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The Extended Watson Lemma and the Asymptotic Expansion of the L^p Norm

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1. INTRODUCTION

For a real analytic function θ which satisfies certain conditions, Watson [6] derived a series in p which expresses the asymptotic behavior of the integral

$$\int_0^b e^{-px^r}\theta(x)\,dx$$

as $p \to \infty$. The terms of that series depend on the coefficients in the Maclaurin expansion of θ.

This paper extends Watson's work, under certain additional conditions on θ, to the integral

$$\int_0^b e^{-px^r}\theta^p(x)\,dx.$$

An immediate consequence of the new result is an asymptotic series for the pth power of the L^p norm of an analytic function. As an application, this series is used to extend the formula for the asymptotic behavior of the pth power integrals of sine and of cosine to noninteger values of p.

Finally, some results are presented to describe the nature of the occurrences of a given Taylor series coefficient of an analytic function in the asymptotic expansion for the pth power of that function's L^p norm. These theorems are useful in developing a characterization of the strict approximation of Descloux [4] (cf. [1], for which the present paper corrects a computational error).

2. The Extended Watson Lemma

In 1918, Watson proved the following result.

THEOREM 1. *Let $\theta(t)$ be an analytic function of t, regular in a neighborhood of the origin and let*

$$\theta(t) = \sum_{m=0}^{\infty} a_m t^m$$

be its Taylor series. Also let

$$|\theta(t)| \leqslant K e^{\beta t^r}, \qquad 0 \leqslant t \leqslant b,$$

where K, β, and r are positive numbers independent of t. Then

$$\int_0^b e^{-zt^r} \theta(t)\, dt \sim (1/r) \sum_{m=0}^{\infty} a_m \Gamma((m+1)/r)\, z^{-(m+1)/r}$$

as $|z| \to \infty$ in a closed sector excluding the negative real axis.

A proof of Watson's lemma can be found in [2] or [3]. It follows the original proof which appeared in [6].

We now prove an extension of Theorem 1 under some special hypotheses.

THEOREM 2. *Suppose $\theta(x)$ is an analytic function for $|x| < a + \delta$ where $a > 0$, $\delta > 0$. Let $\theta(x)$ have the Taylor series*

$$\theta(x) = 1 + \sum_{m=k}^{\infty} a_m x^m \qquad (k \geqslant 1).$$

Suppose one can pick $A \leqslant a$ and $b > A$ such that, on $[0, A]$, $|\theta(x) - 1| < \frac{1}{2}$ and on $[A, b]$

$$|\theta(x)| \leqslant K e^{\beta x^r},$$

where K, β, and r are positive and independent of x. Write

$$\theta^p(x) = 1 + \sum_{m=k}^{\infty} b_m x^m.$$

Then, if $r < k$, if $\alpha \geqslant \beta$ for b finite and $\alpha > \beta$ for b infinite and if $\ln K < (\alpha - \beta) A^r$, *then*

$$\int_0^b e^{-p\alpha x^r} \theta^p(x)\, dx \sim \frac{1}{r} \Gamma\left(\frac{1}{r}\right) (\alpha p)^{-1/r} + \frac{1}{r} \sum_{m=k}^{\infty} b_m \Gamma\left(\frac{m+1}{r}\right) (\alpha p)^{-(m+1)/r}$$

as $p \to \infty$ through positive reals.

The proof of Theorem 2 depends on the following four lemmas.

LEMMA 1. $\int_0^\infty e^{-p\alpha x^r} x^m \, dx = (1/r)\, \Gamma((m+1)/r)(\alpha p)^{-(m+1)/r}$.

Proof. This useful result is easily proved by the change of variables $t = p\alpha x^r$.

LEMMA 2. *With A, K, β, and r as in the statement of Theorem 2,*

$$\left| \int_A^b e^{-p\alpha x^r} \theta^p(x)\, dx \right| \to 0$$

faster than any power of $1/p$ as $p \to \infty$.

