Combinatorial R Matrices for a Family of Crystals: $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ Cases

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For coherent families of crystals of affine Lie algebras of type $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ we describe the combinatorial R matrix using column insertion algorithms for B, C, D Young tableaux. This is a continuation of previous work by the authors (2000, *in* "Physical Combinatorics" (M. Kashiwara and T. Miwa, Eds.), Birkhäuser, Boston). © 2002 Elsevier Science

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1. INTRODUCTION

A combinatorial R matrix is the $q = 0$ limit of the quantum R matrix for a quantum affine algebra $U_q(\mathfrak{g})$, where q is the deformation parameter and $q = 1$ means non-deformed. It is defined on the tensor product of two affine crystals $Aff(B) \otimes Aff(B')$ (see Section 2 for notations) and consists of an isomorphism and an energy function. It was first introduced in [KMN] for the *homogeneous* case where one has $B = B'$. In this case the

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isomorphism is trivial. The energy function was used to describe the path realization of the crystals of highest weight representations of quantum affine algebras. The definition of the energy function was extended in [NY] to the *inhomogeneous* case, i.e., $B \neq B'$, to study the charge of the Kostka−Foulkes polynomials [Ma, LS, KR].

In [KKM] the theory of coherent families of perfect crystals was developed for quantum affine algebras of type $A_n^{(1)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $D_{n+1}^{(2)}$. An element of these crystals is written as an array of nonnegative integers and an explicit description of the energy functions is given in terms of piecewise linear functions of its entries for the homogeneous case. Unfortunately this description is not applicable to the inhomogeneous cases. The purpose of this paper is to give an explicit description of the isomorphism and energy function for the inhomogeneous cases.

The main tool of our description is an insertion algorithm that is a certain procedure on Young tableaux. The insertion algorithm itself had been invented in the context of the Robinson−Schensted correspondence [F, Sc] long before the crystal basis theory was initiated, and subsequently generalized, in, e.g., [Ber, P, Su]. As far as $A_n^{(1)}$ crystals are concerned, the isomorphisms and energy functions were obtained in terms of usual (type *A*) Young tableaux and insertion algorithms thereof [S, SW]. In contrast, no similar description for the combinatorial *R* matrix had been made for other quantum affine algebras, since an insertion algorithm suitable for the B, C, D tableaux given in [KN] was known only recently [B1, B2, L]. In the previous work [HKOT] the authors gave a description for type $C_n^{(1)}$ and $A_{2n-1}^{(2)}$. There we used the $\mathfrak{sp}$-version of semistandard tableaux defined in [KN] and the column insertion algorithm presented in [B1] on these tableaux. In this paper we study the remaining types, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, $B_n^{(1)}$, and $D_n^{(1)}$. We use $\mathfrak{sp}$ and $\mathfrak{so}$ versions of semistandard tableaux defined in [KN] and the column insertion algorithms presented in [B1, B2] on these tableaux.

The layout of this paper is as follows. In Section 2 we give a brief review of the basic notions in the theory of crystals and give the definition of a combinatorial *R* matrix. We first give the description for types $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ in Section 3. In Sections 3.1 and 3.2 we recall the definitions of crystal B_l for type $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ respectively, and give a description of its elements in terms of one-row tableaux. We introduce the map ω from these crystals to the crystal of type $C_n^{(1)}$; hence the procedure is reduced to that of the latter case which we have already developed in [HKOT]. In Section 3.3 we list up elementary operations of column insertions and their inverses for type *C* tableaux with at most two rows. In Section 3.4 we give the main theorem and give the description of the isomorphism and energy function for type $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$, and in Section 3.5 we give examples. The $B_n^{(1)}$ and $D_n^{(1)}$ cases are treated in Section 4. The layout is parallel to

Section 3. In Sections 4.3 and 4.4, however, we also prove the column bumping lemmas (Lemmas 4.4 and 4.9) for B and D tableaux, since a route in the tableau made from inserted letters (bumping route) has some importance in the main theorem.

2. CRYSTALS AND COMBINATORIAL R MATRIX

Let us recall basic notions in the theory of crystals.

See [KMN, KKM] for details. Let $I = \{0, 1, \ldots, n\}$ be the index set. Let B be a P_{cl}-weighted crystal; i.e., B is a finite set equipped with the *crystal structure* that is given by the maps $\tilde{e}_i$ and $\tilde{f}_i$ from $B \sqcup \{0\}$ to $B \sqcup \{0\}$ and maps ε_i and φ_i from B to $\mathbb{Z}_{\geq 0}$. It is always assumed that $\tilde{e}_i 0 = \tilde{f}_i 0 = 0$ and $\tilde{f}_i b = b'$ means $\tilde{e}_i b' = b$.

The crystal B is identified with a colored oriented graph (*crystal graph*) if one draws an arrow as $b \xrightarrow{i} b'$ for $\tilde{f}_i b = b'$. Such an arrow is called *i*-arrow. Pick any i and neglect all the *j*-arrows with $j \neq i$. One then finds that all the connected components are *strings* of finite lengths; i.e., there is no loop or branch. Fix a string and take any node b in the string. Then the maps $\varepsilon_i(b)$, $\varphi_i(b)$ have the following meaning. Along the string you can go forward by $\varphi_i(b)$ steps to an end following the arrows and backward by $\varepsilon_i(b)$ steps against the arrows.

Given two crystals B and B', let $B \otimes B'$ be a crystal defined as follows. As a set it is identified with $B \times B'$. The actions of the operators $\tilde{e}_i$, $\tilde{f}_i$ on $B \otimes B'$ are given by

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

Here $0 \otimes b'$ and $b \otimes 0$ should be understood as 0. All crystals B and the tensor products of them $B \otimes B'$ are connected as a graph.

Let $Aff(B) = \{z^d b \mid b \in B, d \in \mathbb{Z}\}$ be an affinization of B [KMN], where z is an indeterminate. The crystal $Aff(B)$ is equipped with the crystal structure, where the actions of $\tilde{e}_i, \tilde{f}_i$ are defined as $\tilde{e}_i \cdot z^d b = z^{d+\delta_{i0}}(\tilde{e}_i b)$, $\tilde{f}_i \cdot z^d b = z^{d-\delta_{i0}}(\tilde{f}_i b)$. The *combinatorial R matrix* is given by

$$R: Aff(B) \otimes Aff(B') \to Aff(B') \otimes Aff(B)$$

$$z^d b \otimes z^{d'} b' \mapsto z^{d'+H(b \otimes b')} \tilde{b}' \otimes z^{d-H(b \otimes b')} \tilde{b},$$

where $\iota(b \otimes b') = \tilde{b}' \otimes \tilde{b}$ under the isomorphism $\iota \colon B \otimes B' \xrightarrow{\sim} B' \otimes B$. $H(b \otimes b')$ is called the *energy function* and determined up to a global additive constant by

$$H\big(\tilde{e}_i(b \otimes b')\big)$$

$$= \begin{cases} H(b \otimes b') + 1 & \text{if } i = 0, \ \varphi_0(b) \geq \varepsilon_0(b'), \ \varphi_0(\tilde{b}') \geq \varepsilon_0(\tilde{b}), \\ H(b \otimes b') - 1 & \text{if } i = 0, \ \varphi_0(b) < \varepsilon_0(b'), \ \varphi_0(\tilde{b}') < \varepsilon_0(\tilde{b}), \\ H(b \otimes b') & \text{otherwise}, \end{cases}$$

since $B \otimes B'$ is connected. By definition ι satisfies $\tilde{e}_i \iota = \iota \tilde{e}_i$ and $\tilde{f}_i \iota = \iota \tilde{f}_i$ on $B \otimes B'$. The definition of the energy function assures the intertwining property of R; i.e., $\tilde{e}_i R = R \tilde{e}_i$ and $\tilde{f}_i R = R \tilde{f}_i$ on $Aff(B) \otimes Aff(B')$. In the remaining part of this paper we do not stick to the formalism on $Aff(B) \otimes Aff(B')$ and rather treat the isomorphism and energy function separately.

3. $U_q'(A_{2n}^{(2)})$ AND $U_q'(D_{n+1}^{(2)})$ CRYSTAL CASES

3.1. *Definitions*: $U_q'(A_{2n}^{(2)})$ *Case*

Given a positive integer l, we consider a $U_q'(A_{2n}^{(2)})$ crystal denoted by B_l that is defined in [KKM]. B_l's are the crystal bases of the irreducible finite-dimensional representations of the quantum affine algebra $U_q'(A_{2n}^{(2)})$. As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \,\middle|\, x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \right.$$
$$\left. \sum_{i=1}^{n} (x_i + \bar{x}_i) \in \{l, l-1, \ldots, 0\} \right\}.$$

For its crystal structure see [KKM]. B_l is isomorphic to $\bigoplus_{0 \leq j \leq l} B(j\Lambda_1)$ as a $U_q(C_n)$ crystal, where $B(j\Lambda_1)$ is the crystal associated with the irreducible representation of $U_q(C_n)$ with highest weight $j\Lambda_1$. The $U_q(C_n)$ crystal $B(j\Lambda_1)$ has a description in terms of semistandard C tableaux [KN]. The entries are $1, \ldots, n$ and $\bar{1}, \ldots, \bar{n}$ with the total order

$$1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

For an element b of $B(j\Lambda_1)$ let us denote by $\mathscr{T}(b)$ the tableau associated with b. Thus for $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B(j\Lambda_1)$ the tableau $\mathscr{T}(b)$ is depicted by

$$\mathscr{T}(b) = \boxed{\overbrace{1 \cdots 1}^{x_1} \,\Big|\, \cdots \,\Big|\, \overbrace{n \cdots n}^{x_n} \,\Big|\, \overbrace{\bar{n} \cdots \bar{n}}^{\bar{x}_n} \,\Big|\, \cdots \,\Big|\, \overbrace{\bar{1} \cdots \bar{1}}^{\bar{x}_1}}. \tag{3.1}$$

The length of this one-row tableau is equal to j, namely $\sum_{i=1}^{n}(x_i + \bar{x}_i) = j$. Here and in the remaining part of this paper we denote

$$\overbrace{\boxed{i \mid i \mid \cdots \mid i}}^{x} \text{ by } \overbrace{\boxed{i \cdots i}}^{x} \text{ or more simply by } \overbrace{\boxed{i \cdots i}}^{x}.$$

To describe our rule for the combinatorial R matrix we shall depict the elements of B_l by one-row tableaux with length $2l$. We do this by duplicating each letters and then by supplying pairs of 0 and $\bar{0}$. Adding 0 and $\bar{0}$ into the set of entries of the tableaux, we assume the total order $0 < 1 < \cdots < \bar{1} < \bar{0}$.

Let us introduce the map ω from the $U_q'(A_{2n}^{(2)})$ crystal B_l to the $U_q'(C_n^{(1)})$ crystal[1] B_{2l}. This ω sends $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1)$ to $\omega(b) = (2x_1, \ldots, 2x_n, 2\bar{x}_n, \ldots, 2\bar{x}_1)$. On the other hand let us introduce the symbol $\mathbb{T}(b')$ for a $U_q'(C_n^{(1)})$ crystal element $b' \in B_{l'}$ [HKOT] that represents a one-row tableau with length l'. Putting these two symbols together we have

$$\mathbb{T}(\omega(b)) = \boxed{\overbrace{0 \cdots 0}^{x_\emptyset} \mid \overbrace{1 \cdots 1}^{2x_1} \mid \cdots \mid \overbrace{n \cdots n}^{2x_n} \mid \overbrace{\bar{n} \cdots \bar{n}}^{2\bar{x}_n} \mid \cdots \mid \overbrace{\bar{1} \cdots \bar{1}}^{2\bar{x}_1} \mid \overbrace{\bar{0} \cdots \bar{0}}^{\bar{x}_\emptyset}},$$
$$(3.2)$$

where $x_\emptyset = \bar{x}_\emptyset = l - \sum_{i=1}^{n}(x_i + \bar{x}_i)$. We shall use this tableau in our description of the combinatorial R matrix (Rule 3.10).

From now on we shall devote ourselves to describing several important properties of the map ω. Our goal here is Lemma 3.3, which our description of the combinatorial R matrix relies on. For this purpose we also use the symbol ω for the following map for $\tilde{e}_i, \tilde{f}_i$:

$$\omega(\tilde{e}_i) = (\tilde{e}_i')^{2 - \delta_{i,0}},$$
$$\omega(\tilde{f}_i) = (\tilde{f}_i')^{2 - \delta_{i,0}}.$$

Hereafter we attach prime $'$ to the notations for the $U_q'(C_n^{(1)})$ crystals, e.g., $\tilde{e}_i', \varphi_i'$ and so on.

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[1] Here we adopted the notation B_{2l} that we have used in the previous work [HKOT]. This B_{2l} was originally denoted by B_l in [KKM].

LEMMA 3.1.

$$\omega(\tilde{e}_i b) = \omega(\tilde{e}_i)\,\omega(b),$$
$$\omega(\tilde{f}_i b) = \omega(\tilde{f}_i)\,\omega(b);$$

i.e., the ω commutes with actions of the operators on B_l.

Let us give a proof in $\tilde{e}_0$ case. (For the other $\tilde{e}_i$'s (and also for $\tilde{f}_i$'s) the proof is similar.)

Proof. Let $b = (x_1, \ldots, \bar{x}_l) \in B_l$ ($U_q'(A_{2n}^{(2)})$ crystal). We have [KKM]

$$\tilde{e}_0 b = \begin{cases} (x_1 - 1, x_2, \ldots, \bar{x}_1) & \text{if } x_1 \geq \bar{x}_1 + 1, \\ (x_1, \ldots, \bar{x}_2, \bar{x}_1 + 1) & \text{if } x_1 \leq \bar{x}_1. \end{cases}$$

This means that

$$\omega(\tilde{e}_0 b) = \begin{cases} (2x_1 - 2, 2x_2, \ldots, 2\bar{x}_1) & \text{if } 2x_1 \geq 2\bar{x}_1 + 2, \\ (2x_1, \ldots, 2\bar{x}_2, 2\bar{x}_1 + 2) & \text{if } 2x_1 \leq 2\bar{x}_1. \end{cases}$$

On the other hand, let $b' = (x_1', \ldots, \bar{x}_l') \in B_{l'}'$ ($U_q'(C_n^{(1)})$ crystal). We have [KKM]

$$\tilde{e}_0' b' = \begin{cases} (x_1' - 2, x_2', \ldots, \bar{x}_1') & \text{if } x_1' \geq \bar{x}_1' + 2, \\ (x_1' - 1, x_2', \ldots, \bar{x}_2', \bar{x}_1' + 1) & \text{if } x_1' = \bar{x}_1' + 1, \\ (x_1', \ldots, \bar{x}_2', \bar{x}_1' + 2) & \text{if } x_1' \leq \bar{x}_1'. \end{cases}$$

Thus putting $l' = 2l$ and $b' = \omega(b)$ we obtain $\omega(\tilde{e}_0)\omega(b) = \tilde{e}_0' b' = \omega(\tilde{e}_0 b)$. (The second choice in the above equation does not occur.) ∎

Let us denote $\omega(b_1) \otimes \omega(b_2)$ by $\omega(b_1 \otimes b_2)$ for $b_1 \otimes b_2 \in B_l \otimes B_k$.

