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A Geometric Presentation of the λ_2-Modules of $C_n(q)$ and $D_n(q)$

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The fixed-point sheaf of the λ_2-module V for each of the groups $C_n(q)$ and $D_n(q)$ is constructed. It is shown that the 0-homology module of the sheaf is isomorphic to V. This gives a presentation of V by geometric generators and relations. © 1994 Academic Press, Inc.

INTRODUCTION

The following is proven:

THEOREM. *Let V be the λ_2-module of the group G, where either*

(i) $G \simeq C_n(q)$, $n \geq 3$, or

(ii) $G \simeq D_n(q)$, $n \geq 4$,

and let $\mathcal{F}_V$ be the irreducible sheaf it determines. Then $H_0(\mathcal{F}_V) \simeq V$.

Before discussing the theorem we note the following:

(1) The field $\mathbf{F}_q$ can be replaced with any perfect field of prime characteristic or a field of algebraic numbers.

(2) The module V arises naturally as a quotient of the alternating square of the natural module N of G. For $G \simeq D_n(q), V \simeq \mathrm{Alt}^2(N)$; for $G \simeq C_n(q)$, V is a codimension-1 quotient of $\mathrm{Alt}^2(N)$. Properties of the alternating square are not, however, used in the proof of the theorem.

(3) V is irreducible except:

(i) $G \simeq C_n(q)$ and p/n,

(ii) $G \simeq D_n(q)$ and $p = 2$.

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In these cases V is indecomposible and has an irreducible submodule W (of codimension 1 or 2) [CPS]; then $\mathscr{F}_V \simeq \mathscr{F}_W$ [RS].

In local group theory one often obtains information about group modules from local-geometric information. Ronan and Smith [RS] considered the case of a Chevalley group G acting on an indecomposable module V in the natural characteristic. They construct a coefficient system $\mathscr{F}_V$ by attaching subspaces of V to the Tits building complex associated with G. A general result is that V is recovered as a quotient of the zero-homology module of $\mathscr{F}_V$. In many cases $\dim(H_0(\mathscr{F}_V))$ is close to that of V or it can be shown that $H_0(\mathscr{F}_V) \simeq V$. Many results were established in this regard in [RS] as well as in papers by Cohen and Smith [CS], Segev and Smith [SS], Smith and Völklein [SV], and Völklein [V].

In the background section where the construction of $\mathscr{F}_V$ is described, many facts are provided regarding the C_n and D_n root systems, the weight decomposition of V, and the structure of the C_n and D_n buildings, with references to the works of Humphreys [H] and Carter [C]. The presheaf terms and the relations among them are established in detail using some sheaf results of [RS] and elementary weight theory [H], [CR].

In the second section the theorem is proven using the same procedures as those in [CS] regarding the natural module of $F_4(q)$. Following the apartment method of Ronan and Smith, one obtains an upper bound on the dimension by exhibiting a generating set defined on a fixed apartment. It is argued that it is sufficient to show that this set generates analogous sets defined on any neighboring apartment. The crucial spanning arguments are reduced to analyzing rank-2 substructures which "connect" neighboring sets of generators. The latter analysis follows Völklein's method of reduction of rank where the cases are reduced to a set of root-orbits induced by the action of the Weyl group stabilizer of the weight λ_2. Once these cases are all classified, the generating set is further refined so it can be seen that its dimension is bounded by $\dim(V)$. Then using the [RS] result that V is a quotient of $H_0(\mathscr{F}_V)$ the theorem will follow. The C_n case was computed by the author while a graduate student at the University of Illinois at Chicago with thesis advisor Stephen Smith. The D_n case was checked independently by Tony Fisher while he was a graduate student at the University of Chicago.

1. Construction of the Fixed-Point Sheaf

The C_n and D_n Root Systems

For a reference for the standard facts provided here, see, for example, Part III of Humphreys [H]. The n-node Dynkin diagrams are labelled as

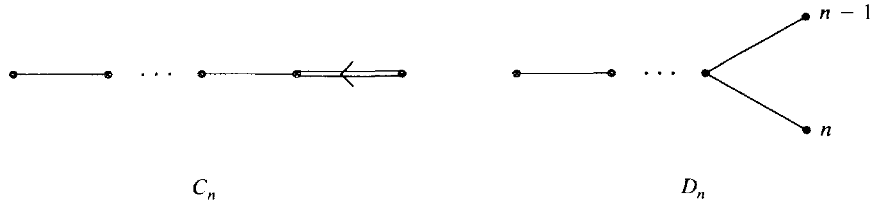

FIG. 1. The Dynkin diagrams of types C_n and D_n.

shown in Fig. 1. We obtain representations of both root systems by considering the following sets of vectors in $\mathbf{R}^n$,

$$\Phi_L := \{\pm 2e_i : 1 \le i \le n\}$$
$$\Phi_S := \{\pm(e_i \pm e_j) : 1 \le i < j \le n\},$$

where e_i and e_j are the standard basis vectors of $\mathbf{R}^n$. Letting Φ_C and Φ_D denote the root systems of C_n and D_n, respectively, we have $\Phi_C = \Phi_L \cup \Phi_S$ and $\Phi_D = \Phi_S$. We can obtain sets of fundamental roots as follows:

$$\Pi_C = \{e_i - e_{i+1}, 2e_n : 1 \le i \le n-1\} \qquad \text{(a basis of } \Phi_C\text{)}$$
$$\Pi_D = \{e_i - e_{i+1}, e_{n-1}, e_n : 1 \le i \le n-1\} \qquad \text{(a basis of } \Phi_D\text{)}.$$

We denote fundamental roots by $\alpha_i := e_i - e_{i+1}$ for $i < n$ and $\alpha_n := 2e_n$ in Π_C and $\alpha_n := e_{n-1} + e_n$ in Π_D. In any root system all roots can be written as $\mathbf{Z}$-linear combinations of the fundamental roots where the coefficients are either all non-negative (Φ^+) or all non-positive (Φ^-). In some situations it is convenient to represent roots as strings of these coefficients (using the ordering of the fundamental roots). We list all of the positive roots (Φ^+) for both root systems in Table I. These roots along with their negatives comprise all Φ_C and Φ_D. Whenever possible we will use the symbol Φ to denote either root system. Note that $\lambda := e_1 + e_2$ (122...221 in Φ_C and 122...211 in Φ_D) is the high short root, i.e., the sum of its coefficients is maximal among the short roots (Φ_S).