Proof.

$$\begin{aligned}
\left| \int_A^b e^{-p\alpha x^r} \theta^p(x)\, dx \right| & \\
&\leqslant \int_A^b e^{-p\alpha x^r} K^p e^{p\beta x^r}\, dx \\
&= K^p \int_0^{b-A} e^{-p(\alpha-\beta)(y+A)^r}\, dy, \qquad \text{where} \qquad b - A = \infty \text{ if } b = \infty, \\
&\leqslant K^p e^{-p(\alpha-\beta)a^r} \int_0^{b-A} e^{-p(\alpha-\beta)y^r}\, dy \\
&\leqslant K^p e^{-p(\alpha-\beta)a^r} \frac{1}{r}\, \Gamma\left(\frac{1}{r}\right) (\alpha-\beta)^{-1/r} p^{-1/r} \qquad \text{if} \quad b = \infty \text{ (Lemma 1)}, \\
&\leqslant K^p e^{-p(\alpha-\beta)a^r} b \qquad \text{if} \quad b < \infty.
\end{aligned}$$

In either case, as $p \to \infty$, the expression approaches 0 faster than any power of $1/p$ because $0 < K/e^{(\alpha-\beta)a^r} < 1$.

LEMMA 3. *On $[0, A]$ where $|\theta(x) - 1| < \frac{1}{2}$,*

$$\theta^p(x) = 1 + \sum_{i=1}^{[p]} \frac{p(p-1)\cdots(p-i+1)}{i!} y^i + R_p$$

with $y = \theta(x) - 1$ and $|\int_0^A e^{-p\alpha x^r} R_p\, dx| \to 0$ faster than any power of $1/p$ as $p \to \infty$.

Proof. Write $\theta^p(x) = (1+y)^p$ and use Taylor's theorem with remainder to show $R_p = 0$ if p is an integer. Otherwise

$$\begin{aligned}
|R_p| &= \frac{p(p-1)\cdots(p-[p])}{([p]+1)!} |1+z|^{p-[p]-1} |y|^{[p]+1} \\
&< 1^{[p]+1} 2^{[p]+1-p} 2^{-[p]-1} = 2^{-p},
\end{aligned}$$

since

$$\frac{p-i}{[p]+1-i} < 1 \quad (0 \leqslant i \leqslant [p]) \qquad \text{and} \qquad |z| < |y| < \frac{1}{2}.$$

Furthermore,

$$\left| \int_0^A e^{-p\alpha x^p} R_p \, dx \right| < A \, | R_p | < A 2^{-p},$$

which approaches 0 in the desired fashion.

LEMMA 4. *Fix M a positive integer. Then, for all integers i such that $M/(k-r) \leqslant i \leqslant p$,*

$$\frac{p(p-1)\cdots(p-i+1)}{i!} \int_0^A e^{-p\alpha x^r} y^i \, dx = o(p^{-M/r}).$$

Proof. Since $y = a_k x^k + a_{k+1} x^{k+1} + \cdots$ and $A \leqslant a$, there exists c such that $|y| < cx^k$ on $[0, A]$. Then

$$\begin{aligned}
&\left| \frac{p(p-1)\cdots(p-i+1)}{i!} \int_0^A e^{-p\alpha x^r} y^i \, dx \right| \\
&\leqslant \frac{p^i}{i!} c^i \int_0^A e^{-p\alpha x^r} x^{ik} \, dx \\
&< \frac{p^i c^i}{i!\, r} \Gamma\left(\frac{ik+1}{r}\right) (\alpha p)^{-(ik+1)/r} \\
&= \frac{c^i}{i!\, r} \Gamma\left(\frac{ik+1}{r}\right) \alpha^{-(ik+1)/r} p^{-i(k-r)/r-1/r} \\
&= o(p^{-M/r}) \qquad \text{for} \quad i(k-r) \geqslant M.
\end{aligned}$$

Proof of Theorem 2. To complete the proof of Theorem 2, we need only examine the integrals of the terms in

$$1 + \sum_{i=1}^{[M/(k-1)]-1} \frac{p(p-1)\cdots(p-i+1)}{i!} y^i.$$

First we write the product

$$p(p-1)\cdots(p-i+1) = \sum_{j=1}^{i} \sigma_{i-j,i} p^j,$$

where $\sigma_{v,i}$ is the vth elementary symmetric function on $-1, -2, \ldots, 1-i$.

