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Group theoretic methods applied to Burgers' equation

Mina B. Abd-el-Malek[a,*], Samy M.A. El-Mansi[b]

[a] *Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt*

[b] *Department of Physical Sciences, Faculty of Engineering, Mansoura University, Mansoura, Egypt*

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Abstract

In this study, the group-theoretic methods for calculating the solution of Burgers' equation with appropriate boundary- and initial-conditions is presented. The application of a one-parameter group reduces the number of independent variables by one, and consequently the governing partial differential equation with the boundary- and initial-conditions to an ordinary differential equation with the appropriate corresponding conditions. The obtained differential equation is solved analytically and the solution obtained in closed form, for a specific choice of boundary condition. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Historical review

The history of Burgers' equation dates back to 1915 when Bateman [7] derived it in a physical context. One of the most interesting solutions of Burgers' equation, in a series form, is due to Fay [17] when it was derived in the acoustic framework. In 1940, Burgers gave special solutions to it and emphasized its importance, and in 1948, Burgers [14] concluded its form as a model in the theory of turbulence. In connection with Burgers' equation, Lagerstrom et al. [22] in 1949 had discovered that the Burgers' equation can be transformed to the linear heat equation which was published by Cole [15] in 1951. This transformation, at about the same time, was discovered independently by Hopf [20], and from which it is known as Cole–Hopf transformation. The Fay series was rediscovered by

* Corresponding author.

E-mail address: minab@aucegypt.edu (M.B. Abd-el-Malek)

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Cole [15] as an approximate solution of the Burgers' equation for a sinusoidal initial condition. Then, independently, Lighthill [23] in 1956 and Blackstock [12] in 1964 employed Burgers' equation in studying the propagation of the one-dimensional acoustic signals of finite amplitude, while in 1958, Hayes [19] used the equation in the discussions of shock structure in Navier–Stokes fluids. Benton [8] and [9] in 1966 and 1967, found an exact solution of the equation. Rodin [30] in 1970, found a Riccati solution for the Burgers' equation without using any auxiliary conditions.

Benton and Platzman [10] in 1972, published 35 distinct solutions to the initial-value problem for Burgers' equation in the infinite domain as well as two other solutions for the initial- and boundary-value problem in the finite domain.

In 1972, Ames [5] discussed how the Morgan–Michal method could be applied for determining the proper groups for Burgers' equation, without taking into consideration the auxiliary conditions.

In 1980, Varoglu and Finn [35] have applied a new finite-element method to solve the Burgers' equation which was based on the combination of the space–time elements and the characteristics. In 1983, Weiss et al. [37] defined the Painlevé property of the partial differential equation, from which they determined the integrability, the Bäcklund transforms, and the Lax pairs of the Burgers' equation as well as KdV equation and the modified KdV equation. Then during the period 1983–1985, Boisvert et al. [13], Nucci [26] and Ames [6] assumed a particular Lie-group with arbitrary functions of time which permit the transformation of the time-dependent equation into the corresponding time-independent ones, and from which any solution of the steady equations generates an infinite number time-dependent solutions.

Vorus [36], in 1989, studied the perturbed vortex sheet from its steady-flow state, by a sinusoidal excitation that was applied at the separation point. The obtained differential equation is the Burgers' equation on a moving medium. To avoid the nonlinearity in the transformed upstream boundary condition, he integrated the equation first and its condition in its spatial variable and then applied the Cole–Hopf transformation. In 1989, Shtelen [34] succeeded to derive the Cole–Hopf transformation, from the group theoretic point of view, and in the same year, Fushchich et al. [18] showed that the Burgers' equation is invariant with respect to space and time translations, scale changes, Galilean transformation, and projective transformation. Also, they concluded the remarkable similarity between the symmetry algebra of Burgers' equation and that for the heat equation. But all these invariants were derived without taking into consideration the auxiliary condition.

In 1993, Peralta-Fabi and Plaschko [29] studied the stability and the bifurcation of solutions to the controlled Burgers' equation by adding an integral term, representing a nonlocal behaviour, to the normal form of the Burgers' equation describing flow through porous media. They applied the perturbation technique, up to fourth order, to find out the critical value of the viscosity.

