doi:10.1006/jmaa.2001.7760, available online at http://www.idealibrary.com on IDEAL®

Neocompact Sets and the Fixed Point Property[1]

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Submitted by William A. Kirk

Received June 26, 2001

We use a newly introduced concept of neocompactness to study problems from metric fixed point theory. In particular, we give a sufficient condition for a super-reflexive Banach space X to have the fixed point property and obtain shorter proofs of some well-known results in that theory. © 2002 Elsevier Science (USA)

1. INTRODUCTION

The notion of neocompact sets was introduced by Fajardo and Keisler in [15] as a generalization of ideas from [27]. With the help of that new concept, the authors were able to present some essential features of nonstandard analysis in a conventional framework. In [14, 15, 28], they gave a large number of examples of how to apply this elegant technique in solving existence problems in probability theory and stochastic analysis (see also [9]). But, as it was written in [28], this method "is intended to be more than a proof technique—it has the potential to suggest new conjectures and new proofs in a wide variety of settings."

Nonstandard methods came to Banach space theory from the work of Luxemburg [35], where the notion of nonstandard hull was introduced. Another approach, based on the concept of Banach space ultraproducts, was proposed by Bretagnolle, Dacunha-Castelle, and Krivine (see [6, 10]). It seems that a nonstandard analysis approach has some conceptual advantages over the ultraproduct method because it provides us with techniques which are not very easy to express in the ultraproduct setting. But the more

[1] Research was supported in part by KBN Grant 2 PO3A 029 15.

0022-247X/02 $35.00

constructible nature of Banach space ultraproducts made this approach more popular among specialists. In 1980, Maurey [36] applied the Banach space ultraproduct construction to solve several difficult problems in metric fixed point theory. His methods have been extended by numerous authors to obtain many strong and deep results in this theory (see, for instance, [3, 12, 17, 32, 34, 38]). Let us also note that there is one more "nonstandard" technique, based on the concept of ultranets, which is effectively used in fixed point theory (see [3, 20, 30, 31, 33]).

In the present paper, we study some problems concerning the fixed point property for nonexpansive mappings (see Section 2 for the definitions). Our aim is to join techniques from metric fixed point theory with the strength of the neometric approach to signal new possibilities in that theory.

Many results of the paper may be translated into ultraproduct language. However, we show several examples where the neometric methods yield stronger results with much less effort.

In Section 2 we recall basic definitions and some of the previous results from fixed point theory for nonexpansive mappings. For more detailed exposition of metric fixed point theory, we refer the reader to [3, 4, 20].

Section 3 contains a brief summary of neometric methods taken from [13–16, 28]. We use these methods in Section 4 to give a new insight into some old problems in metric fixed point theory. As corollaries, we obtain well-known results such as the Goebel–Karlovitz lemma, Lin's lemma, and Maurey's result about metric convexity of Fix $\widetilde{T}$.

In Section 5 we develop our approach in the case of superreflexive spaces. In particular, we give a sufficient condition for a superreflexive Banach space X to have the fixed point property which generalizes the results from [40].

It is our hope that the results presented here will be useful to pursue further study on interactions between nonstandard analysis and fixed point theory.

2. FIXED POINT THEORY FOR NONEXPANSIVE MAPPINGS

Let C be a nonempty, bounded, closed, and convex subset of a Banach space E. A mapping $T\colon C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We say that E has the fixed point property (FPP, in short) if every such mapping has a fixed point. We say that E has the weak fixed point property (wc-FPP, in short) if we additionally assume that C is weakly compact. It is clear that in the case of reflexive spaces, both definitions coincide.

In 1965, Browder [7] proved that Hilbert spaces have FPP. In the same year, Browder [8] and Göhde [22] showed independently that uniformly convex spaces have FPP, and Kirk [29] proved a more general result stating that all Banach spaces with the so-called normal structure have wc-FPP. Whether or not every Banach space enjoyed the weak fixed point property remained an open question until 1980, when Alspach [1] discovered an example of a nonexpansive self-mapping defined on a weakly compact convex subset of $L_1[0,1]$ without fixed points. At the same time, Maurey [36] used the Banach space ultraproduct construction to prove the fixed point property for all reflexive subspaces of $L_1[0,1]$. He also showed [12] that isometries in superreflexive spaces always have FPP. Quite recently, Dowling and Lennard [11] have proved that every nonreflexive subspace of $L_1[0,1]$ fails FPP, and they have developed their techniques in a series of papers. However, the following two important questions in metric fixed point theory remain open:

(1) Does reflexivity imply the fixed point property?

(2) Does the fixed point property imply reflexivity?

