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# Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities

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**Abstract**

We consider solutions to the Helmholtz equation in two and three dimensions. Based on layer potential techniques we provide for such solutions a rigorous systematic derivation of complete asymptotic expansions of perturbations resulting from the presence of diametrically small inhomogeneities with constitutive parameters different from those of the background medium. It is expected that our results will find important applications for developing effective algorithms for reconstructing small dielectric inhomogeneities from boundary measurements.


*Keywords:* Helmholtz equation; Small inhomogeneities; Asymptotic expansions; Generalized polarization tensors

## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with a connected Lipschitz boundary $\partial\Omega$. Let $\nu$ denote the unit outward normal to $\partial\Omega$. Suppose that $\Omega$ contains a small inhomogeneity $D$ of the form $D = z + \delta B$, where $B$ is a bounded Lipschitz domain in $\mathbb{R}^d$

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containing the origin and $\delta$ is the order of magnitude of the diameter of the inhomogeneity. We assume that the domain $D$ is separated apart from the boundary, i.e., there exists a constant $c_0 > 0$ such that $\operatorname{dist}(z, \partial\Omega) \geqslant c_0 > 0$. Let $u$ denote the solution to the Helmholtz equation

$$\nabla \cdot \left( \frac{1}{\mu_\delta} \nabla u \right) + \omega^2 \varepsilon_\delta u = 0 \quad \text{in } \Omega, \tag{1.1}$$

with the boundary condition $u = f$ on $\partial\Omega$, where $\omega > 0$ is a given frequency. Here $\mu_\delta$ and $\varepsilon_\delta$ denote the constitutive parameters of the inhomogeneity defined by

$$\mu_\delta(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \bar{D}, \\ \mu, & x \in D, \end{cases} \tag{1.2}$$

$$\varepsilon_\delta(x) = \begin{cases} \varepsilon_0, & x \in \Omega \setminus \bar{D}, \\ \varepsilon, & x \in D, \end{cases} \tag{1.3}$$

where $\mu, \mu_0, \varepsilon,$ and $\varepsilon_0$ are positive constants. If we allow the degenerate case $\delta = 0$, then the functions $\mu_\delta(x)$ and $\varepsilon_\delta(x)$ equal the constants $\mu_0$ and $\varepsilon_0$. Problem (1.1) can be written as

$$\begin{cases} (\Delta + \omega^2 \varepsilon_0 \mu_0) u = 0 & \text{in } \Omega \setminus \bar{D}, \\ (\Delta + \omega^2 \varepsilon \mu) u = 0 & \text{in } D, \\ \frac{1}{\mu} \frac{\partial u}{\partial \nu}\big|_- - \frac{1}{\mu_0} \frac{\partial u}{\partial \nu}\big|_+ = 0 & \text{on } \partial D, \\ u|_- - u|_+ = 0 & \text{on } \partial D, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Here the subscripts $+$ and $-$ indicate the limit from outside and from inside $D$, respectively. In order to insure well-posedness (also for the $\delta$-dependent case for $\delta$ sufficiently small [15]) we shall assume that $\omega^2 \varepsilon_0 \mu_0$ is not an eigenvalue for the operator $-\Delta$ in $L^2(\Omega)$ with the Dirichlet boundary conditions.

The main achievement of this paper is a rigorous derivation, based on layer potential techniques, of a complete asymptotic expansion of $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ as $\delta \to 0$ for $d = 2, 3$. The leading order term in this asymptotic formula has been derived by Vogelius and Volkov [15], see also [5,10] for previous results on the conductivity problem and [4] where the second order term in the asymptotic expansions of solutions to the Helmholtz equation is obtained.

The proof of our asymptotic expansion is radically different from the variational ones in [4,15]. It is based on layer potential techniques and a new decomposition formula of the solution to the Helmholtz equation. Our decomposition formula generalizes that due to Kang and Seo [12] for the conductivity problem. In that case the steady-state voltage potential is decomposed into a harmonic part and a refraction part.

It is expected that our results will find important applications for developing effective algorithms for reconstructing small dielectric inhomogeneities from boundary measurements which can be applied in medical imagining, breast cancer, tumor, and land mine. By use of higher-order terms in the asymptotic expansions of the boundary perturbations due to the presence of the dielectric inhomogeneities the reconstruction technique described in [2] could be carried out to recuperate the locations of the inhomogeneities with a higher resolution and capture further properties of their geometries (namely, their generalized polarization tensors defined in [1]).

In our recent paper [2] we have used the leading order term in the expansion derived in this paper for efficiently determining the locations and/or shapes of the small dielectric inhomogeneities from boundary measurements at a fixed frequency by reducing the reconstruction problem of the small inhomogeneities to the calculation of an inverse Fourier transform. Our algorithm uses plane wave sources for identifying the small electromagnetic inhomogeneities. A different approach based on projections on three planes was proposed and successfully tested by Volkov in [16].

The extension of the techniques used in [4,15] to construct complete asymptotic expansions seems to be laborious. The present work represents a natural completion of [1]. It is organized as follows. In Section 2 we prove one preliminary result on the unique solvability of a system of two integral equations. In Section 3 we give slightly different representations of the solution of (1.4). In Section 4 we provide a rigorous derivation of high-order terms in its asymptotic expansion. Our derivations are valid for inhomogeneities with Lipschitz boundaries.

## 2. Preliminary result

Let $k_0 := \omega\sqrt{\varepsilon_0\mu_0}$ and $k := \omega\sqrt{\varepsilon\mu}$. Let $\Phi_k(x)$ be the fundamental solution for $\Delta + k^2$, that is for $x \neq 0$,

$$\Phi_k(x) = \begin{cases} -\frac{i}{4}H_0^1(k|x-y|), & d=2, \\ -\frac{e^{ik|x-y|}}{4\pi|x-y|}, & d=3, \end{cases}$$

where $H_0^1$ is the Hankel function of the first kind of order 0 [7]. We have,

$$-\frac{i}{4}H_0^1\big(k|x-y|\big) = \frac{1}{2\pi}\log|x-y| + \tau + \sum_{n=1}^{+\infty}\big(b_n \log k|x-y| + c_n\big)\big(k|x-y|\big)^{2n}, \tag{2.1}$$

where the constant $\tau = (1/2\pi)\log k + \gamma - i/4$, $\gamma$ is the Euler constant. Let for $x \neq 0$,

$$\Phi(x) = \Phi_0(x) = \begin{cases} \frac{1}{2\pi}\log|x-y|, & d=2, \\ \frac{1}{4\pi|x-y|}, & d=3. \end{cases}$$

For a bounded domain $D$ in $\mathbb{R}^d$ and $k > 0$ let $\mathcal{S}_D^k$ and $\mathcal{D}_D^k$ be the single and double layer potentials defined by $\Phi_k$, that is,

$$\mathcal{S}_D^k\varphi(x) = \int_{\partial D} \Phi_k(x-y)\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_D^k\varphi(x) = \int_{\partial D} \frac{\partial \Phi_k(x-y)}{\partial \nu(y)}\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D.$$

It is well known, see [7, Theorem 3.1], that

$$\frac{\partial(\mathcal{S}_D^k\varphi)}{\partial\nu}\bigg|_{\pm}(x) = \left(\pm\frac{1}{2}I + \big(\mathcal{K}_D^k\big)^*\right)\varphi(x), \quad \text{a.e. } x \in \partial D, \tag{2.2}$$

$$\big(\mathcal{D}_D^k\varphi\big)\big|_{\pm} = \left(\mp\frac{1}{2}I + \mathcal{K}_D^k\right)\varphi(x), \quad \text{a.e. } x \in \partial D, \tag{2.3}$$

for $\varphi \in L^2(\partial\Omega)$, where $\mathcal{K}_D^k$ is the operator defined by

$$\mathcal{K}_D^k\varphi(x) = \text{p.v.}\int_{\partial D} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\varphi(y)\,d\sigma(y), \tag{2.4}$$

and $(\mathcal{K}_D^k)^*$ is the $L^2$-adjoint of $\mathcal{K}_D^k$. The operator $\mathcal{K}_D^k$ is known to be bounded on $L^2(\partial D)$ [6].