LEMMA 3.2.

$$\omega(\tilde{e}_i(b_1 \otimes b_2)) = \omega(\tilde{e}_i)\,\omega(b_1 \otimes b_2),$$
$$\omega(\tilde{f}_i(b_1 \otimes b_2)) = \omega(\tilde{f}_i)\,\omega(b_1 \otimes b_2).$$

Namely the ω commutes with actions of the operators on $B_l \otimes B_k$.

Proof. Let us check the latter. Suppose we have $\varphi_i(b_1) \geq \varepsilon_i(b_2) + 1$. Then $\tilde{f}_i(b_1 \otimes b_2) = (\tilde{f}_i b_1) \otimes b_2$. In this case we have $\varphi_i'(\omega(b_1)) \geq \varepsilon_i'(\omega(b_2)) + 2 - \delta_{i,0}$, since

$$\varphi_i'(\omega(b)) = (2 - \delta_{i,0})\varphi_i(b),$$
$$\varepsilon_i'(\omega(b)) = (2 - \delta_{i,0})\varepsilon_i(b).$$

Therefore we obtain

$$
\begin{aligned}
\omega\big(\tilde{f}_i\big)\omega(b_1 \otimes b_2) &= \big(\tilde{f}_i'\big)^{2-\delta_{i,0}}\big(\omega(b_1) \otimes \omega(b_2)\big) \\
&= \Big(\big(\tilde{f}_i'\big)^{2-\delta_{i,0}}\omega(b_1)\Big) \otimes \omega(b_2) \\
&= \Big(\omega\big(\tilde{f}_i\big)\omega(b_1)\Big) \otimes \omega(b_2) \\
&= \omega\big(\tilde{f}_i b_1\big) \otimes \omega(b_2) \\
&= \omega\big(\tilde{f}_i b_1 \otimes b_2\big).
\end{aligned}
$$

The other case when $\varphi_i(b_1) \leq \varepsilon_i(b_2)$ is similar. ∎

Finally we obtain the following important properties of the map ω.

LEMMA 3.3. (i) *If* $b_1 \otimes b_2$ *is mapped to* $b_2' \otimes b_1'$ *under the isomorphism of the* $U_q'(A_{2n}^{(2)})$ *crystals* $B_l \otimes B_k \simeq B_k \otimes B_l$, *then* $\omega(b_1) \otimes \omega(b_2)$ *is mapped to* $\omega(b_2') \otimes \omega(b_1')$ *under the isomorphism of the* $U_q'(C_n^{(1)})$ *crystals* $B_{2l} \otimes B_{2k} \simeq B_{2k} \otimes B_{2l}$.
(ii) *Up to a global additive constant, the value of the energy function* $H_{B_l B_k}(b_1 \otimes b_2)$ *for the* $U_q'(A_{2n}^{(2)})$ *crystal* $B_l \otimes B_k$ *is equal to the value of the energy function* $H'_{B_{2l} B_{2k}}(\omega(b_1) \otimes \omega(b_2))$ *for the* $U_q'(C_n^{(1)})$ *crystal* $B_{2l} \otimes B_{2k}$.

Proof. First we consider (i). Since the crystal graph of $B_l \otimes B_k$ is connected, it remains to check (i) for any specific element in $B_l \otimes B_k \simeq B_k \otimes B_l$. We can do it by taking $(l,0,\ldots,0) \otimes (k,0,\ldots,0) \mapsto (k,0,\ldots,0) \otimes (l,0,\ldots,0)$ as the specific element, for which (i) certainly holds.

We proceed to (ii). We can set

$$
\begin{aligned}
H_{B_l B_k}\big((l,0,\ldots,0) &\otimes (k,0,\ldots,0)\big) \\
&= H'_{B_{2l} B_{2k}}\big(\omega((l,0,\ldots,0)) \otimes \omega((k,0,\ldots,0))\big).
\end{aligned}
$$

Suppose $\tilde{e}_i(b_1 \otimes b_2) \neq 0$. Recall the defining relations of the energy function $H_{B_l B_k}$:

$$
\begin{aligned}
&H_{B_l B_k}\big(\tilde{e}_i(b_1 \otimes b_2)\big) \\
&= \begin{cases}
H_{B_l B_k}(b_1 \otimes b_2) + 1 & \text{if } i = 0,\ \varphi_0(b_1) \geq \varepsilon_0(b_2),\ \varphi_0(b_2') \geq \varepsilon_0(b_1'), \\
H_{B_l B_k}(b_1 \otimes b_2) - 1 & \text{if } i = 0,\ \varphi_0(b_1) < \varepsilon_0(b_2),\ \varphi_0(b_2') < \varepsilon_0(b_1'), \\
H_{B_l B_k}(b_1 \otimes b_2) & \text{otherwise}.
\end{cases}
\end{aligned}
$$

Claim (ii) holds if for any i and $b_1 \otimes b_2$ with $\tilde{e}_i(b_1 \otimes b_2) \neq 0$, we have

$$H'_{B_{2l}B_{2k}}\big(\omega\big(\tilde{e}_i(b_1 \otimes b_2)\big)\big) - H'_{B_{2l}B_{2k}}\big(\omega(b_1 \otimes b_2)\big)$$
$$= H_{B_l B_k}\big(\tilde{e}_i(b_1 \otimes b_2)\big) - H_{B_l B_k}(b_1 \otimes b_2). \tag{3.3}$$

The $i = 0$ case is verified as follows. Since ω commutes with crystal actions we have $\omega(\tilde{e}_0(b_1 \otimes b_2)) = \omega(\tilde{e}_0)(\omega(b_1) \otimes \omega(b_2)) = \tilde{e}'_0(\omega(b_1 \otimes b_2))$. On the other hand ω preserves the inequalities in the classification conditions in the defining relations of the energy function; i.e., $\varphi'_0(\omega(b_1)) \geq \varepsilon'_0(\omega(b_2)) \Leftrightarrow \varphi_0(b_1) \geq \varepsilon_0(b_2)$, and so on. Thus (3.3) follows from the defining relations of the $H'_{B_{2l}B_{2k}}$. The $i \neq 0$ case is easier. This completes the proof. ∎

Since ω is injective we obtain the converse of (i).

COROLLARY 3.4.   *If $\omega(b_1) \otimes \omega(b_2)$ is mapped to $\omega(b'_2) \otimes \omega(b'_1)$ under the isomorphism of the $U'_q(C_n^{(1)})$ crystals $B_{2l} \otimes B_{2k} \simeq B_{2k} \otimes B_{2l}$, then $b_1 \otimes b_2$ is mapped to $b'_2 \otimes b'_1$ under the isomorphism of the $U'_q(A_{2n}^{(2)})$ crystals $B_l \otimes B_k \simeq B_k \otimes B_l$.*

3.2. *Definitions*: $U'_q(D_{n+1}^{(2)})$ *Crystal Case*

Given a positive integer l, we consider a $U'_q(D_{n+1}^{(2)})$ crystal denoted by B_l that is defined in [KKM]. B_l's are the crystal bases of the irreducible finite-dimensional representation of the quantum affine algebra $U'_q(D_{n+1}^{(2)})$. As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, x_\circ, \bar{x}_n, \ldots, \bar{x}_1) \,\middle|\, \begin{array}{l} x_\circ = 0 \text{ or } 1, \ x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0} \\ x_\circ + \sum_{i=1}^n (x_i + \bar{x}_i) \in \{l, l-1, \ldots, 0\} \end{array} \right\}.$$

For its crystal structure see [KKM]. B_l is isomorphic to $\bigoplus_{0 \leq j \leq l} B(j\Lambda_1)$ as a $U_q(B_n)$ crystal, where $B(j\Lambda_1)$ is the crystal associated with the irreducible representation of $U_q(B_n)$ with highest weight $j\Lambda_1$. The $U_q(B_n)$ crystal $B(j\Lambda_1)$ has a description in terms of semistandard B-tableaux [KN]. The entries are $1, \ldots, n, \bar{1}, \ldots, \bar{n}$ and $\circ$ with the total order

$$1 < 2 < \cdots < n < \circ < \bar{n} < \cdots < \bar{2} < \bar{1}.$$

In this paper we use the symbol $\circ$ for the member of entries of the semistandard B tableaux that is conventionally denoted by 0. For $b = (x_1, \ldots, x_n, x_\circ, \bar{x}_n, \ldots, \bar{x}_1) \in B(j\Lambda_1)$ the tableau $\mathscr{T}(b)$ is depicted by

$$\mathscr{T}(b) = \boxed{\overbrace{1 \cdots 1}^{x_1}} \boxed{\cdots} \boxed{\overbrace{n \cdots n}^{x_n}} \boxed{\overbrace{\circ}^{x_\circ}} \boxed{\overbrace{\bar{n} \cdots \bar{n}}^{\bar{x}_n}} \boxed{\cdots} \boxed{\overbrace{\bar{1} \cdots \bar{1}}^{\bar{x}_1}}.$$

The length of this one-row tableau is equal to j, namely $x_\circ + \sum_{i=1}^n (x_i + \bar{x}_i) = j$.

To describe our rule for the combinatorial R matrix we shall depict the elements of B_l by one-row C tableaux with length $2l$. We introduce the map ω from the $U_q'(D_{n+1}^{(2)})$ crystal B_l to the $U_q'(C_n^{(1)})$ crystal B_{2l}. ω sends $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1)$ to $\omega(b) = (2x_1, \ldots, 2x_{n-1}, 2x_n + x_\circ, 2\bar{x}_n + x_\circ, 2\bar{x}_{n-1}, \ldots, 2\bar{x}_1)$. By using the symbol $\mathbb{T}$ introduced in the previous subsection the tableau $\mathbb{T}(\omega(b))$ is depicted by

$$\mathbb{T}(\omega(b)) = \boxed{\overset{x_\emptyset}{\overbrace{0 \cdots 0}} \; \overset{2x_1}{\overbrace{1 \cdots 1}} \; \cdots \; \overset{2x_n + x_\circ}{\overbrace{n \cdots n}} \; \overset{2\bar{x}_n + x_\circ}{\overbrace{\bar{n} \cdots \bar{n}}} \; \cdots \; \overset{2\bar{x}_1}{\overbrace{\bar{1} \cdots \bar{1}}} \; \overset{\bar{x}_\emptyset}{\overbrace{\bar{0} \cdots \bar{0}}}},$$

where $x_\emptyset = \bar{x}_\emptyset = l - x_\circ - \sum_{i=1}^n (x_i + \bar{x}_i)$.

Our description of the combinatorial R matrix (Theorem 3.11) is based on the following lemma.

LEMMA 3.5. (i) *If $b_1 \otimes b_2$ is mapped to $b_2' \otimes b_1'$ under the isomorphism of the $U_q'(D_{n+1}^{(2)})$ crystals $B_l \otimes B_k \simeq B_k \otimes B_l$, then $\omega(b_1) \otimes \omega(b_2)$ is mapped to $\omega(b_2') \otimes \omega(b_1')$ under the isomorphism of the $U_q'(C_n^{(1)})$ crystals $B_{2l} \otimes B_{2k} \simeq B_{2k} \otimes B_{2l}$.*

(ii) *Up to a global additive constant, the value of the energy function $H_{B_l B_k}(b_1 \otimes b_2)$ for the $U_q'(D_{n+1}^{(2)})$ crystal $B_l \otimes B_k$ is equal to the value of the energy function $H_{B_{2l} B_{2k}}'(\omega(b_1) \otimes \omega(b_2))$ for the $U_q'(C_n^{(1)})$ crystal $B_{2l} \otimes B_{2k}$.*

Proof. To distinguish the notations we attach prime $'$ to the notations for the $U_q'(C_n^{(1)})$ crystals. Then we have

$$\varphi_i'(\omega(b)) = (2 - \delta_{i,0} - \delta_{i,n})\varphi_i(b),$$

$$\varepsilon_i'(\omega(b)) = (2 - \delta_{i,0} - \delta_{i,n})\varepsilon_i(b).$$

Let us define the action of ω on the operators by

$$\omega(\tilde{e}_i) = (\tilde{e}_i')^{2 - \delta_{i,0} - \delta_{i,n}},$$

$$\omega(\tilde{f}_i) = (\tilde{f}_i')^{2 - \delta_{i,0} - \delta_{i,n}}.$$

By repeating an argument similar to that in the previous subsection we obtain formally the same assertions of Lemmas 3.1, 3.2, and 3.3. This completes the proof. ∎

Since ω is injective we obtain the converse of (i).

COROLLARY 3.6. *If $\omega(b_1) \otimes \omega(b_2)$ is mapped to $\omega(b_2') \otimes \omega(b_1')$ under the isomorphism of the $U_q'(C_n^{(1)})$ crystals $B_{2l} \otimes B_{2k} \simeq B_{2k} \otimes B_{2l}$, then $b_1 \otimes b_2$*

is mapped to $b'_2 \otimes b'_1$ under the isomorphism of the $U'_q(D^{(2)}_{n+1})$ crystals $B_l \otimes B_k \simeq B_k \otimes B_l$.

3.3. *Column Insertion and Inverse Insertion for C_n*

Set an alphabet $\mathscr{X} = \mathscr{A} \sqcup \bar{\mathscr{A}}$, $\mathscr{A} = \{0, 1, \ldots, n\}$ and $\bar{\mathscr{A}} = \{\bar{0}, \bar{1}, \ldots, \bar{n}\}$, with the total order $0 < 1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1} < \bar{0}$.[2]

3.3.1. *Semistandard C Tableaux*

Let us consider a *semistandard C tableau* made by the letters from this alphabet. We follow [KN] for its definition. We present the definition here but restrict ourselves to the special cases that are sufficient for our purpose. Namely we consider only those tableaux that have no more than two rows in their shapes. Thus they have the forms

$$
\boxed{\begin{array}{|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \cdots & \alpha_j \\ \hline \end{array}} \quad \text{or} \quad \boxed{\begin{array}{|c|c|c|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \cdots & \alpha_i & \alpha_{i+1} & \cdots & \alpha_j \\ \hline \beta_1 & \beta_2 & \cdots & \beta_i \\ \cline{1-4} \end{array}} \,,
$$
$$(3.4)$$

and the letters inside the boxes should obey the following conditions:

$$\alpha_1 \le \cdots \le \alpha_j, \qquad \beta_1 \le \cdots \le \beta_i, \tag{3.5}$$

$$\alpha_a < \beta_a, \tag{3.6}$$

$$(\alpha_a, \beta_a) \ne (0, \bar{0}), \tag{3.7}$$

$$(\alpha_a, \alpha_{a+1}, \beta_{a+1}) \ne (x, x, \bar{x}), \qquad (\alpha_a, \beta_a, \beta_{a+1}) \ne (x, \bar{x}, \bar{x}). \tag{3.8}$$

Here we assume $1 \le x \le n$. The last conditions (3.8) are referred to as the absence of the (x, x)-configurations.