Fixed point theory for nonexpansive mappings starts with the following consequence of the Banach contraction principle (see, for instance, [20]):

LEMMA 2.1. *Let C be a nonempty, closed, convex, and bounded subset of a Banach space E and let $T: C \to C$ be a nonexpansive mapping. Then*

$$\inf\{\|x - Tx\|: x \in C\} = 0.$$

Lemma 2.1 asserts that T has "almost fixed points." Consequently, there exist sequences $\langle x_n \rangle$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Such sequences are said to be approximate fixed point sequences. It is easy to see that T has a fixed point if the set C is compact.

Assume now that C is weakly compact and put

$$\mathcal{F} = \{K \subset C: K \text{ is nonempty, closed, convex, and } T(K) \subset K\}.$$

From the weak compactness of C, and decreasing chain of elements in $\mathcal{F}$ has a nonempty intersection which belongs to $\mathcal{F}$. By Zorn's lemma, there exists a minimal (in the sense of inclusion) convex and weakly compact set $K \subset C$ which is invariant under T: $T(K) \subset K$. Such sets are called minimal invariant for T. If a mapping $T: C \to C$ has a fixed point x, then clearly $\{x\}$ is a minimal T-invariant set. In view of Alspach's example [1], there exist minimal invariant sets which are not singletons. It turns out that the geometrical structure of such sets is rather odd. For instance, a minimal set K must be diametral, that is,

$$\sup_{y\in K} \|x - y\| = \operatorname{diam} K$$

for all $x \in K$. This observation led Kirk [29] to prove his celebrated fixed point theorem. Since then, a very fruitful approach to the fixed point problem is to use special features of minimal invariant sets and to look for a contradiction. We shall return to this theme in Section 4. For recent results concerning minimal invariant sets, we refer the reader to [19] and [21].

3. NEOCOMPACT SETS AND LONG SEQUENCES

A family of neocompact sets is a generalization of the family of compact sets and shares many of its properties. Let us first recall basic definitions from [15].

By the product $\mathcal{M} \times \mathcal{N}$ of two metric spaces $\mathcal{M} = (M, \rho)$ and $\mathcal{N} = (N, \sigma)$ we mean the pair $(M \times N, \rho \times \sigma)$, where

$$(\rho \times \sigma)(x, y) = \max\{\rho(x_1, y_1), \sigma(x_2, y_2)\}.$$

DEFINITION 3.1. Let $\mathbf{M}$ be a collection of complete metric spaces $\mathcal{M}$ which is closed under finite cartesian products, and, for each $\mathcal{M} \in \mathbf{M}$, let $\mathcal{B}(\mathcal{M})$ be a collection of subsets of $\mathbf{M}$, which we call basic sets. By a neocompact family over $(\mathbf{M}, \mathcal{B})$ we mean the triple $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ such that, for each $\mathcal{M} \in \mathbf{M}$, $\mathcal{C}(\mathcal{M})$ is a collection of subsets of $\mathcal{M}$ with the following properties, where $\mathcal{M}, \mathcal{N}$ vary over $\mathbf{M}$:

(a) $\mathcal{B}(\mathcal{M}) \subset \mathcal{C}(\mathcal{M})$.

(b) $\mathcal{C}(\mathcal{M})$ is closed under finite unions; that is, if $A, B \in \mathcal{C}(\mathcal{M})$, then $A \cup B \in \mathcal{C}(\mathcal{M})$.

(c) $\mathcal{C}(\mathcal{M})$ is closed under finite and countable intersections.

(d) If $C \in \mathcal{C}(\mathcal{M})$ and $D \in \mathcal{C}(\mathcal{N})$, then $C \times D \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$.

(e) If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$, then the set $\{x\colon (\exists\, y \in \mathcal{N})(x, y) \in C\}$ belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite cartesian product.

(f) If $C \in \mathcal{C}(\mathcal{M} \times \mathcal{N})$ and D is a nonempty set in $\mathcal{B}(\mathcal{N})$, then $\{x\colon (\forall\, y \in D)(x, y) \in C\}$ belongs to $\mathcal{C}(\mathcal{M})$, and the analogous rule holds for each factor in a finite cartesian product.

The sets in $\mathcal{C}(\mathcal{M})$ are called neocompact sets.

DEFINITION 3.2. We say that a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ has the countable compactness property, or is countably compact, if, for each $\mathcal{M} \in \mathbf{M}$, every decreasing chain $C_0 \supset C_1 \supset \cdots$ of nonempty sets in $\mathcal{C}(\mathcal{M})$ has a nonempty intersection $\bigcap_n C_n$.

DEFINITION 3.3. A set $C \subset \mathcal{M}$ is neoclosed in $\mathcal{M}$ if $C \cap D$ is neocompact in $\mathcal{M}$ for every neocompact set D in $\mathcal{M}$. The complement of a neoclosed set is called neoopen.

