make some appropriate limitations to \(g(x)\) and \(h(x)\) in the following theorem such that (2.1) and (2.2) hold, then get three corollaries according to the kinds of \(g(x)\). From these corollaries we can conclude a series of interesting results, which contain Theorems A1–A3 of Gut [9].

Theorem 2.1. Let \(g(x)\) and \(h(x)\) be positive and differentiable functions defined on \([n_0, \infty)\), which are both strictly increasing to \(\infty\), \(\varphi(x) = g'(h(x))h'(x)\) is monotone, and if \(\varphi(x)\) is monotone nondecreasing, we assume \(\lim_{n \to \infty} (\varphi(n + 1)/\varphi(n)) = 1\). And assume that there exists \(a \geq 0\) such that, in (2.3), \(G_0(\varepsilon)\) satisfies (2.1). Finally, suppose that either of the following two conditions is satisfied:

\[
\begin{align*}
\text{(1)} & \quad \sum_{n=n_0}^\infty \varphi(n)(\log n)^{-1/2} < \infty. \tag{2.5} \\
\text{(2)} & \quad \lim_{M \to \infty} \lim_{\varepsilon \downarrow a} \varepsilon G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^\infty g(y) e^{-\varepsilon^2 y^2 / 2} dy = 0, \tag{2.6} \\
& \quad \lim_{M \to \infty} \lim_{\varepsilon \downarrow a} \varepsilon^{-q} G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^\infty y^{-q} dg(y) = 0 \quad \text{for some } q \geq 2, \tag{2.7}
\end{align*}
\]
where \(g^{-1}(x)\) is the inverse function of \(g(x)\).

Then (2.2) holds. Furthermore, when \(a > 0\) or \(a = 0\) and there exist two constants \(b > 0\) and \(M_0 > 1\), such that for any \(\varepsilon > 0\) small enough
\[
\varepsilon g^{-1}\left(G_0(\varepsilon)M_0\right) \geq b \tag{2.8}
\]
holds, then \(E\mu(n)\) in (2.2) can be replaced by \(\log n\).
Corollary 2.1. In Theorem 2.1, \( \lim_{n \to \infty} (\psi(n + 1)/\psi(n)) = 1 \) is a mild condition. Regular varying functions \( R \) and extended regular varying functions \( E \) satisfy this condition (see [2]).

Choose \( g(x) = x^r l(x) \), \( r \geq 0 \), where \( l \in R_0 \) is a slowly varying function. By (2.3), set \( G(\varepsilon) = e^{e^r (l(\varepsilon^{-1}))^{-1}} (E|N|^r)^{-1}, \varepsilon > 0 = a \); then we have

Corollary 2.1. Let \( h(x) \) be a positive and differentiable function defined on \([n_0, \infty)\), which is strictly increasing to \( \infty \), \( \phi(x) = r(h(x))^r - 1 h'(x) \) be monotone, and if \( \psi(x) \) is monotone nondecreasing, we assume \( \lim_{n \to \infty} (\psi(n + 1)/\psi(n)) = 1 \). Further, let \( I \) be bounded away from \( 0 \) and \( \infty \) on every compact subset of \([n_0, \infty)\). Then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^r l(\varepsilon^{-1})^{-1} \sum_{n=n_0}^{\infty} \phi(n) P \left( |\mu(n) - \log n| > \varepsilon (\log n)^{1/2} h(n) \right) = r^{-1} E|N|^r. \tag{2.9}
\]

Especially, we can get Theorems 3.1 and 3.3 (Theorems A1 and A3) in Gut [9], as long as \( l(x) \equiv 1, r > 0, h(x) = (\log x)^{1/p-1/2}, 1 \leq p < 2, \delta = r(2-p)/(2p) - 1; l(x) = (\log x)^{\delta}, r = 1, h(x) = (\log x)^{1/2}; l(x) = (\log x)^{\delta}, r = 1, h(x) = (\log \log x)^{1/2}; \) and \( l(x) \equiv 1, r = 2, h(x) = (\log \log \log x)^{1/2}, \) respectively.