**Theorem 2.1.** *Suppose that $k_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ on $D$. For each $(F,G) \in H^1(\partial D) \times L^2(\partial D)$, there exists a unique solution $(f,g) \in L^2(\partial D) \times L^2(\partial D)$ to the integral equation*

$$\begin{cases} \mathcal{S}_D^k f - \mathcal{S}_D^{k_0} g = F, \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_D^k f)}{\partial\nu}\big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^{k_0} g)}{\partial\nu}\big|_+ = G, \end{cases} \quad \text{on } \partial D. \tag{2.5}$$

*There exists a constant $C$ independent of $F$ and $G$ such that*

$$\|f\|_{L^2(\partial D)} + \|g\|_{L^2(\partial D)} \leqslant C\big(\|F\|_{H^1(\partial D)} + \|G\|_{L^2(\partial D)}\big). \tag{2.6}$$

**Proof.** We only give the proof for $\mu_0 \neq \mu$ leaving the modification of the arguments presented here in the general case to the reader. Let $X := L^2(\partial D) \times L^2(\partial D)$ and $Y := H^1(\partial D) \times L^2(\partial D)$, and define an operator $T : X \to Y$ by

$$T(f,g) := \left(\mathcal{S}_D^k f - \mathcal{S}_D^{k_0} g, \frac{1}{\mu}\frac{\partial(\mathcal{S}_D^k f)}{\partial\nu}\bigg|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^{k_0} g)}{\partial\nu}\bigg|_+\right).$$

We also define $T_0$ by

$$T_0(f,g) := \left(\mathcal{S}_D^0 f - \mathcal{S}_D^0 g, \frac{1}{\mu}\frac{\partial(\mathcal{S}_D^0 f)}{\partial\nu}\bigg|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^0 g)}{\partial\nu}\bigg|_+\right).$$

One can easily see that $\mathcal{S}_D^{k_0} - \mathcal{S}_D^0 : L^2(\partial D) \to H^1(\partial D)$ is a compact operator, and so is $\frac{\partial}{\partial\nu}\mathcal{S}_D^{k_0}|_\pm - \frac{\partial}{\partial\nu}\mathcal{S}_D^0|_\pm : L^2(\partial D) \to L^2(\partial D)$. Therefore, $T - T_0$ is a compact operator from $X$ into $Y$. If $\mu_0 \neq \mu$, then it is proved in [9] that $T_0 : X \to Y$ is invertible (see also [13]). Thus by the Fredholm alternatives, it is enough to prove that $T$ is injective.

Suppose that $T(f,g) = 0$. Then the function $u$ defined by

$$u(x) := \begin{cases} \mathcal{S}_D^{k_0} g(x), & \text{if } x \in \mathbb{R}^d \setminus D, \\ \mathcal{S}_D^k f(x), & \text{if } x \in D, \end{cases}$$

is the unique solution of the transmission problem

$$\begin{cases} (\Delta + \omega^2 \varepsilon_0 \mu_0) u = 0 & \text{in } \mathbb{R}^d \setminus \bar{D}, \\ (\Delta + \omega^2 \varepsilon \mu) u = 0 & \text{in } D, \\ \frac{1}{\mu} \frac{\partial u}{\partial \nu}\big|_- - \frac{1}{\mu_0} \frac{\partial u}{\partial \nu}\big|_+ = 0 & \text{on } \partial D, \\ u|_- - u|_+ = 0 & \text{on } \partial D, \end{cases}$$

subject to the radiation condition

$$\frac{x}{|x|} \cdot \nabla u(x) - i k_0 u(x) = O\big(|x|^{-(d+1)/2}\big), \quad |x| \to \infty. \tag{2.7}$$

By the uniqueness of a solution to the interface problem for the Helmholtz equation, see for instance [7], we conclude that $f = g = 0$. This completes the proof of solvability of (2.5). The estimate (2.6) is a consequence of solvability and the closed graph theorem. □

Note that in the three-dimensional case, using classical results on the low wave number asymptotics for the Helmholtz equation and single layer potential [14], it can easily be proved that $f$ and $g$ have limits in $L^2(\partial D)$ as $k_0$ and $k$ go to zero, and thus the constant $C$ in (2.6) can be chosen independently of $k_0$ and $k$. This remark will be of use to us in establishing Proposition 4.1 in the three-dimensional case.

## 3. Representation of solutions

In this section we present two representations of the solution of (1.4). A similar representation formula for the transmission problem for the harmonic equation was found in [12,13].

**Theorem 3.1.** *Suppose that $k_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ on $D$. Let $u$ be the solution of* (1.4) *and $g := \frac{\partial u}{\partial \nu}|_{\partial \Omega}$. Define*

$$H(x) := -\mathcal{S}_\Omega^{k_0}(g)(x) + \mathcal{D}_\Omega^{k_0}(f)(x), \quad x \in \mathbb{R}^d \setminus \partial\Omega, \tag{3.1}$$

*and $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ be the unique solution of*

$$\begin{cases} \mathcal{S}_D^k \varphi - \mathcal{S}_D^{k_0} \psi = H \\ \frac{1}{\mu} \frac{\partial (\mathcal{S}_D^k \varphi)}{\partial \nu}\Big|_- - \frac{1}{\mu_0} \frac{\partial (\mathcal{S}_D^{k_0} \psi)}{\partial \nu}\Big|_+ = \frac{1}{\mu_0} \frac{\partial H}{\partial \nu} \end{cases} \quad \text{on } \partial D. \tag{3.2}$$

*Then $u$ can be represented as*

$$u(x) = \begin{cases} H(x) + \mathcal{S}_D^{k_0} \psi(x), & x \in \Omega \setminus \bar{D}, \\ \mathcal{S}_D^k \varphi(x), & x \in D. \end{cases} \tag{3.3}$$

*Moreover, there exists $C > 0$ independent of $H$ such that*

$$\|\varphi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \leqslant C\big(\|H\|_{L^2(\partial D)} + \|\nabla H\|_{L^2(\partial D)}\big). \tag{3.4}$$

**Proof.** Note that $u$ defined by (3.3) satisfies the differential equations and the transmission condition on $\partial D$ in (1.4). Thus in order to prove (3.3), it suffices to prove $\frac{\partial u}{\partial \nu} = g$ on $\partial\Omega$. Let $f := u|_{\partial\Omega}$ and consider the following two phase transmission problem:

$$\begin{cases} (\Delta + k_0^2)v = 0 & \text{in } (\Omega \setminus \bar{D}) \cup (\mathbb{R}^d \setminus \bar{\Omega}), \\ (\Delta + k^2)v = 0 & \text{in } D, \\ v|_- - v|_+ = 0,\ \frac{1}{\mu}\frac{\partial v}{\partial \nu}\big|_- - \frac{1}{\mu_0}\frac{\partial v}{\partial \nu}\big|_+ = 0 & \text{on } \partial D, \\ v|_- - v|_+ = f,\ \frac{\partial v}{\partial \nu}\big|_- - \frac{\partial v}{\partial \nu}\big|_+ = g & \text{on } \partial\Omega, \\ \frac{x}{|x|} \cdot \nabla v(x) - ik_0 v(x) = O(|x|^{-(d+1)/2}), & |x| \to \infty. \end{cases} \tag{3.5}$$