A set $C \subset \mathcal{M}$ is neoseparable in $\mathcal{M}$ if it is the closure of the union of a countable collection of basic subsets of $\mathcal{M}$.

Let $D \subset \mathcal{M}$. A function $f \colon D \to \mathcal{N}$ is neocontinuous from $\mathcal{M}$ to $\mathcal{N}$ if, for every neocompact set $A \subset D$ in $\mathcal{M}$, the restriction $f \mid A = \{(x, f(x)) \colon x \in A\}$ of f to A is neocompact in $\mathcal{M} \times \mathcal{N}$.

DEFINITION 3.4. We call a neocompact family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ a neometric family, and call its members neometric spaces, if the metric space of reals is a subspace of some $\mathcal{R} \in \mathbf{M}$, and all the coordinate projections and distance functions in $\mathbf{M}$ are neocontinuous.

DEFINITION 3.5. A neometric family $(\mathbf{M}, \mathcal{B}, \mathcal{C})$ is said to have the diagonal intersection property if, for every sequence $\langle A_n \rangle$ of neocompact sets and $\langle \varepsilon_n \rangle$ of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, the "diagonal intersection" $\bigcap_n (A_n)^{\varepsilon_n}$ is neocompact (here $A^{\varepsilon} = \{x \colon \rho(x, A) \leq \varepsilon\}$).

The classical example of a countably compact neometric family is the standard neometric family $(\mathbf{S}, \mathcal{B}, \mathcal{C})$, where $\mathbf{S}$ is the family of all complete metric spaces, and

$$\mathcal{B}(\mathcal{M}) = \mathcal{C}(\mathcal{M}) = \{A \subset \mathcal{M} \colon A \text{ is compact}\}.$$

Several other neometric families were introduced in [14].

We are particularly interested in the so-called huge neometric family. To understand well the rest of this section, some knowledge of nonstandard analysis is needed. But, in future considerations, we shall try not to use nonstandard techniques directly, but properties of neocompact sets and long sequences only.

We fix an $\aleph_1$-saturated nonstandard universe. Recall that for a given internal *metric space $(\overline{M}, \bar{\rho})$, the standard part of an element $X \in \overline{M}$ is the equivalence class ${}^{\circ}X = \{Y \in \overline{M} \colon \bar{\rho}(X, Y) \approx 0\}$, and if $C \subset \overline{M}$, then ${}^{\circ}C = \{{}^{\circ}X \colon X \in C\}$. For each $U \in \overline{M}$, the nonstandard hull $\mathcal{H}(\overline{M}, U)$ is the set $\{{}^{\circ}X \colon \bar{\rho}(X, U) \text{ is finite}\}$ with the metric $\rho({}^{\circ}X, {}^{\circ}Y) = \operatorname{st}(\bar{\rho}(X, Y))$. It is well known that each nonstandard hull is a complete metric space. The monad of a subset $A \subset \mathcal{H}(\overline{M}, U)$ is the set

$$\operatorname{monad}(A) = \{X \in \overline{M} \colon {}^{\circ}X \in A\}.$$

By a Π^0_1 subset of set $\overline{M}$ we mean the intersection of a countable collection of internal subsets of $\overline{M}$. The next definition comes from [14].

DEFINITION 3.6. The huge neometric family $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is defined as follows. $\mathbf{H}$ is the class of the metric spaces $(\mathcal{M}, \rho)$ such that $\mathcal{M}$ is a closed subset of some nonstandard hull $\mathcal{H}(\overline{M}, U)$. For each $\mathcal{M} \in \mathbf{H}$, the collections of basic and neocompact subsets of $\mathcal{M}$ are

$$\mathcal{B}(\mathcal{M}) = \{B \subset \mathcal{M}\colon B = {}^\circ A \text{ for some internal set } A \subset \text{ monad } (\mathcal{M})\},$$
$$\mathcal{C}(\mathcal{M}) = \{C \subset \mathcal{M}\colon C = {}^\circ A \text{ for some } \Pi_1^0 \text{ set } A \subset \text{ monad } (\mathcal{M})\}.$$

It is proved in [14] that $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ is a countably compact neometric family with the diagonal intersection property.

Note that the concept of the nonstandard hull of a Banach space is closely related to the Banach space ultraproduct. If E is a Banach space and $A \subset E$, we shall write $\widetilde{E}$ for the nonstandard hull $\mathcal{H}({}^*E, 0)$ and $\widetilde{A}$ for the set ${}^\circ({}^*A)$. It is easy to see that for each Banach space E in the original superstructure, E is a closed subset of $\widetilde{E}$ if we identify each $x \in E$ with ${}^\circ({}^*x)$. Hence $E \in \mathbf{H}$. Moreover, $\widetilde{A}$ is a basic set for every bounded $A \subset E$. If we recall that the *balls and *spheres in ${}^*E$ are internal, we immediately obtain that the balls and spheres in $\widetilde{E}$ are basic, too.