Weight Theory of V

Again, for more complete details we refer the reader to Humphreys [H]. For convenience we will refer to the λ_2-module of $C_n(q)$ as V_C and the λ_2-module of $D_n(q)$ as V_D. The high short root $\lambda = e_1 + e_2$ is the high weight of V_C and V_D. Each root system has a dual basis $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ to Π, that is, $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ spans $\mathbf{R}^n$, and $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$, where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbf{R}^n$, and $\lambda = \lambda_2$. In both cases the set of non-zero weights corresponds to Φ_S. For any $\gamma \in \Phi_S$, let V_γ denote the

TABLE I
Positive Roots of C_n and D_n

i, j-Range	Vector form	String in Φ_C	String in Φ_D
$1 \leq i < j \leq n$	$e_i - e_j$	$\overset{(i)}{} \quad \overset{(j)}{}$ 00...011...100...0	$\overset{(i)}{} \quad \overset{(j)}{}$ 00...011...100...0
$1 \leq i < j \leq n-2$	$e_i + e_j$	$\overset{(i)}{} \quad \overset{(j)}{}$ 00...011...122...21	$\overset{(i)}{} \quad \overset{(j)}{}$ 00...011...122...211
$1 \leq i \leq n-2$	$e_i + e_{n-1}$	$\overset{(i)}{}$ 00...011...121	$\overset{(i)}{}$ 00...011...111
$1 \leq i \leq n-1$	$e_i + e_n$	$\overset{(i)}{}$ 00...011...111	$\overset{(i)}{}$ 00...011...101
$1 \leq i \leq n$	$2e_i$	$\overset{(i)}{}$ 00...022...221	—

1-dimensional weight space of γ; then $V = GV_\gamma$. The complete decomposition of the module into weight spaces includes an $(n-1)$-dimensional 0-weight space in V_C and an n-dimensional 0-weight space in V_D. Since $|\Phi_S| = 4\binom{n}{2}$

$$4\binom{n}{2} + n = \frac{4n(n-1)}{2} + n = \frac{2n(2n-2)}{2} + n$$
$$= \frac{2n(2n-1)}{2} - \frac{2n}{2} + n = \binom{2n}{2}$$

we obtain

$$\dim(V_C) = \binom{2n}{2} - 1$$
$$\dim(V_D) = \binom{2n}{2}$$

The C_n and D_n Buildings

For a reference for this information see, for example, Chapters 2 and 15 of Carter [C]. The building Δ is a simplicial complex built up from the geometry of the natural module of G. Each simplex in Δ is identified with a flag of subspaces of V stabilized by a parabolic subgroup of G. The incidence structure of Δ mirrors the structure of the containment lattice of parabolic subgroups of G. That is, if σ_1 and σ_2 are simplices in Δ corresponding to parabolics P_1 and P_2 in G, then σ_1 is a face of σ_2 if and only if P_2 is a subgroup of P_1. Each vertex type in Δ corresponds to a conjugacy class of maximal parabolic subgroups which are indexed by the n nodes of the Dynkin diagram, while maximal simplices or chambers correspond to Borel subgroups of G. Geometrically, the ith node corresponds to an $(i-1)$-simplex, except in type D_2, where both node $n-1$ and n correspond to $(n-1)$-simplices [T].

We obtain a representative parabolic from each conjugacy class in the following way: Fix a Cartan subgroup $H < G$. For each $\alpha \in \Phi$, let X_α denote the root group of α relative to H. Then for each subset $\Pi' \subset \Pi$, let $\Phi(\Pi') = \Phi \cap \mathbf{Z}\Pi'$ and define

$$P(\Pi') := \langle H, X_\alpha : \alpha \in \Phi \cap (\mathbf{Z}^+\Pi \cup \mathbf{Z}^-\Pi') \rangle$$

$$U(\Pi') := \langle X_\alpha : \alpha \in \Phi^+ \setminus \mathbf{Z}^+\Pi' \rangle$$

$$L(\Pi') := \langle H, X_\alpha : \alpha \in \Phi(\Pi') \rangle.$$

Each parabolic in G is conjugate to some $P(\Pi') = U(\Pi') \cdot L(\Pi')$, where $U(\Pi')$ is the unipotent radical of $P(\Pi')$ and $L(\Pi')$ a Levi complement. Maximal parabolics are determined by subsets $\Pi_i := \Pi \setminus \{\alpha_i\}$ for $1 \leq i \leq n$.

Let $N = N_G(H)$. The set of simplices corresponding to all of the N conjugates of $P(\Pi')$ over all subsets $\Pi' \subset \Pi$ comprises the apartment A in Δ. The apartment has the structure of a Coxeter complex for the Weyl group $W \simeq N/H$.

The Fixed-Point Sheaf

The fixed-point sheaf $\mathcal{F}_V$ is constructed on the building Δ in the following way. At each simplex σ corresponding to the parabolic $P = U \cdot L$, attach the subspace spanned by vectors which are fixed by the unipotent radical U. We denote this sheaf space by $\mathcal{F}_\sigma$. We construct a chain-complex on the sheaf spaces by composing the usual oriented boundary map on the building complex with the natural inclusion maps among the sheaf spaces (see [RS] for more complete details), and form homology quotients. Our concern here is with $H_0(\mathcal{F}_V)$.

To compute $H_0(\mathcal{F}_V)$ we need to study $C_0 = \oplus \mathcal{F}_\sigma$ and $C_1 = \oplus \mathcal{F}_\sigma$ in $\mathcal{F}_V$. The sizes of the sheaf spaces and the relations among them are established by identifying these spaces as fundamental modules of Chevalley groups of lower rank. To accomplish this we utilize a result of Smith [S] that V^U, the fixed space of the unipotent radical, is an irreducible module of the Levi complement.

We analyze C_0 by considering the maximal subsets of Π.

PROPOSITION 1. *For $G \simeq C_n(q)$ $(D_n(q))$, $V^{U(\Pi_1)} = L(\Pi_1)V_1$ is isomorphic to the natural symplectic (orthogonal) module of $C_{n-1}(q)$ $(D_{n-1}(q))$. Therefore* $\dim(V^{U(\Pi_1)}) = 2(n-1)$.

