Extremal graphs for the list-coloring version of a theorem of Nordhaus and Gaddum

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Abstract

We characterize the graphs $G$ such that $Ch(G) + Ch(\bar{G}) = n + 1$, where $Ch(G)$ is the choice number (list-chromatic number) of $G$ and $n$ is its number of vertices.

Keywords: Graph coloring; List coloring

1. Introduction

We consider undirected, finite, simple graphs. A coloring of a graph $G = (V, E)$ is a mapping $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E$. A coloring which uses at most $k$ colors is called a $k$-coloring. The chromatic number $\chi(G)$ is the smallest integer $k$ such that $G$ admits a $k$-coloring. A graph is called $k$-colorable if it admits a $k$-coloring. Deciding whether a graph admits a $k$-coloring is an NP-complete problem [4] for any fixed $k \geq 3$.

Vizing [12], as well as Erdős et al. [2] introduced a variant of the coloring problem as follows. Suppose that each vertex $v$ is assigned a list $L(v)$ of allowed colors; we then want to find a vertex-coloring $c$ such that $c(v) \in L(v)$ for all $v \in V$. In the case where such a $c$ exists we will say that the graph $G$ is $L$-colorable; we may also say that $c$ is an $L$-coloring of $G$. Graph $G$ is $k$-choosable if $G$ is $L$-colorable for every assignment $L$ that satisfies $|L(v)| \geq k$ for all $v \in V$. The choice number or list-chromatic number $Ch(G)$ of $G$ is the smallest $k$ such that $G$ is $k$-choosable. It is easy to see from...
this definition that every graph $G$ satisfies $Ch(G) \geq \chi(G)$. A well-known theorem of Nordhaus and Gaddum [10] states that $\chi(G) + \chi(\widehat{G}) \leq n + 1$ holds for every graph $G$ on $n$ vertices. As shown in [2] this inequality can be extended to the choice number:

**Theorem 1** (Erdős et al. [2]). Every graph $G$ on $n$ vertices satisfies $Ch(G) + Ch(\widehat{G}) \leq n + 1$.

For a short proof, see [5,11]. The graphs attaining equality in the Nordhaus–Gaddum theorem were characterized by Finck [3], who proved that there are exactly two types of such graphs, the types (a) and (b) defined as follows.

- A graph $G=(V,E)$ has type (a) if it has a vertex $v$ such that $V \setminus v$ can be partitioned into subsets $K$ and $S$ with the properties that $K \cup \{v\}$ induces a clique of $G$ and $S \cup \{v\}$ induces a stable set of $G$ (adjacency between $K$ and $S$ is arbitrary). Note that if $G$ has type (a) then so does its complementary graph $\widehat{G}$.
- A graph $G=(V,E)$ has type (b) if it has a subset $C$ of five vertices such that $V \setminus C$ can be partitioned into subsets $K$ and $S$ with the properties that $K$ induces a clique, $S$ induces a stable set, $C$ induces a 5-cycle, and every vertex of $C$ is adjacent to every vertex of $K$ and to no vertex of $S$ (adjacency between $K$ and $S$ is arbitrary). Note that if $G$ has type (b) then so does its complementary graph.

It is easy to see that Finck’s graphs have no induced $2K_2$ and no induced $C_4$ (see also [1,8]).

The aim of this paper is to characterize the graphs $G=(V,E)$ that satisfy $Ch(G) + Ch(\widehat{G}) = |V| + 1$; this solves Problem 1.11 in [11]. In short we will call such graphs **extremal**. In contrast with Finck’s characterization, we will see that every graph $H$ can be an induced subgraph of an extremal graph $G$.

Some notions must be introduced before we can state the main result.

The *join of two graphs*: Given two vertex-disjoint graphs $G_1, G_2$, the graph $G_1 \oplus G_2$ with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2) \cup \{xy\mid x \in V(G_1), y \in V(G_2)\}$ is called the join of $G_1$ and $G_2$. It is easy to see that $\chi(G_1 \oplus G_2) = \chi(G_1) + \chi(G_2)$. In contrast the choice number does not behave so simply; for instance, if $G_1$ and $G_2$ are edgeless graphs on respectively $q$ and $q^d$ vertices, then obviously $Ch(G_1) = Ch(G_2) = 1$, but it is known (see [9]) that $Ch(G_1 \oplus G_2) = q + 1$, i.e., the complete bipartite graph $K_{q,q^d}$ is not $q$-choosable.