We claim that (3.5) has a unique solution. In fact, if $f = g = 0$, then one can show as before that $v = 0$ in $\mathbb{R}^d \setminus \bar{D}$. Thus $v = \frac{\partial v}{\partial \nu}|_- = 0$ on $\partial D$. By the unique continuation for the operator $\Delta + k^2$, we have $v = 0$ in $D$, and hence $v \equiv 0$ in $\mathbb{R}^d$. Note that $v_j$, $j = 1, 2$, defined by

$$v_1(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases} \qquad v_2(x) = \begin{cases} H(x) + \mathcal{S}_D^{k_0}\psi(x), & x \in \Omega \setminus \bar{D}, \\ \mathcal{S}_D^k \varphi(x), & x \in D, \end{cases}$$

are two solutions of (3.5), and hence $v_1 \equiv v_2$. This completes the proof. □

**Proposition 3.2.** *For each integer $n$ there exists $C_n$ independent of $D$ such that*

$$\|H\|_{\mathcal{C}^n(\bar{D})} \leqslant C_n \|f\|_{H^{1/2}(\partial\Omega)}. \tag{3.6}$$

**Proof.** Let $g := \frac{\partial u}{\partial \nu}|_{\partial\Omega}$. By the definition (3.1), it is easy to see that

$$\|H\|_{\mathcal{C}^n(\bar{D})} \leqslant C\big(\|g\|_{H^{-1/2}(\partial\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}\big),$$

where $C$ depends only on $n$ and $\operatorname{dist}(D, \partial\Omega)$. Therefore, it is enough to show that

$$\|g\|_{H^{-1/2}(\partial\Omega)} \leqslant C\|f\|_{H^{1/2}(\partial\Omega)}$$

for some $C$ independent of $D$.

Let $\varphi$ be a $\mathcal{C}^\infty$ function which is 0 in a neighborhood of $D$ and 1 in a neighborhood of $\partial\Omega$. Let $v \in H^{1/2}(\partial\Omega)$ and define $\tilde{v} \in H^1(\Omega)$ to be the unique solution to $\Delta\tilde{v} = 0$ in $\Omega$ and $\tilde{v} = v$ on $\partial\Omega$. Let $\langle\, ,\rangle$ denote $H^{-1/2} - H^{1/2}$ pairing on $\partial\Omega$. Then

$$\begin{aligned} \langle g, v\rangle &= \int_\Omega \Delta(\varphi u)\tilde{v}\, dx + \int_\Omega \nabla(\varphi u) \cdot \nabla\tilde{v}\, dx \\ &= \int_\Omega \Delta\varphi u\tilde{v}\, dx + 2\int_\Omega \nabla\varphi \cdot \nabla u\tilde{v}\, dx - k_0^2\int_\Omega \varphi u\tilde{v}\, dx + \int_\Omega \nabla(\varphi u) \cdot \nabla\tilde{v}\, dx. \end{aligned}$$

Therefore, it follows from the Cauchy–Schwartz inequality that

$$\big|\langle g, v\rangle\big| \leqslant C\|u\|_{H^1(\Omega\setminus\bar{D})}\|\tilde{v}\|_{H^1(\Omega)} \leqslant C\|u\|_{H^1(\Omega\setminus\bar{D})}\|v\|_{H^{1/2}(\partial\Omega)}.$$

Since $v \in H^{1/2}(\partial\Omega)$ is arbitrary, we get

$$\|g\|_{H^{-1/2}(\partial\Omega)} \leqslant C\|u\|_{H^1(\Omega\setminus\bar{D})}. \tag{3.7}$$

Note that the constant $C$ depends only on $\mathrm{dist}(D, \partial\Omega)$. On the other hand, since $\omega^2$ is not a Dirichlet eigenvalue for the Helmholtz equation (1.4) in $\Omega$, we can prove that

$$\|u\|_{H^1(\Omega)} \leqslant C\|f\|_{H^{1/2}(\partial\Omega)},$$

where $C$ depends only on $\mu_0, \mu, \varepsilon_0$, and $\varepsilon$. It then follows from (3.7) that

$$\|g\|_{H^{-1/2}(\partial\Omega)} \leqslant C\|f\|_{H^{1/2}(\partial\Omega)}.$$

This completes the proof. □

We now transform the representation (3.3) into the one using the Green function and the background solution. The background solution $u_0$ is the solution of

$$\begin{cases} (\Delta + k_0^2)u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Let $G(x, y)$ be the Dirichlet Green function for $\Delta + k_0^2$ in $\Omega$, i.e., for each $y \in \Omega$, $G$ is the solution of

$$\begin{cases} (\Delta + k_0^2)G(x, y) = \delta_y(x), & x \in \Omega, \\ G(x, y) = 0, & x \in \partial\Omega. \end{cases}$$

Then,

$$u_0(x) = \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x, y) f(y)\, d\sigma(y), \quad x \in \Omega.$$

Define one more notation: For a Lipschitz domain $D \subset \Omega$ and $\varphi \in L^2(\partial D)$, we define

$$G_D\varphi(x) := \int_{\partial D} G(x, y)\varphi(y)\, d\sigma(y), \quad x \in \bar{\Omega}.$$

Our second representation is the following theorem.

**Theorem 3.3.** *Let $\psi$ be the function defined in* (3.2)*. Then*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \frac{\partial (G_D\psi)}{\partial \nu}(x), \quad x \in \partial\Omega. \tag{3.9}$$

We need a few facts to prove Theorem 3.3. We first observe an easy identity: if $x \in \mathbb{R}^d \setminus \Omega$ and $z \in \Omega$,

$$\int_{\partial\Omega} \Phi_{k_0}(x - y) \frac{\partial G(z, y)}{\partial \nu(y)}\bigg|_{\partial\Omega} d\sigma(y) = \Phi_{k_0}(x, z), \quad x \in \mathbb{R}^d \setminus \Omega,\ z \in \Omega. \tag{3.10}$$

As a consequence of (3.10), we have

$$\left(\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial G(\cdot, y)}{\partial \nu(y)}\bigg|_{\partial\Omega}\right)(x) = \frac{\partial \Phi_{k_0}(x, y)}{\partial \nu(x)}, \quad x \in \partial\Omega. \tag{3.11}$$

**Lemma 3.4.** *If $k_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ on $\Omega$, then $\frac{1}{2}I + (\mathcal{K}_\Omega^{k_0})^* : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is injective.*

**Proof.** Suppose that $\varphi \in L^2(\partial\Omega)$ and $(\frac{1}{2}I + (\mathcal{K}_\Omega^{k_0})^*)\varphi = 0$. Define $u(x) := \mathcal{S}_\Omega^{k_0}\varphi(x)$, $x \in \mathbb{R}^d \setminus \bar{\Omega}$. Then $u$ is a radiating solution of $(\Delta + k_0^2)u = 0$ in $\mathbb{R}^d \setminus \bar{\Omega}$, and satisfies $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = (\frac{1}{2}I + (\mathcal{K}_\Omega^{k_0})^*)\varphi = 0$. Therefore, by the uniqueness for the exterior Neumann problem [7], we obtain $\mathcal{S}_\Omega^{k_0}\varphi(x) = 0$, $x \in \mathbb{R}^d \setminus \bar{\Omega}$. Since $k_0^2$ is not a Dirichlet eigenvalue for $-\Delta$ on $\Omega$, we can prove that $\varphi = 0$ in the same way as before. This completes the proof. □

We are now ready to prove Theorem 3.3.