Proof. By definition $L(\Pi_1) = \langle H, X_\alpha : \alpha \in \Phi(\Pi_1)\rangle$. $\Phi(\Pi_1)$ is a root system of type C_{n-1} (D_{n-1}) with ordered fundamental basis $\Pi_1 = (\alpha_2, \ldots, \alpha_n)$; thus there is a natural surjection of $L(\Pi_1)$ onto $C_{n-1}(q)$ $(D_{n-1}(q))$ with the kernel of that surjection acting by scalars on $L(\Pi_1)V_\lambda$. Since λ corresponds to λ_2 in the dual basis to Π of $\mathbf{R}^n$, λ is the first fundamental weight in the C_{n-1} (D_{n-1}) subsystem, and therefore the high weight of the natural module of $C_{n-1}(q)$ $(D_{n-1}(q))$. ∎

PROPOSITION 2

(i) *For* (a) $G \simeq C_n(q)$, $2 \le i \le n$, *or* (b) $G \simeq D_n(q)$, $2 \le i \le n-2$, $V^{(U(\Pi_i)} = L(\Pi_i)V_1$ *is isomorphic to the alternating square of the natural module of $A_{i-1}(q)$. Therefore* $\dim(V^{U(\pi_i)}) = \binom{i}{2}$.

(ii) *For $G \simeq D_n(q)$ and $i = n-1$ or n, $L(\Pi_i)V_1$ is isomorphic to the alternating square of $A_{n-1}(q)$ and therefore* $\dim(V^{U(\Pi_i)}) = \binom{n}{2}$.

Proof. For $i < n$ and $G \simeq C_n(q)$ or $i < n-1$ and $G \simeq D_n(q)$, $L(\Pi_i) = L(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}) \times L(\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n)$ which maps naturally onto $A_{i-1}(q) \times C_{n-i-1}(q)$ or $A_{i-1}(q) \times D_{n-i-1}(q)$. Since V_λ is fixed by the right factor, $L(\Pi_i)V_\lambda = L(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})V_\lambda$. λ corresponds to the second fundamental weight relative to $\Phi(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$, and A_{i-1} root subsystem; therefore $L(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})V_\lambda$ is the second fundamental module of $A_{i-1}(q)$, the alternating square of the natural module.

For $G \simeq C_n(q)$ $(i = n)$, or $G \simeq D_n(q)$ $(i = n-1)$ the subdiagram determined by Π_i is connected and of type A_{n-1}. Again λ corresponds to the second fundamental weight of $\Phi(\Pi_i)$ and $L(\Pi_i)V_\lambda$ is the exterior power of the natural module of $A_{n-1}(q)$. ∎

We next determine the relations among these subspaces relevant to a description of $C_1(\mathcal{F}_V)$.

PROPOSITION 3

(i) *For* (a) $2 \le i < j \le n$ *and* $G \simeq C_n(q)$; *or* (b) $2 \le i < j \le n$, *with* $i \le n-2$ *and* $G \simeq D_n(q)$, $V^{U(\Pi_i)} \supset V^{U(\Pi_j)}$.

(ii) *For* (a) $2 \le i \le n$ *and* $G \simeq C_n(q)$; *or* (b) $2 \le i \le n$, *with* $i \ne n-1$ *and* $G \simeq D_n(q)$, $V^{U(\Pi_1)} \cap V^{U(\Pi_i)}$ *is isomorphic to the natural module of $A_{i-2}(q)$. Therefore* $\dim(V^{U(\Pi_i)} \cap V^{U(\Pi_1)}) = i - 1$.

(iii) *For* $G = D_n(q)$, $V^{U(\Pi_{n-1})} \cap V^{U(\Pi_1)}$ *is isomorphic to the natural module of* $A_{n-2}(q)$ *and therefore has dimension* $n - 1$ *while* $V^{U(\Pi_{n-1})} \cap V^{U(\Pi_n)}$ *is isomorphic to the exterior power of the natural module of* $A_{n-2}(q)$ *and therefore has dimension* $\binom{n-1}{2}$.

Proof.

(i) This follows from the observation made in the proof of Proposition 2 that the non-scalar action of $L(\Pi_i)$ on V_λ is determined by that of $L(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$. Since $i < j$, $L(\alpha_1, \alpha_2, \ldots, \alpha_{i-1}) < L(\alpha_1, \alpha_2, \ldots, \alpha_{j-1})$ and therefore $V^{U(\Pi_i)} = L(\Pi_i)V_\lambda \subset L(\Pi_j)V_\lambda = V^{U(\Pi_j)}$ (the outer equalities due to [S]).

(ii) $V^{U(\Pi_1)} \cap V^{U(\Pi_i)} = V^{U(\Pi_1 \cap \Pi_i)}$ (since $\Phi^+ \setminus (N(\Pi_1 \cap \Pi_i) = (\Phi^+ \setminus N\Pi_1) \cup (\Phi^+ \setminus N\Pi_i))$ and thus we need only determine the module $L(\Pi_1 \cap \Pi_i)V_\lambda$. $L(\Pi_1 \cap \Pi_i) = L(\alpha_2, \alpha_3, \ldots, \alpha_{i-1}) \times L(\alpha_{i+1}, \ldots, \alpha_n)$ with the non-scalar action on V_λ determined by the left factor. We are therefore