Choose \( g(x) = e^{e^2}, r > 0 \), then by (2.3), set \( G(\varepsilon) = (2r)^{1/2} (2r)^{-1/2}, \varepsilon > (2r)^{1/2} = a \), hence we have

Corollary 2.2. Let \( h(x) \) be a positive and differentiable function defined on \([n_0, \infty)\), which is strictly increasing to \( \infty \), \( \phi(x) = 2 e^{e^2 h(x) h'(x)} \) be monotone, and if \( \psi(x) \) is monotone nondecreasing, we assume \( \lim_{n \to \infty} (\psi(n + 1)/\psi(n)) = 1 \). Finally, assume that \( h(x) \) satisfies condition (1). Then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{2r - 1/2} \sum_{n=n_0}^{\infty} e^{e^{2r h(n) h'(n) \phi(n)}} P \left( |\mu(n) - \log n| > \varepsilon (\log n)^{1/2} h(n) \right) = (2r)^{-1/2}. \tag{2.10}
\]

Especially, let \( h(x) = (\log \log \log x)^{1/2} \); then we get Theorem 3.2 (Theorem A2) in Gut [9].

Let \( h(x) = (\log \log x)^{1/2} \) and \( 0 < r < 1/2 \), then

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{2r - 1/2} \sum_{n=3}^{\infty} n^{-1} (\log n)^{r-1} P \left( |\mu(n) - \log n| > \varepsilon (\log n \log \log n)^{1/2} \right) = 2^{1/2} r^{-1/2}. \tag{2.11}
\]

In the end, we will give a new kind of convergence rate. Choose \( g(x) = e^{e^x}, r > 0 \), by (2.3) set \( G(\varepsilon) = e^{-\varepsilon^r/(2e^r)}/2, \varepsilon > 0 = a \), then it follows that

Corollary 2.3. Let \( h(x) \) be a positive and differentiable function defined on \([n_0, \infty)\), which is strictly increasing to \( \infty \), \( \phi(x) = r e^{e^2 h(x) h'(x)} \) be monotone, and if \( \psi(x) \) is monotone non-

\[
\text{...}
\]
decreasing, we assume \( \lim_{n \to \infty} \frac{(\varphi(n+1)/\varphi(n))}{\varphi(n)} = 1 \). Finally, assume that \( h(x) \) satisfies condition (1). Then
\[
\lim_{\varepsilon \downarrow 0} \frac{2^2}{(2\varepsilon)^2} \sum_{n=n_0}^{\infty} e^{\varepsilon h(n)} h'(n) P\left( |\mu(n) - \log n| > \varepsilon (\log n)^{1/2} h(n) \right) = 2r^{-1}. \tag{2.12}
\]

Especially, let \( h(x) = \log \log x \) and \( 0 < r < 1/2 \); then
\[
\lim_{\varepsilon \downarrow 0} \frac{2^2}{(2\varepsilon)^2} \sum_{n=2}^{\infty} n^{-1}(\log n)^{r-1} P\left( |\mu(n) - \log n| > \varepsilon (\log n)^{1/2} \log \log n \right) = 2r^{-1}. \tag{2.13}
\]