**Proof of Theorem 3.3.** Let $g := \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ and $g_0 := \frac{\partial u_0}{\partial \nu}$ for convenience. By the divergence theorem, we get

$$u_0(x) = -\mathcal{S}_\Omega^{k_0}(g_0)(x) + \mathcal{D}_\Omega^{k_0}(f)(x), \quad x \in \Omega.$$

It then follows from (3.1) that

$$H(x) = -\mathcal{S}_\Omega^{k_0}(g)(x) + \mathcal{S}_\Omega^{k_0}(g_0)(x) + u_0(x), \quad x \in \Omega.$$

By substituting (3.3) into above equation, we obtain

$$H(x) = -\mathcal{S}_\Omega^{k_0}\left(\frac{\partial H}{\partial \nu}\bigg|_{\partial\Omega} + \frac{\partial(\mathcal{S}_D^{k_0}\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right)(x) + \mathcal{S}_\Omega^{k_0}(g_0)(x) + u_0(x), \quad x \in \Omega. \tag{3.12}$$

We then get from (2.2),

$$\frac{\partial H}{\partial \nu} = -\left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial H}{\partial \nu}\bigg|_{\partial\Omega} + \frac{\partial(\mathcal{S}_D^{k_0}\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) + \left(\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)(g_0) \quad \text{on } \partial\Omega. \tag{3.13}$$

By (3.11), we get for $x \in \partial\Omega$,

$$\frac{\partial(\mathcal{S}_D^{k_0}\psi)}{\partial \nu}(x) = \int_{\partial D} \frac{\partial \Phi_{k_0}(x,y)}{\partial \nu(x)}\psi(y)\,d\sigma(y) = \left(\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial(G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right)(x). \tag{3.14}$$

Thus we obtain

$$\left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial(\mathcal{S}_D^{k_0}\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) = \left(\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial(G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right)\right) \quad \text{on } \partial\Omega.$$

It then follows from (3.13) that

$$\left(\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial H}{\partial \nu}\bigg|_{\partial\Omega} + \left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) - g_0\right) = 0$$
on $\partial\Omega$,

and hence, by Lemma 3.4, we obtain

$$\frac{\partial H}{\partial \nu}\bigg|_{\partial\Omega} + \left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) - g_0 = 0 \quad \text{on } \partial\Omega. \tag{3.15}$$

By substituting this equation into (3.3), we get

$$\frac{\partial u}{\partial \nu} = \frac{\partial u_0}{\partial \nu} - \left(-\frac{1}{2}I + \left(\mathcal{K}_\Omega^{k_0}\right)^*\right)\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) + \frac{\partial (\mathcal{S}_D^{k_0}\psi)}{\partial \nu} \quad \text{on } \partial\Omega.$$

By (3.14), we have (3.9) and the proof is complete. □

Observe that, by (2.2), (3.15) is equivalent to

$$\frac{\partial}{\partial \nu}\left(H + \mathcal{S}_\Omega^{k_0}\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right) - u_0\right)\bigg|_- = 0 \quad \text{on } \partial\Omega.$$

On the other hand, by (3.10), $\mathcal{S}_\Omega^{k_0}(\frac{\partial (G_D\psi)}{\partial \nu}|_{\partial\Omega})(x) = \mathcal{S}_\Omega^{k_0}\psi(x)$, $x \in \partial\Omega$. Thus, by (3.3), we obtain

$$H(x) + \mathcal{S}_\Omega^{k_0}\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right)(x) - u_0(x) = 0, \quad x \in \partial\Omega.$$

Then, by the unique continuation for $\Delta + k_0^2$, we obtain the following lemma.

**Lemma 3.5.**

$$H(x) = u_0(x) - \mathcal{S}_\Omega^{k_0}\left(\frac{\partial (G_D\psi)}{\partial \nu}\bigg|_{\partial\Omega}\right)(x), \quad x \in \Omega. \tag{3.16}$$

## 4. Derivation of asymptotic formula

Suppose that the domain $D$ is of the form $D = \delta B + z$, and let $u$ be the solution of (1.4). $u_0$ is the background solution as before. In this section we derive an asymptotic expansion of $\frac{\partial u}{\partial \nu}|_{\partial\Omega}$ as $\delta \to 0$ in terms of $u_0$.

We first derive an estimate of the form (3.4) with the constant $C$ independent of $\delta$.

**Proposition 4.1.** *Let $D = \delta B + z$ and $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ be the unique solution of* (3.2). *There exists $\delta_0$ such that for all $\delta \leqslant \delta_0$, there exists $C$ independent of $\delta$ such that*

$$\|\varphi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \leqslant C\left(\delta^{-1}\|H\|_{L^2(\partial D)} + \|\nabla H\|_{L^2(\partial D)}\right). \tag{4.1}$$

**Proof.** After the scaling $x = z + \delta y$, (3.2) takes the form for $d = 2, 3$,

$$\begin{cases} \mathcal{S}_B^{k\delta}\varphi_\delta - \mathcal{S}_B^{k_0\delta}\psi_\delta = \frac{1}{\delta}H_\delta, \\ \frac{1}{\mu}\frac{\partial (\mathcal{S}_B^{k\delta}\varphi_\delta)}{\partial \nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial (\mathcal{S}_B^{k_0\delta}\psi_\delta)}{\partial \nu}\Big|_+ = \frac{1}{\delta\mu_0}\frac{\partial H_\delta}{\partial \nu}, \end{cases} \quad \text{on } \partial B, \tag{4.2}$$

where $\varphi_\delta(y) = \varphi(z + \delta y)$, $y \in \partial B$, etc., and the single layer potentials $\mathcal{S}_B^{k\delta}$ and $\mathcal{S}_B^{k_0\delta}$ are defined by fundamental solutions $\Phi_{k\delta}$ and $\Phi_{k_0\delta}$, respectively. If $d = 3$, it then follows from Theorem 2.1 and the remark just after that for $\delta$ small enough the following estimate holds:

$$\|\varphi_\delta\|_{L^2(\partial B)} + \|\psi_\delta\|_{L^2(\partial B)} \leqslant C\delta^{-1}\|H_\delta\|_{H^1(\partial B)}, \tag{4.3}$$

for some $C$ independent of $\delta$ (but depending on $B$). By scaling back, we obtain (4.1). This argument cannot be applied to the two-dimensional case because of the fact that the fundamental solutions $\Phi_{k\delta}$ and $\Phi_{k_0\delta}$ do not converge to $\Phi_0$ as $\delta$ goes to zero.