3.3.2. *Column Insertion for C_n [B1]*

We give a list of column insertions on semistandard C tableaux that are sufficient for our purpose (Rule 3.10). First of all let us explain the relation between the *insertion* and the *inverse insertion*. Since we are deliberately avoiding the occurrence of the *bumping sliding transition* [B1], the situation is basically the same as that for the usual tableau case [F, Appendix A.2]. Namely, when a letter α was inserted into the tableau T, we obtain a new

---

[2] We also introduce 0 and $\bar{0}$ in the alphabet. Compare with [B1, KN].

tableau T' whose shape is one more box larger than the shape of T. If we know the location of the new box we can reverse the insertion process to retrieve the original tableau T and letter α. This is the inverse insertion process. These processes go on column by column. Thus, from now on let us pay attention to a particular column C in the tableau. Suppose we have inserted a letter α into C. Suppose then we have obtained a column C' and a letter α' bumped out from the column. If we inversely insert the letter α' into the column C', we retrieve the original column C and letter α.

For the alphabet $\mathscr{X}$, we follow the convention that Greek letters $\alpha, \beta, \ldots$ belong to $\mathscr{X}$ while Latin letters $x, y, \ldots$ (resp. $\bar{x}, \bar{y}, \ldots$) belong to $\mathscr{A}$ (resp. $\bar{\mathscr{A}}$). The pictorial equations in the list should be interpreted as follows. (We take up two examples.)

• In Case B0, the letter[3] α is inserted into the column with only one box that has letter β in it. The α is set in the box and the β is bumped out to the right-hand column.

• In Case B1, the letter β is inserted into the column with two boxes that have letters α and γ in them. The β is set in the lower box and the γ is bumped out to the right-hand column.

Other equations should be interpreted in a similar way.[4] We note that there is no overlapping case in the list. Note also that it does not exhaust all patterns of the column insertions that insert a letter into a column with at most two boxes. In Rule 3.10 the letters are always inserted in weakly decreasing order. Since only such cases are necessary for our purpose we excluded those cases that may appear if one inserts letters in strictly increasing order. For instance it does not cover the case of insertion $\boxed{2} \rightarrow \boxed{\frac{1}{2}}$. In Rule 3.10 we do not encounter such a case.

A0 $\quad \boxed{\alpha} \rightarrow \emptyset = \boxed{\alpha}$,

A1 $\quad \boxed{\beta} \rightarrow \boxed{\alpha} = \boxed{\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}}$ $\qquad\qquad$ if $\alpha < \beta$,

B0 $\quad \boxed{\alpha} \rightarrow \boxed{\beta} = \boxed{\alpha}\,\boxed{\beta} \rightarrow$ $\qquad\quad$ if $\alpha \le \beta$,

B1 $\quad \boxed{\beta} \rightarrow \boxed{\begin{smallmatrix} \alpha \\ \gamma \end{smallmatrix}} = \boxed{\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}}\,\boxed{\gamma} \rightarrow$ $\qquad$ if $\alpha < \beta \le \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

---

[3] By abuse of notation we identify a letter with the one-box tableau having the letter in it.
[4] This interpretation is also eligible for the lists of type B and D cases in Sections 4.3.2 and 4.4.2.

B2 $\boxed{\alpha} \rightarrow \dfrac{\boxed{\beta}}{\boxed{\gamma}} = \dfrac{\boxed{\alpha}}{\boxed{\gamma}} \ \boxed{\beta} \rightarrow$ 　　　if $\alpha \le \beta < \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

B3 $\boxed{\beta} \rightarrow \dfrac{\boxed{x}}{\boxed{\bar{x}}} = \dfrac{\boxed{x-1}}{\boxed{\beta}} \ \boxed{x-1} \rightarrow$ 　if $x \le \beta \le \bar{x}$ and $x \ne 0$,

B4 $\boxed{x} \rightarrow \dfrac{\boxed{\beta}}{\boxed{\bar{x}}} = \dfrac{\boxed{x+1}}{\boxed{x+1}} \ \boxed{\beta} \rightarrow$ 　　if $x < \beta < \bar{x}$ and $x \ne n$.

3.3.3. *Column Insertion and $U_q(\bar{\mathfrak{g}})$ Crystal Morphism*

In this subsection we illustrate the relation between column insertion and the crystal morphism that was given by Baker [B1, B2]. A crystal morphism is a (not necessarily one-to-one) map between two crystals that commutes with the actions of crystals. See, for instance, [KKM] for a precise definition. A $U_q(\bar{\mathfrak{g}})$ crystal morphism is a morphism that commutes with the actions of $\tilde{e}_i$ and $\tilde{f}_i$ for $i \ne 0$. For later use we also include semistandard B and D tableaux in our discussion; therefore we assume $\bar{\mathfrak{g}} = B_n$, C_n, or D_n. See Section 4.3.1 (resp. 4.4.1) for the definition of semistandard B (resp. D) tableaux.

Let T be a semistandard B, C, or D tableau. For this T we denote by $w(T)$ the Japanese reading word of T; i.e., $w(T)$ is a sequence of letters that is created by reading all letters on T from the rightmost column to the leftmost one, and in each column, from top to bottom. For instance,

$$w\left(\boxed{\alpha_1 \ \alpha_2 \ \cdots \ \alpha_j} \right) = \alpha_j \cdots \alpha_2 \alpha_1,$$

$$w\left(\begin{array}{|c|c|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \cdots & \alpha_i & \alpha_{i+1} & \cdots & \alpha_j \\ \hline \beta_1 & \beta_2 & \cdots & \beta_i \\ \cline{1-4} \end{array} \right) = \alpha_j \cdots \alpha_{i+1} \alpha_i \beta_i \cdots \alpha_2 \beta_2 \alpha_1 \beta_1.$$

Let T and T' be two tableaux. We define the product tableau $T * T'$ by

$$T * T' = \left(\tau_1 \rightarrow \ \cdots \left(\tau_{j-1} \rightarrow \left(\tau_j \rightarrow T \right) \right) \cdots \right),$$

where

$$w(T') = \tau_j \tau_{j-1} \cdots \tau_1.$$

The symbol $\rightarrow$ represents the column insertions in [B1, B2] which we partly describe in Sections 3.3.2, 4.3.2, and 4.4.2. (Note that the author of [B1, B2] uses $\leftarrow$ instead of $\rightarrow$.)

For a dominant integral weight λ of the $\bar{\mathfrak{g}}$ root system, let $B(\lambda)$ be the $U_q(\bar{\mathfrak{g}})$ crystal associated with the irreducible highest weight representation

$V(\lambda)$. The elements of $B(\lambda)$ can be represented by semistandard $\bar{g}$ tableaux of shape λ [KN].

PROPOSITION 3.7 [B1, B2].   *Let $B(\mu) \otimes B(\nu) \simeq \bigoplus_j B(\lambda_j)^{\oplus m_j}$ be the tensor product decomposition of crystals. Here λ_j's are distinct highest weights and $m_j (\geq 1)$ is the multiplicity of $B(\lambda_j)$. Forgetting the multiplicities we have the canonical morphism from $B(\mu) \otimes B(\nu)$ to $\bigoplus_j B(\lambda_j)$. Define ψ by*

$$\psi(b_1 \otimes b_2) = b_1 * b_2.$$

Then ψ gives the unique $U_q(\bar{g})$ crystal morphism from $B(\mu) \otimes B(\nu)$ to $\bigoplus_j B(\lambda_j)$.

See Examples 3.8, 3.9, 4.1, 4.6, and 4.7.

3.3.4. *Column Insertion and $U_q(C_n)$ Crystal Morphism*

To illustrate Proposition 3.7, let us check a morphism of the $U_q(C_2)$ crystal $B(\Lambda_2) \otimes B(\Lambda_1)$ by taking two examples. Let ψ be the map that sends $\boxed{\begin{array}{c}\alpha\\\beta\end{array}} \otimes \boxed{\gamma}$ to the tableau which is made by the column insertion $\boxed{\gamma} \to \boxed{\begin{array}{c}\alpha\\\beta\end{array}}$.

EXAMPLE 3.8.

Here the left (resp. right) ψ is given by Case B3 (resp. B1) column insertion.

EXAMPLE 3.9.

Here the left (resp. right) ψ is given by Case B4 (resp. B2) column insertion.

3.3.5. *Inverse Insertion for C_n* [B1]

In this subsection we give a list of inverse column insertions on semi-standard C tableaux that are sufficient for our purpose (Rule 3.10). The pictorial equations in the list should be interpreted as follows. (We take two examples.)

• In Case C0, the letter β is inversely inserted into the column with only one box that has letter α in it. The β is set in the box and the α is bumped out to the left-hand column.

• In Case C1, the letter γ is inversely inserted into the column with two boxes that have letters α and β in them. The γ is set in the lower box and the β is bumped out to the left-hand column.

Other equations illustrate analogous procedures.

$$
\begin{array}{lll}
\textbf{C0} & \boxed{\alpha}\ \boxed{\beta} \to\ =\ \boxed{\alpha} \to \boxed{\beta} & \text{if } \alpha \le \beta, \\[2mm]
\textbf{C1} & \begin{array}{c}\boxed{\alpha}\\ \boxed{\beta}\end{array}\boxed{\gamma} \to\ =\ \boxed{\beta} \to \begin{array}{c}\boxed{\alpha}\\ \boxed{\gamma}\end{array} & \text{if } \alpha < \beta \le \gamma \text{ and } (\alpha,\gamma) \ne (x,\bar{x}), \\[2mm]
\textbf{C2} & \begin{array}{c}\boxed{\alpha}\\ \boxed{\gamma}\end{array}\boxed{\beta} \to\ =\ \boxed{\alpha} \to \begin{array}{c}\boxed{\beta}\\ \boxed{\gamma}\end{array} & \text{if } \alpha \le \beta < \gamma \text{ and } (\alpha,\gamma) \ne (x,\bar{x}), \\[2mm]
\textbf{C3} & \begin{array}{c}\boxed{x}\\ \boxed{\beta}\end{array}\boxed{\bar{x}} \to\ =\ \boxed{\beta} \to \begin{array}{c}\boxed{x+1}\\ \boxed{x+1}\end{array} & \text{if } x < \beta < \bar{x} \text{ and } x \ne n, \\[2mm]
\textbf{C4} & \begin{array}{c}\boxed{x}\\ \boxed{\bar{x}}\end{array}\boxed{\beta} \to\ =\ \boxed{x-1} \to \begin{array}{c}\boxed{\beta}\\ \boxed{x-1}\end{array} & \text{if } x \le \beta \le \bar{x} \text{ and } x \ne 0.
\end{array}
$$

3.4. *Main Theorem*: $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ Cases

Fix $l, k \in \mathbb{Z}_{\ge 1}$. Given $b_1 \otimes b_2 \in B_l \otimes B_k$, we define an $U_q'(C_n^{(1)})$ crystal element $\tilde{b}_2 \otimes \tilde{b}_1 \in B_{2k} \otimes B_{2l}$ and $l', k', m \in \mathbb{Z}_{\ge 0}$ by the following rule.

Rule 3.10. Set $z = \min(\#\boxed{0}$ in $\mathbb{T}(\omega(b_1))$, $\#\boxed{0}$ in $\mathbb{T}(\omega(b_2)))$. Thus $\mathbb{T}(\omega(b_1))$ and $\mathbb{T}(\omega(b_2))$ can be depicted by

$$
\mathbb{T}(\omega(b_1)) = \overset{z}{\boxed{0\cdots 0}}\ \boxed{\ \ T_*\ \ }\ \overset{z}{\boxed{\bar{0}\cdots\bar{0}}},
$$

$$
\mathbb{T}(\omega(b_2)) = \overset{z}{\boxed{0\cdots 0}}\ \boxed{v_1}\cdots\boxed{v_{k'}}\ \overset{z}{\boxed{\bar{0}\cdots\bar{0}}}.
$$

Set $l' = 2l - 2z$ and $k' = 2k - 2z$; hence T_* is a one-row tableau with length l'. Operate the column insertions for semistandard C tableaux and define

$$T^{(0)} := \left(v_1 \to \left(\cdots \left(v_{k'-1} \to \left(v_{k'} \to T_* \right) \right) \cdots \right) \right).$$

It has the form

$$T^{(0)} = \boxed{\begin{array}{c} \boxed{j_1 \ \cdots\cdots \ j_{k'} \ \big| \ i_{m+1} \ \cdots \ i_{l'}} \\ \boxed{i_1 \ \cdots \ i_m} \end{array}}$$

where m is the length of the second row; hence that of the first row is $l' + k' - m$ $(0 \le m \le k')$.