Let $(y^i)_j$ be the polynomial of largest degree less than $M + jr$ which agrees with the sum of the leading terms of the power series expansion of y^i. On $[0, A]$, we can find $c_{i,j}$ with

$$| y^i - (y^i)_j | < c_{i,j} x^{M+jr}.$$

Then,

$$\sum_{j=1}^{i} \sigma_{i-j,i} p^i \int_0^A e^{-p\alpha x^r} y^i \, dx = \sum_{j=1}^{i} \sigma_{i-j,i} p^i \int_0^A e^{-p\alpha x^r} (y^i)_j \, dx + T_i,$$

where

$$\begin{aligned} | T_i | &\leqslant \sum_{j=1}^{i} \sigma_{i-j,i} p^i \int_0^A e^{-p\alpha x^r} c_{i,j} x^{M+jr} \, dx \\ &< \sum_{j=1}^{i} \sigma_{i-j,i} c_{i,j} \frac{1}{r} \Gamma\left(\frac{M+jr}{r}\right) \alpha^{-(M+jr+1)/r} p^{j-(M+jr+1)/r} \\ &= o(p^{-M/r}). \end{aligned}$$

The final result now follows by applying Theorem 1 to just those terms of

$$\theta^p(x) = 1 + \sum_{m=k}^{\infty} b_m x^m$$

of the form $Bp^j x^m$ for which $m - jr < M$. For such terms,

$$\begin{aligned} Bp^j \int_0^A e^{-p\alpha x^r} x^m \, dx &\sim Bp^j \int_0^\infty e^{-p\alpha x^r} x^m \, dx \\ &= (B/r)\, \Gamma((m+1)/r)\, \alpha^{-(m+1)/r} p^{-(m+1-jr)/r}. \end{aligned}$$

This final expression has an exponent of $1/p$, namely, $(m + 1 - jr)/r$, which is no greater than M/r.

3. The pth Power Integral

An important application of Theorem 2 yields an asymptotic expansion for the pth power of the L^p norm of an analytic function.

Theorem 3. *Let $\psi(x)$ be a nonnegative function, analytic for $0 \leqslant x \leqslant a$. Denote the Chebyshev norm of ψ by $\| \psi \|$. Assume ψ attains that norm at $x = 0$ only. Then $\phi = \psi / \| \psi \|$ has the Taylor series*

$$\phi(x) = 1 - \alpha x^r + \sum_{m=r+1}^{\infty} \alpha_m x^m$$

with $\alpha > 0$ and $r \geqslant 1$. Define $\theta(x) = e^{\alpha x^r}\phi(x)$. Let $Q(\psi)$ be the asymptotic expansion Theorem 2 yields for $\int_0^a e^{-\alpha x^r}\theta^p(x)\,dx$. Then $\int_0^a \psi^p(x)\,dx \sim \|\psi\|^p Q(\psi)$ as $p \to \infty$.

Proof. Since $\|\phi\| = 1$ and ϕ attains that norm at $x = 0$ only, $\phi(x) < 1$ for all $x > 0$. Thus the Taylor series for ϕ has the form claimed.

By a simple computation, the Taylor series for θ has the form

$$\theta(x) = 1 + \sum_{m=r+1}^{\infty} a_m x^m.$$

Pick $A \leqslant a$ such that $\theta(x) > \frac{1}{2}$ on $[0, A]$. Let $K = \sup\{\phi(x) \mid x \in [A, a]\}$. Trivially, $K < 1$; i.e., $\ln K < 0$ and $|\theta(x)| \leqslant Ke^{\alpha x^r}$ for $A \leqslant x < a$. Thus, by Theorem 2 with $\alpha = \beta$, $k = r + 1$,

$$\int_0^a \psi^p(x)\,dx = \|\psi\|^p \int_0^a e^{-p\alpha x^r}\theta^p(x)\,dx \sim \|\psi\|^p Q(\psi).$$