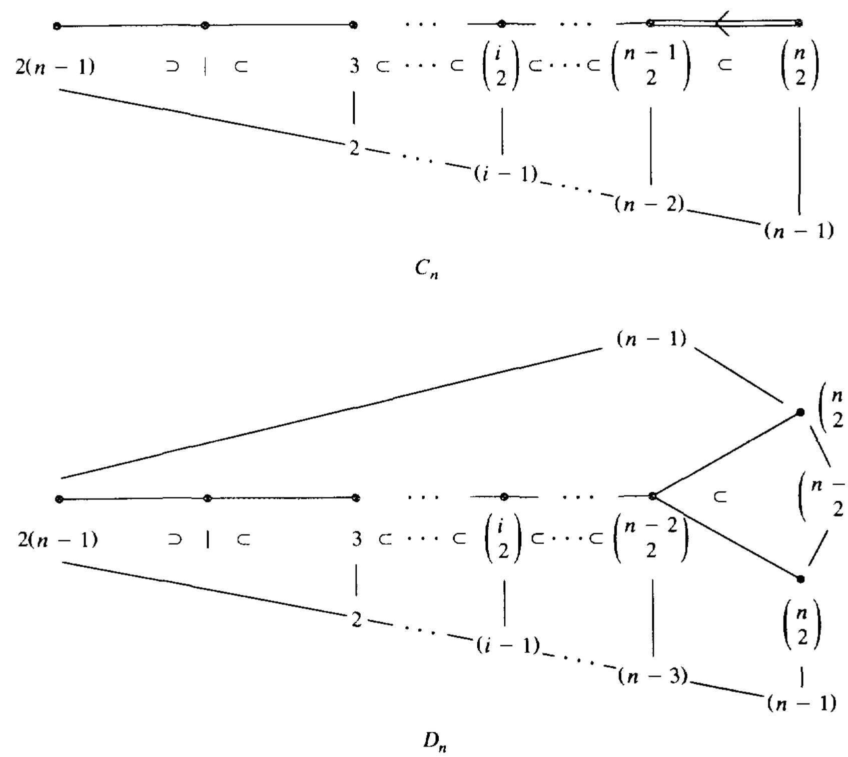

FIG. 2. Dimensions of vertex sheaf spaces and their intersections.

TABLE II
The Weight Decompositions of Representative Vertex Spaces

i	Weights of $V_C^{U(\Pi_i)}$	Weights of $V_D^{U(\Pi_i)}$
1	$e_1 + e_l \ (1 \le l \le n)$	$e_1 + e_l \ (2 \le l \le n)$
$2, 3, \ldots, n-2, n$	$e_k + e_l \ (1 \le k < l \le i)$	$e_k + e_l \ (1 \le k < l \le i)$
$n-1$	$e_k + e_l \ (1 \le k < l \le n-1)$	$e_l + e_l \ (1 \le l \le n-1)$

considering the module of the first fundamental weight of an A_{i-2} subsystem. Thus, $L(\Pi_1 \cap \Pi_i)V_\lambda$ is isomorphic to the natural module of $A_{i-2}(q)$ and the result follows.

(iii) As in (ii) it is sufficient to consider $L(\Pi_{n-1} \cap \Pi_1)V_\lambda$ and $L(\Pi_{n-1} \cap \Pi_n)V_\lambda$. With $G \simeq D_n(q)$, $H(\Pi_{n-1} \cap \Pi_1)$ is an A_{n-2} subsystem with $\lambda = \lambda_2$ the first fundamental weight, while $\Phi(\Pi_{n-1} \cap \Pi_n)$ is an A_{n-2} subsystem with λ the second fundamental weight. Again the results follow from our knowledge of the fundamental weight modules of $A_{n-1}(q)$. ∎

The diagram of the sheaf spaces and the relations among them is given in Fig. 2. With the dimensions established the list of roots can be used to determine the weights involved with the sheaf spaces relative to a choice of H. For $\alpha \in \Phi_S$ and $\beta \in \Phi$, if $\beta \neq -\alpha$ and $\beta + \alpha \notin \Phi_S$ then X_β fixes V_α. It is straightforward to check this, and we summarize the results in Table II.

2. CALCULATION OF $H_0(\mathscr{F}_V)$

Introduction

We are proving that $H_0(\mathscr{F}_V)$ which we will abbreviate by $\hat{V}$ is isomorphic to V. It is sufficient to show that $\dim(\hat{V}) \le \dim(V)$ since V is a quotient of $\hat{V}$ [RS]. To get an upper bound on the dimension of $\hat{V}$ we construct a generating set of 1-space vertices—that is, conjugates of V_λ. Following the apartment method of [RS], we base this construction at a fixed apartment A in Δ. We argue that it spans every 1-space vertex in Δ by showing that it spans the analogous set on every neighboring apartment. Völklein's [V] reduction of rank method is employed to reduce the crucial spanning arguments to classifying the rank-2 substructures which arise in this context. This is facilitated by a complete analysis of the Weyl group action on pairs of roots (α, β) with $\alpha \in \Phi_S$.

Construction of the Generating Set

We continue to fix an apartment A relative to a choice of Cartan subgroup H. Then $W = N_G(H)/H$ is the Weyl group which acts on the root system $\Phi(A)$. V_λ denotes the sheaf space attached to the vertex corresponding to $P(\Pi_2)$. For each $\alpha \in \Phi_S(A)$ let $V_\alpha = V_\lambda^w$, where $w\lambda = \alpha$ for some $w \in W$. We define subgroups of G generated by sets of root groups corresponding to rank-1 and -2 subsystems of Φ. Subspaces of $\mathscr{F}_V$ are generated by V_α under the action of a subgroup corresponding to a root subsystem containing α.

Definition

(i) We construct rank-1 and -2 subgroups as follows:

(a) $L_\alpha := \langle X_{\pm\alpha} \rangle$ $(\alpha \in \Phi(A))$.

(b) $L_{\alpha,\beta} := \langle X_\gamma : \gamma \in \Phi(\alpha, \beta) \rangle$ $(\alpha, \beta) \in \Phi_S(A) \times \Phi(A)$.

(ii) Subspaces generated by the subgroup action are defined by:

(a) $L(V_\alpha) := \langle V_\alpha^g : g \in L_\alpha \rangle$ $(\alpha \in \Phi_S(A))$

(b) $L_{\alpha,\beta}(V_\alpha) := \langle V_\alpha^g : g \in L_{\alpha,\beta} \rangle$ $(\alpha, \beta) \in \Phi_S(A) \times \Phi(A)$

(c) $L(A) := \langle L(V_\alpha) : \alpha \in \Phi_S(A) \rangle$.

Regarding these definitions we note that the action of L_α on V_α parallels that of $SL_2(q)$ acting on the high weight space of the adjoint module of 2×2 traceless matrices; there are $q + 1$ 1-spaces in $L(V_\alpha) = L(V_{-\alpha})$. We will eventually show that $L(V_\alpha)$ is 3-dimensional in homology and therefore $\dim(L(A)) \leq 3(|\Phi_S|/2) = 6\binom{n}{2}$.