The function $f(G)$: For any graph $G=(V,E)$, let $f(G)$ be the smallest integer $k$ such that $Ch(G \oplus S_k) > |V|$, where $S_k$ denotes the edgeless graph on $k$ vertices. As observed above, the complete bipartite graph $K_{q,q^d}$ is not $q$-choosable, thus if $G$ is any graph on $q$ vertices and $S_{q^d}$ is the edgeless graph on $q^d$ vertices then $Ch(G \oplus S_{q^d}) > q$ since $K_{q,q^d}$ is a subgraph of $G \oplus S_{q^d}$. This implies that $f(G)$ is a well-defined integer for every graph $G$.

Since $K_{q,q^d}$ is not $q$-choosable, and removing any edge from it results in a $q$-choosable graph (see [9]), we see that $f(S) = q^d$ holds for every edgeless graph $S$ on $q$ vertices. It is easy to see that $f(K) = 1$ for every complete graph $K$, because $K_a \oplus S_1 = K_{a+1}$ and $Ch(K_a) = n$. On the other hand, if $G$ is not a complete graph and $a,b$ are any two
non-adjacent vertices then \( f(G + ab) \leq f(G) \), where \( G + ab \) is the graph obtained by adding the edge \( ab \). Thus,

For every graph \( G \) on \( q \) vertices, we have \( 1 \leq f(G) \leq q^q \). \( \text{(1)} \)

Types \( F_1, \bar{F}_1, \) and \( F_2 \): We define three new types of graphs that will be central to our main result.

- A graph \( G \) is of type \( F_1 \) if its vertex-set can be partitioned into three sets \( S, H, S_f \) such that \( S \cup S_f \) is a stable set of \( G \), every vertex of \( S_f \) is adjacent to every vertex of \( H \) (so \( H \oplus S_f \) is an induced subgraph of \( G \)), \( |S_f| \geq f(H) \), every vertex of \( S \) has at least one non-neighbor in \( H \). Here \( H \) can be any graph. We may write \( G = F_1(S, H, S_f) \).
- A graph \( G \) is of type \( \bar{F}_1 \) if and only if it is the complement \( \bar{G} \) of a graph of type \( F_1 \), and we will write \( G = \bar{F}_1(S, H, S_f) \) whenever \( \bar{G} = F_1(S, H, S_f) \).
- A graph \( G \) is of type \( F_2 \) if and only if its vertex-set can be partitioned into a clique \( K \), a stable set \( S \) and a 5-cycle \( C \) such that every vertex of the 5-cycle is adjacent to every vertex of \( K \) and to no vertex of \( S \). We may write \( G = F_2(S, K, C) \). Observe that the complement of a graph of Type \( F_2 \) is itself of type \( F_2 \).

With our notation, Finck’s graphs are the graphs \( G = F_1(S, K, S_f) \), where \( K \) is a complete graph and \( |S_f| \geq 1 \), and the graphs of type \( F_2 \).

Our main result is:

**Theorem 2.** A graph \( G \) on \( n \) vertices satisfies \( Ch(G) + Ch(\bar{G}) = n + 1 \) if and only if \( G \) is of type \( F_1, \bar{F}_1 \) or \( F_2 \).

We prove this theorem in Section 2. Note that any graph \( H \) can be used in types \( F_1 \) and \( \bar{F}_1 \); thus every graph can be an induced subgraph of a graph \( G \) that satisfies \( Ch(G) + Ch(\bar{G}) = n + 1 \).

The next lemma summarizes some simple facts about graphs of type \( F_1, \bar{F}_1 \), and \( F_2 \).

**Lemma 3.**

- Every graph \( G \) of type \( F_1(S, H, S_f) \) satisfies \( Ch(G) = |V(H)| + 1 \).
- Every graph \( G \) of type \( \bar{F}_1(S, H, S_f) \) satisfies \( Ch(G) = |S \cup S_f| \).
- Every graph \( G \) of type \( F_2(S, K, C) \) satisfies \( Ch(G) = |V(K)| + 3 \).