The list of basic facts about the huge neometric family is given in [28]. In the next sections, we shall especially use the following one.

Proposition 3.7. *Let $\mathcal{M}, \mathcal{N} \in \mathbf{H}$, let $C \subset \mathcal{M}$ be a neocompact set, and let $f\colon C \to \mathcal{N}$. Then f is neocontinuous if and only if there is an internal function F such that ${}^\circ F(X) = f({}^\circ X)$ for all $X \in \operatorname{monad} C$.*

A very useful tool for studying central notions of the huge neometric family is the concept of long sequences. We briefly recall some basic definitions and results from [13].

Definition 3.8. A function $\langle x_n \rangle$ mapping $\mathbb{N}$ into a set S will be called a sequence in S, and a (possibly external) function $\langle x_J \rangle$ mapping ${}^*\mathbb{N}$ into S will be called a long sequence in S.

Definition 3.9. We say that a statement $\phi(J)$ holds a.e., or that $\phi(J)$ holds for all sufficiently small infinite J, if there is an infinite hyperinteger K such that $\phi(J)$ is true for all infinite hyperintegers $J \le K$.

Proposition 3.10 (countable completeness). *The set of all $S \subset {}^*\mathbb{N}$ such that $J \in S$ a.e. is a countably complete filter.*

Definition 3.11. If $\langle x_J \rangle$ is a long sequence in $\mathcal{M} \in \mathbf{H}$ and $\langle X_J \rangle$ is an internal long sequence in $\overline{M}$ such that $x_J = {}^\circ X_J$ for all finite J and all sufficiently small infinite J, we say that $\langle X_J \rangle$ lifts $\langle x_J \rangle$. By an $\mathcal{M}$-sequence we shall mean a long sequence $\langle x_J \rangle$ in $\mathcal{M}$ which has a lifting. A (short) sequence $\langle x_n \rangle$ of elements of $\mathcal{M}$ will be said to be $\mathcal{M}$-extendible if it is the restriction to ${}^*\mathbb{N}$ of some $\mathcal{M}$-sequence $\langle x_J \rangle$, and $\langle x_J \rangle$ will be called an $\mathcal{M}$-extension of $\langle x_n \rangle$.

PROPOSITION 3.12. *If* $\lim_{n\to\infty} x_n = b$ *in* $\mathcal{M}$, *then* $\langle x_n\rangle$ *is* $\mathcal{M}$*-extendible and* $x_J = b$ *a.e.*

PROPOSITION 3.13. *If a set* $C \subset \mathcal{M}$ *is neocompact, then every (short) sequence* $\langle x_n\rangle$ *in* C *has an* $\mathcal{M}$*-extension to a long sequence* $\langle x_J\rangle$ *in* C.

PROPOSITION 3.14. *Suppose* $C \subset \mathcal{M}$, C *is neoclosed, and* $\langle x_J\rangle$ *is an* $\mathcal{M}$*-sequence such that* $\lim_{n\to\infty} \rho(x_n, C) = 0$. *Then* $x_J \in C$ *a.e.*

PROPOSITION 3.15. *Let* $C \subset \mathcal{M}$ *and let* $f\colon C \to \mathcal{N}$ *be neocontinuous from* $\mathcal{M}$ *to* $\mathcal{N}$. *If a sequence* $\langle x_n\rangle$ *in* C *is* $\mathcal{M}$*-extendible, then* $\langle f(x_n)\rangle$ *is* $\mathcal{N}$*-extendible to an* $\mathcal{N}$*-sequence* $\langle y_J\rangle$, *and* $f(x_J) = y_J$ *a.e.*

PROPOSITION 3.16. *Let* C *be neoseparable in* $\mathcal{M}$, *and let* $\langle x_J\rangle$ *be an* $\mathcal{M}$*-sequence such that* $x_J \in C$ *a.e. Then for each* $k \in \mathbb{N}$, $x_n \in C^{\frac{1}{k}}$ *for all but finitely many* $n \in \mathbb{N}$.

4. GENERAL RESULTS

Let us first recall that the relative Chebyshev radius $r_G(A)$ is given by

$$r_G(A) = \inf_{y\in G} \sup_{x\in A} \|x - y\|.$$

Here A is a bounded subset of a Banach space E and $G \subset E$.

