Bi-Algebraic Groups

ZENSHO NAKAO*.

California State University, Long Beach, California 90840

Communicated by N. Jacobson
Received November 1, 1977

We obtain several necessary and sufficient conditions for a pro-affine bi-algebraic group to be an algebraic group.

1. INTRODUCTION

Throughout the discussion $k$ will denote a fixed algebraically closed ground field for varieties and algebraic groups. All the algebras are assumed to be commutative and to have an identity element. If $B$ is a subalgebra of $A$ then it is assumed that the identity element of $B$ coincides with that of $A$. Every algebra homomorphism is required to send the identity element of one algebra to that of another. All unadorned tensor products are taken over $k$.

A bi-algebraic group $(G, A(G))$ is an almost affine variety [3], i.e., a pro-affine algebraic variety whose coordinate ring $A(G)$ is an integral extension of an affine subalgebra (i.e., an almost affine algebra [3]), which has the structure of an abstract group such that both the left and right translations by the group elements are morphisms of variety structure.

In this paper we investigate when a bi-algebraic group is a pro-affine algebraic group. More specifically, the problem can be stated as follows: Suppose that $G$ is a bi-algebraic group over $k$. (1) Is the group multiplication $G \times G \rightarrow G$ a morphism? (2) Is the group inversion $G \rightarrow G$ also a morphism? (3) Are questions (1) and (2) independent of each other?

This set of problems was first formulated in the category of affine bi-algebraic groups by R. Palais [7, Sect. 10] in 1975, and he obtained some partial results. Later, in 1976, A. Magid [4] studied the problems in the same (i.e., affine) setting, and proved that the answer to (1) is in the affirmative (if $k$ is infinite and perfect, which always holds in our case).

* This paper is based on part of the author's doctoral dissertation at the University of Oklahoma (1977) under the direction of Professor Andy Magid.

† Current address: Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901.
We will examine the problems in a more general setting where the given bi-algebraic group is almost affine as a variety. Refer to (3.1) for the principal result. What we are trying to achieve is an algebraic analogue of Montgomery’s theorem [5] which states that every complete metric separable bi-topological group is a topological group.

2. Preliminaries on Almost Affine Varieties

We rely heavily on the results of Palais [7] for definitions and techniques of proof in obtaining the first several propositions which extend the affine case to the almost affine case.

Recall from [7] the following notational convention: If X and Y are sets, and \( f: X \times Y \to k \) is a function, then \( f_x \) and \( f_y \) both denote functions \( Y \to k \) and \( X \to k \) given by \( f_x(y) = f(x, y) \) and \( f_y(x) = f(x, y) \), respectively, for \( x \in X \) and \( y \in Y \). (We are simply fixing one of the two variables at a time.)

We also introduce a definition for later reference: Suppose \( f: V \times V \to k \) is a function, where \( V \) is a variety (affine or almost affine). Then we call \( f \) a separately morphic function on \( V \times V \) if \( f_x \) and \( f_y \) are morphic functions on \( V \), i.e., they are elements of the coordinate ring \( A(V) \) of \( V \), for all \( x, y \in V \).

**Lemma 2.1.** Let \( x \) be any set and let \( B \) be a finite dimensional subspace of the vector space \( k^X \) of all \( k \)-valued functions on \( X \). Then \( \{ x^0 \} \) (the evaluation at \( x \)): \( x \in X \) spans the dual space \( B^* \) of \( B \), so in particular there exist \( \{ x_1, ..., x_n \} \) in \( X \) such that \( \{(x_1)^0, ..., (x_n)^0\} \) is a basis for \( B^* \). Choose such \( \{ x_1, ..., x_n \} \) and let \( \{ \xi_1, ..., \xi_n \} \) be the dual basis for \( B^{**} = B \). If \( Y \) is any set and \( f: X \times Y \to k \) is any function such that \( f_y \) lies in \( B \) for all \( y \in Y \) then \( f \) can be expressed as \( \sum \xi_i \otimes (f_{x_i}) \).