**Proof.** Let \( G \) be a graph of type \( F_1(S, H, S_f) \). Let \( q = |V(H)| \). By the definition of \( f(H) \), we have \( Ch(G) \geq q + 1 \). Since \( S \cup S_f \) induces a clique in \( \bar{G} \), we have \( Ch(\bar{G}) \geq |S \cup S_f| \). Summing up these two inequalities yields \( Ch(G) + Ch(\bar{G}) \geq n + 1 \), where \( n \) is the number of vertices of \( G \). Comparing with Theorem 1, we obtain that both inequalities are equalities, which proves the first two items of the lemma.

Now let \( G \) be of type \( F_2(S, K, C) \) and \( n \) be its number of vertices. We have \( Ch(G) \geq Ch(K \oplus C_S) \geq \chi(K \oplus C_S) = |K| + 3 \). Likewise, since \( \bar{G} \) is of type \( F_2(S, K, C) \), we have \( Ch(\bar{G}) \geq |S| + 3 \). Therefore \( Ch(G) + Ch(\bar{G}) \geq |S| + 3 + |K| + 3 = n + 1 \).
Comparing with Theorem 1, we obtain that both inequalities are equalities, which proves the third item of the lemma. \(\square\)

The following simple observation from [11] will be very useful for our proofs.

**Lemma 4.** Let \(G = (V, E)\) be a graph, \(L\) be an assignment of lists of colors on the vertex set of \(G\), and \(v\) be a vertex of \(G\) such that \(d(v) < |L(v)|\). Then \(G\) is \(L\)-colorable if and only if so is \(G \setminus v\). \(\square\)

An extension of this remark can be formulated as a list-coloring version of a well-known theorem of Brooks:

**Theorem 5** (Erdős et al. [2] and Vizing [12]). Let \(G\) be a graph with maximum degree \(\Delta\). If \(Ch(G) > \Delta\), then either some connected component of \(G\) is a complete graph \(K_{\Delta+1}\) or \(\Delta = 2\) and some connected component of \(G\) is an odd cycle. \(\square\)

Now we give a property of \(f(H)\).

**Lemma 6.** If \(H\) is not a complete graph and \(|V(H)| = n\), then \(Ch(H \oplus S_n) \leq n\), in other words \(f(H) > n\).

**Proof.** Suppose that the lemma is false and let \(H\) be a counterexample of minimal order. Write \(q = |V(H)|\). So \(H \oplus S_q\) is not \(q\)-choosable. Let \(a, b\) be two non-adjacent vertices of \(H\). The graph \(H\) must have at least three vertices, for otherwise \(H = S_2\), thus \(f(H) = 2^2 = 4\), and \(H\) is not a counterexample to the lemma.

Let \(L\) be an assignment of lists of colors on the vertex set of \(H \oplus S_q\) which satisfies \(|L(v)| = q\) for all \(v \in V(H \oplus S_q)\) and such that \(H \oplus S_q\) is not \(L\)-colorable.

First suppose that there exist vertices \(x \in V(H) \setminus \{a, b\}\) and \(y \in S_q\) such that \(L(x) \neq L(y)\). Pick a color \(z\) from \(L(x) \setminus L(y)\). For each vertex \(v \in (V(H) \setminus x) \cup (S_q \setminus y)\), set \(L'(v) = L(v) \setminus \{z\}\). By the minimality of \(H\), and because \(H \setminus x\) is not a complete graph, the graph \((H \setminus x) \oplus (S_q \setminus y)\) admits an \(L'\)-coloring. By Lemma 4, we can extend this coloring by assigning to \(y\) a color from \(L(y)\) different from those assigned to its \(q - 1\) neighbours other than \(x\), and by assigning color \(z\) to \(x\) (recall that \(z \notin L(y)\)). Thus we get an \(L\)-coloring of \(H \oplus S_q\), a contradiction.

Now we may assume that the set \(L(v)\) is the same for every \(v \in (V(H) \setminus \{a, b\}) \cup S_q\), say it is equal to \(\{1, \ldots, q\}\). Assign color 1 to every vertex of \(S_q\), and color the \(q - 2\) vertices of \(H \setminus \{a, b\}\) with colors 2, \ldots, \(q - 1\). Thus we are using \(q - 1\) colors to color \(S_q \cup (H \setminus \{a, b\})\), and we can extend this coloring to \(a\) and \(b\) since \(|L(a)| = |L(b)| = q\) and \(a, b\) are not adjacent; this is a contradiction again. \(\square\)

A direct consequence of this lemma is the following corollary.