Our main effort is to show that $\hat{V} = L(A)$. We follow the "apartment method" of [RS] by focusing on the apartments which are neighbors to A. Two apartments in Δ are neighbors if they intersect in a half-apartment. We can reduce showing $\hat{V} = L(A)$ to the following:

Lemma 1. *Let A and A' be neighboring apartments. Then $L(A') \subset L(A)$.*

Sufficiency of Lemma 1 *to show that $\hat{V} = L(A)$.* From our account of the weight decomposition of representative vertex spaces (see Table II) in Δ, it is clear that the higher dimensional vertex spaces are generated in homology by the Weyl conjugates of V_λ. $L(A)$ includes all such 1-spaces attached to vertices of the apartment A. Therefore showing $\hat{V} = L(A)$ reduces to showing that $L(A') \subset L(A)$ for any apartment A' in Δ. We know, however, that the apartments in Δ are connected by the neighbor relation (see Lemma 4.4 of [RS]) and are finite in number. It follows by induction that it is sufficient to consider $L(A')$ for A' a neighbor to A.

Reduction to Rank-2

To prove the lemma we first note some facts about neighboring apartments. If A and A' are neighbors, the half-apartment $A \cap A'$ is fixed by a root group $X_\beta < G$ with $\beta \in \Phi(A)$. Furthermore, X_β is transitive on the set of apartments which share $A \cap A'$ [RS]. Then $A' = A^{g_\beta}$ for some $g_\beta \in X_\beta$, and $L(A') = L(A)^{g_\beta} = \langle L(V_\alpha)^{g_\beta} | \alpha \in \Phi_S(A)\rangle$.

$L(V_\alpha)^{g_\beta} \subset L_{\alpha,\beta}(V_\alpha)$; thus we classify these rank-2 substructures. We obtain generators from $L(A)$ by appealing to rank-2 results of [RS], [V].

Orbits of W_λ

Since W is transitive on Φ_S, each pair of roots (α, β) with $\alpha \in \Phi_S$ is conjugate to (λ, β'), where β' can be chosen as a particular orbit representative under the action of W_λ, the Weyl group stabilizer of λ. We will see that this reduces the process of classifying the rank-2 structures $L_{\alpha,\beta}(V_\alpha)$ to examining at most six cases.

We begin the calculation of the W_λ-orbits of Φ by noting that W is generated by fundamental reflections s_i, $i = 1, 2, \ldots, n$, where $s_i(\beta) = \beta - (2\langle\alpha_i, \beta\rangle / \langle\alpha_i, \alpha_i\rangle)$. Since $\langle\alpha_i, \lambda\rangle = \delta_{i2}$ (see Section 1: Weight Theory of V), then $s_i(\lambda) = \lambda$ for $i \neq 2$. Then W_λ is the (maximal) subgroup of W generated by $\{s_i : i \neq 2\}$.

The Weyl group acting on pairs of roots preserves the inner product; thus every member of a W_λ orbit has the same inner product with λ. The orbits of W_λ are nearly determined by the range of inner products.

LEMMA 2. *For $n > 3$ in either system W_λ induces six orbits on Φ_S. Four of these orbits are composed of roots with inner product ± 2 or ± 1 with λ and two of the orbits have inner product 0 with λ. Otherwise, for $n = 3$ and $G \simeq C_n(q)$, Φ_S has exactly five orbits.*

Proof. (1) $\langle\beta, \lambda\rangle = \pm 2$:

$$W_\lambda(\lambda) = \{\lambda\}; \qquad W_\lambda(-\lambda) = \{-\lambda\}.$$

This is the case when $\beta = \pm\lambda$. By definition W_λ fixes λ and as the Weyl group action commutes with $\mathbf{F}_q$, W_λ fixes $-\lambda$ as well.

(2) $\langle\beta, \lambda\rangle = \pm 1$:

$$W_\lambda(\alpha_2) = \{e_i \pm e_j : i = 1, 2 : j = 3, 4, \ldots, n\}$$

$$W_\lambda(-\alpha_2) = \{-(e_i \pm e_j) : i = 1, 2; j = 3, 4, \ldots, n\}.$$

As previously noted $(\lambda, \alpha_2) = 1$. We check directly that for $3 \leq j \leq n - 1$, $\alpha_j = e_j - e_{j+1}$ and

$$s_j(e_2 - e_j) = (e_2 - e_j) - \frac{2\langle e_j - e_{j+1}, e_2 - e_j\rangle}{\langle e_j - e_{j+1}, e_j - e_{j+1}\rangle}(e_j - e_{j+1}) = (e_2 - e_{j+1})$$

$$s_1(e_2 - e_j) = (e_2 - e_j) - \frac{2\langle e_1 - e_2, e_2 - e_j\rangle}{\langle e_1 - e_2, e_1 - e_2\rangle}(e_1 - e_2) = (e_1 - e_j).$$

To account for roots of the form $e_i + e_j$ we note for the C_n-root system where $\alpha_n = 2e_n$:

$$s_n(e_2 - e_n) = (e_2 - e_n) - \frac{2\langle 2e_n, e_2 - e_n\rangle}{\langle 2e_n, 2e_n\rangle}(2e_n) = (e_2 + e_n)$$

$$s_{n-1}(e_2 + e_n) = (e_2 + e_n) - \frac{2\langle e_{n-1} - e_n, e_2 + e_n\rangle}{\langle e_{n-1} - e_n, e_{n-1} - e_n\rangle}(e_{n-1} - e_n)$$

$$= e + e_{n-1}.$$

For the D_n-root system, $(\alpha_n = e_{n-1} + e_n)$:

$$s_n(e_2 - e_n) = (e_2 - e_n) - \frac{2\langle 2e_{n-1} + e_n, e_2 - e_n\rangle}{\langle e_{n-1} + e_n, e_{n-1} + e_n\rangle}(e_{n-1} + e_n)$$

$$= (e_2 + e_{n-1}).$$

In both systems for $3 \leq j \leq n - 1$,

$$s_n(e_2 - e_{n-1}) = (e_2 - e_{n-1}) - \frac{2\langle e_{n-1} + e_n, e_2 - e_{n-1}\rangle}{\langle e_{n-1} + e_n, e_{n-1} + e_n\rangle}(e_{n-1} + e_n)$$

$$= (e_2 + e_n)$$

$$s_{j-1}(e_2 + e_j) = e_2 + e_j - \frac{2\langle e_{j-1} - e_j, e_2 + e_j\rangle}{\langle e_{j-1} - e_j, e_{j-1} - e_j\langle}(e_{j-1} - e_j) = e_2 + e_{j-1}$$

$$s_1(e_2 + e_j) = e_2 + e_j - \frac{2\langle e_1 - e_2, e_2 + e_j\rangle}{\langle e_1 - e_2, e_1 - e_2\rangle}(e_1 - e_2) = e_1 + e_j.$$

Similar relations hold for $-\alpha_2$.