From now on, we work in an $\aleph_1$-saturated nonstandard universe and let $(\mathbf{H}, \mathcal{B}, \mathcal{C})$ be its huge neometric family. Let us consider a weakly compact set C in a Banach space $E \in \mathbf{H}$. Then C is also weakly compact in the nonstandard hull $\widetilde{E}$. Note that $\widetilde{C} = \{{}^\circ Y\colon Y \in {}^*C\}$ is not necessarily weakly compact. The following lemma is the starting point of our considerations.

LEMMA 4.1. *Let* C *be a convex, weakly compact subset of a Banach space* E. *Then* $r_{\widetilde{C}}(C) = r_C(C)$.

Proof. Obviously $r_{\widetilde{C}}(C) \le r_C(C)$. Assume that $r_{\widetilde{C}}(C) < r < r_C(C)$ for some $r > 0$ and notice that

$$\bigcap_{x\in C} B_E(x, r) \cap C = \emptyset,$$

where $B_E(x, r)$ is the closed ball in E centered at x with radius r. From the weak compactness of C we have

$$\bigcap_{k=1}^{n} B_E(x_i, r) \cap C = \emptyset$$

for some $x_1, \dots, x_n \in C$. Hence, for every $x \in C$, there exists $i \in \{1, \dots, n\}$ such that $\|x - x_i\| > r$. By transfer, and passing to the nonstandard hull $\widetilde{E} = \mathcal{H}({}^*E, 0)$, for every $x \in \widetilde{C}$, there exists $i \in \{1, \dots, n\}$ such that $\|x - x_i\| \ge r$. Therefore $r_{\widetilde{C}}(C) \ge r_{\widetilde{C}}(\{x_1, \dots, x_n\}) \ge r$ and we obtain a contradiction. ∎

Remark 4.2. Let us note, especially for those readers who prefer the ultraproduct approach, that the transfer principle may be replaced in that proof by the argument that the nonstandard hull $\widetilde{E}$ is finitely representable in E.

Assume now that a Banach space E does not have the weak fixed point property. Then, as in Section 2, there exists a nonexpansive mapping T and a weakly compact, convex set K, which is minimal invariant for T. Without loss of generality we may assume that $\operatorname{diam} K = 1$. If we denote by $\widetilde{T}: \widetilde{K} \to \widetilde{K}$ a natural extension of T and bear in mind Lemma 2.1, we immediately obtain that Fix $\widetilde{T}$, the set of fixed points of $\widetilde{T}$, is nonempty. Using Lemma 4.1, we get a simple proof of the following well-known result which will be useful in our work (see [3, 20, 34, 39]).

PROPOSITION 4.3. *Under the above assumptions, for every $x \in K$ and $w \in$ Fix $\widetilde{T}$,*

$$\|x - w\| = 1.$$

Proof. Let us first notice that if $w \in \operatorname{Fix} \widetilde{T}$ and $r > 0$, then $B_{\widetilde{E}}(w, r)$ is invariant under $\widetilde{T}$. Set $x \in K$ and assume that $\|x - w\| = r < 1$. Then the intersection $B_{\widetilde{E}}(w, r) \cap K$ is a nonempty $\widetilde{T}$-invariant subset of K. But K is minimal and hence $K \subset B_{\widetilde{E}}(w, r)$. Thus, from Lemma 4.1, $r_K(K) = r_{\widetilde{K}}(K) \leq r$. This contradicts the fact that K is diametral [29]:

$$r_K(K) = \operatorname{diam} K = 1.$$

∎

Now we can start our machinery. We shall always assume that K is minimal invariant for T and $\operatorname{diam} K = 1$.

THEOREM 4.4 (Lin [34]). *If $\langle w_n \rangle$ is an approximate fixed point sequence for $\widetilde{T}$ in $\widetilde{K}$, then* $\lim_{n\to\infty} \|w_n - x\| = 1$ *for all $x \in K$.*

Proof. Let us first notice that $\widetilde{K}$ is a neocompact (even a basic) set and $\widetilde{T}: \widetilde{K} \to \widetilde{K}$ is a neocontinuous function. If we write $u_n = Tw_n$, it follows from Proposition 3.13 that $u_J \in \widetilde{K}$ a.e. and, from Proposition 3.15, $u_J = Tw_J$ a.e. But the metric $\rho(x, y) = \|x - y\|$ is also neocontinuous and compositions of neocontinuous functions are neocontinuous. Hence, and from Proposition 3.12, $w_J = u_J$ a.e. and consequently, $w_J \in \operatorname{Fix} \widetilde{T}$ for all sufficiently small infinite J. Fix $x \in K$ and notice that $\|x - w_J\| = 1$ a.e. by Proposition 4.3. Moreover, it is not difficult to see that the unit sphere $S_{\widetilde{E}}(x, 1)$ is a basic, hence a neoseparable, set. If we now use Proposition 3.16, we obtain the desired conclusion. ∎

Let us notice that the well-known Goebel–Karlovitz lemma is now a direct consequence of the above result.