**Corollary 7.** \(f(H) = 1\) if and only if \(H\) is a complete graph.

The next theorem will be also useful.
that is, \( G \) and we must have equality throughout, in particular we have \( Ch \) three cases.

Let \( Corollary 9. \) Let \( G \) be a graph which admits a \( k \)-coloring such that each color class has size at most \( 3 \) and there is at most two color classes of size \( 3 \). Then \( G \) is \( k \)-choosable.

Proof. The result is clear if \( k = 1 \) and it is a direct consequence of Theorem 8 when \( k \geq 3 \). If \( k = 2 \) then \( G \) is contained in the complete bipartite \( K_{3,2} \) which is \( 2 \)-choosable. \( \square \)

2. Proof of Theorem 2

We first observe that every graph \( G=(V,E) \) of type \( F_1, F_2 \) satisfies \( Ch(G) + Ch(\tilde{G}) = |V| + 1 \); this follows directly from Lemma 3.

Now we prove that each graph \( G=(V,E) \) satisfying

\[
Ch(G) + Ch(\tilde{G}) = |V(G)| + 1 \tag{2}
\]

is of type \( F_1, F_2 \). Our proof works by induction on the number \( n = |V| \) of vertices of \( G \). Clearly, the statement is true for \( n \leq 2 \). Let \( G \) be a graph satisfying (2) of order \( n + 1 \geq 3 \), that is

\[
Ch(G) + Ch(\tilde{G}) = n + 2. \tag{3}
\]

Consider an arbitrary vertex \( p \) of \( G \). Clearly we have \( Ch(G) \leq Ch(G \setminus p) + 1 \) and \( Ch(\tilde{G}) \leq Ch(\tilde{G} \setminus p) + 1 \). Let \( r \) be the degree of \( p \) in \( G \); note that \( p \) has degree \( n - r \) in \( \tilde{G} \). We observe that if \( r < Ch(G \setminus p) \) then \( Ch(G) = Ch(G \setminus p) \). Likewise if \( n - r < Ch(\tilde{G} \setminus p) \) then \( Ch(\tilde{G}) = Ch(\tilde{G} \setminus p) \).

Since \( Ch(G \setminus p) + Ch(\tilde{G} \setminus p) \leq n + 1 \) by Theorem 1 applied to \( G \setminus p \), we must have either \( Ch(G \setminus p) \leq r \) or \( Ch(\tilde{G} \setminus p) \leq n - r \). This leads us to distinguishing between three cases.

Case 1: Some vertex \( p \) of \( G \) is such that \( Ch(G \setminus p) \leq r \) and \( Ch(\tilde{G} \setminus p) > n - r \).

Applying Lemma 4 to \( \tilde{G} \) and \( p \), and since \( Ch(\tilde{G} \setminus p) > n - r \), we obtain \( Ch(\tilde{G}) = Ch(\tilde{G} \setminus p) \); thus, and by (3):

\[ n + 2 = Ch(G) + Ch(\tilde{G}) \leq Ch(G \setminus p) + 1 + Ch(\tilde{G} \setminus p) \leq n + 2 \]

and we must have equality throughout, in particular we have \( Ch(G) = Ch(G \setminus p) + 1 \), that is, \( G \setminus p \) satisfies (2). So, by the induction hypothesis, \( G \setminus p \) is of type \( F_1, F_2 \), and we consider these three cases separately.

Assume that \( G \setminus p \) is of type \( F_1(S,H,S_f) \). Then \( Ch(G \setminus p) = q + 1 \), where \( q = |V(H)| \), and consequently \( Ch(G) = q + 2 \). Consider the subgraph \( H' \) of \( G \) induced by \( V(H) \cup \{p\} \) and the sets \( S_f' = N(p) \cap S_f \) and \( S = S \cup S_f \setminus S_f' \). We claim that \( G \) is of type \( F_1(S',H',S_f') \).