3. Proofs

Proof of Theorem 2.1. At first we discuss the relations between the integral and the series, which are both composed of the same function \( \varphi(x) \psi(\varepsilon h(x)) \). If \( \varphi(x) \) is nonincreasing, then \( \varphi(x) \psi(\varepsilon h(x)) \), \( \varepsilon > a \), is nonincreasing, hence by (2.3), (2.1) and integration by parts, we have
\[
\int_{n_1+1}^{\infty} \varphi(x) \psi(\varepsilon h(x)) \, dx \leq \sum_{n=n_1+1}^{\infty} \varphi(n) \psi(\varepsilon h(n)) \leq \int_{n_0}^{\infty} \varphi(x) \psi(\varepsilon h(x)) \, dx < \infty. \tag{3.1}
\]
If \( \varphi(x) \) is nondecreasing, then by \( \lim_{n \to \infty} \frac{(\varphi(n+1)/\varphi(n))}{\varphi(n)} = 1 \), for any \( 0 < \delta < 1 \), there exists \( n_1 = n_1(\delta) \), when \( n \geq n_1 \), \( \varphi(n+1)/\varphi(n) < 1 + \delta \) and \( \varphi(n)/\varphi(n+1) > 1 - \delta \). Thus we have that
\[
(1 + \delta)^{-1} \int_{n_1+1}^{\infty} \varphi(x) \psi(\varepsilon h(x)) \, dx \leq \sum_{n=n_1+1}^{\infty} \varphi(n) \psi(\varepsilon h(n)) \leq (1 - \delta)^{-1} \int_{n_1}^{\infty} \varphi(x) \psi(\varepsilon h(x)) \, dx. \tag{3.2}
\]
Hence by (3.1), (3.2), (2.1), (2.3), integration by parts and \( G(\varepsilon) \sim (G_0(\varepsilon))^{-1} \), \( \varepsilon \downarrow a \), we have
\[
(1 + \delta)^{-1} \leq \lim \inf_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n=n_0}^{\infty} \varphi(n) \psi(\varepsilon h(n)) \leq \lim \sup_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n=n_0}^{\infty} \varphi(n) \psi(\varepsilon h(n)) \leq (1 - \delta)^{-1}. \tag{3.3}
\]
Let \( \delta \downarrow 0 \); then we conclude
\[
\lim_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n=n_0}^{\infty} \varphi(n) \psi(\varepsilon h(n)) = 1. \tag{3.4}
\]
By [9], we know that
\[
(\mu(n) - E\mu(n)) (\log n)^{-1/2} \xrightarrow{d} N, \quad n \to \infty,
\] (3.5)
and by [13], we know that
\[
\Delta_n = \sup_x \left| P \left( \left| \mu(n) - E\mu(n) \right| > x \right) - \psi \left( x ( \log n )^{-1/2} \right) \right| \leq 3.8 ( \log n )^{-1/2}, \quad n \geq 2.
\] (3.6)
Together with condition (1), we get (2.2) immediately.

Next assume that condition (2) is satisfied. Denote the inverse functions of \( g(x) \) and \( h(x) \) by \( g^{-1}(x) \) and \( h^{-1}(x) \), respectively. For any \( M > 1 \), set \( b(\varepsilon) = h^{-1} \circ g^{-1}(G_0(\varepsilon) M) \), \( \varepsilon > a \); then \( b(\varepsilon) \to \infty, \varepsilon \downarrow a \).

First, we give the following lemma.

**Lemma 3.1.** Let \( g(x), h(x), \psi(x) \), \( x \geq n_0 \) be the same as in Theorem 2.1. Then
\[
g \circ h(x + 1) \leq 4g \circ h(x)
\]
holds for all \( x \geq n_0 \), when \( \psi(x) \) is nonincreasing; and holds for \( x \) large enough, when \( \psi(x) \) is nondecreasing.

**Proof.** If \( \psi(x) \) is nondecreasing, then by \( \psi(n + 1)/\psi(n) \to 1, n \to \infty \), it is easy to see that \( \psi(n + 1)/\psi(n) \to 1, x \to \infty, x > 0 \). Using L’Hospital’s rule, we have that
\[
\lim_{x \to \infty} \frac{g \circ h(x + 1)}{g \circ h(x)} = \lim_{x \to \infty} \frac{\psi(x + 1)}{\psi(x)} = 1.
\]
Hence there exists \( a < \varepsilon_0 \), when \( a < \varepsilon < \varepsilon_0 \), for any \( x > 0 \), \( 4g \circ h(x) \geq g \circ h(x + 1) \).