Next we bump out l' letters from the tableau $T^{(0)}$ by the type C reverse bumping algorithm in Section 3.3.5. In general, an inverse column insertion starts at a rightmost box in a row. After an inverse column insertion we obtain a tableau which has the shape with one box deleted; i.e., the box where we started the reverse bumping is removed from the original shape. We have labeled the boxes by $i_{l'}, i_{l'-1}, \ldots, i_1$ at which we start the inverse column insertions. Namely, for the boxes containing $i_{l'}, i_{l'-1}, \ldots, i_1$ in the above tableau, we do it first for $i_{l'}$, then $i_{l'-1}$, and so on. Correspondingly, let w_1 be the first letter that is bumped out from the leftmost column and let w_2 be the second and so on. Denote by $T^{(i)}$ the resulting tableau when w_i is bumped out $(1 \le i \le l')$. Note that $w_1 \le w_2 \le \cdots \le w_{l'}$. Now the $U_q'(C_n^{(1)})$ crystal elements $\tilde{b}_1 \in B_{2l}$ and $\tilde{b}_2 \in B_{2k}$ are uniquely specified by

$$\mathbb{T}(\tilde{b}_2) = \boxed{\overset{z}{\overbrace{0 \cdots 0}} \quad T^{(l')} \quad \overset{z}{\overbrace{\bar{0} \cdots \bar{0}}}},$$

$$\mathbb{T}(\tilde{b}_1) = \boxed{\overset{z}{\overbrace{0 \cdots 0}} \, w_1 \, \cdots \, w_{l'} \, \overset{z}{\overbrace{\bar{0} \cdots \bar{0}}}}.$$

We normalize the energy function as $H_{B_l B_k}(b_1 \otimes b_2) = 0$ for

$$\mathscr{T}(b_1) = \overset{l}{\overbrace{\boxed{1 \cdots 1}}} \quad \text{and} \quad \mathscr{T}(b_2) = \overset{k}{\overbrace{\boxed{\bar{1} \cdots \bar{1}}}}$$

irrespective of $l < k$ or $l \geq k$. Our main result for $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ is

THEOREM 3.11. *Given $b_1 \otimes b_2 \in B_l \otimes B_k$, find the $U_q'(C_n^{(1)})$ crystal element $\tilde{b}_2 \otimes \tilde{b}_1 \in B_{2k} \otimes B_{2l}$ and l', k', m by Rule 3.10. Let $\iota \colon B_l \otimes B_k \xrightarrow{\sim} B_k \otimes B_l$ be the isomorphism of $U_q'(A_{2n}^{(2)})$ (or $U_q'(D_{n+1}^{(2)})$) crystal. Then $\tilde{b}_2 \otimes \tilde{b}_1$ is in the image of the $B_k \otimes B_l$ by the injective map ω and we have*

$$\iota(b_1 \otimes b_2) = \omega^{-1}\big(\tilde{b}_2 \otimes \tilde{b}_1\big),$$

$$H_{B_l B_k}(b_1 \otimes b_2) = \min(l', k') - m.$$

By Theorem 3.4 of [HKOT] (the corresponding theorem for the $C_n^{(1)}$ case), one can immediately obtain this theorem using Lemmas 3.3 and 3.5 and their corollaries.

3.5. *Examples*

EXAMPLE 3.12.   Let us consider $B_3 \otimes B_2 \simeq B_2 \otimes B_3$ for $A_4^{(2)}$. Let b be an element of B_3 (resp. B_2). It is depicted by a one-row tableau $\mathcal{T}(b)$ with length 0, 1, 2, or 3 (resp. 0, 1 or 2):

Here we have chosen three examples. One can check that they are mapped to each other under the isomorphism of the $U_q'(A_4^{(2)})$ crystals by explicitly writing down the crystal graphs of $B_3 \otimes B_2$ and $B_2 \otimes B_3$.

First we shall show that the use of the tableau $\mathcal{T}(b)$ given by (3.1) is not enough for our purpose, while the less simpler tableau $\mathbb{T}(\omega(b))$ given by (3.2) suffices. Recall that by neglecting its zero arrows any $U_q'(A_4^{(2)})$ crystal graph decomposes into $U_q(C_2)$ crystal graphs. Thus if $b_1 \otimes b_2$ is mapped to $b_2' \otimes b_1'$ under the isomorphism of the $U_q'(A_4^{(2)})$ crystals, they should also be mapped to each other under an isomorphism of $U_q(C_2)$ crystals. In this example this $U_q(C_2)$ crystal isomorphism can be checked in terms of the tableau $\mathcal{T}(b)$ in the following way. Given $b_1 \otimes b_2$ let us construct the product tableau $\mathcal{T}(b_1) * \mathcal{T}(b_2)$) according to the original insertion rule in [B1] where in particular we have $(\bar{1} \to \boxed{1}) = \emptyset$.[5] One can see that both sides of the above three mappings then yield a common tableau $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array}$. This

---

[5] See the first footnote of Section 3.3.

means that they are mapped to each other under isomorphisms of $U_q(C_2)$ crystals. Thus we see that they are satisfying the necessary condition for the isomorphism of $U_q'(A_4^{(2)})$ crystals. However, we also see that this method of constructing $\mathscr{T}(b_1) * \mathscr{T}(b_2)$ is not strong enough to determine the $U_q'(A_4^{(2)})$ crystal isomorphism. Theorem 3.11 asserts that we are able to determine the $U_q'(A_4^{(2)})$ crystal isomorphism by means of the tableau $\mathbb{T}(\omega(b_1)) * \mathbb{T}(\omega(b_2))$. Namely the above three mappings are embedded into the following mappings in $B_6 \otimes B_4 \simeq B_4 \otimes B_6$ for the $U_q'(C_2^{(1)})$ crystals:

$$\boxed{0}\,\boxed{0}\,\boxed{1}\,\boxed{1}\,\boxed{\bar{0}}\,\boxed{\bar{0}} \otimes \boxed{2}\,\boxed{2}\,\boxed{2}\,\boxed{2} \;\dashrightarrow\; \boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{1} \otimes \boxed{2}\,\boxed{2}\,\boxed{2}\,\boxed{2}\,\boxed{\bar{1}}\,\boxed{\bar{1}}$$

$$\boxed{0}\,\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{2}\,\boxed{\bar{0}} \otimes \boxed{0}\,\boxed{2}\,\boxed{2}\,\boxed{\bar{0}} \;\dashrightarrow\; \boxed{0}\,\boxed{1}\,\boxed{1}\,\boxed{\bar{0}} \otimes \boxed{0}\,\boxed{2}\,\boxed{2}\,\boxed{2}\,\boxed{2}\,\boxed{\bar{0}}$$

$$\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{2} \otimes \boxed{2}\,\boxed{2}\,\boxed{\bar{1}}\,\boxed{\bar{1}} \;\dashrightarrow\; \boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{2} \otimes \boxed{0}\,\boxed{0}\,\boxed{2}\,\boxed{2}\,\boxed{\bar{0}}\,\boxed{\bar{0}}$$

We adopted a rule that the column insertion ($\bar{1} \to \boxed{1}$) does not vanish [HKOT]. Accordingly the both sides of the first mapping give the tableau
$\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & \bar{0} & \bar{0} \\\hline 2 & 2 & 2 & 2 \\\cline{1-4}\end{array}$. By deleting a $0,\bar{0}$ pair, those of the second one give the tableau $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\\hline 2 & 2 \\\cline{1-2}\end{array}$. Those of the third one give the tableau
$\begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & 2 & 2 \\\hline 2 & 2 & \bar{0} & \bar{0} \\\cline{1-4}\end{array}$. They are distinct. The right hand side is uniquely determined from the left hand side.

Second, let us illustrate in more detail the procedure of Rule 3.10. Take the last example. From the left hand side we proceed the column insertions as follows:

$$\bar{1} \;\to\; \boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{1}\,\boxed{2}\,\boxed{2} \;=\; \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\\hline \bar{1} \\\cline{1-1}\end{array}$$

$$\bar{1} \;\to\; \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\\hline \bar{1} \\\cline{1-1}\end{array} \;=\; \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 1 & 2 & 2 \\\hline \bar{1} & \bar{0} \\\cline{1-2}\end{array}$$

$$2 \;\to\; \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 1 & 2 & 2 \\\hline \bar{1} & \bar{0} \\\cline{1-2}\end{array} \;=\; \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 1 & 2 & 2 \\\hline 2 & \bar{1} & \bar{0} \\\cline{1-3}\end{array}$$

$$2 \;\to\; \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 1 & 1 & 2 & 2 \\\hline 2 & \bar{1} & \bar{0} \\\cline{1-3}\end{array} \;=\; \begin{array}{|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 1 & 2 & 2 \\\hline 2 & 2 & \bar{0} & \bar{0} \\\cline{1-4}\end{array}$$

The reverse bumping procedure goes as follows:

$$T^{(0)} \;=\; \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & \bar{0} & \bar{0} \\ \cline{1-4} \end{array}$$

$$T^{(1)} \;=\; \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 1 & 2 & 2 \\ \hline 2 & 2 & \bar{0} & \bar{0} \\ \cline{1-4} \end{array} \;,\qquad w_1 = 0$$

$$T^{(2)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 2 & \bar{0} & \bar{0} \\ \hline \end{array} \;,\qquad w_2 = 0$$

$$T^{(3)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & \bar{0} & \bar{0} \\ \cline{1-3} \end{array} \;,\qquad w_3 = 2$$

$$T^{(4)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \bar{0} & \bar{0} \\ \cline{1-2} \end{array} \;,\qquad w_4 = 2$$

$$T^{(5)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \bar{0} \\ \cline{1-1} \end{array} \;,\qquad w_5 = \bar{0}$$

$$T^{(6)} \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array} \;,\qquad w_6 = \bar{0}.$$

Thus we obtained the right hand side. We assign $H_{B_3, B_2} = 0$ to this element since we have $l' = 6$, $k' = 4$, and $m = 4$ in this case.

EXAMPLE 3.13. $B_3 \otimes B_2 \simeq B_2 \otimes B_3$ for $D_3^{(2)}$.

Here we have chosen three examples. According to the rule of the type B column insertion in [B2] we obtain $(\bar{1} \to \boxed{1}) = \emptyset$. In this rule we find that both sides of the above three mappings give a common tableau $\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}\,\boxed{\circ}$.

Theorem 3.11 asserts that we are able to determine the isomorphism of $U_q'(D_3^{(2)})$ crystals by means of the tableau $\mathbb{T}(\omega(b))$. The above three mappings are embedded into the following mappings in $B_6 \otimes B_4 \simeq B_4 \otimes B_6$

for the $U_q'(C_2^{(1)})$ crystals:

$$\boxed{0\,|\,0\,|\,1\,|\,1\,|\,\bar{0}\,|\,\bar{0}} \otimes \boxed{2\,|\,2\,|\,2\,|\,\bar{2}} \xrightarrow{\sim} \boxed{1\,|\,1\,|\,1\,|\,1} \otimes \boxed{2\,|\,2\,|\,2\,|\,\bar{2}\,|\,\bar{1}\,|\,\bar{1}}$$

$$\boxed{0\,|\,1\,|\,1\,|\,2\,|\,\bar{2}\,|\,\bar{0}} \otimes \boxed{0\,|\,2\,|\,2\,|\,\bar{0}} \xrightarrow{\sim} \boxed{0\,|\,1\,|\,1\,|\,\bar{0}} \otimes \boxed{0\,|\,2\,|\,2\,|\,2\,|\,\bar{2}\,|\,\bar{0}}$$

$$\boxed{1\,|\,1\,|\,1\,|\,1\,|\,2\,|\,\bar{2}} \otimes \boxed{2\,|\,2\,|\,\bar{1}\,|\,\bar{1}} \xrightarrow{\sim} \boxed{1\,|\,1\,|\,2\,|\,\bar{2}} \otimes \boxed{0\,|\,0\,|\,2\,|\,2\,|\,\bar{0}\,|\,\bar{0}}$$

Both sides of the first mapping give the tableau $\begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & \bar{0} & \bar{0} \\ \hline 2 & 2 & 2 & \bar{2} & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array}$. By

deleting a $0, \bar{0}$ pair, those of the second one give the tableau $\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & \bar{2} \\ \hline 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$.

Those of the third one give the tableau $\begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 1 & 1 & 2 & \bar{2} \\ \hline 2 & 2 & \bar{0} & \bar{0} & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array}$. They are distinct.

The right hand side is uniquely determined from the left hand side.

4. $U_q'(B_n^{(1)})$ AND $U_q'(D_n^{(1)})$ CRYSTAL CASES

4.1. *Definitions*: $U_q'(B_n^{(1)})$ *Crystal Case*

Given a positive integer l, let us denote by B_l the $U_q'(B_n^{(1)})$ crystal defined in [KKM]. As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, x_\circ, \bar{x}_n, \ldots, \bar{x}_1) \,\middle|\, x_\circ = 0 \text{ or } 1, \right.$$

$$\left. x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \ x_\circ + \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}.$$

For its crystal structure see [KKM]. B_l is isomorphic to $B(l\Lambda_1)$ as a $U_q(B_n)$ crystal. We depict the element $b = (x_1, \ldots, x_n, x_\circ, \bar{x}_n, \ldots, \bar{x}_1) \in B_l$ by the tableau

$$\mathcal{T}(b) = \overbrace{\boxed{1 \cdots 1}}^{x_1} \cdots \overbrace{\boxed{n \cdots n}}^{x_n} \overset{x_\circ}{\boxed{\circ}} \overbrace{\boxed{\bar{n} \cdots \bar{n}}}^{\bar{x}_n} \cdots \overbrace{\boxed{\bar{1} \cdots \bar{1}}}^{\bar{x}_1}.$$

The length of this one-row tableau is equal to l; namely $x_\circ + \sum_{i=1}^{n}(x_i + \bar{x}_i) = l$.

4.2. *Definitions*: $U_q'(D_n^{(1)})$ *Crystal Case*

Given a positive integer l, let us denote by B_l the $U_q'(D_n^{(1)})$ crystal defined in [KKM]. As a set B_l reads

$$B_l = \left\{ (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \middle| x_n = 0 \text{ or } \bar{x}_n = 0, \right.$$

$$\left. x_i, \bar{x}_i \in \mathbb{Z}_{\geq 0}, \ \sum_{i=1}^{n} (x_i + \bar{x}_i) = l \right\}.$$

For its crystal structure see [KKM]. B_l is isomorphic to $B(l\Lambda_1)$ as a $U_q(D_n)$ crystal. We depict the element $b = (x_1, \ldots, x_n, \bar{x}_n, \ldots, \bar{x}_1) \in B_l$ by the tableau

$$\mathcal{T}(b) = \overbrace{\boxed{1 \cdots 1}}^{x_1} \cdots \overbrace{\boxed{n \cdots n}}^{x_n}\overbrace{\boxed{\bar{n} \cdots \bar{n}}}^{\bar{x}_n} \cdots \overbrace{\boxed{\bar{1} \cdots \bar{1}}}^{\bar{x}_1}.$$

The length of this one-row tableau is equal to l; namely $\sum_{i=1}^{n}(x_i + \bar{x}_i) = l$.

4.3. *Column Insertion and Inverse Insertion for* B_n

Set an alphabet $\mathscr{X} = \mathscr{A} \sqcup \{\circ\} \sqcup \bar{\mathscr{A}}$, $\mathscr{A} = \{1, \ldots, n\}$ and $\bar{\mathscr{A}} = \{\bar{1}, \ldots, \bar{n}\}$, which the total order $1 < 2 < \cdots < n < \circ < \bar{n} < \cdots < \bar{2} < \bar{1}$.