In the two-dimensional case, we further consider the system of integral equations

$$\begin{cases} \mathcal{S}_B^0\tilde{\varphi}_\delta + \tau\int_{\partial B}\tilde{\varphi}_\delta - \mathcal{S}_B^0\tilde{\psi}_\delta - \tau\int_{\partial B}\tilde{\psi}_\delta = \frac{1}{\delta}H_\delta, \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_B^0\tilde{\varphi}_\delta)}{\partial\nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_B^0\tilde{\psi}_\delta)}{\partial\nu}\Big|_+ = \frac{1}{\delta\mu_0}\frac{\partial H_\delta}{\partial\nu}, \end{cases} \quad \text{on } \partial B. \tag{4.4}$$

Here the constant $\tau$ is defined in (2.1). Recall that according to [11] the integral equation

$$\mathcal{S}_B^0 h + \tau\int_{\partial B} h = g$$

has a unique solution $h \in L^2(\partial B)$ for any $g \in H^1(\partial B)$. Moreover, there exists a constant $C$ independent of $\delta$ such that

$$\left\|\mathcal{S}_B^{k\delta}h - \mathcal{S}_B^0 h - \tau\int_{\partial B} h\right\|_{H^1(\partial B)} \leqslant C\big(\delta^2|\log\delta|\big)\|h\|_{L^2(\partial B)}.$$

Applying the results of [11], we can immediately prove that $(\tilde{\varphi}_\delta - \varphi_\delta)/\|\varphi_\delta\|_{L^2(\partial B)}$ and $(\tilde{\psi}_\delta - \psi_\delta)/\|\psi_\delta\|_{L^2(\partial B)}$ converge to zero as $\delta$ goes to zero. But

$$\|\tilde{\varphi}_\delta\|_{L^2(\partial B)} + \|\tilde{\psi}_\delta\|_{L^2(\partial B)} \leqslant C\delta^{-1}\|H_\delta\|_{H^1(\partial B)},$$

for some constant $C$ independent of $\delta$ and thus, after scaling back, (4.1) holds. □

Let $H$ be the function defined in (3.1). Fix an integer $n$ and define

$$H_n(x) = \sum_{|\alpha|=0}^{n}\frac{\partial^\alpha H(z)}{\alpha!}(x - z)^\alpha,$$

and let $(\varphi_n, \psi_n)$ be the unique solution of

$$\begin{cases} \mathcal{S}_D^k\varphi_n - \mathcal{S}_D^{k_0}\psi_n = H_{n+1}, \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_D^k\varphi_n)}{\partial\nu}\Big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^{k_0}\psi_n)}{\partial\nu}\Big|_+ = \frac{1}{\mu_0}\frac{\partial H_{n+1}}{\partial\nu}, \end{cases} \quad \text{on } \partial D. \tag{4.5}$$

Then $(\varphi - \varphi_n, \psi - \psi_n)$ is the unique solution of (4.5) with the right-hand sides defined by $H - H_{n+1}$. Therefore, by (4.1), we get

$$\begin{aligned} &\|\varphi - \varphi_n\|_{L^2(\partial D)} + \|\psi - \psi_n\|_{L^2(\partial D)} \\ &\quad \leqslant C\big(\delta^{-1}\|H - H_{n+1}\|_{L^2(\partial D)} + \|\nabla(H - H_{n+1})\|_{L^2(\partial D)}\big). \end{aligned} \tag{4.6}$$

By the definition of $H_{n+1}$, we have

$$\|H - H_{n+1}\|_{L^2(\partial D)} \leqslant C|\partial D|^{1/2}\|H - H_{n+1}\|_{L^\infty(\partial D)} \leqslant C|\partial D|^{1/2}\delta^{n+2}\|H\|_{\mathcal{C}^{n+2}(\bar{D})},$$

and

$$\big\|\nabla(H - H_{n+1})\big\|_{L^2(\partial D)} \leqslant C|\partial D|^{1/2}\delta^{n+1}\|H\|_{\mathcal{C}^{n+1}(\bar{D})}.$$

It then follows from (4.6) and Proposition 3.2 that

$$\|\varphi - \varphi_n\|_{L^2(\partial D)} + \|\psi - \psi_n\|_{L^2(\partial D)} \leqslant C|\partial D|^{1/2}\delta^{n+1}. \tag{4.7}$$

By (3.9), we obtain

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \frac{\partial(G_D\psi_n)}{\partial \nu}(x) + \frac{\partial(G_D(\psi - \psi_n))}{\partial \nu}(x), \quad x \in \partial\Omega. \tag{4.8}$$

Since $\operatorname{dist}(D, \partial\Omega) \geqslant c_0$, $\sup_{x\in\partial\Omega,\, y\in\partial D}|\frac{\partial G}{\partial \nu}(x, y)| \leqslant C$ for some $C$. Hence, for each $x \in \partial\Omega$, we have from (4.7),

$$\left|\frac{\partial(G_D(\psi - \psi_n))}{\partial \nu}(x)\right| \leqslant \left[\int_{\partial D}\left|\frac{\partial G(x, y)}{\partial \nu(x)}\right|^2 d\sigma(y)\right]^{1/2}\|\psi - \psi_n\|_{L^2(\partial D)}$$
$$\leqslant C|\partial D|^{1/2}|\partial D|^{1/2}\delta^{n+1} \leqslant C'\delta^{n+d},$$

where $C$ and $C'$ are independent of $x \in \partial\Omega$ and $\delta$. Thus we conclude that

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \frac{\partial(G_D\psi_n)}{\partial \nu}(x) + O\big(\delta^{n+d}\big), \quad \text{uniformly in } x \in \partial\Omega. \tag{4.9}$$

For each multi-index $\alpha$, define $(\varphi_\alpha, \psi_\alpha)$ to be the unique solution to

$$\begin{cases} \mathcal{S}_B^{k\delta}\varphi_\alpha - \mathcal{S}_B^{k_0\delta}\psi_\alpha = x^\alpha, \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_B^{k\delta}\varphi_\alpha)}{\partial \nu}\big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_B^{k_0\delta}\psi_\alpha)}{\partial \nu}\big|_+ = \frac{1}{\mu_0}\frac{\partial x^\alpha}{\partial \nu}, \end{cases} \quad \text{on } \partial B. \tag{4.10}$$

Then, we claim that

$$\varphi_n(x) = \sum_{|\alpha|=0}^{n+1}\delta^{|\alpha|-1}\frac{\partial^\alpha H(z)}{\alpha!}\varphi_\alpha\big(\delta^{-1}(x - z)\big),$$
$$\psi_n(x) = \sum_{|\alpha|=0}^{n+1}\delta^{|\alpha|-1}\frac{\partial^\alpha H(z)}{\alpha!}\psi_\alpha\big(\delta^{-1}(x - z)\big).$$

In fact, they follow from the uniqueness of the solution to the integral equation (2.5) and the relation

$$\mathcal{S}_D^{k_0}\left(\sum_{|\alpha|=0}^{n+1}\delta^{|\alpha|-1}\frac{\partial^\alpha H(z)}{\alpha!}\varphi_\alpha\big(\delta^{-1}(\cdot - z)\big)\right)(x)$$
$$= \sum_{|\alpha|=0}^{n+1}\delta^{|\alpha|}\frac{\partial^\alpha H(z)}{\alpha!}\big(\mathcal{S}_B^{k_0\delta}\varphi_\alpha\big)\big(\delta^{-1}(x - z)\big),$$

for $x \in \partial D$. It then follows from (4.9) that

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \sum_{|\alpha|=0}^{n+1} \delta^{|\alpha|-1} \frac{\partial^\alpha H(z)}{\alpha!} \frac{\partial}{\partial \nu} G_D\bigl(\psi_\alpha\bigl(\delta^{-1}(\cdot - z)\bigr)\bigr)(x) + O\bigl(\delta^{n+d}\bigr), \tag{4.11}$$

uniformly in $x \in \partial \Omega$. Note that

$$\begin{aligned} G_D\bigl(\psi_\alpha\bigl(\delta^{-1}(\cdot - z)\bigr)\bigr)(x) &= \int_{\partial D} G(x, y)\psi_\alpha\bigl(\delta^{-1}(y - z)\bigr)\, d\sigma(y) \\ &= \delta^{d-1} \int_{\partial B} G(x, \delta w + z)\psi_\alpha(w)\, d\sigma(w). \end{aligned}$$

Moreover, for $x$ near $\partial\Omega$, $z \in \Omega$, $w \in \partial B$, and sufficiently small $\delta$, we have

$$G(x, \delta w + z) = \sum_{|\beta|=0}^{\infty} \frac{\delta^{|\beta|}}{\beta!} \partial_z^\beta G(x, z) w^\beta.$$

Therefore, we get

$$G_D\bigl(\psi_\alpha\bigl(\delta^{-1}(\cdot - z)\bigr)\bigr)(x) = \sum_{|\beta|=0}^{\infty} \frac{\delta^{|\beta|+d-1}}{\beta!} \partial_z^\beta G(x, z) \int_{\partial B} w^\beta \psi_\alpha(w)\, d\sigma(w).$$

Define, for multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^d$,

$$W_{\alpha\beta} := \int_{\partial B} w^\beta \psi_\alpha(w)\, d\sigma(w). \tag{4.12}$$

Then we obtain the following theorem from (4.11).