If \( \psi(x) \) is nonincreasing, then \( g \circ h(x) \) is a concave function, thus for any \( x > 0 \),
\[
g \circ h \left( \frac{1}{2} (x + 1) + \frac{1}{2} \right) = g \circ h \left( x + \frac{1}{2} \right)
\]
\[
\geq \frac{1}{2} \left( 4g \circ h(x + 1) + \frac{1}{2} g \circ h(x) \right) \geq \frac{1}{2} \left( 4g \circ h(x + 1) \right).
\]
Thus
\[
g \circ h(x) = g \circ h \left( x - \frac{1}{2} + \frac{1}{2} \right) \geq \frac{1}{2} \left( 4g \circ h \left( x + \frac{1}{2} \right) \right) \geq \frac{1}{2} \left( 4g \circ h(x + 1) \right).
\]
Next we continue the proof of Theorem 2.1. If \( \psi(x) \) is nondecreasing, then by Lemma 3.1,
\[
4G_0(\varepsilon) M = 4g \circ h(b(\varepsilon)) \geq g \circ h(b(\varepsilon) + 1) \geq \int_{n_0}^{b(\varepsilon) + 1} dg \circ h(x)
\]
\[
= \int_{n_0}^{b(\varepsilon) + 1} \psi(x) \, dx \geq \sum_{n_0 \leq a \leq b(\varepsilon)} \int_{n}^{n+1} \psi(x) \, dx \geq \sum_{n_0 \leq a \leq b(\varepsilon)} \psi(n)
\]
\[
\sum_{n=n_0}^{n_1-1} \varphi(n) + \sum_{n_1 \leq n \leq b(\varepsilon)} \varphi(n) \geq \sum_{n=n_0}^{n_1-1} \varphi(n) + (1 - \delta) \sum_{n_1 \leq n \leq b(\varepsilon)} \varphi(n + 1)
\]
\[
\geq (1 - \delta) \left( \sum_{n_0 \leq n \leq b(\varepsilon) + 1} \varphi(n) - \varphi(n_1) \right). \tag{3.7}
\]

If \( \varphi(x) \) is nonincreasing, similarly we have
\[
4G_0(\varepsilon)M \geq \sum_{n_0 \leq n \leq b(\varepsilon) + 1} \varphi(n) - \varphi(n_0). \tag{3.8}
\]

By (3.6)–(3.8) and Toeplitz lemma (see also Lemma 6.10 in [12]), we get
\[
\lim_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n_0 \leq n \leq b(\varepsilon) + 1} \varphi(n) \Delta_n = 0. \tag{3.9}
\]

Similar to (3.1) and (3.2), for any \( 0 < \delta < 1 \), there exists \( \varepsilon_0 > a \), when \( a < \varepsilon < \varepsilon_0 \), by integration by parts and (2.1),
\[
\sum_{n > b(\varepsilon) + 1} \varphi(n) \psi(\varepsilon h(n)) \leq (1 - \delta)^{-1} \left( \int_{\varepsilon b(\varepsilon)}^{\infty} \varphi(x) \psi(\varepsilon h(x)) dx + \varphi\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right) \psi\left(\varepsilon h\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right)\right) \right)
\]
\[
= (1 - \delta)^{-1} \left( \int_{h(b(\varepsilon))}^{\infty} \psi(\varepsilon y) dg(y) + \varphi\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right) \psi\left(\varepsilon h\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right)\right) \right)
\]
\[
\leq (1 - \delta)^{-1} \left( \sqrt{\frac{2}{\pi}} \int_{h^{-1}(G_0(\varepsilon)M)}^{\infty} g(y) e^{-y^2/2} dy + \varphi\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right) \psi\left(\varepsilon h\left(\left\lfloor b(\varepsilon) \right\rfloor + 2\right)\right) \right). \tag{3.10}
\]

By (2.6) of condition (2) and (3.1), we get
\[
\lim_{M \to \infty} \lim_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n > b(\varepsilon) + 1} \varphi(n) \psi(\varepsilon h(n)) = 0. \tag{3.11}
\]

In the following, we prove that
\[
\lim_{M \to \infty} \lim_{\varepsilon \downarrow a} G(\varepsilon) \sum_{n > b(\varepsilon) + 1} \varphi(n) P\left(\left| \mu(n) - E\mu(n) \right| > \varepsilon (\log n)^{1/2} h(n) \right) = 0. \tag{3.12}
\]

Let
\[
I_k = \begin{cases} 
1 & \text{if } X_k \text{ is record}, \\
0 & \text{otherwise},
\end{cases}
\]
so that \( \mu(n) = \sum_{k=1}^{n} I_k, n \geq 1 \). In view of [13] (see also Lemma 6.3.3 in [6]), we know \( P(I_k = 1) = 1 - P(I_k = 0) = k^{-1}, k \geq 1 \) and \( \{I_k: k \geq 1\} \) are independent random variables. Hence by Rosenthal inequality (see also Theorem 2.9 in [12]), we have that for any \( q \geq 2 \),

\[
P(\left| \mu(n) - E \mu(n) \right| > \varepsilon \log n)^{1/2} h(n)) \]

\[
= O(1) \varepsilon^{-q} (\log n)^{-q/2} (h(n))^{-q} \left( \sum_{k=1}^{n} E \left| I_k - \frac{1}{k} \right| + \left( \sum_{k=1}^{n} E \left( I_k - \frac{1}{k} \right)^2 \right)^{q/2} \right)
\]

\[
= O(1) \varepsilon^{-q} (h(n))^{-q}.
\]