4.3.1. *Semistandard B Tableaux*

Let us consider a *semistandard B tableau* made by the letters from this alphabet. We follow [KN] for its definition. We present the definition here but restrict ourselves to special cases that are sufficient for our purpose. Namely we consider only those tableaux that have no more than two rows in their shapes. Thus they have the forms as in (3.4), with the letters inside the boxes now chosen from the alphabet given as above. The letters obey the conditions (3.5) and the absence of the (x, x)-configuration (3.8) where we now assume $1 \leq x < n$. They also obey the following conditions:

$$\alpha_a < \beta_a \quad \text{or} \quad (\alpha_a, \beta_a) = (\circ, \circ), \tag{4.1}$$

$$(\alpha_a, \beta_a) \neq (1, \bar{1}), \tag{4.2}$$

$$(\alpha_a, \beta_{a+1}) \neq (n, \bar{n}), (n, \circ), (\circ, \circ), (\circ, \bar{n}). \tag{4.3}$$

The last conditions (4.3) are referred to as the absence of the (n, n)-configurations.

4.3.2. *Column Insertion for B_n* [B2]

We give a partial list of patterns of column insertions on the semistandard B tableaux that are sufficient for our purpose. For the interpretation of the pictorial equations in the list, see the remarks in Section 3.3.2. Note that this list does not exhaust all cases. It does not contain, for instance, the insertion $\boxed{\circ} \to \boxed{\substack{\circ \\ \circ}}$. As far as this case is concerned, we see that in Rule 4.11 neither $\mathcal{T}(b_1)$ nor $\mathcal{T}(b_2)$ has more than one $\circ$. Thus we do not encounter a situation where more than two $\circ$'s appear in the procedure. (See Proposition 4.13.)

A0 $\quad \boxed{\alpha} \dashrightarrow \varnothing = \boxed{\alpha},$

A1 $\quad \boxed{\beta} \to \boxed{\alpha} = \boxed{\substack{\alpha \\ \beta}} \qquad\qquad$ if $\alpha < \beta$ or $(\alpha, \beta) = (\circ, \circ),$

B0 $\quad \boxed{\alpha} \to \boxed{\beta} = \boxed{\alpha}\,\boxed{\beta} \to \qquad$ if $\alpha \le \beta$ and $(\alpha, \beta) \neq (\circ, \circ),$

B1 $\quad \boxed{\beta} \to \boxed{\substack{\alpha \\ \gamma}} = \boxed{\substack{\alpha \\ \beta}}\,\boxed{\gamma} \to \qquad$ if $\alpha < \beta \le \gamma$ and $(\alpha, \gamma) \neq (x, \bar{x})$ and $(\beta, \gamma) \neq (\circ, \circ),$

B2 $\quad \boxed{\alpha} \to \boxed{\substack{\beta \\ \gamma}} = \boxed{\substack{\alpha \\ \gamma}}\,\boxed{\beta} \to \qquad$ if $\alpha \le \beta < \gamma$ and $(\alpha, \gamma) \neq (x, \bar{x})$ and $(\alpha, \beta) \neq (\circ, \circ),$

B3 $\quad \dashv \to \boxed{\substack{\circ \\ \circ}} = \boxed{\substack{\circ \\ x}} \vdash$

B4 $\quad \boxed{x} \to \boxed{\circ} = \boxed{\circ}\,\boxed{\circ} \to ,$

B5 $\quad \boxed{\beta} \to \boxed{\substack{x \\ \bar{x}}} = \boxed{\substack{x-1 \\ \beta}}\,\boxed{x-1} \to$ if $x \le \beta \le \bar{x}$ and $x \neq 1,$

B6 $\quad \boxed{x} \to \boxed{\substack{\beta \\ \bar{x}}} = \boxed{\substack{x+1 \\ x+1}}\,\boxed{\beta} \to \qquad$ if $x < \beta < \bar{x}$ and $x \neq n,$

B7 $\quad \boxed{n} \to \boxed{\substack{\circ \\ \bar{n}}} = \boxed{\substack{n \\ \circ}}\,\boxed{\bar{n}} \to .$

4.3.3. *Column Insertion and $U_q(B_n)$ Crystal Morphism*

To illustrate Proposition 3.7 let us check a morphism of the $U_q(B_3)$ crystal $B(\Lambda_2) \otimes B(\Lambda_1)$ by taking an example. Let ψ be the map that is similarly defined as in Section 3.3.2 for the type C case.

EXAMPLE 4.1.

$$\begin{array}{c}\boxed{\begin{smallmatrix}\circ\\3\end{smallmatrix}}\otimes\boxed{\circ}\end{array}\xrightarrow{\tilde{e}_3}\boxed{\begin{smallmatrix}\circ\\3\end{smallmatrix}}\otimes\boxed{3}\xrightarrow{\tilde{e}_3}\boxed{\begin{smallmatrix}\circ\\\circ\end{smallmatrix}}\otimes\boxed{3}\xrightarrow{\tilde{e}_3}\boxed{\begin{smallmatrix}3\\\circ\end{smallmatrix}}\otimes\boxed{3}$$

$$\Big\downarrow\psi\qquad\qquad\Big\downarrow\psi\qquad\qquad\Big\downarrow\psi\qquad\qquad\Big\downarrow\psi$$

$$\begin{array}{c}\boxed{\circ\ \bar{3}}\\\boxed{\circ}\end{array}\xrightarrow{\tilde{e}_3}\begin{array}{c}\boxed{3\ \bar{3}}\\\boxed{\circ}\end{array}\xrightarrow{\tilde{e}_3}\begin{array}{c}\boxed{3\ \circ}\\\boxed{\circ}\end{array}\xrightarrow{\tilde{e}_3}\begin{array}{c}\boxed{3\ 3}\\\boxed{\circ}\end{array}$$
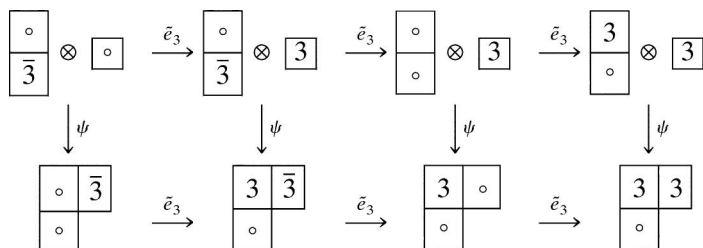
Here the ψ's are given by Case B3, B7, B4, and B2 column insertions, respectively, from left to right.

4.3.4. *Inverse Insertion for B_n* [B2]

We give a list of inverse column insertions on semistandard B tableaux that are sufficient for our purpose. For the interpretation of the pictorial equations in the list, see the remarks in Section 3.3.5.

|			
C0	$\boxed{\alpha}\,\boxed{\beta}\to\ =\boxed{\alpha}\to\boxed{\beta}$		if $\alpha\le\beta$ and $(\alpha,\beta)\ne(\circ,\circ)$,
C1	$\begin{array}{c}\boxed{\alpha}\\\boxed{\beta}\end{array}\boxed{\gamma}\to\ =\boxed{\beta}\to\begin{array}{c}\boxed{\alpha}\\\boxed{\gamma}\end{array}$		if $\alpha<\beta\le\gamma$ and $(\alpha,\gamma)\ne(x,\bar{x})$ and $(\beta,\gamma)\ne(\circ,\circ)$,
C2	$\begin{array}{c}\boxed{\alpha}\\\boxed{\gamma}\end{array}\boxed{\beta}\to\ =\boxed{\alpha}\to\begin{array}{c}\boxed{\beta}\\\boxed{\gamma}\end{array}$		if $\alpha\le\beta<\gamma$ and $(\alpha,\gamma)\ne(x,\bar{x})$ and $(\alpha,\beta)\ne(\circ,\circ)$,
C3	$\begin{array}{c}\boxed{\circ}\\\boxed{\circ}\end{array}\boxed{\bar{x}}\to\ =\boxed{\circ}\to\begin{array}{c}\boxed{\circ}\\\boxed{\bar{x}}\end{array},$		
C4	$\begin{array}{c}\boxed{x}\\\boxed{\circ}\end{array}\boxed{\circ}\to\ =\boxed{x}\to\begin{array}{c}\boxed{\circ}\\\boxed{\circ}\end{array},$		
C5	$\begin{array}{c}\boxed{x}\\\boxed{\beta}\end{array}\boxed{\bar{x}}\to\ =\boxed{\beta}\to\begin{array}{c}\boxed{x+1}\\\boxed{x+1}\end{array}$		if $x<\beta<\bar{x}$ and $x\ne n$,
C6	$\begin{array}{c}\boxed{x}\\\boxed{\bar{x}}\end{array}\boxed{\beta}\to\ =\boxed{x-1}\to\begin{array}{c}\boxed{\beta}\\\boxed{x-1}\end{array}$		if $x\le\beta\le\bar{x}$ and $x\ne 1$,
C7	$\begin{array}{c}\boxed{n}\\\boxed{\circ}\end{array}\boxed{\bar{n}}\to\ =\boxed{n}\to\begin{array}{c}\boxed{\circ}\\\boxed{\bar{n}}\end{array}.$		

4.3.5. *Column Bumping Lemma for B_n*

The aim of this subsection is to give a simple result on successive insertions of two letters into a tableau (Corollary 4.5). This result will be used in the proof of the main theorem (Theorem 4.12). This corollary follows from Lemma 4.4. This lemma is (a special case of) the *column bumping lemma*, whose claim is almost the same as that for the original lemma for the usual tableaux [F, Exercise 3 of Appendix A].

We restrict ourselves to the situation where by column insertions there only appear semistandard B tableaux with at most two rows. We consider a column insertion of a letter α into a tableau T. We insert the α into the leftmost column of T. According to the rules, the α is set in the column, and bump a letter if possible. The bumped letter is then inserted in the right column. The procedure continues until we come to Case A0 or A1.

When a letter is inserted in a tableau, we can define a *bumping route*. It is a collection of boxes in the new tableau that has those letters set by the insertion. In each column there is at most one such box. Thus we regard the bumping route as a path that goes from the left to the right. In the classification of the column insertions, we regard that the inserted letter is set in the first row in Cases A0, B0, B2, B4, B6, and B7, and that it is set in the second row in the other cases.

EXAMPLE 4.2.   Here we give an example of column insertion and its resulting bumping route in a B_3 tableau:
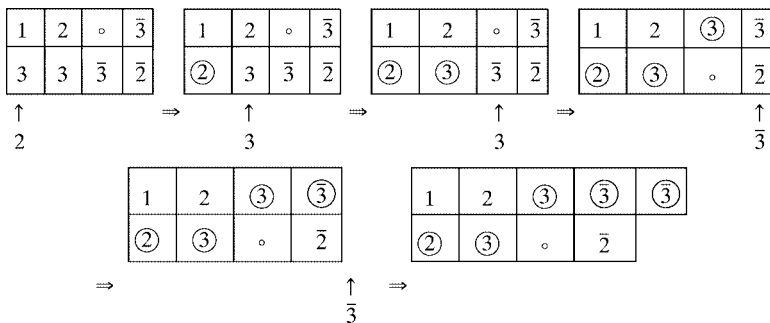
LEMMA 4.3.   *The bumping route does not move down.*

Proof.   It suffices to consider the bumping processes occurring on pairs of neighboring columns in the tableau. Our strategy is as follows. We are to show that if in the left column the inserted letter sets into the first row then the same occurs in the right column as well. Let us classify the situations on the neighboring columns into five cases.

1. Suppose that in the following column insertion Case B0 has occurred in the first column:

$$\boxed{\alpha} \rightarrow \boxed{\beta}\ \boxed{\gamma} \tag{4.4}$$

Then in the second column Case B0 occurs and Case A1 does not happen.

The semistandard condition for B tableaux imposes that $(\beta, \gamma) \neq (\circ, \circ)$ and $\beta \leq \gamma$. Thus if β is bumped out from the left column, it certainly bumps γ out of the right column.

2. Suppose that in the following column insertion one of the Cases B2, B4, or B6 has occurred in the first column:

$$\boxed{\alpha} \rightarrow \begin{array}{cc} \boxed{\beta} & \boxed{\gamma} \\ \boxed{\delta} & \end{array} \tag{4.5}$$

Then in the second column Case B0 occurs and Case A1 does not happen.

The reason is as follows. Whichever one of the B2, B4, or B6 may have occurred in the first column, the letter bumped out from the first column is always β. And again we have the semistandard condition between β and γ.

3. In the following column insertion Case B7 occurs in the first column:

$$\boxed{n} \rightarrow \begin{array}{cc} \boxed{\circ} & \boxed{\gamma} \\ \boxed{\bar{n}} & \end{array} \tag{4.6}$$

Then in the second column Case B0 occurs and Case A1 does not happen.

The letter bumped out from the first column is $\bar{n}$. Due to the semistandard condition we have $\gamma \geq \bar{n}$; hence the claim follows.

4. Suppose that in the following column insertion one of the Cases B2, B4, or B6 has occurred in the first column:

$$\alpha \;\rightarrow\; \boxed{\begin{array}{|c|c|} \hline \beta & \gamma \\ \hline \delta & \varepsilon \\ \hline \end{array}} \tag{4.7}$$

Then in the second column Cases B1, B3, and B5 do not happen.

The reason is as follows. The letter bumped out from the first column is always β.

Since $\beta \le \gamma$, B1 does not happen. Since $(\beta, \gamma) \ne (\circ, \circ)$, B3 does not happen. B5 does not happen since $(\beta, \gamma, \varepsilon) \ne (x, x, \bar{x})$, i.e., due to the absence of the (x, x)-configuration (3.8).

5. In the following column insertion Case B7 occurs in the first column:

$$n \;\rightarrow\; \boxed{\begin{array}{|c|c|} \hline \circ & \gamma \\ \hline \bar{n} & \varepsilon \\ \hline \end{array}} \tag{4.8}$$

Then in the second column Cases B1, B3, and B5 do not happen.

The letter bumped out from the first column is $\bar{n}$. Due to the semistandard condition we have $\gamma \ge \bar{n}$; hence the claim follows. ∎

LEMMA 4.4. *Let $\alpha' \le \alpha$ and $(\alpha, \alpha') \ne (\circ, \circ)$. Let R be the bumping route that is made when α is inserted into T, and let R' be the bumping route that is made when α' is inserted into $(\alpha \rightarrow T)$. Then R' does not lie below R.*

Proof. First we consider the case where the bumping route lies only in the first row. Suppose that, when α was inserted into the tableau T, it was set in the first row in the first column. We are to show that when α' is inserted, it will be also set in the first row in the first column. If T is an empty set (resp. has only one row), the insertion of α should have been A0 (resp. B0). In either case we have B0 when α' is inserted; hence the claim is true. Suppose T has two rows. By assumption B2, B4, B6, or B7 has occurred when α was inserted. We see that if B4, B6, or B7 has occurred, then B2 will occur when α' is inserted. Thus it is enough to show that if B2 has occurred, then B1, B3, or B5 does not happen when α' is inserted. Since $\alpha' \le \alpha$, B1 does not happen. Since $(\alpha, \alpha') \ne (\circ, \circ)$, B3 does not happen. B5 does not happen, since the first column does not have the entry $\frac{x}{x}$ as the result of B2 type insertion of α.