Ozis and Ozdes [28], in 1996, applied the direct variational method, from which they succeeded to generate an approximate solution in the form of a sequence solution which converges to the exact solution. In 1997, Mazzia and Mazzia [24], converted Burgers' equation into a system of ordinary differential equations, and then applied the transverse scheme (where the approximate solutions are defined on the lines along the time axis) in combination with the boundary value methods.

1.2. Physical applications

The nonhomogeneous form of the Burgers' equation has physical applications in many fields as in the approximate theory of flow through shock wave propagation in a viscous fluid and in the modelling of turbulence flow in a channel, see [16,24,27,28,38]. Also, it describes one-dimensional two-phase (oil/water) flow under gravity in a semi-infinite porous reservoir with constant boundary injection, see Rogers and Ames [31] and Rogers et al. [32]. One of the most interesting applications of Burgers' equation is the traffic flow problem, for more details see Kevorkian [21] and Whitham [38].

1.3. Present work

In case of zero initial condition, unfortunately, the Cole–Hopf transformation, due to its nature, does not give the appropriate boundary condition, see Kevorkian [21]. In the present work we applied the one-parameter group transformation to the Burgers' equation associated with the initial- and boundary-conditions. Under the transformation, the partial differential equations with auxiliary conditions are reduced to an ordinary differential equation with the appropriate corresponding conditions. The differential equation is solved analytically and the solution has been obtained in closed form.

Group theoretic methods provide a powerful tool because they are not based on linear operators, superposition, or any other aspects of linear solution techniques. Therefore, these methods are applicable to nonlinear differential models.

The concept of a symmetry of a differential equation was introduced by Sophus Lie at the end of the 19th century while he was searching for a general theory of solving differential equations, see [33]. Although Lie has obtained numerous results for partial differential equations, only the concept of a similarity transformation, introduced as a systematic method about 50 years after Lie's death by Birkhoff [11], caused broader applications and led to a better understanding of various apparently unrelated results, see Schwarz [33].

Throughout the history of similarity analysis, a variety of problems in science and engineering has been solved. Among these we find the nonlinear temperature variation across the lake depth neglecting the effect of external heat sources by Abd-el-Malek [1] in 1997, the steady and unsteady laminar boundary-layer flow of a nonisothermal vertical circular cylinder by Abd-el-Malek and Badran [3,4] in 1990 and 1991. Many physical applications are illustrated in [2,18,31,32].

2. Mathematical formulation

Assuming a unity kinematic viscosity, the governing equation, for the one-dimensional velocity field $w(x,t)$, is given by

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2}, \quad x > 0,\ t > 0 \tag{2.1}$$

with the following conditions:

(i) Initial condition:

$$w(x,0) = 0. \tag{2.2}$$

(ii) Boundary conditions:

$$\begin{aligned} &\text{(a) } w(0,t) = \alpha q(t), \quad t > 0,\ \alpha \neq 0.\\ &\text{(b) } \lim_{x\to\infty} w(x,t) = 0. \end{aligned} \tag{2.3}$$

Write

$$w(x,t)=u(x,t)q(t), \tag{2.4}$$

where $q(t)$ is an unknown function. Its proper form will be determined later. Thus differential equation (2.1) takes the form

$$q(t)\frac{\partial u}{\partial t}+u\frac{\mathrm{d}q}{\mathrm{d}t}+q^2u\frac{\partial u}{\partial x}=q\frac{\partial^2 u}{\partial x^2}, \quad x>0,\ t>0 \tag{2.5}$$

and the initial and boundary conditions take the form:
(i) Initial condition:

$$u(x,0)=0. \tag{2.6}$$

(ii) Boundary conditions:

$$\begin{aligned}&\text{(a) } u(0,t)=\alpha,\ t>0,\ \alpha\neq 0.\\ &\text{(b) } \lim_{x\to\infty}u(x,t)=0.\end{aligned} \tag{2.7}$$

3. Solution of the problem

Our method of solution depends on the application of a one-parameter group transformation to partial differential equation (2.5). Under this transformation the two independent variables will be reduced by one and differential equation (2.5) transforms into an ordinary differential equation.