COROLLARY 4.5 (Goebel [18], Karlovitz [26]). *If $\langle x_n \rangle$ is an approximate fixed point sequence for T in K, then* $\lim_{n\to\infty} \|x_n - x\| = 1$ *for all* $x \in K$.

Further considerations of this section are motivated by the following observation. Assume that C is a bounded and convex subset of a Banach space E and $T\colon C \to C$ is a nonexpansive mapping.

PROPOSITION 4.6. *If $C \subset D$ for some neocompact set $D \subset \widetilde{E}$, then* $\operatorname{Fix} \widetilde{T} \cap D \neq \emptyset$.

Even more can be said.

PROPOSITION 4.7. *Let D be a neocompact set and let $\langle w_n \rangle$ be an approximate fixed point sequence for $\widetilde{T}$ in $\widetilde{C}$ such that* $\operatorname{dist}(w_n, D) \to 0$ *if* $w_n \to \infty$. *Then* $\operatorname{Fix} \widetilde{T} \cap D \neq \emptyset$.

Proof. We concluded from Proposition 3.14 that $w_J \in D$ a.e. But $\langle w_n \rangle$ is an approximate fixed point sequence, so $w_J \in \operatorname{Fix} \widetilde{T}$ a.e. This completes the proof. ∎

Note that another proof of the above result is possible with the use of the so-called approximation theorem (see [15]).

In particular, Proposition 4.7 is true for a minimal invariant set K. It follows that, in order to prove the weak fixed point property for E, it is enough to separate the set K from $\operatorname{Fix} \widetilde{T}$ by a neocompact set. Therefore, we should know as much as possible about the structure of these sets. The following theorem gives us some information about the structure of $\operatorname{Fix} \widetilde{T}$.

THEOREM 4.8. *Suppose $T\colon C \to C$ is nonexpansive, suppose C is a bounded and convex subset of E, and assume that*

$$\bigcap_{n=1}^{\infty} B_{\widetilde{E}}(z_n, r_n) \cap \widetilde{C} \neq \emptyset$$

for some $z_1, z_2, z_3, \ldots \in \operatorname{Fix} \widetilde{T}$ *and* $r_1, r_2, r_3, \ldots > 0$. *Then*

$$\bigcap_{n=1}^{\infty} B_{\widetilde{E}}(z_n, r_n) \cap \operatorname{Fix} \widetilde{T} \neq \emptyset.$$

Proof. Let us first notice that the set $D = \bigcap_{n=1}^{\infty} B_{\widetilde{E}}(z_n, r_n) \cap \widetilde{C}$ is a closed convex subset of $\widetilde{C}$ which is invariant under $\widetilde{T}$, and hence there exists an approximate fixed point sequence $\langle w_n \rangle \subset D$. As before, $w_j \in \operatorname{Fix} \widetilde{T}$ a.e. But the balls $B_{\widetilde{E}}(z_n, r_n)$ are neocompact (see remarks after Definition 3.6) and consequently D is neocompact. Therefore, $w_j \in D$ a.e. and the proof is complete. ∎

In spite of its short proof, Theorem 4.8 is strong enough to generalize the well-known result of Maurey which is often used in fixed point theory. Recall [37] that a set A of a Banach space E is said to be metrically convex if, for every $x, y \in A$ with $x \neq y$, there exists $z \in A$, $x \neq z \neq y$, such that $\|x - z\| + \|y - z\| = \|x - y\|$. Equivalently, assuming that A is closed, for every $x, y \in A$ there exists $z \in A$ such that $\|x - z\| = \|y - z\| = \frac{1}{2}\|x - y\|$ (see also [20, Lemma 2.2]).

COROLLARY 4.9 (Maurey [36]). *The set* $\operatorname{Fix} \widetilde{T}$ *is metrically convex.*

Proof. It is enough to consider $z_1, z_2 \in \operatorname{Fix} \widetilde{T}$ and $r_1 = r_2 = \frac{1}{2}\|z_1 - z_2\|$. ∎

Proposition 4.8 suggests the following definition:

DEFINITION 4.10. A set A of a Banach space E is said to be ω_1-metrically convex if, for every sequence $\langle x_n \rangle \subset A$ and $\langle r_n \rangle \subset \mathbb{R}_+$, the condition $\bigcap_{n=1}^{\infty} B_E(x_n, r_n) \cap \overline{\operatorname{conv}}\, A \neq \emptyset$ implies $\bigcap_{n=1}^{\infty} B_E(x_n, r_n) \cap A \neq \emptyset$.