(3.13)

Similar to (3.10), for any \( 0 < \delta < 1 \), there exists \( \varepsilon_0 > a \), when \( a < \varepsilon < \varepsilon_0 \), we have

\[
\sum_{n=b(\varepsilon) + 1}^{\infty} \psi(n) (h(n))^{-q} \leq (1 - \delta)^{-1} \left( \int_{b(\varepsilon)}^{\infty} \psi(x)(h(x))^{-q} \, dx + \psi([b(\varepsilon)] + 2)(h([b(\varepsilon)] + 2))^{-q} \right),
\]

(3.14)

together with (2.7) of condition (2), (3.12) holds.

If (2.8) is satisfied, it is obvious that in above proof \( E \mu(n) \) can be replaced by \( \log n \). \( \square \)

**Proof of Corollary 2.1.** Since \( l \in \mathcal{R}_0 \) is bounded away from 0 and \( \infty \) on every compact subset of \([n_0, \infty)\), then by Potter’s theorem (see also Theorem 1.5.6 in [2]), we know that for every \( \delta > 0 \) there exists \( A = A(\delta) > 0 \) such that

\[
\frac{l(y)}{l(x)} \leq A \max \left\{ \left( \frac{y}{x} \right)^{\delta}, \left( \frac{x}{y} \right)^{\delta} \right\}, \quad x, y > n_0.
\]

(3.15)

Set \( u(z) = A \sqrt{2/\pi} \varepsilon^{-\delta} e^{-z^2/2} \); then it is easy to see that \( \int_0^{\infty} u(z) \, dz < \infty \). By (3.15) we have

\[
\sqrt{\frac{2}{\pi}} \varepsilon^{-\delta} \int (z - \varepsilon h(n_0)) \leq A \sqrt{\frac{2}{\pi}} \varepsilon^{-\delta} e^{-z^2/2} I(z > \varepsilon h(n_0)) \leq A \sqrt{\frac{2}{\pi}} \varepsilon^{-\delta} e^{-z^2/2} I(z > \varepsilon h(n_0)) \leq u(z),
\]

hence by the dominant convergence theorem and properties of slowly varying functions

\[
\lim_{r \to 0} \int \frac{2}{\pi} \varepsilon^{-\delta} e^{-z^2/2} \frac{y'' l(y) e^{-z^2/2}}{l(y)} \, dy = \int \frac{2}{\pi} \varepsilon^{-\delta} e^{-z^2/2} \frac{z' l(z - \varepsilon h(n_0)) e^{-z^2/2}}{l(z - \varepsilon h(n_0))} \, dz
\]

\[
= \sqrt{\frac{2}{\pi}} \int_0^{\infty} z' e^{-z^2/2} \, dz = E |N|^r.
\]

Hence we can choose \( G(\varepsilon) = e^{-\varepsilon l(\varepsilon^{-1})} \). Theorem 2.1 tells us that we only need to check condition (2).

By \( g \in \mathcal{R}_c \) and Theorem 1.5.12 in [2], we know that \( g^{-1} \in \mathcal{R}_{1/r} \), hence by Potter’s theorem we get that for any \( A > 0, \delta > 0 \), when \( \varepsilon \) is small enough,
\[ g^{-1}(g(\varepsilon^{-1})ME|N'|) \leq B(ME|N'|)^{\varepsilon^{-1}}, \]  
(3.16)

Together with (3.15), we know that when \( \varepsilon \) is small enough,

\[
\varepsilon G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} g(y) e^{-y^2/2} \, dy = G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} g(y) y^{-q} \, dy \leq \int_{2e^{-1}(g(\varepsilon^{-1})ME|N'|)}^{\infty} z^{r+\delta} e^{-z^2/2} \, dz \to 0, \quad M \to \infty,
\]
i.e., (2.6) is satisfied.