3.1. The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter a of the form

$$G:\bar{S}=C^S(a)S+K^S(a), \tag{3.1}$$

where S stands for x,t,u,q and the C's and K's are real-valued and at least differentiable in the real argument a.

3.2. The invariance analysis

To transform the differential equation, transformations of the derivatives of u and q are obtained from G via chain-rule operations:

$$\bar{S}_{\bar{i}}=\left(\frac{C^S}{C^i}\right)S_i, \qquad \bar{S}_{\bar{i}\bar{j}}=\left(\frac{C^S}{C^iC^j}\right)S_{ij}, \quad i=x,y,\ j=x,y, \tag{3.2}$$

where S stands for u and q.

Eq. (2.5) is said to be invariantly transformed for some function $H(a)$, whenever

$$\bar{q}\bar{u}_{\bar{t}}+\bar{u}\bar{q}_{\bar{t}}+\bar{q}^2\bar{u}\bar{u}_{\bar{x}}-\bar{q}\bar{u}_{\bar{x}\bar{x}}=H(a)[qu_t+uq_t+q^2uu_x-qu_{xx}]. \tag{3.3}$$

Substitution from (3.1) into (3.3) yields

$$(C^q q + K^q)\left(\frac{C^u}{C^t}\right) u_t + (C^u u + K^u)\left(\frac{C^q}{C^t}\right) q_t + (C^q q + K^q)^2 (C^u u + K^u)\left(\frac{C^u}{C^x}\right) u_x$$
$$- (C^q q + K^q)\left(\frac{C^u}{(C^x)^2}\right) u_{xx} = H(a)[q u_t + u q_t + q^2 u u_x - q u_{xx}] \tag{3.4}$$

from which

$$\left[\frac{C^q C^u}{C^t}\right] q u_t + \left[\frac{C^u C^q}{C^t}\right] u q_t + \left[\frac{(C^u C^q)^2}{C^x}\right] u q^2 u_x - \left[\frac{C^q C^u}{(C^x)^2}\right] q u_{xx} + R(a)$$
$$= H(a)[q u_t + u q_t + q^2 u u_x - q u_{xx}], \tag{3.5}$$

where

$$R(a) = \left[\frac{K^q C^u}{C^t}\right] u_t + \left[\frac{K^u C^q}{C^t}\right] q_t + \left[K^u (C^q q + K^q)^2 \left(\frac{C^u}{C^x}\right)\right] u_x - \left[\frac{K^q C^u}{(C^x)^2}\right] u_{xx}.$$

The invariance of (3.5) implies $R(a) \equiv 0$. This is satisfied by putting

$$K^q = K^u = 0 \tag{3.6}$$

and

$$\left[\frac{C^q C^u}{C^t}\right] = \left[\frac{(C^q C^u)^2}{C^x}\right] = \left[\frac{C^q C^u}{(C^x)^2}\right] = H(a)$$

which yields

$$C^t = (C^x)^2, \qquad C^q C^u = \frac{1}{C^x}. \tag{3.7}$$

Moreover, boundary conditions (2.7) are also invariant in form, implying that

$$K^x = K^u = 0 \quad \text{and} \quad C^u = 1 \tag{3.8}$$

and the invariance of initial condition (2.6) implies that

$$K^t = 0. \tag{3.9}$$

Finally, we get the one-parameter group G which transforms invariantly differential equation (2.5), as well as initial condition (2.6), and the boundary conditions (2.7). The group G is of the form

$$G: \begin{cases} \bar{x} = C^x x, \\ \bar{t} = (C^x)^2 t, \\ \bar{u} = u, \\ \bar{q} = \left(\dfrac{1}{C^x}\right) q. \end{cases} \tag{3.10}$$

3.3. *The complete set of absolute invariant*

Our aim is to make use of the group methods to represent the problem in the form of an ordinary differential equation. Then we have to proceed in our analysis to obtain a complete set of absolute invariants.