(3) $\langle \beta, \lambda \rangle = 0$

(a) $W_\lambda(\alpha_1) = \{\pm\alpha_1\}$

For $\alpha_1 = e_1 - e_2$, $S_1(\alpha_1) = -\alpha_1$ and $S_j(\alpha_1) = \alpha_1$ for $3 \leq j \leq n$.

(b) $W_\lambda(\alpha_3) = \{\pm(e_i \pm e_j) : 3 \leq i < j \leq n\}$.

For the C_n system we note that for $3 \leq i \leq n$

$$s_n(e_i - e_n) = e_i - e_n - \frac{2\langle 2e_n, e_i - e_n\rangle}{\langle 2e_n, 2e_n\rangle} 2e_n = e_i + e_n$$

$$s_{n-1}(e_i + e_n) = e_i + e_{n-1}.$$

For the D_n system,

$$s_n(e_i - e_n) = e_i - e_n - \frac{2\langle e_{n-1} + e_n, e_i - e_n\rangle}{\langle e_{n-1} + e_n, e_{n-1} + e_n\rangle}(e_{n-1} + e_n) = e_i + e_{n-1}$$

$$s_n(e_i - e_{n-1}) = e_i + e_n.$$

In either system for $3 \leq i < j \leq n - 2$

$$s_j(e_i - e_{i+1}) = (e_i + e_j)$$

and for $3 \leq i < j \leq n - 1$

$$s_j(e_i - e_j) = (e_i - e_{j+1})$$

$$s_i(e_i - e_j) = (e_{i+1} - e_j).$$

Finally, as $s_3(e_3 - e_4) = -(e_3 - e_4)$, we obtain analogous relations for the negatives as well. We have now accounted for all roots of Φ_S, and the lemma follows. ▮

LEMMA 3. *Φ_L is partitioned into exactly three orbits corresponding to inner products* 0, 2, *and* -2. *(Note that $G \simeq C_n(q)$.)*

Proof.

(1) $\langle \beta, \lambda \rangle = 0$. $W_\lambda(\alpha_n) = \{\pm 2e_i : i \geq 3\}$. For $3 \leq i < n$:

$$s_i(2e_{i+1}) = 2e_{i+1} - \frac{2\langle e_i - e_{i+1}, 2e_{i+1}\rangle}{\langle e_i - e_{i+1}, e_i - e_{i+1}\rangle}(e_i - e_{i+1}) = 2e_i.$$

(2) $\langle \beta, \lambda \rangle = \pm 2$.

(a) $W_\lambda(2e_2) = \{2e_1, 2e_2\}$;

(b) $W_\lambda(-2e_2) = \{-2e_1, -2e_2\}$.

$$s_1(2e_2) = 2e_2 - \frac{2\langle e_1 - e_2, 2e_2 \rangle}{\langle e_1 - e_2, e_1 - e_2 \rangle}(e_1 - e_2) = 2e_1$$

and similarly $s_1(-2e_2) = -2e_1$. We have now accounted for all roots in Φ_L, and the lemma follows. ▮

Rank-2 Analysis

With the orbits of W_λ established we can now classify the structures of the form $L_{\alpha,\beta}(V_\alpha)$ ($\alpha \in \Phi_S, \beta \in \Phi$) and find generators for them in $L(A)$.

LEMMA 4. *Let $G \simeq C_n(q)$ and let $(\alpha, \beta) \in \Phi_S \times \Phi_C$. Then if either*

(a) $\langle \alpha, \beta \rangle = \pm 2$ with $\beta \in \Phi_L$, or

(b) (α, β) is conjugate to $(\lambda, \pm \alpha_1)$ via the Weyl group then the following hold:

(i) $\Phi(\alpha, \beta)$ is a $(B)C_2$ root system with fundamental roots conjugate to α_1 and $2e_2$.

(ii) $L_{\alpha,\beta}(V_\alpha)$ is a quotient of the sheaf of the natural 4-dimensional symplectic module of $C_2(q)$.

(iii) $L_{\alpha,\beta}(V_\alpha)$ is at most 5-dimensional generated by $\{V_{\pm\bar{\lambda}}, L(V_{\bar{\alpha}_1})\} \subset L(A)$, where $\bar{\alpha}_1$ is the short fundamental root and $\bar{\lambda}$ is the high short root of $\Phi(\alpha, \beta)$.

Proof. By Weyl group conjugation we may assume that $\alpha = \lambda$. For β long, using conjugation by W_λ and part (2) in the proof of Lemma 3 we may assume $\beta = \pm 2e_1$; otherwise we assume $\beta = \alpha_1$. In either case $\Phi(\lambda, \beta) = \{\pm\alpha_1, \pm 2e_1 \pm 2e_2, \pm \lambda\}$ is a rank-2 root system with roots of different length. Fundamental roots can be chosen as α_1 and $2e_2$, thus establishing part (i).

To prove (ii), we construct the fixed-point sheaf of $L_{\lambda,\beta}$ acting on $L_{\lambda,\beta}(V_\lambda)$. As in Section 1 we fix a Cartan subgroup $\bar{H} = H \cap L_{\lambda,\beta}$ and construct representative parabolic subgroups and their Levi decompositions (for convenience we denote each fundamental root as a root string relative to the fundamental basis $(\alpha_1, 2e_2)$, as was done in Table I):

$$\alpha_1 = 10, \qquad 2e_2 = 01, \qquad 2e_1 = 21, \qquad \lambda = 11.$$

Let

$$P_1 = U_1 \cdot L_1 \qquad \text{with } U_1 = \langle X_{10}, X_{11}, X_{21} \rangle,\ L_1 = \langle \overline{H}, X_{\pm 01} \rangle$$
$$P_2 = U_2 \cdot L_2 \qquad \text{with } U_2 = \langle X_{01}, X_{11}, X_{21} \rangle,\ L_2 = \langle \overline{H}, X_{\pm 10} \rangle$$
$$P_{12} = U_{12} L_{12} \qquad \text{with } U_{12} = \langle X_{10}, X_{01}, X_{11}, X_{21} \rangle,\ L_{12} = \overline{H}.$$

We determine the fixed spaces of the U_i by again utilizing the theorem of Smith [S] and focus on the Levi complement. For L_1, we note that $\pm 01 = \pm 2e_2 = \pm 022\ldots 21 \in \Phi(\Pi_1)$. By Proposition 1 we know that $L(\Pi_1)$ acts on V_λ to generate a copy of the natural symplectic module of $C_{n-1}(q)$. Using the weight decomposition of this known module we see that L_1 acts naturally on the 2-space generated by weight spaces V_λ and $V_{\lambda - 2e_2}$ $(= V_{\alpha_1})$, and these are fixed by U_1. Analyzing geometrically, since $L_1 < L(\Pi_1)$, we see that all of the conjugates of V_λ by L_1 are incident to the sheaf term at the simplex corresponding to $P(\Pi_1)$; these conjugates are therefore generated in homology by V_λ and V_{α_1}.