Second, we consider the case where the bumping route R lies across the first and the second rows. Suppose that from the leftmost column to the $(i - 1)$th column the bumping route lies in the second row, and from

the ith column to the rightmost column it lies in the first row. Let us call the position of the vertical line between the $(i-1)$th and the ith columns the *crossing point* of R. It is unique due to Lemma 4.3. We call an analogous position of R' its crossing point. We are to show that the crossing point of R' does not locate strictly right to the crossing point of R. Let the situation around the crossing point of R be

$$
\begin{array}{|c|c|}\hline & \eta \\\hline \xi & \\\hline\end{array}
\quad \text{or} \quad
\begin{array}{|c|c|}\hline & \eta \\\hline \xi & \\\hline\end{array}\ . \tag{4.9}
$$

While the insertion of α led to these configurations, let η' be the letter that was bumped out from the left column.

Claim 1: $\xi \le \eta' \le \eta$ and $(\xi, \eta) \ne (\circ, \circ)$.   To see this note that in the left column, B1, B3, or B5 has occurred when α was inserted. We have $\xi \le \eta'$ and $(\xi, \eta') \ne (\circ, \circ)$ (B1), or $\xi < \eta'$ (B3, B5). In the right column A0, B0, B2, B4, B6, or B7 has subsequently occurred. We have $\eta' = \eta$ (A0, B0, B2, B4, B7), or $\eta' < \eta$ (B6). In any case we have $\xi \le \eta' \le \eta$ and $(\xi, \eta) \ne (\circ, \circ)$.

Claim 2.   In (4.9) the following configurations do not exist:

$$
\begin{array}{|c|c|}\hline x & \bar{x} \\\hline \bar{x} & \\\hline\end{array},\quad
\begin{array}{|c|c|}\hline x & \bar{x} \\\hline \bar{x} & \\\hline\end{array},\quad
\begin{array}{|c|c|}\hline & x \\\hline x & \bar{x} \\\hline\end{array},\quad
\begin{array}{|c|c|}\hline & \circ \\\hline \circ & \\\hline\end{array},\quad \text{or} \quad
\begin{array}{|c|c|}\hline & \circ \\\hline \circ & \\\hline\end{array}. \tag{4.10}
$$

Due to Claim 1, the first and the second configurations can exist only if B1 with $\alpha = x$, $\gamma = \eta'$ happens in the left column and $\xi = \eta' = \eta = \bar{x}$. But $(\alpha, \gamma) = (x, \bar{x})$ is not compatible with B1. The third configuration can exist only if B6 happens in the right column and $\xi = \eta' = \eta = x$ by Claim 1. But we see from the proof of Claim 1 that B6 actually happens only when $\eta' < \eta$. The fourth and the fifth are forbidden since $(\xi, \eta) \ne (\circ, \circ)$ by Claim 1. Claim 2 is proved.

Let the situation around the crossing point of R be one of (4.9) excluding (4.10). When inserting α', suppose in the left column of the crossing point, B1, B3, or B5 has occurred. Let ξ' be the letter bumped out therefrom.

Claim 3: $\xi' \le \eta$ and $(\xi', \eta) \ne (\circ, \circ)$.   We divide the check into two cases.

(a) If B1 or B3 has occurred in the left column, we have $\xi' = \xi$. Thus the assertion follows from Claim 1. (b) If B5 has occurred, the left column had the entry $\begin{smallmatrix} x \\ \bar{x} \end{smallmatrix}$ and we have $\xi' = \overline{x-1}$, $\xi = \bar{x}$. Claim 1 tells $\xi = \bar{x} \le \eta$,

and Claim 2 does $\eta \neq \bar{x}$. Therefore we have $\xi' = \overline{x-1} \leq \eta$. $(\xi', \eta) \neq (\circ, \circ)$ is obvious. Claim 3 is proved.

Now we are ready to finish the proof of the main assertion. Assume the same situation as Claim 3. We should verify that A1, B1, B3, and B5 do not occur in the right column. Claim 3 immediately prohibits A1, B1, and B3 in the right column. Suppose that B5 happens in the right column. It means that $\eta \in \{1, \ldots, n\}$, $\xi' \geq \eta$ and the right column had the entry $\frac{\eta}{\eta}$. Since $\xi' \leq \eta$ by Claim 3, we find $\xi' = \eta$; therefore $\xi' \in \{1, \ldots, n\}$. Such ξ' can be bumped out from B1 process only in the left column and not from B3 or B5. It follows that $\xi' = \xi$. This leads to the third configuration in (4.10), and hence a contradiction.

Finally we consider the case where the bumping route R lies only in the second row. If R' lies below R the tableau should have more than two rows, which is prohibited by Proposition 4.13. ∎

COROLLARY 4.5. *Let $\alpha' \leq \alpha$ and $(\alpha, \alpha') \neq (\circ, \circ)$. Suppose that a new box is added at the end of the first row when α is inserted into T. Then a new box is added also at the end of the first row when α' is inserted into $(\alpha \to T)$.*

4.4. *Column Insertion and Inverse Insertion for D_n*

Set an alphabet $\mathscr{X} = \mathscr{A} \sqcup \overline{\mathscr{A}}$, $\mathscr{A} = \{1, \ldots, n\}$ and $\overline{\mathscr{A}} = \{\bar{1}, \ldots, \bar{n}\}$, with the partial order $1 < 2 < \cdots < \frac{n}{\bar{n}} < \cdots < \bar{2} < \bar{1}$.

4.4.1. *Semistandard D Tableaux*

Let us consider a *semistandard D tableau* made by the letters from this alphabet. We follow [KN] for its definition. We present the definition here but restrict ourselves to special cases that are sufficient for our purpose. Namely we consider only those tableaux that have no more than two rows in their shapes. Thus they have the forms as in (3.4), with the letters inside the boxes being chosen from the alphabet given as above. The letters obey the conditions (3.5),[6] (4.2) and the absence of the (x, x)-configuration (3.8) where we now assume $1 \leq x < n$. They also obey the following conditions:

$$\alpha_a < \beta_a \quad \text{or} \quad (\alpha_a, \beta_a) = (n, \bar{n}) \quad \text{or} \quad (\alpha_a, \beta_a) = (\bar{n}, n), \quad (4.11)$$

$$(\alpha_a, \alpha_{a+1}, \beta_a, \beta_{a+1}) \neq (n-1, n, n, \overline{n-1}), (n-1, \bar{n}, \bar{n}, \overline{n-1}), \quad (4.12)$$

$$(\alpha_a, \beta_{a+1}) \neq (n, n), (n, \bar{n}), (\bar{n}, n), (\bar{n}, \bar{n}). \quad (4.13)$$

The conditions (4.13) are referred to as the absence of the (n, n)-configurations.

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[6] Note that there is no order between n and $\bar{n}$.
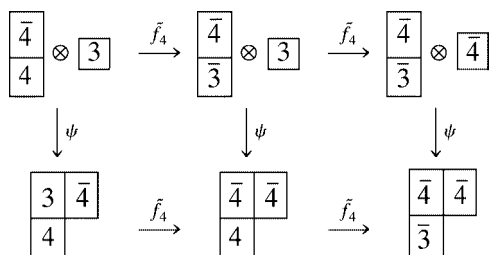
4.4.2. *Column Insertion for D_n* [B2]

We give a list of column insertions on semistandard D tableaux that are sufficient for our purpose. For the interpretation of the pictorial equations in the list, see the remarks in Section 3.3.2. Note that this list does not exhaust all cases. It does not contain, for instance, the insertion $\boxed{n} \to \boxed{\genfrac{}{}{0pt}{}{n}{\bar{n}}}$. (See Proposition 4.13.)

A0 $\boxed{\alpha} \to \varnothing \; = \; \boxed{\alpha}$,

A1 $\boxed{\beta} \to \boxed{\alpha} \; = \; \boxed{\genfrac{}{}{0pt}{}{\alpha}{\beta}}$ if $\alpha < \beta$ or $(\alpha, \beta) = (n, \bar{n})$ or $(\bar{n}, n)$,

B0 $\boxed{\alpha} \to \boxed{\beta} \; = \; \boxed{\alpha}\,\boxed{\beta} \to$ if $\alpha \le \beta$,

B1 $\boxed{\beta} \to \boxed{\genfrac{}{}{0pt}{}{\alpha}{\gamma}} \; = \; \boxed{\genfrac{}{}{0pt}{}{\alpha}{\beta}}\,\boxed{\gamma} \to$ if $\alpha < \beta \le \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

B2 $\boxed{\alpha} \to \boxed{\genfrac{}{}{0pt}{}{\beta}{\gamma}} \; = \; \boxed{\genfrac{}{}{0pt}{}{\alpha}{\gamma}}\,\boxed{\beta} \to$ if $\alpha \le \beta < \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

B3 $\boxed{\beta} \to \boxed{\genfrac{}{}{0pt}{}{x}{\bar{x}}} \; = \; \boxed{\genfrac{}{}{0pt}{}{x-1}{\beta}}\,\boxed{x-1} \to$ if $x \le \beta \le \bar{x}$ and $x \ne n, 1$,

B4 $\boxed{x} \to \boxed{\genfrac{}{}{0pt}{}{\beta}{\bar{x}}} \; = \; \boxed{\genfrac{}{}{0pt}{}{x+1}{x+1}}\,\boxed{\beta} \to$ if $x < \beta < \bar{x}$ and $x \ne n-1, n$,

B5 $\boxed{x} \to \boxed{\genfrac{}{}{0pt}{}{\mu_1}{\mu_2}} \; = \; \boxed{\genfrac{}{}{0pt}{}{x}{\mu_2}}\,\boxed{\mu_1} \to$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$ and $x \ne n$,

B6 $\boxed{\mu_2} \to \boxed{\genfrac{}{}{0pt}{}{\mu_1}{\bar{x}}} \; = \; \boxed{\genfrac{}{}{0pt}{}{\mu_1}{\mu_2}}\,\boxed{\bar{x}} \to$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$ and $\bar{x} \ne \bar{n}$,

B7 $\boxed{n-1} \to \boxed{\genfrac{}{}{0pt}{}{\mu}{n-1}} \; = \; \boxed{\genfrac{}{}{0pt}{}{\mu}{\bar{\mu}}}\,\boxed{\mu} \to$ if $\mu = n$ or $\bar{n}$ ($\bar{\mu} := n$ if $\mu = \bar{n}$),

B8 $\boxed{\mu_2} \to \boxed{\genfrac{}{}{0pt}{}{\mu_1}{\mu_2}} \; = \; \boxed{\genfrac{}{}{0pt}{}{n-1}{\mu_2}}\,\boxed{n-1} \to$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$.
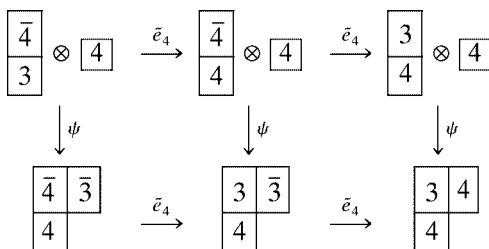
4.4.3. *Column Insertion and $U_q(D_n)$ Crystal Morphism*

To illustrate Proposition 3.7 let us check a morphism of the $U_q(D_4)$ crystal $B(\Lambda_2) \otimes B(\Lambda_1)$ by taking two examples. Let ψ be the map that is similarly defined as in Section 3.3.2 for the type C case.

EXAMPLE 4.6.

$$
\begin{array}{c}
\boxed{\begin{smallmatrix}\bar{4}\\4\end{smallmatrix}} \otimes \boxed{3} \xrightarrow{\tilde{f}_4} \boxed{\begin{smallmatrix}\bar{4}\\\bar{3}\end{smallmatrix}} \otimes \boxed{3} \xrightarrow{\tilde{f}_4} \boxed{\begin{smallmatrix}\bar{4}\\\bar{3}\end{smallmatrix}} \otimes \boxed{\bar{4}} \\[4pt]
\downarrow\psi \qquad\qquad \downarrow\psi \qquad\qquad \downarrow\psi \\[4pt]
\boxed{\begin{smallmatrix}3 & \bar{4}\\ 4 & \end{smallmatrix}} \xrightarrow{\tilde{f}_4} \boxed{\begin{smallmatrix}\bar{4} & \bar{4}\\ 4 & \end{smallmatrix}} \xrightarrow{\tilde{f}_4} \boxed{\begin{smallmatrix}\bar{4} & \bar{4}\\ \bar{3} & \end{smallmatrix}}
\end{array}
$$

Here the ψ's are given by Case B5, B7, and B2 column insertions, respectively.

EXAMPLE 4.7.

$$
\begin{array}{c}
\boxed{\begin{smallmatrix}\bar{4}\\3\end{smallmatrix}} \otimes \boxed{4} \xrightarrow{\tilde{e}_4} \boxed{\begin{smallmatrix}\bar{4}\\4\end{smallmatrix}} \otimes \boxed{4} \xrightarrow{\tilde{e}_4} \boxed{\begin{smallmatrix}3\\4\end{smallmatrix}} \otimes \boxed{4} \\[4pt]
\downarrow\psi \qquad\qquad \downarrow\psi \qquad\qquad \downarrow\psi \\[4pt]
\boxed{\begin{smallmatrix}\bar{4} & 3\\ 4 & \end{smallmatrix}} \xrightarrow{\tilde{e}_4} \boxed{\begin{smallmatrix}3 & 3\\ 4 & \end{smallmatrix}} \xrightarrow{\tilde{e}_4} \boxed{\begin{smallmatrix}3 & 4\\ 4 & \end{smallmatrix}}
\end{array}
$$

Here the ψ's are given by Case B6, B8, and B1 column insertions, respectively.

4.4.4. *Inverse Insertion for D_n* [B2]

We give a list of inverse column insertions on semistandard D tableaux that are sufficient for our purpose. For the interpretation of the pictorial equations in the list, see the remarks in Section 3.3.5.