If $\eta \equiv \eta(x,t)$ is the absolute invariant of the independent variables, then

$$g_j(x,t;u,q) = F_j[\eta(x,t)], \quad j = 1,2 \tag{3.11}$$

are the two absolute invariants corresponding to u and q. The application of a basic theorem in group theory, see [25], states that *a function $g(x,t;u,q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*:

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \qquad S_i \equiv x,t,u,q, \tag{3.12}$$

where

$$\alpha_i = \frac{\partial C^{S_i}}{\partial a}(a^0) \quad \text{and} \quad \beta_i = \frac{\partial K^{S_i}}{\partial a}(a^0), \quad i = 1,2,3,4 \tag{3.13}$$

and a^0 denotes the value of "a" which yields the identity element of the group.

From which we get $\alpha_3 = 0$ and $\beta_i = 0$; $i = 1,2,3,4$.

Owing to Eq. (3.12), $\eta(x,t)$ is an absolute invariant if it satisfies

$$\alpha_1 x \frac{\partial \eta}{\partial x} + \alpha_2 t \frac{\partial \eta}{\partial t} = 0 \tag{3.14}$$

which has a solution in the form

$$\eta(x,t) = \frac{x}{t^{\beta}}, \qquad \beta = \frac{\alpha_1}{\alpha_2} > 0. \tag{3.15}$$

By a similar analysis the absolute invariants of the dependent variables w and q are $q(t) = \Gamma(t)\theta(\eta)$.

Since $q(t)$ and $\Gamma(t)$ are independent of x, while η is a function of x and t, then $\theta(\eta)$ must be a constant, say $\theta(\eta) = 1$, from which

$$q(t) = \Gamma(t) \tag{3.16}$$

and the second absolute invariant is

$$u(x,t) = F(\eta). \tag{3.17}$$

4. The reduction to an ordinary differential equation

Substituting from (3.15)–(3.17) into Eq. (2.5), we get

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2} + [\beta x t^{\beta-1}] \frac{\mathrm{d}F}{\mathrm{d}\eta} - [\Gamma t^{\beta}] F \frac{\mathrm{d}F}{\mathrm{d}\eta} - \left[\frac{t^{2\beta}}{\Gamma} \frac{\mathrm{d}\Gamma}{\mathrm{d}t} \right] F = 0. \tag{4.1}$$

For (4.1) to be reduced to an expression in the single independent invariant η, it is necessary that the coefficients should be constants or functions of η alone. Thus

$$\beta x t^{\beta-1} = c_1, \tag{4.2.1}$$

$$\Gamma t^{\beta} = c_2, \tag{4.2.2}$$

$$\left(\frac{t^{2\beta}}{\Gamma} \right) \frac{\mathrm{d}\Gamma}{\mathrm{d}t} = c_3. \tag{4.2.3}$$

The only possible values of β and c_3 are $\beta = 0.5$ and $c_3 = -0.5$. Hence from (4.2.3)

$$\Gamma(t) = \frac{1}{\sqrt{t}}.$$

From which Eq. (4.1) takes the form

$$2\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2} + (\eta - 2F)\frac{\mathrm{d}F}{\mathrm{d}\eta} + F = 0. \tag{4.4}$$

Under the similarity variable η, the boundary conditions are

$$F(0) = \alpha, \tag{4.5}$$

$$F(\infty) = 0. \tag{4.6}$$

5. Analytical solution

Eq. (4.4) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{\mathrm{d}F}{\mathrm{d}\eta} - \frac{1}{2}F^2 + \frac{1}{2}\eta F\right) = 0. \tag{5.1}$$

Integrating (5.1), we get

$$\frac{\mathrm{d}F}{\mathrm{d}\eta} - \frac{1}{2}F^2 + \frac{1}{2}\eta F = k_1, \tag{5.2}$$

where k_1 is a constant.

Eq. (5.2) is a Riccati equation, which has a special solution in the form

$$F_1(\eta) = k_2\eta, \tag{5.3}$$

which leads to $k_1 = k_2$, and the possible values of k_2 are 0 and 1.