Indeed, \( S' \cup S_f' = S \cup S_f \) induces a stable set in \( G \). Every vertex \( x \) in \( S' \) either is in \( S \),
and thus has a non-neighbor in $H$, or is in $S_f \setminus N(p)$, and thus is not adjacent to $p$. By the construction, $G$ contains $H' \oplus S_f'$. It remains to prove that $|S_f'| \geq f(H')$. Suppose on the contrary that $|S_f'| < f(H')$; we will prove that $G$ is $(q+1)$-list-colorable, which is a contradiction. Let $L$ be an assignment of sets of colors on the vertices of $G$ such that $|L(v)| \geq q + 1$. By the definition of $S'$ every vertex in $S'$ has degree at most $q$. Thus, and by Lemma 4 applied to each vertex of $S'$, we need only prove that $G \setminus S'$ is $L$-colorable. However, $G \setminus S' = H' \oplus S_f'$ is indeed $L$-colorable by the definition of $f(H')$ and because $|S_f'| < f(H')$. Therefore $G$ is of type $F_1(S', H', S_f')$ and we are done.

Now assume that $G \setminus p$ is of type $F_1(S, H, S_f)$. Let $|S| = s$, $|S_f| = g \geq f(H)$ and $|V(H)| = q$. We claim that $Ch(G \setminus p) = s + g$. Indeed, when $f(H) = 1$, Lemma 6 implies that $H$ induces a stable set in $G$, and so every vertex of $H$ has degree at most $s$ in $G \setminus p$. The graph $(G \setminus p) \setminus H$ is $(s + g)$-choosable as it is a clique of size $s + g$. Hence and by Lemma 4, $G \setminus p$ is $(s + g)$-choosable. On the other hand, when $f(H) > 1$, Lemma 6 implies $g \geq f(H) \geq q + 1$. Moreover every vertex in $H$ has degree at most $s + q - 1 < s + g$ in $G \setminus p$. So we can conclude as above. Thus we obtain that $Ch(G \setminus p) = s + g$ as claimed, and consequently $Ch(G) = s + g + 1$.

Suppose that $H$ is not a complete subgraph of $G$. By Lemma 6, $g = |S_f| \geq f(H) \geq q + 1$. We claim that vertex $p$ is adjacent to all of $S \cup S_f$ in $G$. Indeed, in the opposite case, we will prove that $G$ is $(s + g)$-choosable. Since $g \geq q + 1$, every vertex in $H$ has degree at most $s + q < s + g$ in $G$, and so, by Lemma 4, we need only prove that $G' = G \setminus H$ is $(s + g)$-choosable. Let $x$ be a vertex in $S \cup S_f$ and not adjacent to $p$. Vertex $x$ has degree $s + g - 1$ in $G'$. Again by Lemma 4, it is sufficient to see that $G' \setminus x$ is $(s + g)$-choosable; but this is clear because $G' \setminus x$ has precisely $s + g$ vertices.

Thus we obtain that $p$ is adjacent to all of $S \cup S_f$ in $G$. Now, if $N(p) \cap V(H) = \emptyset$ then $G$ is of type $F_1(S, H, S_f + p)$; else $G$ is of type $F_1(S + p, H, S_f)$.

Assume now that $H$ is a complete subgraph of $G$. If $p$ is adjacent to all vertices in $S \cup S_f$ then as before $G$ is of type $F_1$. Else, let $x$ be a vertex of $S \cup S_f$ not adjacent to $p$. We claim that $g = 1$. Indeed, in the opposite case, we will prove that $G$ is $(s + g)$-choosable. Every vertex in $H$ has degree at most $s + 1 < s + g$ in $G$ and so, by Lemma 4, we need only prove that $G' = G \setminus H$ is $(s + g)$-choosable. Since $x$ has degree $s + g - 1$, it is sufficient to prove that $G' \setminus x$ is $(s + g)$-choosable; but this is true since $G' \setminus x$ has $s + g$ vertices.