To compute the module $L_2 V_\lambda$ we note that $\pm 10 = \pm\alpha_1 \in \Phi(\Pi_2)$ and therefore $L_2 < L(\Pi_2)$, which fixes V_λ by Proposition 2. Finally, $P_1 \cap P_2 = P_{12}$ is a Borel subgroup of $L_{\lambda,\beta}$, which stabilizes the flag $V_\lambda \subset \langle V_\lambda, V_{\alpha_1} \rangle$. The resulting configuration of spaces is precisely the presheaf of the natural module of $\Omega_S(q)$, proving (ii).

For part (iii) we appeal to [RS, 4.1, 4.3], where the 0-homology of the fixed-point sheaf of the natural module of $C_2(q)$ was computed. It was determined to be at most 5-dimensional, generated by the two pairs of short root 1-spaces, $V_{\pm\bar{\alpha}_1}$ and $V_{\pm\bar{\lambda}}$, along with possibly an additional 1-space that can be chosen in $L(V_{\bar{\alpha}_1})$. By the universal property of H_0 [RS, 2.3], $L_{\lambda,\beta}(V_\lambda)$ is a homomorphic image of the 0-homology of the C_2 sheaf. Thus $L_{\lambda,\beta}(V_2)$ is generated by $L(V_{\bar{\alpha}_1})$ and $V_{\pm\bar{\lambda}}$. ∎

LEMMA 5. *Let $\alpha, \beta \in \Phi_S(A)$ with $\langle \alpha, \beta \rangle = \pm 1$. Then*

(i) *$\Phi(\alpha, \beta)$ is an A_2-root subsystem*

(ii) *$L_{\alpha,\beta}(V_\alpha)$ is isomorphic to the sheaf of the A_2 adjoint module*

(iii) *$L_{\alpha,\beta}(V_\alpha)$ is 8-dimensional, generated by $V_{\pm\bar{\lambda}}$, $L(V_{\bar{\alpha}_1})$, and $L(V_{\bar{\alpha}_2})$, where $\bar{\lambda}$ is the high root, and $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are the fundamental roots of $\Phi(\alpha, \beta)$.*

Proof. Using Weyl conjugacy and part (2) of the proof of Lemma 2, we may assume that $\alpha = \lambda$ and $\beta = \pm\alpha_2$. Since $\lambda - \alpha_2 = e_1 + e_3$, and $\lambda - 2\alpha_2$ is not a root, we obtain $\Phi(\lambda, \beta) = \{\pm\lambda, \pm\alpha_2, \pm(e_1 + e_3)\}$ an A_2 system with fundamental roots α_2 and $e_1 + e_3$, establishing (i).

To prove (ii) and (iii), we construct the fixed point sheaf of $L_{\lambda,\beta}$ acting on $L_{\lambda,\beta}(V_\lambda)$:

Set $\bar{H} = H \cap L_{\lambda,\beta}, \bar{\alpha}_1 = \alpha_2, \bar{\alpha}_2 = e_1 + e_3, \bar{\lambda} = \lambda$ and for convenience we use the root string codes: $\bar{\alpha}_1 = 10, \bar{\alpha}_2 = 01, \bar{\lambda} = 11$. We then define representative parabolics as

$$P_1 := U_1 \cdot L_1 \qquad \text{with } U_1 := \langle X_{10}, X_{11}\rangle, L_1 := \langle \bar{H}, X_{\pm 01}\rangle$$

$$P_2 := U_2 \cdot L_2 \qquad \text{with } U_2 := \langle X_{01}, X_{11}\rangle, L_2 := \langle \bar{H}, X_{\pm 10}\rangle$$

$$P_{12} := U_{12} \cdot L_{12} \qquad \text{with } U_{12} := \langle X_{10}, X_{01}, X_{11}\rangle, L_{12} := \bar{H}.$$

Since $10 = \alpha_2$ is in $\Pi_1 \cap \Pi_3$ then $L_2 < L(\Pi_1) \cap L(\Pi_3)$, which maps naturally onto $A_1(q) \times C_{n-4}(q)$ or $A_1(q) \times D_{n-4}(q)$. From our discussion in the proof of Proposition 2, we see that the right factor acts trivially on V_λ, while the left factor generates the natural 2-space stabilized by $L(\Pi_1) \cap L(\Pi_3)$ (see Proposition 3). This 2-space is spanned by V_λ and $V_{(e_1+e_3)}$, and these are fixed by U_2. Geometrically we see that all L_2 conjugates of V_λ are incident to the vertices in Δ corresponding to $P(\Pi_1)$ and $P(\Pi_3)$.

We can then conjugate by a suitable element of W_λ which transposes α_2 and $e_1 + e_3$ to obtain analogous conclusions for U_1 and L_1. That is, L_1 stabilizes the 2-space generated by V_λ and V_{α_2}.

Finally, P_{12} fixes V_λ. The resulting configuration corresponds to the presheaf of the adjoint module of $A_2(q)$. By [SV], H_0 of this module is actually isomorphic to the adjoint module. It is generated by the three pairs of opposite root 1-spaces along with two additional 1-spaces selected from $L(V_{\bar{\alpha}_1})$ and $L(V_{\bar{\alpha}_2})$, thus establishing the lemma. ∎

The above lemma gives us a way to determine the homological dimension of each $L(V_\alpha)$. Although there are $q + 1$ 1-spaces in each $L(V_\alpha)$, the relations reduce the dimension.