Reduction (5.1): Corresponds to $k_2 = 0$

Eq. (5.2) takes the form

$$\frac{\mathrm{d}F}{\mathrm{d}\eta} + \frac{1}{2}\eta F = \frac{1}{2}F^2 \tag{5.4}$$

which is the Bernoulli's equation and has the solution

$$F(\eta) = \frac{\mathrm{e}^{-\eta^2/4}}{k_3 - \frac{1}{2}\int_0^n \mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi}. \tag{5.5}$$

From (4.5), $k_3 = 1/\alpha$.

Eq. (5.1) has the solution

$$F(\eta) = \frac{\mathrm{e}^{-\eta^2/4}}{1/\alpha - \frac{1}{2}\int_0^n \mathrm{e}^{-\xi^2/4}\,\mathrm{d}\xi} \tag{5.6}$$

which can be written in the form

$$F(\eta) = \frac{\mathrm{e}^{-\eta^2/4}}{1/\alpha - \sqrt{\pi}/2\,\mathrm{erf}(\eta/2)}, \tag{5.7}$$

hence we get

$$w(x,t)=\frac{1}{\sqrt{t}}\left[\frac{e^{-x^2/4t}}{1/\alpha-\sqrt{\pi}/2\,\mathrm{erf}(x/2\sqrt{t})}\right],\quad x>0,\ t>0,\ \alpha\neq 0. \tag{5.8}$$

Reduction (5.2): Corresponds to $k_2=1$
Write

$$F(\eta)=\eta+F_2(\eta). \tag{5.9}$$

Eq. (5.2) takes the form

$$\frac{dF_2}{d\eta}-\frac{1}{2}\eta F_2=\frac{1}{2}F_2^2 \tag{5.10}$$

which has the solution

$$F_2(\eta)=\frac{e^{\eta^2/4}}{k_4-\frac{1}{2}\int_0^\eta e^{\xi^2/4}\,d\xi}. \tag{5.11}$$

Hence from (5.3), (5.4) and (5.11) we get

$$F\eta=\eta+\frac{e^{\eta^2/4}}{k_4-\frac{1}{2}\int_0^\eta e^{\xi^2/4}\,d\xi}. \tag{5.12}$$

From (4.5), $k_4=1/\alpha$, and (4.6) is satisfied, hence we have

$$F(\eta)=\eta+\frac{e^{\eta^2/4}}{1/\alpha-\frac{1}{2}\int_0^\eta e^{\xi^2/4}\,d\xi}. \tag{5.13}$$

Then the form of $w(x,t)$ is

$$w(x,t)=\frac{x}{t}+\frac{1}{\sqrt{t}}\left(\frac{e^{x^2/4t}}{1/\alpha-\frac{1}{2}\int_0^{x/\sqrt{t}} e^{\xi^2/4}\,d\xi}\right). \tag{5.14}$$

6. Applications to the KdV and MKdV equations

The one-parameter group transformation, defined in (3.1) can be used to study the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (MKdV) equations, with boundary and initial conditions.

6.1. Korteweg–de Vries equation

Consider the one-dimensional KdV equation

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}+\sigma\frac{\partial^3 w}{\partial x^3}=0 \tag{6.1}$$

with the following conditions:
(i) Initial condition:

$$w(x,0)=0. \tag{6.2}$$

(ii) Boundary conditions:

$$\begin{aligned}&\text{(a) } w(0,t)=q(t),\\&\text{(b) } w_x(0,t)=0,\\&\text{(c) } w_{xx}(0,t)=0.\end{aligned} \tag{6.3}$$