C0 $\quad \boxed{\alpha}\,\boxed{\beta} \to \; = \; \boxed{\alpha} \to \boxed{\beta} \qquad$ if $\alpha \le \beta$,

C1 $\quad \boxed{\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}}\,\boxed{\gamma} \to \; = \; \boxed{\beta} \to \boxed{\begin{smallmatrix}\alpha\\\gamma\end{smallmatrix}} \qquad$ if $\alpha < \beta \le \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

C2 $\quad \boxed{\begin{smallmatrix}\alpha\\\gamma\end{smallmatrix}}\,\boxed{\beta} \to \; = \; \boxed{\alpha} \to \boxed{\begin{smallmatrix}\beta\\\gamma\end{smallmatrix}} \qquad$ if $\alpha \le \beta < \gamma$ and $(\alpha, \gamma) \ne (x, \bar{x})$,

C3 $\quad \boxed{\begin{smallmatrix}x\\\beta\end{smallmatrix}}\,\boxed{\bar{x}} \to \; = \; \boxed{\beta} \to \boxed{\begin{smallmatrix}x+1\\x+1\end{smallmatrix}} \qquad$ if $x < \beta < \bar{x}$ and $x \ne n-1, n$,

C4 $\quad \boxed{\begin{smallmatrix}x\\\bar{x}\end{smallmatrix}}\,\boxed{\beta} \to \; = \; \boxed{x-1} \to \boxed{\begin{smallmatrix}\beta\\x-1\end{smallmatrix}} \qquad$ if $x \le \beta \le \bar{x}$ and $x \ne n, 1$,

C5 $\boxed{\begin{array}{c}x\\ \mu_2\end{array}}\ \boxed{\mu_1} \rightarrow\ =\ \boxed{x} \rightarrow \boxed{\begin{array}{c}\mu_1\\ \mu_2\end{array}}$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$ and $x \neq n$,

C6 $\boxed{\begin{array}{c}\mu_1\\ \mu_2\end{array}}\ \boxed{\bar{x}} \rightarrow\ =\ \boxed{\mu_2} \rightarrow \boxed{\begin{array}{c}\mu_1\\ \bar{x}\end{array}}$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$ and $\bar{x} \neq \bar{n}$,

C7 $\boxed{\begin{array}{c}\mu_1\\ \mu_2\end{array}}\ \boxed{\mu_1} \rightarrow\ =\ \boxed{n-1} \rightarrow \boxed{\begin{array}{c}\mu_1\\ n-1\end{array}}$ if $(\mu_1, \mu_2) = (n, \bar{n})$ or $(\bar{n}, n)$,

C8 $\boxed{\begin{array}{c}n-1\\ \mu\end{array}}\ \boxed{n-1} \rightarrow\ =\ \boxed{\mu} \rightarrow \boxed{\begin{array}{c}\bar{\mu}\\ \mu\end{array}}$ if $\mu = n$ or $\bar{n}$ ($\bar{\mu} := n$ if $\mu = \bar{n}$).

4.4.5. Column Bumping Lemma for D_n

The aim of this subsection is to give a simple result on successive insertions of two letters into a tableau (Corollary 4.10). This result will be used in the proof of the main theorem (Theorem 4.12). This corollary follows from the column bumping lemma (Lemma 4.9).

We restrict ourselves to the situation where by column insertions there only appear semistandard D tableaux with at most two rows. In the classification of the column insertions, we regard that the inserted letter is set in the first row in Cases A0, B0, B2, B4, B5, and B7, and that it is set in the second row in the other cases. Then the bumping route is defined in the same way as in Section 4.3.5.

LEMMA 4.8. *The bumping route does not move down.*

Proof.   It is enough to consider the following three cases.

1. Suppose that in the following column insertion Case B0 has occurred in the first column:

$$\boxed{\alpha} \rightarrow \boxed{\beta}\ \boxed{\gamma} \tag{4.14}$$

Then in the second column Case B0 occurs and Case A1 does not happen.

The semistandard condition for D tableau imposes that $(\beta, \gamma) \neq (n, \bar{n})$, $(\bar{n}, n)$ and $\beta \leq \gamma$. Thus if β is bumped out from the left column, it certainly bumps γ out of the right column.

2. Suppose that in the following column insertion one of the Cases B2, B4, B5, or B7 has occurred in the first column:

$$\boxed{\alpha} \rightarrow \begin{array}{c}\boxed{\beta}\ \boxed{\gamma}\\ \boxed{\delta}\end{array} \tag{4.15}$$

Then in the second column Case B0 occurs and Case A1 does not happen.

The reason is as follows. Whichever one of the B2, B4, B5, or B7 may have occurred in the first column, the letter bumped out from the first column is always β. And we have the semistandard condition between β and γ.

3. Suppose that in the following column insertion one of the Cases B2, B4, B5, or B7 has occurred in the first column.

$$
\boxed{\alpha} \rightarrow
\begin{array}{|c|c|}
\hline
\beta & \gamma \\
\hline
\delta & \varepsilon \\
\hline
\end{array}
\tag{4.16}
$$

Then in the second column Cases B1, B3, B6, and B8 do not happen.

The reason is as follows. As in the previous case the letter bumped out from the first column is always β.

Since $\beta \leq \gamma$, B1 does not happen. Since $(\beta, \gamma) \neq (n, \bar{n}), (\bar{n}, n)$, B6 and B8 do not happen. B3 does not happen since $(\beta, \gamma, \varepsilon) \neq (x, x, \bar{x})$, i.e., due to the absence of the (x, x)-configuration (3.8). ∎

LEMMA 4.9. *Let $\alpha' \leq \alpha$; in particular $(\alpha, \alpha') \neq (n, \bar{n}), (\bar{n}, n)$. Let R be the bumping route that is made when α is inserted into T, and let R' be the bumping route that is made when α' is inserted into $(\alpha \rightarrow T)$. Then R' does not lie below R.*

Proof. First we consider the case where the bumping route lies only in the first row. Suppose that, when α was inserted into the tableau T, it was set in the first row in the first column. We are to show that when α' is inserted, it will also be set in the first row in the first column. If T is an empty set (resp. has only one row), the insertion of α should have been A0 (resp. B0). In either case we have B0 when α' is inserted; hence the claim is true. Suppose T has two rows. By assumption B2, B4, B5, or B7 has occurred when α was inserted. We see that (a) If B7 has occurred, then B5 will occur when α' is inserted; (b) if B5 or B4 has occurred, then B2 will occur when α' is inserted. Thus it is enough to show that if B2 has occurred, then B1, B3, B6, and B8 do not happen when α' is inserted. Since $\alpha' \leq \alpha$, B1 does not happen. Since $(\alpha, \alpha') \neq (n, \bar{n}), (\bar{n}, n)$, B6 and B8 do not happen. B3 does not happen, since the first column does not have the entry $\frac{x}{\bar{x}}$ as the result of B2 type insertion of α.

Second, we consider the case where the bumping route R lies across the first and the second rows. Suppose that from the leftmost column to the $(i-1)$th column the bumping route lies in the second row, and from the ith column to the rightmost column it lies in the first row. As in the type B

case let us call the position of the vertical line between the $(i-1)$th and the ith columns the crossing point of R. It is unique due to Lemma 4.8. We call an analogous position of R' its crossing point. We are to show that the crossing point of R' does not locate strictly right to the crossing point of R. Let the situation around the crossing point of R be

$$
\begin{array}{c}
\begin{array}{|c|c|}
\hline
 & \eta \\
\hline
\xi & \\
\hline
\end{array}
\quad \text{or} \quad
\begin{array}{|c|c|}
\hline
 & \eta \\
\hline
\xi & \\
\hline
\end{array}
\end{array}
\tag{4.17}
$$

While the insertion of α that led to these configurations, let η' be the letter that was bumped out from the left column.

Claim 1: $\xi \le \eta$ and $(\xi, \eta) \ne (n, \bar{n}), (\bar{n}, n)$.   To see this note that in the left column, B1, B3, B6, or B8 has occurred when α was inserted. We have $\xi \le \eta'$ (B1) or $\xi < \eta'$ (B3, B6, B8). In the right column A0, B0, B2, B4, B5, or B7 has subsequently occurred. We have $\eta' = \eta$ (A0, B0, B2, B5), or $\eta' < \eta$ (B4, B7). In any case we have $\xi \le \eta$ and $(\xi, \eta) \ne (n, \bar{n}), (\bar{n}, n)$.

Claim 2.   In (4.17) the following configurations do not exist:

$$
\begin{array}{|c|c|}
\hline
x & \bar{x} \\
\hline
\bar{x} & \\
\hline
\end{array}
\ ,\quad
\begin{array}{|c|c|}
\hline
x & \bar{x} \\
\hline
\bar{x} & \\
\hline
\end{array}
\ ,\quad
\begin{array}{|c|c|}
\hline
 & x \\
\hline
x & \bar{x} \\
\hline
\end{array}
\ ,\quad \text{or} \quad
\begin{array}{|c|c|}
\hline
 & \bar{n} \\
\hline
\bar{n} & n \\
\hline
\end{array}
\ .
\tag{4.18}
$$

Due to Claim 1, the first and the second configurations can exist only if B1 with $\alpha = x$, $\gamma = \eta'$ happens in the left column and $\xi = \eta' = \eta = \bar{x}$. But $(\alpha, \gamma) = (x, \bar{x})$ is not compatible with B1. The third (resp. fourth) configuration can exist only if B4 (resp. B7) happens in the right column and $\xi = \eta' = \eta = x$ (resp. $= \bar{n}$) by Claim 1. But we see from the proof of Claim 1 that B4 (resp. B7) actually happens only when $\eta' < \eta$. Claim 2 is proved.

Let the situation around the crossing point of R be one of (4.17) excluding (4.18). When inserting α', suppose in the left column of the crossing point, B1, B3, B6, or B8 has occurred. Let ξ' be the letter bumped out therefrom.

Claim 3: $\xi' \le \eta$ and $(\xi', \eta) \ne (n, \bar{n}), (\bar{n}, n)$.   We divide the check into three cases. (a) If B1 or B6 has occurred in the left column, we have $\xi' = \xi$. Thus the assertion follows from Claim 1. (b) If B3 has occurred, the left column had the entry $\begin{smallmatrix} x \\ \bar{x} \end{smallmatrix}$ and we have $\xi' = \overline{x-1}$, $\xi = \bar{x}$. Claim 1 tells $\xi = \bar{x} \le \eta$, and Claim 2 does $\eta \ne \bar{x}$. Therefore we have $\xi' = \overline{x-1} \le \eta$. (c) If B8 has occurred we have $\xi' = \overline{n-1}$ for either $\xi = n$ or $\xi = \bar{n}$. If $\xi = n$ (resp. $\bar{n}$) the entry on the left of η was $\bar{n}$ (resp. n); therefore

$\eta \geq \bar{n}$ (resp. n). On the other hand Claim 1 tells $(\xi, \eta) \neq (n, \bar{n})$, $(\bar{n}, n)$. Thus we have $\eta \geq \overline{n-1}$. Claim 3 is proved.

Now we are ready to finish the proof of the main assertion. Assume the same situation as Claim 3. We should verify that A1, B1, B3, B6, and B8 do not occur in the right column. Claim 3 immediately prohibits A1, B1, B6, and B8 in the right column. Suppose that B3 happens in the right column. It means that $\eta \in \{1, \ldots, n\}$, $\xi' \geq \eta$ and the right column had the entry $\frac{\eta}{\eta}$. Since $\xi' \leq \eta$ by Claim 3, we find $\xi' = \eta$; therefore $\xi' \in \{1, \ldots, n\}$. Such ξ' can be bumped out from the B1 process only in the left column and not from B3, B6, or B8. It follows that $\xi' = \xi$. This leads to the third configuration in (4.18), and hence a contradiction.

Finally, we consider the case where the bumping route R lies only in the second row. If R' lies below R the tableau should have more than two rows, which is prohibited by Proposition 4.13. ∎

COROLLARY 4.10.   *Let $\alpha' \leq \alpha$; in particular $(\alpha, \alpha') \neq (n, \bar{n})$, $(\bar{n}, n)$. Suppose that a new box is added at the end of the first row when α is inserted into T. Then a new box is added also at the end of the first row when α' is inserted into $(\alpha \to T)$.*

4.5. *Main Theorem*: $B_n^{(1)}$ and $D_n^{(1)}$ Cases

Given $b_1 \otimes b_2 \in B_l \otimes B_k$, we define the element $b_2' \otimes b_1' \in B_k \otimes B_l$ and $l', k', m \in \mathbb{Z}_{\geq 0}$ by the following rule.

Rule 4.11.   Set $z = \min(\sharp \boxed{1}$ in $\mathscr{T}(b_1), \sharp \boxed{\bar{1}}$ in $\mathscr{T}(b_2))$. Thus $\mathscr{T}(b_1)$ and $\mathscr{T}(b_2)$ can be depicted by

$$\mathscr{T}(b_1) = \boxed{\overset{z}{\boxed{1 \cdots 1}} \quad T_*} \, , \qquad \mathscr{T}(b_2) = \boxed{v_1 \Big| \cdots \Big| v_{k'} \Big| \overset{z}{\boxed{\bar{1} \cdots \bar{1}}}} \, .$$

Let $l' = l - z$ and $k' = k - z$; hence T_* is a one-row tableau with length l'. Operate the column insertions and define

$$T^{(0)} := \left(v_1 \to \left(\cdots \left(v_{k'-1} \to (v_{k'} \to T_*) \right) \cdots \right) \right). \tag{4.19}$$

It has the form (See Proposition 4.13.)

$$T^{(0)} = \boxed{\begin{array}{c} \boxed{j_1 \ \cdots\cdots \ j_{k'} \Big| i_{m+1} \ \cdots \ i_{l'}} \\ \boxed{i_1 \ \cdots \ i_m} \end{array}} \tag{4.20}$$

where m is the length of the second row; hence that of the first row is $l' + k' - m$. $(0 \leq m \leq k'.)$

Next we bump out l' letters from the tableau $T^{(0)}$ by the reverse bumping algorithm. For the boxes containing $i_{l'}, i_{l'-1}, \ldots, i_1$ in the above tableau, we do it first for $i_{l'}$ then $i_{l'-1}$ and so on. Correspondingly, let w_1 be the first letter that is bumped out from the leftmost column, let w_2 be the second, and so on. Denote by $T^{(i)}$ the resulting tableau when w_i is bumped out $(1 \leq i \leq l')$. Now $b_1' \in B_l$ and $b_2' \in B_k$ are uniquely specified by

$$\mathcal{T}(b_2') = \boxed{\overset{z}{\boxed{1 \cdots 1}} \ \ T^{(l')}} \ , \qquad \mathcal{T}(b_1') = \boxed{w_1 \ \Big| \cdots \Big| \ w_{l'} \ \overset{z}{\boxed{\overline{1} \cdots \overline{1}}}} \ .$$

We normalize the energy function as $H_{B_l B_k}(b_1 \otimes b_2) = 0$ for

$$\mathcal{T}(b_1) = \overset{l}{\boxed{1 \cdots 1}} \quad \text{and} \quad \mathcal{T}(b_2) = \overset{k}{\boxed{\overline{1} \cdots \overline{1}}}$$

irrespective of $l < k$ or $l \geq k$. Our main result for $U_q'(B_n^{(1)})$ and $U_q'(D_n^{(1)})$ is the following.