Thus we obtain $|S_f| = f(H) = 1$, which implies that $T = H \cup S_f$ is a stable set of $G$, and $Ch(G) = s + 2$. Let $S_f'$ be the maximal subset of $T$ such that $S \cup \{p\} \cup S_f'$ is an induced subgraph of $G$. Let $H'$ be the subgraph of $G$ induced by $S \cup \{p\}$. We claim that $|S_f'| \geq f(H')$. Indeed in the opposite case we will prove that $Ch(G) = s + 1$. Note that every vertex in $U = T \setminus S_f'$ has degree at most $s$, and so, by Lemma 4, we need only prove that $G \setminus U = H' \oplus S_f'$ is $(s + 1)$-choosable; but this is true by the definition of $f(H')$. Therefore we see that $G$ is of type $F_1(U, H', S_f')$ and we are done.

Assume now that $G \setminus p$ is of type $F_2(S, K, C_5)$. Thus $Ch(G \setminus p) = k + 3$, where $k = |K|$, and so $Ch(G) = k + 4$. We claim that $p$ is adjacent to all vertices in $K \cup C_5$. This will imply that $G$ is of type $F_2(S, K + p, C_5)$ as desired. Suppose on the contrary that $p$ is not adjacent to some vertex in $K \cup C_5$. We will prove that $Ch(G) = k + 3$, a contradiction.
If there exists a vertex \( x \in C_5 \) that is not adjacent to \( p \), then the degree of \( x \) in \( G \) is at most \( k+2 \). Thus, by Lemma 4, we need only prove that \( G' = G \setminus x \) is \((k+3)\)-choosable.

Let \( y, z \) be the two neighbors of \( x \) in \( C_5 \). The degree of \( y \) and \( z \) in \( G' \) is at most \( k+2 \). Then, similarly, we need only prove that \( G' \setminus \{y, z\} \) is \((k+3)\)-choosable; but this is obvious because \( G' \setminus \{y, z\} \) has \( k+3 \) vertices.

Suppose now that \( p \) is not adjacent to some vertex \( x \) in \( K \). We will prove again that \( Ch(G) = k + 3 \), a contradiction. Let \( L \) be an assignment of lists of colors to the vertices of \( G \) such that \( |L(v)| \geq k+3 \) for all \( v \in V \).

If \( L(x) \cap L(p) \neq \emptyset \), pick a color \( \alpha \in L(x) \cap L(p) \) and assign it to \( x \) and \( p \). Write \( G' = G \setminus \{x, p\} \). Define an assignment \( L' \) on \( G' \) by \( L'(v) = L(v) \setminus \{\alpha\} \) for all \( v \in V(G') \). The assignment \( L' \) satisfies \( |L'(v)| \geq k + 2 \) for all \( v \in V(G') \). Moreover, the graph \( G' \) is of type \( F_2(S, K \setminus x, C_5) \), thus it is \((k+2)\)-choosable. So \( G' \) admits an \( L' \)-coloring, which extends to an \( L \)-coloring of \( G \).

We may now assume that \( L(p) \cap L(x) = \emptyset \). Consider any \( y \in C_5 \). We may choose a color \( \alpha \in L(x) \cup L(p) \) such that \( |L(y) \setminus \{\alpha\}| \geq k + 3 \). We may assume that \( \alpha \in L(p) \) (if \( \alpha \in L(x) \) the rest of the proof here is identical). Assign color \( \alpha \) to \( p \). Write \( G' = G \setminus p \) and define an assignment \( L' \) on \( G' \) by \( L'(v) = L(v) \setminus \{\alpha\} \) for all \( v \in V(G') \). Observe that \( |L'(v)| \geq k + 2 \) for every vertex of \( G' \), and \( |L'(y)| \geq k + 3 \). Moreover the degree of \( y \) in \( G' \) is \( k + 2 \). So we may conclude as above that \( G' \) is \( L' \)-choosable and consequently that \( G \) is \((k+3)\)-choosable.

**Case 2:** Some vertex \( p \) of \( G \) is such that \( Ch(G \setminus p) > r \) and \( Ch(\bar{G} \setminus p) \leq n - r \). This case is similar to Case 1 by complementarity.

**Case 3:** For every vertex \( p \) of \( G \), we have \( Ch(G \setminus p) \leq d_\varnothing(p) \) and \( Ch(\bar{G} \setminus p) \leq d_\bar{\varnothing}(p) \).