Combining Propositions 4.3 and 4.6 and Theorem 4.8, we obtain

PROPOSITION 4.11. *Let C be a convex, weakly compact subset of a Banach space E and let A be an ω_1-metrically convex subset of the nonstandard hull $\widetilde{E}$ with* $\operatorname{diam} A = \operatorname{diam} C = 1$ *and* $\|x - y\| = 1$ *for every $x \in C$ and $y \in A$. If for every such pair of sets C and A, there exists a neocompact set D with $C \subset D$ and $A \cap D = \emptyset$, then E has the weak fixed point property.*

A large library of neocompact sets and neocontinuous functions is given in [9] and [15]. Let us point out that the assumptions of Proposition 4.11 are easily satisfied if E has normal structure. Indeed, in this case we have $K \subset D$, where $D = B_{\widetilde{E}}(x_0, r)$ for some $x_0 \in K$ and $r < 1$.

In the next section, we show how to simplify the assumptions of the above theorem in the case of superreflexive spaces.

5. THE CASE OF SUPERREFLEXIVE SPACES

We start this section by recalling the following characterization of superreflexive spaces due to Henson and Moore ([23, 24]; see also [25]).

THEOREM 5.1. *For a Banach space E, the following assertions are equivalent*:

(1) *E is superreflexive.*
(2) *$\widetilde{E}$ is reflexive.*
(3) *$\widetilde{E}$ is superreflexive.*
(4) $\widetilde{E'} \cong (\widetilde{E})'$.

In view of Proposition 3.7, the above result leads to the following observation:

PROPOSITION 5.2. *Let E be a superreflexive space. Then every functional $f \in (\widetilde{E})'$ is a neocontinuous function from $\widetilde{E}$ to $\mathbb{R}$.*

Proof. Let $f \in (\widetilde{E})'$. Since $\widetilde{E'} \cong (\widetilde{E})'$, there exists $F \in {}^*(E')$ such that $f({}^\circ X) = {}^\circ F(X)$ for every $X \in {}^*E$. ∎

Unless otherwise stated, we will further assume that E is a superreflexive space and C is a closed and convex (hence weakly compact) subset of E.

Let $T: C \to C$ be a nonexpansive mapping and let $\langle x_n \rangle$ be an approximate fixed point sequence for T. There is no loss of generality in assuming that $\langle x_n \rangle$ converges weakly to $0 \in C$.

PROPOSITION 5.3. *Under the above assumptions, for every $F \in (\widetilde{E})'$, there exists $x \in \operatorname{Fix} \widetilde{T}$ such that $F(x) = 0$.*

Proof. Since $\langle x_n \rangle$ tends weakly to zero, $\lim_{n\to\infty} F(x_n) = 0$ and, from Propositions 3.12, 3.15, and 5.2, $F(x_J) = 0$ a.e. On the other hand, $x_J \in \operatorname{Fix} \widetilde{T}$ a.e. and the proof is complete. ∎

It is natural to ask whether we can replace functionals in Proposition 5.3 by some other functions. In the following lemma, E denotes an arbitrary Banach space.

LEMMA 5.4. *Let S be a weakly compact subset of a nonstandard hull $\widetilde{E}$ and let $f: S \to \mathbb{R}$ be a weakly continuous function. Then there exists a sequence $\langle f_n \rangle$ of neocontinuous and weakly continuous functions which tends to f uniformly on S: $\lim_{n\to\infty} \sup_{x\in S} \|f_n(x) - f(x)\| = 0$.*

Proof. Denote by $C(S)$ the Banach algebra of all weakly continuous, real-valued functions defined on S with the usual supremum norm and let X be the subalgebra of $C(S)$ generated by the unit function and all neocontinuous linear functionals. It is rather routine to check that every $g \in X$ is neocontinuous because internal functions are preserved under finite sum and product operations. Moreover, X separates points of S. By the Stone–Weierstrass Theorem, $\operatorname{cl}(X) = C(S)$, which is our claim. ∎

In view of Lemma 5.4, the following improvement of Proposition 5.3 is now a consequence of the countable compactness property (see Definition 3.2) or the countable completeness property (Proposition 3.10).