COROLLARY. *Let $\psi(x)$ be analytic for $0 \leqslant x \leqslant a$. Since ψ has only finitely many zeros and extrema on $[0, a]$, one can write $[0, a]$ as the union of closed intervals $I_1, \ldots, I_n$ with disjoint interiors such that ψ does not change sign on any I_j and such that the only place*

$$\psi_j = \psi \mid I_j$$

achieves its norm $\|\psi_j\|$ is at one of the endpoints of I_j. Suppose, for $j = 1, \ldots, s$, that $\|\psi_j\| = \|\psi\|$. After an appropriate change of variables, $|\psi_j|$ satisfies the conditions of Theorem 3. Let $\bar{\psi}_j$ be this transformed ψ_j. Define

$$Q(\psi) = \sum_{j=1}^{s} Q(\bar{\psi}_j).$$

Then

$$\int_0^a |\psi(x)|^p\,dx = \|\psi\|^p Q(\psi).$$

Proof. For $j = s + 1, \ldots, n$, $\{\|\psi_j\|/\|\psi\|\}^p \to 0$ faster than any power of $1/p$, as $p \to \infty$. Write

$$\int_0^a |\psi(x)|^p\,dx = \sum_{j=1}^{n} \int_{I_j} |\psi(x)|^p\,dx$$

and the result follows.

4. An Application

The well-known formula of Wallis [5] states that for $p \geqslant 2$,

$$\int_0^{\pi/2} \sin^p x\,dx = \int_0^{\pi/2} \cos^p x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(p-1)}{2\cdot 4\cdot 6\cdots p}\,\dfrac{\pi}{2} & \text{if } p \text{ is even,} \\ \dfrac{2\cdot 4\cdot 6\cdots(p-1)}{3\cdot 5\cdot 7\cdots p} & \text{if } p \text{ is odd,} \end{cases}$$

$$= (\pi/2p)^{1/2}\{1 - (1/4p) + (1/32p^2) + o(1/p^3)\}$$

when p is a large positive integer [2, p. 62]. Theorem 3, however, allows us to compute as many terms of the asymptotic expansion as we like and to remove the restriction that p be an integer.

Let $\phi(x) = \cos x$ on $[0, \pi/2]$. Then $r = 2$ and $\alpha = \frac{1}{2}$. The Taylor series for θ is

$$\theta(x) = 1 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{11}{3360}x^8 - \frac{19}{56700}x^{10} - \cdots$$

and for θ^p is

$$\begin{aligned}\theta^p(x) = 1 &- p\left(\frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{11}{3360}x^8 - \frac{19}{56700}x^{10} - \cdots\right) \\ &+ \frac{p^2-p}{2}\left(\frac{1}{144}x^8 + \frac{1}{270}x^{10} + \frac{943}{907200}x^{12} + \cdots\right) \\ &+ \frac{p^3-3p^2+2p}{6}\left(-\frac{1}{1728}x^{12} - \frac{1}{2160}x^{14} - \cdots\right) \\ &+ \frac{p^4-6p^3+11p^2-6p}{24}\left(\frac{1}{20736}x^{16} + \cdots\right) + \cdots.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi/2} \cos^p x\,dx \sim \frac{1}{2}\Big\{&\Gamma\left(\frac{1}{2}\right)\left(\frac{p}{2}\right)^{-1/2} - \left(\frac{p}{12}\right)\Gamma\left(\frac{5}{2}\right)\left(\frac{p}{2}\right)^{-5/2} - \left(\frac{p}{45}\right)\Gamma\left(\frac{7}{2}\right)\left(\frac{p}{2}\right)^{-7/2} \\ &- \frac{11p}{3360}\Gamma\left(\frac{9}{2}\right)\left(\frac{p}{2}\right)^{-9/2} - \frac{19p}{56700}\Gamma\left(\frac{11}{2}\right)\left(\frac{p}{2}\right)^{-11/2} - \cdots \\ &+ \frac{p^2-p}{2}\frac{1}{144}\Gamma\left(\frac{9}{2}\right)\left(\frac{p}{2}\right)^{-9/2} + \frac{p^2-p}{2}\frac{1}{270}\Gamma\left(\frac{11}{2}\right)\left(\frac{p}{2}\right)^{-1/2} \\ &+ \frac{p^2}{2}\frac{943}{907200}\Gamma\left(\frac{13}{2}\right)\left(\frac{p}{2}\right)^{-13/2} + \cdots \\ &- \frac{p^3-3p^2}{6}\frac{1}{1728}\Gamma\left(\frac{13}{2}\right)\left(\frac{p}{2}\right)^{-13/2}\end{aligned}$$