Let \( \Delta \) be the maximum degree in \( G \) and \( p \) be a vertex of \( G \) with degree \( \Delta \). Then:

\[
Ch(G \setminus p) \leq \Delta \quad \text{and} \quad Ch(\bar{G} \setminus p) \leq n - \Delta. \tag{4}
\]

Hence \( Ch(G \setminus p) + Ch(\bar{G} \setminus p) \leq k \). From the last inequality and from (3) it follows that \( Ch(G) = Ch(G \setminus p) + 1 \) and \( Ch(\bar{G}) = Ch(\bar{G} \setminus p) + 1 \). These two equalities together with (3) imply \( Ch(G \setminus p) + Ch(\bar{G} \setminus p) = n \). From this and from (4) we obtain \( Ch(G \setminus p) = \Delta \) and \( Ch(\bar{G} \setminus p) = n - \Delta \), i.e.,

\[
Ch(G) = \Delta + 1 \quad \text{and} \quad Ch(\bar{G}) = n - \Delta + 1. \tag{5}
\]

According to Theorem 5 and to the first part of (5), we have either \( \Delta \leq 1 \), or \( \Delta = 2 \) and \( G \) has a connected component that is an odd chordless cycle, or \( \Delta \geq 3 \) and \( G \) has a connected component that is a clique of cardinality \( \Delta + 1 \). We examine the three cases separately.

Assume \( \Delta = 0 \). Graph \( G \) has no edge, so it is of type \( F_1(\emptyset, \emptyset, V) \).

Assume \( \Delta = 1 \). Graph \( G \) consists of \( e \geq 1 \) pairwise disjoint edges and \( n + 1 - 2e \) isolated vertices. If \( G \) has only one edge \( ab \) then \( G = F_1(V \setminus \{a, b\}, \{a\}, \{b\}) \). If \( e \geq 2 \), then \( \bar{G} \) is the join of \( n + 1 - e \) stable sets of size at most two, and by Corollary 9, \( \bar{G} \) is \((n + 1 - e)\)-choosable, which contradicts (5).
Assume \( \Delta = 2 \) and \( G \) has a connected component that is an odd cycle \( C \). Let \( l \) be the length of \( C \). We distinguish between the cases \( l = 3 \), \( l = 5 \), and \( l \geq 7 \).

First assume \( l = 3 \). If \( E(G) = E(C) \) then \( G \) is of type \( F_1(V \setminus C, C \setminus v, v) \) where \( v \) is any vertex of \( C \). If \( E(G) \neq E(C) \), pick an edge \( ab \in E(G) \setminus E(C) \), and call \( G' \) the graph \((V, E(C) \cup \{ab\})\). Then \( G' \) is the join of \( n - 3 \) stable sets of size at most 2 plus one stable set of size 3. By Corollary 9, \( G' \) is \((n - 2)\)-choosable and consequently \( G \) is \((n - 2)\)-choosable, which contradicts (5).

Now assume \( l = 5 \). If \( E(G) = E(C) \) then \( G \) is of type \( F_2(V \setminus C, \emptyset, C) \) and we are done. If \( E(G) \neq E(C) \) we are going to prove that \( G \) is \((n - 2)\)-choosable, which is a contradiction to (5). To prove this, pick an edge \( ab \in E(G) \setminus E(C) \). We have \(|V \setminus (C \cup \{a, b\})| = n - 6 \), so \( G \) is a subgraph of the join of a clique \( K \) of size \( n - 6 \), a stable set \( \{a, b\} \) and \( C \). Let \( L \) be an assignment of lists of colors to the vertices of \( G \) such that \(|L(v)| \geq n - 2 \) for all \( v \in V \). Assign distinct colors from their respective lists to the vertices of \( K \); call \( D \) the set of \( n - 6 \) colors thus used. Define an assignment \( L' \) on \( G \setminus K \) by setting \( L'(v) = L(v) \setminus D \) for each \( v \in V \setminus K \); so \(|L'(v)| \geq 4 \) for each \( v \in V \setminus K \).

We need only prove that \( G \setminus K \) (whose vertex-set is \( C \cup \{a, b\} \)) is \( L'\)-colorable, which we do in details as follows.

Suppose that \( L'(a) \cap L'(b) \neq \emptyset \). Pick a color \( z \in L'(a) \cap L'(b) \) and assign it to \( a \) and \( b \). Let \( L'' \) be the assignment on \( C \) defined by \( L''(v) = L'(v) \setminus \{z\} \). We have \(|L''(v)| \geq 3 \) for all \( v \in C \), so \( C \) is \( L'' \)-colorable, so \( G \setminus K \) is \( L' \)-colorable and so \( G \) is \( L \)-colorable, a contradiction.