PROPOSITION 5.5. *Let $\langle x_n \rangle$ be an approximate fixed point sequence of a nonexpansive map $T: C \to C$ which is weakly convergent to $0 \in C$. Then, for any weakly continuous map $f: \widetilde{C} \to \mathbb{R}$, there exists $x \in \operatorname{Fix} \widetilde{T}$ such that $f(x) = f(0)$.*

Proof. Let us first notice that $\widetilde{C}$ is weakly compact as the closed and convex subset of a superreflexive space $\widetilde{E}$. By Lemma 5.4, for any weakly continuous $f: \widetilde{C} \to \mathbb{R}$, there exists a sequence $\langle f_n \rangle$ of neocontinuous and weakly continuous functions such that

$$\lim_{n\to\infty} \sup_{x\in\widetilde{C}} |f_n(x) - f(x)| = 0. \tag{$\star$}$$

Fix a sequence $\langle \varepsilon_n \rangle$ of positive real numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and write

$$A_k = \{x \in \operatorname{Fix} \widetilde{T}: |f(x) - f(0)| \le \varepsilon_k\}.$$

It follows from ($\star$) that, for every ε_k, there exists a neocontinuous function f_{n_k} such that $\sup_{x\in\widetilde{C}} |f_{n_k}(x) - f(x)| < \varepsilon_k/3$. If we put

$$C_k = \left\{x \in \operatorname{Fix} \widetilde{T}: |f_{n_k}(x) - f_{n_k}(0)| \le \frac{\varepsilon_k}{3}\right\},$$

we obtain $C_k \subset A_k$ for $k = 1, 2, \ldots$ and consequently, $\bigcap_{k=1}^{\infty} C_k \subset \bigcap_{k=1}^{\infty} A_k = \{x \in \operatorname{Fix} \widetilde{T}: f(x) = f(0)\}$. But $\lim_{i\to\infty} f_{n_k}(x_i) = f_{n_k}(0)$ and hence, $f_{n_k}(x_J) = 0$ a.e. for every fixed n_k. By Proposition 3.10, $\bigcap_{k=1}^{\infty} C_k \ne \emptyset$ and consequently, $\bigcap_{k=1}^{\infty} A_k \ne \emptyset$. This means that $f(x) = f(0)$ for some $x \in \operatorname{Fix} \widetilde{T}$ and the proof is complete. ∎

Let us give examples how to apply the above results.

Recall [3] that a nonexpansive map $\widetilde{T}: C \to C$ is said to be of convex type if $\operatorname{Fix} \widetilde{T}$ is convex.

COROLLARY 5.6 (see also [3]). *Let E be superreflexive. If a nonexpansive mapping $T: C \to C$ is of convex type, then it has a fixed point.*

Proof. As before, we may assume that an approximate fixed point sequence $\langle x_n \rangle \subset C$ converges weakly to $0 \in C$. If $0 \in \operatorname{Fix} \widetilde{T}$, we are done. If $0 \notin \operatorname{Fix} \widetilde{T}$, it is enough to notice that $\operatorname{Fix} \widetilde{T}$ is closed and convex, and apply the Hahn–Banach Theorem. ∎

It is well known that every nonexpansive T is of convex type if $\widetilde{E}$ is strictly convex. But in the case of nonstandard hulls, strict convexity is equivalent to uniform convexity.

The situation is more interesting if we consider a minimal invariant set K for T.

THEOREM 5.7. *Let E be superreflexive and assume that for every ω_1-metrically convex set $A \subset S_{\widetilde{E}}$ with* $\operatorname{diam} A \le 1$, *there exists a weakly continuous function $f: \widetilde{E} \to \mathbb{R}$ such that $f(x) > f(0)$ for every $x \in A$. Then E has the fixed point property.*

Proof. Assume conversely that E does not have the fixed point property and let $T: K \to K$ be a nonexpansive mapping with a minimal invariant set K for T. We may assume that $\langle x_n \rangle \subset K$ is weakly convergent to $0 \in K$ and $\operatorname{diam} K = 1$. Then, from Proposition 4.3, Fix $\widetilde{T} \subset S_{\widetilde{E}}$, diam Fix $\widetilde{T} = 1$, and, from Proposition 4.8, Fix $\widetilde{T}$ is ω_1-metrically convex. By the assumptions, there exists a weakly continuous function $f: \widetilde{E} \to R$ such that $f(x) > f(0)$ for $x \in$ Fix $\widetilde{T}$. This contradicts Proposition 5.5. ∎

In [40], we introduced and studied the so-called (S_m) property for a Banach space E. Let us recall the definition. A Banach space E is said to have the (S_m) property if, for every metrically convex set $A \subset S_E$ with $\operatorname{diam} A \leq 1$, there exists $F \in E'$ such that $F(x) > 0$ for every $x \in A$.

The following result is now a direct consequence of Theorem 5.7.

COROLLARY 5.8 [40]. *If a superreflexive nonstandard hull $\widetilde{E}$ has property (S_m), then E has the fixed point property.*

In [40], the (S_m) property was proved for various classes of Banach spaces including separable and strictly convex, as well as uniformly noncreasy spaces (see [38] for the definition). In view of Corollary 5.8, it seems to be an important open question whether all superreflexive spaces have property (S_m) or, more generally, whether the assumptions of Theorem 5.7 are satisfied in every superreflexive Banach space. Note that no example of a Banach space without the (S_m) property is even known.

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