$$- \frac{p^3}{6} \frac{1}{2160} \Gamma\left(\frac{15}{2}\right)\left(\frac{p}{2}\right)^{-15/2} - \cdots$$

$$+ \frac{p^4}{24} \frac{1}{20736} \Gamma\left(\frac{17}{2}\right)\left(\frac{p}{2}\right)^{-17/2} + \cdots\Big\}$$

$$= \left(\frac{\pi}{2p}\right)^{1/2} \Big\{1 - \frac{1}{4p} + \frac{1}{32p^2} + \frac{5}{128p^3} - \frac{21}{204p^4} + \cdots\Big\}.$$

This result can also be obtained by noting that

$$\int_0^{\pi/2} \cos^p x \, dx = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{\Gamma((p-1)/2)\,\Gamma(\frac{1}{2})}{2\Gamma((p+2)/2)}$$

and writing asymptotic power series for $\Gamma((p+1)/2)$ and $\Gamma((p+2)/2)$. (B, here, is the standard Beta function.)

5. The Coefficients in Q

Although it is virtually impossible to display the coefficient of p^{-v} in $Q(\psi)$ as a function of the Taylor series coefficients of ϕ, we can say something about the nature of the first occurrences of various powers of those Taylor series coefficients.

Theorem 4. *For fixed positive integers m and s, α^s_{r+m} first appears in $Q(\psi)$ in the coefficient of p^{-u} where $u = (sm+1)/r$.*

Proof. Write

$$\theta(x) = 1 + y + \alpha_{r+m} x^{r+m} e^{\alpha r x^r},$$

where $y = a_{r+1}x^{r+1} + \cdots$ but does not involve α_{r+m}. Then

$$\theta^p(x) = (1+y)^p + \sum_{i=1}^{[p]} p(p-1)\cdots(p-i+1)(1+y)^{p-i}$$

$$\times\, \alpha^i_{r+m} x^{i(r+m)} e^{i\alpha x^r} + R$$

$$= \sum_{i=1}^{p} (p^i + \sigma_{1,i-1}p^{i-1} + \cdots + \sigma_{i-1,i-1}p)(1+y)^{p-i}$$

$$\times\, \alpha^i_{r+m} x^{i(r+m)}(1 + i\alpha x^r + \cdots) + (1+y)^p + R,$$

where $\sigma_{j,i-1}$ is the jth elementary symmetric function of the numbers -1, $-2, \ldots, -(i-1)$.

When $(1 + y)^{p-i}$ is written as a polynomial plus remainder, it is apparent that the first term involving α_{r+m}^s, i.e., that term with the lowest power of x, is

$$(p^s + \sigma_{1,s-1}p^{s-1} + \cdots + \sigma_{s-1,s-1}p)\,\alpha_{r+m}^s x^{s(r+m)}.$$

This contributes

$$\frac{1}{r}(p^s + \sigma_{1,s-1}p^{s-1} + \cdots + \sigma_{s-1,s-1}p)\,\alpha_{r+m}^s \Gamma\left(\frac{sr + sm + 1}{r}\right)(\alpha p)^{-(sr+sm+1)/r}$$

to the asymptotic expansion $Q(\psi)$. When the powers of p are collected, the first occurrence of α_{r+m}^s; i.e., the term with the smallest power of $1/p$, is

$$\frac{1}{r}\,\alpha_{r+m}^s \Gamma\left(\frac{sr + sm + 1}{r}\right)\alpha^{-(sr+sm+1)/r}p^{-(sm+1)/r}.$$

COROLLARY. *For fixed m, α_{r+m}^2 first appears in $Q(\psi)$ as $B\alpha_{r+m}^2 p^{-(2m+1)/r}$, where $B > 0$ depends only on α, r and m.*