Next we check (2.7). By integration by parts, Karamata’s theorem (see also Proposition 1.5.10 in [2]) and (3.15) and (3.16), we have that for \( q > \max\{r, 2\}, 0 < \delta < q - r \), when \( \varepsilon \) is small enough,

\[
\varepsilon^{-q} G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} y^{-q} \, dy \leq q \varepsilon^{-q} G(\varepsilon) \int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} g(y) y^{-q+1} \, dy = q \int_{g^{-1}(G_0(\varepsilon)M)}^{g(\varepsilon^{-1})z^{-q-1} \, dz = q (E|N'|)^{-1} \int_{g^{-1}(G_0(\varepsilon)M)}^{\infty} g(\varepsilon^{-1})z^{-q-1} \, dz \leq Aq (E|N'|)^{-1} \int_{2e^{-1}(g(\varepsilon^{-1})ME|N'|)}^{\infty} z^{r-q-1+\delta} \, dz \to 0, \quad M \to \infty,
\]
i.e., (2.7) is satisfied. Hence by Theorem 2.1, we have

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^\varepsilon (l(\varepsilon^{-1}))^{-1} \sum_{n=n_0}^{\infty} \psi(n) P(\left| \mu(n) - E\mu(n) \right| > \varepsilon (\log n)^{1/2} h(n)) = r^{-1} E|N'|.
\]
(3.17)

Furthermore, by (3.15) we know for any \( M_0 > 0 \),

\[
\varepsilon g^{-1}(g(\varepsilon)M_0) \geq \frac{1}{2} \varepsilon g^{-1}(g(\varepsilon)M_0 E|N'|) \geq \frac{1}{2} A^{-1} g^{-1}(g(\varepsilon))(M_0 E|N'|)^{-\delta} = \frac{1}{2} A^{-1}(M_0 E|N'|)^{-\delta} > 0,
\]
i.e., (2.8) is satisfied, hence we can substitute \( \log n \) for \( E\mu(n) \) in (3.17). \( \Box \)
Proof of Corollary 2.2. By $g(x) = e^{rx^2}$, $r > 0$ and (2.3), we know that $G(\varepsilon) \sim (\varepsilon^2 - 2r)^{-1/2}$, $\varepsilon \downarrow a = (2r)^{1/2}$ satisfies (2.1), hence we can choose $G(\varepsilon) = (\varepsilon^2 - 2r)^{1/2} \times (2r)^{-1/2}$, $\varepsilon > (2r)^{1/2}$. Taking account of Theorem 2.1, we have
\[
\lim_{\varepsilon \downarrow (2r)^{1/2}} (\varepsilon^2 - 2r)^{1/2} \sum_{n=n_0}^{\infty} e^{h(n)h'(n)}h(n)h'(n)P\left(|\mu(n) - E\mu(n)| > \varepsilon (\log n)^{1/2}h(n)\right)
= (2r)^{-1/2}.
\]
(3.18)

By $a = (2r)^{1/2} > 0$, we can substitute $\log n$ for $E\mu(n)$ in (3.18). \hfill \Box

Proof of Corollary 2.3. By $g(x) = e^{rx}$, $r > 0$, $x \geq n_0$ and (2.3), we know that $G_0(\varepsilon) \sim 2e^{r^2/(2\varepsilon^2)}$, $\varepsilon \downarrow a = 0$ satisfies (2.1). We can choose $G(\varepsilon) = 2^{-1}e^{-r^2/(2\varepsilon^2)}$, $\varepsilon > 0$, and by Theorem 2.1, we have
\[
\lim_{\varepsilon \downarrow 0} e^{-r^2/(2\varepsilon^2)} \sum_{n=n_0}^{\infty} e^{h(n)h'(n)}h(n)h'(n)P\left(|\mu(n) - E\mu(n)| > \varepsilon (\log n)^{1/2}h(n)\right) = 2r^{-1}.
\]
(3.19)

Furthermore, for any $b > 0$, let $\varepsilon$ be small enough; then
\[
\varepsilon g^{-1}\left(G_0(\varepsilon)M_0\right) = r^{-1}\left((2\varepsilon)^{-1/2} + \varepsilon \log 2M_0\right) > b,
\]
i.e., (2.8) is satisfied, hence $E\mu(n)$ in (3.19) can be replaced by $\log n$. \hfill \Box

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