Proof. See [7, (3.1)].

**Lemma 2.2.** Let \( V \) be a variety (affine or almost affine) over \( k \) and let \( f: V \times V \to k \) be a separately morphic function. If there exists a dense subset \( S \) of \( V \) such that for all \( y \) in \( S \) the morphic functions \( f^y: V \to k \) lie in some fixed finite dimensional subspace of \( A(V) \), then \( f \) is a morphic function on \( V \times V \), i.e., \( f \) is in \( A(V) \otimes A(V) \). Conversely, if \( f \) is a morphic function on \( V \times V \), then the set of all \( f_x \) (or \( f^y \)) where \( x \) (or \( y \)) ranges over \( V \) spans a finite dimensional subspace of \( A(V) \).

Proof. Suppose first that there is a dense subset \( S \) of \( V \) such that \( \{ f^y: y \in S \} \) lies in some fixed finite dimensional subspace of \( A(V) \). Let \( B \) be the linear span of \( \{ f^y: y \in S \} \), which is, a fortiori, a finite dimensional subspace of \( k^V \). Apply Lemma (2.1) to \( f \mid V \times S: V \times V \to k \), and we can choose \( \{ x_1, ..., x_n \} \) from \( V \) and a basis \( \{ \xi_1, ..., \xi_n \} \) for \( B \) such that \( f \mid V \times S = \sum \xi_i \otimes (f_{x_i} \mid S) \), because \( (f \mid V \times S)_{x_i} = f_{x_i} \mid S \). Let \( g = \sum \xi_i \otimes f_{x_i} \). Then \( g \) is a morphic function on \( V \times V \) (for \( \xi_i \) and \( f_{x_i} \) are morphic functions on \( V \)). We claim that \( f = g \). Clearly \( f_{x_i} \mid S = g_{x_i} \mid S \) by definition of \( g \). But \( S \) is dense in \( V \), so \( f_x = g_x \) for all
$x \in V$. Hence $f(x, y) = f_x(y) = g_x(y) = g(x, y)$ for all $(x, y) \in V \times V$, i.e., $f = g$ as claimed, which implies that $f$ is also a morphic function on $V \times V$ since $g$ is.

Next suppose that $f: V \times V \rightarrow k$ is a morphic function. Then we can put $f = \sum g_i \otimes h_i$, where $g_i$ and $h_i$ are elements of $A(V)$. Now let $(x, y)$ be an element of $V \times V$. Then $f_x(y) = f(x, y) = \sum g_i(x) h_i(y)$, so $f_x = \sum g_i(x) h_i$, i.e., $f_x$ lies in the span of $h_i$, hence the span of $\{f_x : x \in V\}$ is a finite dimensional subspace of $A(V)$. (We can argue similarly for the span of $\{f^y : y \in V\}$.)

**Proposition 2.3.** Let $V$ be an almost affine variety over $k$ which cannot be decomposed into a union of $\dim_k A(V)$ or fewer proper closed subvarieties. Then every separately morphic function $f: V \times V \rightarrow k$ is a morphic function.

**Proof.** We can always write $A(V) = \bigcup A_i$ where $\dim_k A_i$ is finite and the union is taken over $I$ with $\operatorname{card}(I) = \dim_k A(V)$. Let $S_i = \{x \in V : f_x \in A_i\}$. Then $V = \bigcup S_i$, where the union is again over $I$, for if $x \in V$ then $f_x \in A(V)$ because $f$ is a separately morphic function, so that $f_x \in A_i$. Hence $x \in S_i$, so $V = \bigcup S_i$ as required.

Now $V = \bigcup \operatorname{Cl}(S_i)$, where $\operatorname{Cl}(S_i)$ denotes the closure of $S_i$, so by assumption some $\operatorname{Cl}(S_i)$ equals $V$. Thus we have found a dense subset $S = S_i$ of $V$ such that for all $x \in S$ the morphic functions $f_x$ lie in a fixed finite dimensional subspace of $A(V)$, namely, $A_i$. So by Lemma (2.2), $f$ is a morphic function on $V \times V$.