Applying the one-group transformation (3.1) we get the following group G which transforms invariantly differential equation (6.1), as well as the initial condition (6.2), and boundary conditions (6.3):

$$G:\begin{cases}\bar{x}=C^{x}x,\\ \bar{t}=(C^{x})^{3}t,\\ \bar{u}=u,\\ \bar{q}=\left(\dfrac{1}{(C^{x})^{2}}\right)q,\end{cases} \tag{6.4}$$

where $u(x,t)=w(x,t)/q(t)$, and the similarity variable is

$$\eta=\frac{x}{t^{1/3}}. \tag{6.5}$$

The reduced ordinary differential equation is

$$3\sigma\frac{\mathrm{d}^{3}F}{\mathrm{d}\eta^{3}}-\eta\frac{\mathrm{d}F}{\mathrm{d}\eta}+3F\frac{\mathrm{d}F}{\mathrm{d}\eta}-2F=0, \tag{6.6}$$

with the auxiliary conditions

$$\begin{aligned}&\text{(i) } F(0)=1,\\&\text{(ii) } \frac{\mathrm{d}F}{\mathrm{d}\eta}(0)=0,\\&\text{(iii) } \frac{\mathrm{d}^{2}F}{\mathrm{d}\eta^{2}}(0)=0,\\&\text{(iv) } F(\infty)=0.\end{aligned} \tag{6.7}$$

For specific values of σ, we can solve Eq. (6.6) analytically or numerically, where

$$F(\eta)=w(x,t)/q(t).$$

6.2. *Modified Korteweg–de Vries equation*

Consider the one-dimensional MKdV equation

$$\frac{\partial w}{\partial t}-\frac{\partial}{\partial x}\left(w^{3}-2\sigma^{2}\frac{\partial^{2}w}{\partial x^{2}}\right)=0 \tag{6.8}$$

with the following conditions:

(i) Initial condition:

$$w(x,0)=0. \tag{6.9}$$

(ii) Boundary conditions:

$$\begin{aligned}&\text{(a) } w(0,t)=q(t),\\ &\text{(b) } w_x(0,t)=0,\\ &\text{(c) } w_{xx}(0,t)=0.\end{aligned} \tag{6.10}$$

Applying the one-group transformation (3.1) we get the following group G which transforms invariantly differential equation (6.8), as well as initial condition (6.9), and the boundary conditions (6.10):

$$G:\begin{cases}\bar{x}=C^x x,\\ \bar{t}=(C^x)^3 t,\\ \bar{u}=u,\\ \bar{q}=\pm\left(\dfrac{1}{(C^x)^2}\right)q,\end{cases} \tag{6.11}$$

where $u(x,t)=w(x,t)/q(t)$, and the similarity variable is

$$\eta=\frac{x}{t^{1/3}}. \tag{6.12}$$

The reduced ordinary differential equation is

$$6\sigma^2\frac{\mathrm{d}^3F}{\mathrm{d}\eta^3}-\eta\frac{\mathrm{d}F}{\mathrm{d}\eta}-9F\frac{\mathrm{d}F}{\mathrm{d}\eta}-F=0 \tag{6.13}$$

with the auxiliary conditions

$$\begin{aligned}&\text{(i) } F(0)=1,\\ &\text{(ii) } \frac{\mathrm{d}F}{\mathrm{d}\eta}(0)=0,\\ &\text{(iii) } \frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}(0)=0,\\ &\text{(iv) } F(\infty)=0.\end{aligned} \tag{6.14}$$

For specific values of σ, we can solve Eq. (6.13) analytically or numerically, where

$$F(\eta)=w(x,t)/q(t).$$

7. Concluding remarks

The most widely applicable method for determining analytical solution of partial differential equation utilizes the underlying group structure that has been applied to the Burgers' equation.

We obtained exact analytical solution, believed to be new, successfully, for a specific form for the boundary conditions which has been determined in an inverse way.

Also the obtained solution corresponds to a specific choice of some arbitrary parameter appearing in the analysis process.

For other possible forms of the boundary conditions, where the obtained ordinary differential equation cannot be solved analytically, numerical solution, via the shooting method, can be obtained.

Conditional symmetries, contact symmetries, and the classical Lie approach will lead better reductions and more solutions to the differential equations only but not for the initial and boundary value problems, since the given conditions limit the reductions.

Also, other transformations like spiral transform and linear transform do not work for leaving the differential equation as well as the initial and boundary conditions invariant.

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