**Theorem 4.2.** *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n-|\beta|+1} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha H(z) \frac{\partial \partial_z^\beta G(x, z)}{\partial \nu(x)} W_{\alpha\beta} + O\bigl(\delta^{n+d}\bigr), \tag{4.13}$$

*where the remainder $O(\delta^{d+n})$ is dominated by $C\delta^{d+n}\|f\|_{H^{1/2}(\partial\Omega)}$ for some $C$ independent of $x \in \partial\Omega$.*

In view of (3.9), we obtain the following expansion:

$$\frac{\partial (G_D \psi)}{\partial \nu}(x) = \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n-|\beta|+1} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha H(z) \frac{\partial \partial_z^\beta G(x, z)}{\partial \nu(x)} W_{\alpha\beta} + O\bigl(\delta^{n+d}\bigr). \tag{4.14}$$

Observe that $\psi_\alpha$, and hence $W_{\alpha\beta}$ depends on $\delta$, and so does $H$. Thus the formula (4.13) is not a genuine asymptotic formula. However, since it is simple and has some potential applicability in solving the inverse problem for Helmholtz equation, we made a record of it as a theorem.

Observe that by the definition (4.10) of $\psi_\alpha$, $\|\psi_\alpha\|_{L^2(\partial B)}$ is bounded, and hence

$$|W_{\alpha\beta}| \leqslant C_{\alpha\beta}, \quad \forall\alpha, \beta, \tag{4.15}$$

where $C_{\alpha\beta}$ is independent of $\delta$. Since $\delta$ is small, one can derive an asymptotic expansion of $(\varphi_\alpha, \psi_\alpha)$ using their definition (4.10). Let us briefly explain this. Let us for simplicity assume that $d = 3$. Let

$$T_\delta \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} \mathcal{S}_B^{k\delta} f - \mathcal{S}_B^{k_0\delta} g \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_B^{k\delta} f)}{\partial\nu}\big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_B^{k_0\delta} g)}{\partial\nu}\big|_+ \end{bmatrix},$$

and let $T_0$ be the operator when $\delta = 0$. Then the solution $(\varphi_\alpha, \psi_\alpha)$ of the integral equation (4.10) is given by

$$\begin{bmatrix} \varphi_\alpha \\ \psi_\alpha \end{bmatrix} = \left[I + T_0^{-1}(T_\delta - T_0)\right]^{-1} T_0^{-1} \begin{bmatrix} x^\alpha \\ \frac{1}{\mu_0}\frac{\partial x^\alpha}{\partial\nu} \end{bmatrix}. \tag{4.16}$$

By expanding $T_\delta - T_0$ in a power series of $\delta$, one can derive the expansions of $\psi_\alpha$ and $W_{\alpha\beta}$. Let, for $\alpha, \beta \in \mathbb{N}^d$, $(\hat{\varphi}_\alpha, \hat{\psi}_\alpha)$ be the leading term of the expansion of $(\varphi_\alpha, \psi_\alpha)$. Then $(\hat{\varphi}_\alpha, \hat{\psi}_\alpha)$ is solution of the integral equation

$$\begin{cases} \mathcal{S}_B^0\hat{\varphi}_\alpha - \mathcal{S}_B^0\hat{\psi}_\alpha = x^\alpha, \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_B^0\hat{\varphi}_\alpha)}{\partial\nu}\big|_- - \frac{1}{\mu_0}\frac{\partial(\mathcal{S}_B^0\hat{\psi}_\alpha)}{\partial\nu}\big|_+ = \frac{1}{\mu_0}\frac{\partial x^\alpha}{\partial\nu}, \end{cases} \quad \text{on } \partial B. \tag{4.17}$$

As a simplest case, let us now take $n = 1$ in (4.13) to find the leading order term in the asymptotic expansion of $\frac{\partial u}{\partial\nu}|_{\partial\Omega}$ as $\delta \to 0$. We first investigate the dependence of $W_{\alpha\beta}$ on $\delta$ for $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$. If $|\alpha| \leqslant 1$, then both sides of the first equation in (4.17) is harmonic in $B$, and hence

$$\mathcal{S}_B^0\hat{\varphi}_\alpha - \mathcal{S}_B^0\hat{\psi}_\alpha = x^\alpha \quad \text{in } B.$$

Therefore we get

$$\frac{\partial(\mathcal{S}_B^0\hat{\varphi}_\alpha)}{\partial\nu}\bigg|_- - \frac{\partial(\mathcal{S}_B^0\hat{\psi}_\alpha)}{\partial\nu}\bigg|_- = \frac{\partial x^\alpha}{\partial\nu} \quad \text{on } \partial B.$$

This identity together with the second equation in (4.17) yields

$$\frac{\mu}{\mu_0}\frac{\partial(\mathcal{S}_B^0\hat{\psi}_\alpha)}{\partial\nu}\bigg|_+ - \frac{\partial(\mathcal{S}_B^0\hat{\psi}_\alpha)}{\partial\nu}\bigg|_- = \left(1 - \frac{\mu}{\mu_0}\right)\frac{\partial x^\alpha}{\partial\nu}.$$

In view of the relation (2.2), we have

$$\frac{\mu}{\mu_0}\left(\frac{1}{2}I + \mathcal{K}_B^*\right)\hat{\psi}_\alpha - \left(-\frac{1}{2}I + \mathcal{K}_B^*\right)\hat{\psi}_\alpha = \left(1 - \frac{\mu}{\mu_0}\right)\frac{\partial x^\alpha}{\partial\nu},$$

where $\mathcal{K}_B^*$ is the operator defined in (2.4) when $k = 0$. Therefore, we have

$$\hat{\psi}_\alpha = (\lambda I - \mathcal{K}_B^*)^{-1}\left(\frac{\partial x^\alpha}{\partial \nu}\bigg|_{\partial B}\right), \qquad \lambda := \frac{\mu/\mu_0 + 1}{2(1-\mu/\mu_0)} = \frac{\mu_0/\mu + 1}{2(\mu_0/\mu - 1)}, \tag{4.18}$$

where invertibility of the operator $\lambda I - \mathcal{K}_B^*$ is proved in [8]. Observe that if $|\alpha| = 0$, then

$$\hat{\psi}_\alpha = 0 \quad \text{and} \quad \mathcal{S}_B^0 \hat{\varphi}_\alpha = 1. \tag{4.19}$$

Hence we obtain $\psi_\alpha = O(\delta)$ and $\mathcal{S}_B^{k\delta}\varphi_\alpha = 1 + O(\delta)$. Moreover, since $\mathcal{S}_B^{k\delta}\varphi_\alpha$ depends on $\delta$ analytically and $(\Delta + k^2\delta^2)\mathcal{S}_B^{k\delta}\varphi_\alpha = 0$ in $B$, we conclude that

$$\psi_\alpha = O(\delta) \quad \text{and} \quad \mathcal{S}_B^{k\delta}\varphi_\alpha = 1 + O\big(\delta^2\big), \quad |\alpha| = 0. \tag{4.20}$$