THEOREM 5. *For fixed m, the only occurrences of α_{r+m} in $Q(\psi)$ before the first appearence of α_{r+m}^2 are all of the form $c\alpha_{r+m}p^{-u}$, where $(m + 1)/r \leqslant u \leqslant (2m + 1)/r$. Furthermore, c depends only on α, r, m, and some of the α_{r+k} whose squares have already appeared.*

Proof. That u lies between $(m + 1)/r$ and $(2m + 1)/r$ is a trivial consequence of Theorem 4. Furthermore, if c depends on α_{r+n}^s for $s \geqslant 2$, Theorem 4 also implies that α_{r+n}^2 has already appeared in a term with a smaller power of $1/p$. Thus, it remains to examine the nature of the occurrences of $\alpha_{r+n}\alpha_{r+m}$ in $Q(\psi)$. We emulate the proof of the last theorem. Write

$$\theta(x) = 1 + z + (\alpha_{r+n}x^{r+n} + \alpha_{r+m}x^{r+m})\,e^{\alpha x^r},$$

where z is a power series which does not involve α_{r+n} or α_{r+m}. Then

$$\theta^p(x) = \sum_{i=1}^{[p]}(p^i + \cdots + \sigma_{i-1,i-1}p)(1 + z)^{p-i}$$

$$\times \sum_{j=0}^{i}\binom{i}{j}\alpha_{r+n}^j\alpha_{r+m}^{i-j}x^{ir+jn+(i-j)m}e^{i\alpha x^r} + (1 + z)^p + R.$$

The expression $\alpha_{r+n}\alpha_{r+m}$ occurs when $i = 2$ and $j = 1$. It is found in

$$(p^2 - p)(1 + z)^{p-2}\,2\alpha_{r+n}\alpha_{r+m}x^{2r+n+m}(1 + 2\alpha x^r + \cdots).$$

The smallest, nonzero exponent of x in $1 + z$ is no less than $r + 1$. Thus, when $(1 + z)^{p-2}$ is written as a polynomial plus remainder, its terms have the form $p^{v-w}\gamma x^{v(r+1)+\epsilon}$, where $v = 0, 1,..., p$, $w = 0, 1,..., v$, $\epsilon \geqslant 0$, and γ is some combination of the coefficients. There are several terms with $\alpha_{r+n}\alpha_{r+m}$ but they all have the form

$$p^{2-\delta}p^{v-w}\bar{\gamma}\alpha_{r+n}\alpha_{r+m}x^{2r+n+v(r+1)+\epsilon+tr},$$

where $\delta = 0, 1$ and $t = 0, 1, 2,...$. They contribute to the p^{-u} term in $Q(\psi)$ when

$$\begin{aligned} u &= -(2 - \delta + v - w) + ((2 + v + t)r + v + \epsilon + n + m + 1)/r) \\ &= \delta + w + t + ((v + \epsilon)/r) + ((n + m + 1)/r). \end{aligned}$$

Since we are interested in $u < (2m + 1)/r$ we need examine only the situation

$$\delta + w + t + ((v + \epsilon)/r) + (n/r) < m/r.$$

But this implies $n < m$ which yields

$$u > \delta + w + t + ((v + \epsilon)/r) + ((2n + 1)/r) \geqslant (2n + 1)/r.$$

Thus, α_{r+n}^2 has already occurred in a term of $Q(\psi)$ with a lower exponent of $1/p$ than u.

6. Conclusion

The corollary to Theorem 3 shows some promise of answering the question of the convergence, as $p \to \infty$, of the path of best L^p approximates in a linear space of functions to some given function outside that space. Descloux [4] has conjectured that, when the functions involved are continuous, piecewise analytic, the path does converge and to an element he named "the strict approximation."

However, in order to show the best L^p approximates converge to the strict approximation it is necessary to prove that the coefficients of Q in the corollary are uniformly bounded in a ball centered at a best approximation. The difficulty in this is that the subintervals I_j of the corollary to Theorem 3 may change from point to point within the ball.

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