COROLLARY 1. *For each $\alpha \in \Phi_S$, $L(V_\alpha)$ is 3-dimensional.*

Proof. We can embed $L(V_\alpha)$ in a rank-2 system $L_{\alpha,\beta}(V_\alpha)$ by selecting a short root β with $\langle \alpha, \beta\rangle = \pm 1$. By Lemma 5, $L_{\alpha,\beta}(V_\alpha)$ is isomorphic to the 8-dimensional adjoint of $A_2(q)$. In the representation of the adjoint module by 3×3 traceless matrices, $L(V_\alpha)$ corresponds to 2×2 traceless matrices. In this context it is clear that $L(V_\alpha)$ is 3-dimensional. ∎

COROLLARY 2. *If $(\alpha, \beta) \in \Phi_S \times \Phi$ with $\langle \alpha, \beta \rangle = 0$ and when $G \simeq C_n(q)$, (α, β) is not conjugate to $(\lambda, \pm \alpha_1)$ (see Lemma 4), then*

(i) *$\Phi(\alpha, \beta)$ is an $A_1 \times A_1$ root system*

(ii) *$L_{\alpha,\beta}(V_\alpha) = L(V_\alpha)$ (and is therefore 3-dimensional by Corollary 1).*

Proof. Again using Lemmas 2 and 3, we may assume that $\alpha = \lambda$ and $\beta = \pm\alpha_3$ $(n > 3)$, $\beta = \alpha_1$ $(G \simeq D_n(q))$, or $\beta = \alpha_n$ $(G \simeq C_n(q))$. In every case $\lambda \pm \beta$ is not a root and therefore $\Phi(\alpha, \beta)$ is an $A_1 \times A_1$ root system. Furthermore, X_λ and $X_{-\lambda}$ commute with X_β and $X_{-\beta}$ and thus $[L_\lambda, L_\beta] = 1$. In all these cases, $\beta \in \Phi(\Pi_2)$ and therefore $L_\beta < L(\Pi_2)$, which fixes V_λ by Proposition 2, part (i). L_β fixes any L_λ conjugate of V_λ since $[L_\lambda, L_\beta] = 1$. Thus $L_{\lambda,\beta}(V_\lambda) = L(V_\lambda)$, and the corollary follows. ∎

Proof of Lemma 1. We have accounted for every possible $(\alpha, \beta) \in \Phi_S \times \Phi$. Every generating space in a neighboring $L(A')$ is contained in a rank-2 structure of the form $L_{\alpha,\beta}(V_\alpha)$, which in turn is generated by 1-spaces from $L(A)$. This is sufficient to prove Lemma 1. ∎

Final Arguments on $\dim(\hat{V})$

With Lemma 1 proven we have $\hat{V} \subset L(A)$. Since $L(V_\alpha)$ is 3-dimensional by Corollary 1, the $\dim(L(A)) \leq 3 \cdot \frac{1}{2}|\Phi_S| = 6\binom{n}{2}$. We can improve on this by noting, for example, that Lemma 5 shows that $L(V_\lambda)$, $L(V_{\alpha_2})$, and $L(V_{\lambda-\alpha_2})$ generate an 8-dimensional subspace, since only $V_{\pm\lambda}$ are needed from $L(V_\lambda)$.

Let $\hat{A} := \langle\{V_{\pm\gamma} : \gamma \in \Phi_S(A) \setminus \pm \Pi\} \cup \{L(V_\gamma) : \gamma \in \Phi_S(A) \cap \Pi\}\rangle$. Using the data from the discussion in Section 1 on the Weight Theory of V, we can show that $\hat{A}$ has the desired dimension. In particular, $\dim(\hat{A}) \leq \binom{2n}{2}$ for $G \simeq D_n(q)$, since there are n simple roots in the D_n-system and $4\binom{n}{2} - 2n$ roots in $\Phi_S \setminus \pm \Pi$, and $4\binom{n}{2} - 2n + 3n = 4\binom{n}{2} + n = \binom{2n}{2}$. Since there are one fewer short simple roots in the C_n-system, then $\dim(\hat{A}) \leq \binom{2n}{2} - 1$ for $G \simeq D_n(q)$. As these are the dimensions of V_D and V_C, respectively, we need only show:

LEMMA 6. $L(A) \subset \hat{A}$.

Proof. We must show that $L(V_\gamma) \subset \hat{A}$ for every $\gamma \in \Phi_S$. We proceed by induction on the height of γ, $\mathrm{ht}(\gamma) := \sum_i |a_i|$ for $\gamma = \sum a_i\alpha_i$ $(\alpha_i \in \Pi)$. If $\mathrm{ht}(\gamma) = 1$ then $\gamma \in \pm\Pi$ and $L(V_\gamma) \subset \hat{A}$ by definition. If $\mathrm{ht}(\gamma) > 1$ it can be checked that $\gamma = \alpha + \beta$ with $\alpha \in \Phi_S$, $\beta \in \Phi$ and $\mathrm{ht}(\alpha), \mathrm{ht}(\beta) < \mathrm{ht}(\gamma)$. Then $\Phi(\alpha, \beta)$ is a non-degenerate rank-2 root system of type A_2 or $(B)C_2$ with fundamental roots α and β, and high root γ. As $L_\gamma \subset L_{\alpha,\beta}$, $L(V_\gamma) \subset L_{\alpha,\beta}(V_\gamma) = L_{\gamma,\beta}(V_\gamma)$.

If β is short then by Lemma 5, $L_{\gamma,\beta}(V_\gamma)$ is generated by $V_{\pm\gamma}$, $L(V_\alpha)$, and $L(V_\beta)$. Since $V_{\pm\gamma}$ are in $\hat{A}$ by definition, and $L(V_\alpha)$ and $L(V_\beta)$ are in $\hat{A}$ by induction, then $L(V_\gamma) \subset L_{\gamma,\beta}(V_\gamma) \subset \hat{A}$.

Similarly if β is long, by Lemma 4, $L_{\gamma,\beta}(V_\gamma)$ is generated by $V_{\pm\gamma}$ and $L(V_\alpha)$, and $L(V_\gamma) \subset L_{\gamma,\beta}(V_\gamma) \subset \hat{A}$. Thus $L(A) \subset \hat{A}$. ▮

In each case we have now established

$$\dim(\hat{V}) = \dim(L(A)) \leq \dim(\hat{A}) \leq \dim(V)$$

and the theorem is proven.

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