It also follows from (4.18) that if $|\alpha| = |\beta| = 1$, then

$$W_{\alpha\beta} = \int_{\partial B} x^\beta \big(\lambda I - \mathcal{K}_B^*\big)^{-1}\left(\frac{\partial x^\alpha}{\partial \nu}\bigg|_{\partial B}\right)(x)\, d\sigma + O(\delta). \tag{4.21}$$

According to [1], the first quantity in the right-hand side of (4.18) is the polarization tensor defined by $M = M(\mu/\mu_0) := (m_{\alpha\beta}(\mu/\mu_0))$ where

$$m_{\alpha\beta}\left(\frac{\mu}{\mu_0}\right) = \left(1 - \frac{\mu}{\mu_0}\right)\left(\delta_{\alpha\beta}|B| + \left(\frac{\mu}{\mu_0} - 1\right)\int_{\partial B} y^\beta \frac{\partial \theta_\alpha}{\partial \nu}\bigg|_+ (y)\, d\sigma(y)\right), \tag{4.22}$$

and $\theta_\alpha$ is the unique solution of the following transmission problem:

$$\begin{cases} \Delta\theta_\alpha(x) = 0, & x \in B \cup \mathbb{R}^d \setminus \bar{B}, \\ \theta_\alpha|_+ - \theta_\alpha|_- = 0 & \text{on } \partial B, \\ \frac{\partial \theta_\alpha}{\partial \nu}\big|_+ - \frac{\mu}{\mu_0}\frac{\partial \theta_\alpha}{\partial \nu}\big|_- = \nu_\alpha & \text{on } \partial B, \\ \theta_\alpha(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Here $\nu_\alpha = \nu \cdot \alpha$ is the $\alpha$-component of the normal vector $\nu$. In summary, we obtained that

$$W_{\alpha\beta} = m_{\alpha\beta}\left(\frac{\mu}{\mu_0}\right) + O(\delta), \quad |\alpha| = |\beta| = 1. \tag{4.23}$$

Suppose that either $\alpha = 0$ or $\beta = 0$. By (2.2) and (4.10), we have

$$\psi_\alpha = \frac{\partial(\mathcal{S}_B^{k_0\delta}\psi_\alpha)}{\partial \nu}\bigg|_+ - \frac{\partial(\mathcal{S}_B^{k_0\delta}\psi_\alpha)}{\partial \nu}\bigg|_- = \frac{\mu_0}{\mu}\frac{\partial(\mathcal{S}_B^{k\delta}\varphi_\alpha)}{\partial \nu}\bigg|_- - \frac{\partial x^\alpha}{\partial \nu} - \frac{\partial(\mathcal{S}_B^{k_0\delta}\psi_\alpha)}{\partial \nu}\bigg|_- . \tag{4.24}$$

It then follows from divergence theorem that

$$\begin{aligned} \int_{\partial B} x^\beta \psi_\alpha\, d\sigma = {} & -k^2\delta^2 \frac{\mu_0}{\mu}\int_B x^\beta \mathcal{S}_B^{k\delta}\varphi_\alpha\, dx + k_0^2\delta^2 \int_B x^\beta \mathcal{S}_B^{k_0\delta}\psi_\alpha\, dx \\ & + \frac{\mu_0}{\mu}\int_{\partial B} \frac{\partial x^\beta}{\partial \nu}\mathcal{S}_B^{k\delta}\varphi_\alpha\, d\sigma - \int_{\partial B} \frac{\partial x^\beta}{\partial \nu}\mathcal{S}_B^{k_0\delta}\psi_\alpha\, d\sigma. \end{aligned} \tag{4.25}$$

From (4.25), we can observe the following:

$$W_{\alpha\beta} = -k^2\delta^2\frac{\mu_0}{\mu}|B| + O\big(\delta^3\big) = -\delta^2\omega^2\epsilon\mu_0|B| + O\big(\delta^3\big), \quad |\alpha| = |\beta| = 0, \tag{4.26}$$

$$W_{\alpha\beta} = O\big(\delta^2\big), \quad |\alpha| = 1,\ |\beta| = 0, \tag{4.27}$$

$$W_{\alpha\beta} = O\big(\delta^2\big), \quad |\alpha| = 0,\ |\beta| = 1. \tag{4.28}$$

In fact, (4.26) and (4.28) follows from (4.20) and (4.25), and (4.27) immediately follows from (4.25). As a consequence of (4.27), (4.28), and (4.14), we obtain

$$\frac{\partial(G_D\psi)}{\partial\nu}(x) = O\big(\delta^d\big), \quad \text{uniformly on } x \in \partial\Omega.$$

Since the center $z$ is apart from $\partial\Omega$, it follows from (3.16) that

$$\big|H(z) - u_0(z)\big| + \big|\nabla H(z) - \nabla u_0(z)\big| = O\big(\delta^d\big). \tag{4.29}$$

We now consider the case $|\alpha| = 2$ and $|\beta| = 0$. In this case, one can show using (4.24) that

$$\int_{\partial B} \psi_\alpha\, d\sigma = -\int_B \Delta x^\alpha\, dx + O\big(\delta^2\big).$$

Therefore, if $|\beta| = 0$, then

$$\sum_{|\alpha|=2} \frac{1}{\alpha!\beta!}\partial^\alpha H(z) W_{\alpha\beta} = -\Delta H(z)|B| + O\big(\delta^2\big) = k_0^2 H(z)|B| + O\big(\delta^2\big). \tag{4.30}$$

So (4.13) together with (4.23)–(4.30) yields the following expansion formula: for $d = 3$ and for any $x \in \partial\Omega$,

$$\frac{\partial u}{\partial\nu}(x) = \frac{\partial u_0}{\partial\nu}(x) + \delta^d\left(\nabla u_0(z) M\left(\frac{\mu}{\mu_0}\right)\frac{\partial\nabla_z G(x,z)}{\partial\nu(x)} + \omega^2\mu_0(\epsilon - \epsilon_0)|B|u_0(z)\frac{\partial G(x,z)}{\partial\nu(x)}\right) + O\big(\delta^{d+1}\big), \tag{4.31}$$

where $M = (m_{\alpha\beta})$ is the polarization tensor defined in (4.22).

Before returning to (4.13) let us make the following important remark. In [1] new concepts of higher order polarization tensors are introduced. These concepts generalize that of classical Pólya-Szegö polarization tensors. These generalized polarization tensors (GPT's) appear naturally in higher order asymptotics of the steady-state voltage potentials under the perturbation of conductor by dielectric inhomogeneities of small diameter. They seem to carry out significant information on the small dielectric inhomogeneities [3]. In this paper, the tensors $W_{\alpha\beta}$ play similar role. As defined in [1] the GPT's are given for $\alpha, \beta \in \mathbb{N}^d$ by

$$M_{\alpha\beta} := \int_{\partial B} w^\beta \hat{\psi}_\alpha(w)\, d\sigma(w),$$

where $\hat{\psi}_\alpha$ is defined by (4.17). The following result makes the connection between $W_{\alpha\beta}$ and $M_{\alpha\beta}$. Its proof is immediate.