THEOREM 4.12. *Given* $b_1 \otimes b_2 \in B_l \otimes B_k$, *find* $b_2' \otimes b_1' \in B_k \otimes B_l$ *and* l', k', m *by Rule* 4.11 *with type B (resp. type D) insertion. Let* $\iota: B_l \otimes B_k \overset{\sim}{\to} B_k \otimes B_l$ *be the isomorphism of* $U_q'(B_n^{(1)})$ (*resp.* $U_q'(D_n^{(1)})$) *crystal. Then we have*

$$\iota(b_1 \otimes b_2) = b_2' \otimes b_1',$$
$$H_{B_l B_k}(b_1 \otimes b_2) = 2\min(l', k') - m.$$

Before giving a proof of this theorem we present two propositions associated with Rule 4.11.

Let the product tableau $\mathcal{T}(b_1) * \mathcal{T}(b_2)$ be given by the $T^{(0)}$ in Eq. (4.19) in Rule 4.11. We assume that it is indeed a (semistandard B or D) tableau.

PROPOSITION 4.13. *The product tableau* $\mathcal{T}(b_1) * \mathcal{T}(b_2)$ *made by* (4.19) *has no more than two rows*.

Proof. Let $T_\bullet$ be a tableau that appears in the intermediate process of the sequence of the column insertions (4.19). Assume that $T_\bullet$ has two

rows. We denote by α the letter which we are going to insert into $T_\bullet$ in the next step of the sequence, and we denote by β the letter which resides in the second row of the leftmost column of $T_\bullet$. It suffices to show that the α does not make a new box in the third row in the leftmost column. In other words it suffices to show that $\alpha \leq \beta$ and $(\alpha, \beta) \neq (\circ, \circ)$ (resp. and in particular $(\alpha, \beta) \neq (n, \bar{n}), (\bar{n}, n)$) in the B_n (resp. D_n) case.

Let us first consider the B_n case. We divide the proof in two cases: (i) $\beta = \circ$, (ii) $\beta \neq \circ$. In case (i) either this $\beta = \circ$ was originally contained in $\mathcal{T}(b_2)$ or this $\beta = \circ$ was made by Case B7 in Section 4.3.2. In any case we see $\alpha \leq n$ (thus $\alpha < \beta$) because of the original arrangement of the letters in $\mathcal{T}(b_2)$. (Note that $\mathcal{T}(b_2)$ did not have more than one $\circ$.) In case (ii) either this β was originally contained in $\mathcal{T}(b_2)$ or this β is an $\overline{x+1}$ which had originally been an $\bar{x}$ in $\mathcal{T}(b_2)$ and then transformed into $\overline{x+1}$ by Case B6 in Section 4.3.2. In any case we see $\alpha \leq \beta$ and $(\alpha, \beta) \neq (\circ, \circ)$.

Second, we consider the D_n case. We divide the proof in two cases: (i) $\beta = n, \bar{n}$, (ii) $\beta \neq n, \bar{n}$. In case (i) either this $\beta = n, \bar{n}$ was originally contained in $\mathcal{T}(b_2)$ or this β was made by Case B7 in Section 4.4.2. In any case we see $\alpha \leq \beta$, in particular $(\alpha, \beta) \neq (n, \bar{n}), (\bar{n}, n)$, because of the original arrangement of the letters in $\mathcal{T}(b_2)$. (Note that $\mathcal{T}(b_2)$ did not contain n and $\bar{n}$ simultaneously.) In case (ii) either this β was originally contained in $\mathcal{T}(b_2)$ or this β is an $\overline{x+1}$ which had originally been an $\bar{x}$ in $\mathcal{T}(b_2)$ and then transformed into $\overline{x+1}$ by Case B4 in Section 4.4.2. In any case we see $\alpha \leq \beta$ and $(\alpha, \beta) \neq (n, \bar{n}), (\bar{n}, n)$. $\blacksquare$

Let $\mathfrak{g} = B_n^{(1)}$ or $D_n^{(1)}$ and $\bar{\mathfrak{g}} = B_n$ or D_n. By neglecting zero arrows, the crystal graph of $B_l \otimes B_k$ decomposes into $U_q(\bar{\mathfrak{g}})$ crystals, where only arrows with indices $i = 1, \ldots, n$ remain. Let us regard $b_1 \in B_l$ as an element of $U_q(\bar{\mathfrak{g}})$ crystal $B(l\Lambda_1)$, and regard $b_2 \in B_k$ as an element of $B(k\Lambda_1)$. Then $b_1 \otimes b_2$ is regarded as an element of $U_q(\bar{\mathfrak{g}})$ crystal $B(l\Lambda_1) \otimes B(k\Lambda_1)$. On the other hand the tableau $\mathcal{T}(b_1) * \mathcal{T}(b_2)$ specifies an element of $B(\lambda)$ which we shall denote by $b_1 * b_2$, where $B(\lambda)$ is a $U_q(\bar{\mathfrak{g}})$ crystal that appears in the decomposition of $B(l\Lambda_1) \otimes B(k\Lambda_1)$.

PROPOSITION 4.14. *The map $\psi: b_1 \otimes b_2 \mapsto b_1 * b_2$ is a $U_q(\bar{\mathfrak{g}})$ crystal morphism; i.e., the actions of $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 1, \ldots, n$ commute with the map ψ.*

This proposition is a special case of Proposition 3.7. Note that, although we have removed the z pairs of 1's and $\bar{1}$'s from the tableaux by hand, this elimination of the letters is also a part of this rule of column insertions (i.e., $(\bar{1} \rightarrow \boxed{1}) = \emptyset$), followed by the sliding (jeu de taquin) rules [B1, B2].

Proof of Theorem 4.12.   First we consider the isomorphism. We are to show:

1.   If $b_1 \otimes b_2$ is mapped to $b_2' \otimes b_1'$ under the isomorphism, then the product tableau $\mathcal{T}(b_1) * \mathcal{T}(b_2)$ is equal to the product tableau $\mathcal{T}(b_2') * \mathcal{T}(b_1')$.

2.   If k and l are specified, we can recover $\mathcal{T}(b_2')$ and $\mathcal{T}(b_1')$ from their product tableau by using the algorithm shown in Rule 4.11. In other words, we can retrieve them by assuming the arrangement of the locations $i_{l'}, \ldots, i_1$ of the boxes in (4.20) from which we start the reverse bumpings.

Claim 2 is verified from Corollary 4.5 or 4.10. We consider Claim 1 in the following. The value of the energy function value will be settled at the same time.

Thanks to the $U_q(\bar{\mathfrak{g}})$ crystal morphism (Proposition 4.14), it suffices to prove the theorem for any element in each connected component of the $U_q(\bar{\mathfrak{g}})$ crystal. We take up the $U_q(\bar{\mathfrak{g}})$ highest weight element as such a particular element. There is a special extreme $U_q(\bar{\mathfrak{g}})$ highest weight element

$$\iota : (l, 0, \ldots, 0) \otimes (k, 0, \ldots, 0) \xrightarrow{\sim} (k, 0, \ldots, 0) \otimes (l, 0, \ldots, 0), \quad (4.21)$$

wherein we find that they are obviously mapped to each other under the $U_q'(\mathfrak{g})$ isomorphism, and that the image of the map is also obviously obtained by Rule 4.11. Let us assume $l \geq k$. (The other case can be treated in a similar way.) Suppose that $b_1 \otimes b_2 \in B_l \otimes B_k$ is a $U_q(\bar{\mathfrak{g}})$ highest element. In general, it has the form

$$b_1 \otimes b_2 = (l, 0, \ldots, 0) \otimes (x_1, x_2, 0, \ldots, 0, \bar{x}_1),$$

where x_1, x_2, and $\bar{x}_1$ are arbitrary as long as $k = x_1 + x_2 + \bar{x}_1$. We are to obtain its image under the isomorphism. Applying

$$\tilde{e}_0^{\bar{x}_1} \tilde{e}_2^{x_2 + \bar{x}_1} \cdots \tilde{e}_{n-1}^{x_2 + \bar{x}_1} \tilde{e}_n^{2x_2 + 2\bar{x}_1} \tilde{e}_{n-1}^{x_2 + \bar{x}_1} \cdots \tilde{e}_2^{x_2 + \bar{x}_1} \tilde{e}_0^{x_2 + \bar{x}_1} \qquad \text{for } \mathfrak{g} = B_n^{(1)}$$

$$\tilde{e}_0^{\bar{x}_1} \tilde{e}_2^{x_2 + \bar{x}_1} \cdots \tilde{e}_{n-1}^{x_2 + \bar{x}_1} \tilde{e}_n^{x_2 + \bar{x}_1} \tilde{e}_{n-2}^{x_2 + \bar{x}_1} \cdots \tilde{e}_2^{x_2 + \bar{x}_1} \tilde{e}_0^{x_2 + \bar{x}_1} \qquad \text{for } \mathfrak{g} = D_n^{(1)}$$

to the both sides of (4.21), we find

$$\iota : (l, 0, \ldots, 0) \otimes (x_1, x_2, 0, \ldots, 0, \bar{x}_1)$$
$$\xrightarrow{\sim} (k, 0, \ldots, 0) \otimes (x_1', x_2, 0, \ldots, 0, \bar{x}_1).$$

Here $x'_1 = l - x_2 - \bar{x}_1$. In the course of the application of $\tilde{e}_i$'s, the value of the energy function has changed as

$$H\big((l,0,\ldots,0) \otimes (x_1, x_2, 0, \ldots, 0, \bar{x}_1)\big)$$
$$= H\big((l,0,\ldots,0) \otimes (k,0,\ldots,0)\big) - x_2 - 2\bar{x}_1.$$

(We have omitted the subscripts of the energy function.) Thus according to our normalization we have $H(b_1 \otimes b_2) = 2(k - \bar{x}_1) - x_2$. (Note that the z in Rule 4.11 is now equal to $\bar{x}_1$; hence we have $k' = k - \bar{x}_1$.) On the other hand for this highest element the column insertions lead to a common tableau

$$T^{(0)} = \boxed{\begin{array}{c} \boxed{1 \cdots\cdots 1} \\ \boxed{2 \cdots 2} \end{array}}$$

whose second row has length x_2 (and first row has the length $l + k - x_2 - 2\bar{x}_1$). This completes the proof. ∎

4.6. *Examples*

EXAMPLE 4.15. $B_5 \otimes B_3 \simeq B_3 \otimes B_5$ for $B_5^{(1)}$.

$$\boxed{5\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \otimes \boxed{5\,|\,\circ\,|\,\bar{5}} \;\tilde{\rightarrow}\; \boxed{5\,|\,\circ\,|\,\bar{5}} \otimes \boxed{5\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}}$$

$$\boxed{5\,|\,5\,|\,\bar{5}\,|\,\bar{5}\,|\,\bar{5}} \otimes \boxed{4\,|\,4\,|\,\circ} \;\tilde{\rightarrow}\; \boxed{\circ\,|\,\bar{5}\,|\,\bar{5}} \otimes \boxed{4\,|\,4\,|\,5\,|\,\bar{5}\,|\,\bar{5}}$$

$$\boxed{1\,|\,1\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \otimes \boxed{\circ\,|\,\bar{1}\,|\,\bar{1}} \;\tilde{\rightarrow}\; \boxed{1\,|\,1\,|\,\circ} \otimes \boxed{\circ\,|\,\bar{5}\,|\,\bar{5}\,|\,\bar{1}\,|\,\bar{1}}$$

Here we have chosen three examples that are specific to type B. The values of the energy function are assigned to be 3, 5, and 1, respectively.

Let us illustrate in more detail the procedure of Rule 4.11 by taking the first example. From the left hand side we proceed the column insertions as follows:

$$\bar{5} \;\rightarrow\; \boxed{5\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \;=\; \begin{array}{l} \boxed{5\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \\ \boxed{\bar{5}} \end{array}$$

$$\circ \;\rightarrow\; \begin{array}{l} \boxed{5\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \\ \boxed{\bar{5}} \end{array} \;=\; \begin{array}{l} \boxed{4\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \\ \boxed{\circ\,|\,\bar{4}} \end{array}$$

$$5 \;\rightarrow\; \begin{array}{l} \boxed{4\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \\ \boxed{\circ\,|\,\bar{4}} \end{array} \;=\; \begin{array}{l} \boxed{4\,|\,5\,|\,\circ\,|\,\bar{5}\,|\,\bar{5}} \\ \boxed{5\,|\,\circ\,|\,\bar{4}} \end{array}$$

The reverse bumping procedure goes as follows:

$$T^{(0)} = \begin{array}{|c|c|c|c|c|} \hline 4 & 5 & \circ & \bar{5} & \bar{5} \\ \hline 5 & \circ & \bar{4} \\ \cline{1-3} \end{array}$$

$$T^{(1)} = \begin{array}{|c|c|c|c|} \hline 4 & \circ & \bar{5} & \bar{5} \\ \hline 5 & \circ & \bar{4} \\ \cline{1-3} \end{array}, \qquad w_1 = 5$$

$$T^{(2)} = \begin{array}{|c|c|c|} \hline 4 & \circ & \bar{5} \\ \hline \circ & \bar{5} & \bar{4} \\ \hline \end{array}, \qquad w_2 = 5$$

$$T^{(3)} = \begin{array}{|c|c|c|} \hline 4 & \circ & \bar{5} \\ \hline \bar{5} & \bar{4} \\ \cline{1-2} \end{array}, \qquad w_3 = \circ$$

$$T^{(4)} = \begin{array}{|c|c|c|} \hline 5 & \circ & \bar{5} \\ \hline \bar{5} \\ \cline{1-1} \end{array}, \qquad w_4 = \bar{5}$$

$$T^{(5)} = \begin{array}{|c|c|c|} \hline 5 & \circ & \bar{5} \\ \hline \end{array}, \qquad w_5 = \bar{5}.$$

Thus we obtained the right hand side. We have $H_{B_5, B_3} = 3$, since $l' = 5$, $k' = 3$, and $m = 3$.

EXAMPLE 4.16. $B_2 \otimes B_1 \simeq B_1 \otimes B_2$ for $D_5^{(1)}$.

$$\begin{array}{ccc} \boxed{4\ \bar{4}} \otimes \boxed{5} & \tilde{\rightarrow} & \boxed{\bar{5}} \otimes \boxed{5\ 5} \\ \boxed{\bar{5}\ \bar{5}} \otimes \boxed{5} & \tilde{\rightarrow} & \boxed{\bar{5}} \otimes \boxed{4\ \bar{4}} \end{array}$$

Here we have chosen two examples that are specific to type D.

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