Suppose now that \( L'(a) \cap L'(b) = \emptyset \). Let \( x \) be a vertex of \( C \). Since \( L'(a) \cap L'(b) = \emptyset \) there exists a color \( z \in L'(a) \cup L'(b) \) such that \(|L'(x) \setminus \{z\}| \geq 4 \). We may assume that \( z \in L'(a) \). Assign color \( z \) to \( a \). Define an assignment \( L'' \) on \( G \setminus (K \cup \{a\}) \) by setting \( L''(v) = L'(v) \setminus \{z\} \) for all \( v \in V \setminus (K \cup \{a\}) \). Remark that \(|L''(v)| \geq 3 \) for all \( v \in C \cup \{b\} \) and that \(|L''(x)| \geq 4 \). Moreover the degree of \( x \) in \( G \setminus (K \cup \{a\}) \) is 3. So, by Lemma 4, it is sufficient to prove that \( G \setminus (K \cup \{a, x\}) \) is \( L'' \)-colorable. Let \( y, z \) be the two neighbors in \( G \) of \( x \) in \( C \). The degree of \( y \) and \( z \) in \( G \setminus (K \cup \{a, x\}) \) is 2; thus, again by Lemma 4, it is sufficient to prove that \( G \setminus (K \cup \{a, x, y, z\}) \) is \( L'' \)-colorable; but this is true since that last graph has three vertices. So \( G \setminus K \) is \( L' \)-colorable and so \( G \) is \( L \)-colorable, a contradiction.

Now assume \( l \geq 7 \). The graph \( G \) is a subgraph of the join of a clique \( K \) of size \( n + 1 - l \) and \( C \). We claim that \( G \) is \((n - 2)\)-choosable, which is a contradiction to (5). To prove this, let \( L \) be any assignment on \( G \) such that \(|L(v)| \geq n - 2 \) for all \( v \in V \). Assign distinct colors from their respective lists \( L(x) \) to the vertices \( x \) of \( K \), and call \( D \) the set of the \( n + 1 - l \) colors thus used. Define an assignment \( L' \) on \( G \setminus K \) by setting \( L'(v) = L(v) \setminus D \) for all \( v \in V \setminus K \). Note that \(|L'(v)| \geq l - 3 \) for all \( v \in V \setminus K \). By Theorem 5, since \( G \setminus K \) is neither an odd chordless cycle nor a complete graph, \( G \setminus K \) is \( \Delta' \)-choosable, where \( \Delta' \) is the maximum degree in \( G \setminus K \). Actually, we have \( \Delta' = l - 3 \). Thus \( G \setminus K \) is \( L' \)-choosable, and \( G \) is \((n - 2)\)-choosable.

Finally, assume that \( \Delta \geq 3 \) and \( G \) has a connected component \( K \) which is a clique of cardinality \( \Delta + 1 \). Write \( H = G \setminus K \) and \( S_f = K \). By the definition of \( f(H) \) and since \( Ch(G) = n + 1 - \Delta = |H| + 1 \), we have \( |S_f| \geq f(H) \) and thus \( G \) is of type \( F_1(\emptyset, H, S_f) \). This completes the proof. \( \square \)
3. Concluding remarks

We have described the graphs $G$ that satisfy $Ch(G) + Ch(\bar{G}) = |V(G)| + 1$ and shown that they cannot be characterized by forbidden induced subgraphs. In the process we needed to introduce the function $f(G)$. We noted that $f(G) = 1$ if and only if $G$ is a clique, that $f(G) \geq |V(G)| + 1$ if and only if $G$ is not a clique, and that $f(G) = |V(G)|^{\overline{V(G)}}$ if $G$ is edgeless. The behavior of $f(G)$ has been studied later in somewhat more details [7], but it seems very hard to compute the value of $f(G)$ for general graphs or even for restricted classes. One may wonder about the complexity status of the problem of determining whether a graph $G$ has $f(G) \leq k$ for a given integer $k$; it is not clear to us so far that the problem is in NP.

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References