**Lemma 4.3.** *Suppose that $a_\alpha$ are constants such that $\sum_\alpha a_\alpha w^\alpha$ is a harmonic polynomial. Then*

$$\sum_\alpha a_\alpha W_{\alpha\beta} \to \sum_\alpha a_\alpha M_{\alpha\beta} \quad \text{as } \delta \to 0.$$

We also note that in the two-dimensional case we should replace the operator $T_\delta$ by

$$\tilde{T}_\delta \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} \mathcal{S}_B^{k\delta} f + \tau \int_{\partial B} f - \mathcal{S}_B^{k_0\delta} g - \tau \int_{\partial B} g \\ \frac{1}{\mu} \frac{\partial(\mathcal{S}_B^{k\delta} f)}{\partial \nu}\big|_- - \frac{1}{\mu_0} \frac{\partial(\mathcal{S}_B^{k_0\delta} g)}{\partial \nu}\big|_+ \end{bmatrix},$$

and $T_0$ by $\tilde{T}_0$ (the $\tilde{T}_\delta$ operator when $\delta = 0$). The results of [11] allow us again to handle the problem in the two-dimensional case. Instead of equation $\mathcal{S}_B^0 \hat{\varphi}_\alpha = 1$ for $|\alpha| = 0$ in (4.19) we deal in this case with the well-posed equation $\mathcal{S}_B^0 \hat{\varphi}_\alpha - \tau \int_{\partial B} \hat{\varphi}_\alpha = 1$. The zero mean-value property of $\hat{\psi}_\alpha$ for $|\alpha| = 1$ can also be easily be deduced from the system of integral equations satisfied by $(\hat{\varphi}_\alpha, \hat{\psi}_\alpha)$ using the fact that $x^\alpha$ is harmonic for $|\alpha| = 1$. So, in the two-dimensional case, we obtain the following expansion formula of Vogelius–Volkov [15]: for any $x \in \partial\Omega$,

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \delta^2 \left( \nabla u_0(z) M\left(\frac{\mu}{\mu_0}\right) \frac{\partial \nabla_z G(x,z)}{\partial \nu(x)} + \omega^2 \mu_0 (\epsilon - \epsilon_0) |B| u_0(z) \frac{\partial G(x,z)}{\partial \nu(x)} \right) + o(\delta^2), \tag{4.32}$$

where $M = (m_{\alpha\beta})$ is the polarization tensor defined in (4.22). In fact, in [15], the formula is expressed in terms of 'free space' Green function $\Phi_k$ instead of the Green function $G$. However, those two formula are the same as one can see using the relation (3.11).

Observing now that the formula (4.13) still contains $\partial^\alpha H$ factors, the remaining task is to convert (4.13) to a formula given solely by $u_0$ and its derivatives. Substitution of (4.14) into (3.16) yields that, for any $x \in \Omega$,

$$H(x) = u_0(x) - \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha H(z) S_\Omega^{k_0} \left( \frac{\partial \partial_z^\beta G(x,z)}{\partial \nu(x)} \right) W_{\alpha\beta} + O(\delta^{n+d}). \tag{4.33}$$

In (4.33) the remainder $O(\delta^{n+d})$ is uniform in the $\mathcal{C}^n$ norm on any compact subset of $\Omega$ for any $n$ and therefore

$$(\partial^\gamma H)(z) + \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n+1-|\beta|} \delta^{|\alpha|+|\beta|} \partial^\alpha H(z) P_{\alpha\beta\gamma} = (\partial^\gamma u_0)(z) + O(\delta^{d+n}), \tag{4.34}$$

for all $\gamma \in \mathbb{N}^d$ with $|\gamma| \leqslant n+1$ where

$$P_{\alpha\beta\gamma} = \frac{1}{\alpha!\beta!} W_{\alpha\beta} \partial^\gamma S_\Omega^{k_0} \left( \frac{\partial \partial_z^\beta G(\cdot,z)}{\partial \nu(x)} \right) \bigg|_{x=z}. \tag{4.35}$$

Following [1], define the operator $\mathcal{P}_\delta$ by

$$\mathcal{P}_\delta : (w_\gamma)_{\gamma\in\mathbb{N}^d,\,|\gamma|\leqslant n} \mapsto \left( w_\gamma + \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n+1-|\beta|} \delta^{|\alpha|+|\beta|} w_\alpha P_{\alpha\beta\gamma} \right)_{\gamma\in\mathbb{N}^d,\ |\gamma|\leqslant n}.$$

Observe from (4.16) that $\mathcal{P}_\delta$ can be written as

$$\mathcal{P}_\delta = I + \delta^d \mathcal{P}_1 + \cdots + \delta^{n+d-1}\mathcal{P}_{n-1} + O\big(\delta^{n+d}\big).$$

Defining as in [1] $\mathcal{Q}_p,\ p = 1, \ldots, n-1$, by

$$\big(I + \delta^d \mathcal{P}_1 + \cdots + \delta^{n+d-1}\mathcal{P}_{n-1}\big)^{-1} = I + \delta^d \mathcal{Q}_1 + \cdots + \delta^{n+d-1}\mathcal{Q}_{n-1} + O\big(\delta^{n+d}\big), \tag{4.36}$$

we finally obtain that

$$\big(\big(\partial^\alpha H\big)(z)\big)_{\alpha\in\mathbb{N}^d,\,|\alpha|\leqslant n+1} = \left(I + \sum_{p=1}^{n} \delta^{d+p-1}\mathcal{Q}_p\right)\big(\big(\partial^\alpha u_0\big)(z)\big)_{\alpha\in\mathbb{N}^d,\,|\alpha|\leqslant n+1} + O\big(\delta^{d+n}\big), \tag{4.37}$$

which yields the main result of this paper.

**Theorem 4.4.** *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \times \left[ \left( \left( I + \sum_{p=1}^{n+2-|\alpha|-|\beta|-d} \delta^{d+p-1}\mathcal{Q}_p \right) \big(\partial^\gamma u_0(z)\big) \right)_\alpha \frac{\partial \partial_z^\beta G(x,z)}{\partial \nu(x)} W_{\alpha\beta} \right] + O\big(\delta^{n+d}\big), \tag{4.38}$$

*where the remainder $O(\delta^{d+n})$ is dominated by $C\delta^{d+n}\|f\|_{H^{1/2}(\partial\Omega)}$ for some $C$ independent of $x \in \partial\Omega$.*

When $n = d$, we have a simpler formula

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \delta^{d-2} \sum_{|\beta|=0}^{d+1} \sum_{|\alpha|=0}^{d+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha u_0(z) \frac{\partial \partial_z^\beta G(x,z)}{\partial \nu(x)} W_{\alpha\beta} + O\big(\delta^{2d}\big). \tag{4.39}$$

Let us now consider the case when there are several well-separated inclusions. The inhomogeneity $D$ takes the form $\bigcup_{s=1}^m (\delta B_s + z_s)$. The magnetic permeability and electric permittivity of the inclusion $\delta B_s + z_s$ are $\mu_s$ and $\epsilon_s$, $s = 1, \ldots, m$. By iterating the formula (4.39), we can derive the following theorem.

**Theorem 4.5.** *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \delta^{d-2} \sum_{s=1}^{m} \sum_{|\beta|=0}^{d+1} \sum_{|\alpha|=0}^{d+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha u_0(z) \frac{\partial \partial_z^\beta G(x, z)}{\partial \nu(x)} W_{\alpha\beta}^s + O(\delta^{2d}). \tag{4.40}$$

*Here $W_{\alpha\beta}^s$ is defined by* (4.12) *with $B, \mu, \epsilon$ replaced by $B_s, \mu_s, \epsilon_s$.*

We conclude this paper by making one final remark. In this paper, we only derive the asymptotic formula for the solution to the Dirichlet problem. However, by the same method, one can derive an asymptotic formula for the Neumann problem as well.

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