The following results, (2.4), (2.5) and (2.6), are slight generalizations of (9.1), (9.2) and (9.3) of Palais [7]. The proofs are obvious modifications of those of Palais.

**Proposition 2.4.** Suppose that $V$ is an irreducible affine variety over $k$ such that the following conditions are satisfied:

(i) If $W$ is a closed subvariety of $V$ of dimension $\dim V - 1$, then $W$ is not an $I$-union (where $\operatorname{card}(I)$ is infinite) of proper closed subvarieties;

(ii) $V$ itself is an $I$-union of proper closed subvarieties. Then $V$ has at most $\operatorname{card}(I)$ distinct irreducible closed subvarieties of dimension $\dim V - 1$.

**Proposition 2.5.** Let $V$ be as in Proposition (2.4), and let $f$ be a polynomial function on $V$. Then the set $E = \{a \in k : \dim f^{-1}(a) > \dim V - 2\}$ has cardinality at most $\operatorname{card}(I)$.

**Proposition 2.6.** Assume $k$ is uncountable. Then no affine variety $V$ over $k$ can be written as a union of fewer than $\operatorname{card}(k)$ proper closed subvarieties.

We remark that Proposition (2.6) above says that every affine variety over an uncountable field is algebraically of the second category, as is stated in [7, (9.3)].

Now we extend Proposition (2.6) to almost affine varieties.
PROPOSITION 2.1. Assume $k$ is uncountable. Then no irreducible almost affine variety $V$ over $k$ can be written as a union of fewer than $\text{card}(k)$ proper closed subvarieties.

Proof. $A(V)$ is an integral extension of an affine subalgebra of $A(V)$, say $A_0$. Let $V_0$ be the irreducible affine variety $\text{Max}(A_0)$. Then there is a surjective morphism $\mu: V \rightarrow V_0$ which corresponds to the inclusion $A_0 \hookrightarrow A(V)$, which is known to be a closed map.

Now suppose to the contrary that $V = \bigcup V_i$ (for $i \in I$), where $V_i$ are proper closed subvarieties of $V$ and $\text{card}(I) < \text{card}(k)$. Then $V_0 = \bigcup \mu(V_i)$. So we must have $\mu(V_i) = V_0$ for some $i \in I$ by our hypothesis that $\text{card}(I) < \text{card}(k)$ because $\mu(V_i)$ are all closed subvarieties of the affine variety $V_0$ (by Proposition (2.6)).

Since $A(V_i) = A(V)/J$ where $J$ is the ideal of morphic functions vanishing on $V_i$, it is integral over an affine algebra $A_0/(J \cap A_0)$, so that we get another surjective morphism from $V_i$ onto an affine variety $W = \text{Max}(A_0/(J \cap A_0))$. The commutative diagram of algebras;

\[
\begin{array}{ccc}
A(V) & \longrightarrow & A(V)/J \\
& & \\
& \uparrow & \\
& & \\
A_0 & \longrightarrow & A_0/(J \cap A_0)
\end{array}
\]

yields the commutative diagram of varieties;

\[
\begin{array}{ccc}
V & \longrightarrow & V_i \\
\downarrow \mu & & \downarrow \mu \\
V_0 & \longrightarrow & W
\end{array}
\]

in which all morphisms in the lower triangle are now surjective. Then the lower horizontal morphism is an isomorphism because it is bijective and the corresponding algebra homomorphism is surjective, which means that $A_0 \cong A_0/(A_0 \cap J)$, hence $A_0 \cap J = 0$.

We claim next that $J = 0$. Let $f$ be an arbitrary morphic function in $J$. Consider the following commutative diagram of affine algebras:

\[
\begin{array}{ccc}
A_0[f] & \longrightarrow & A_0[f]/(J \cap A_0[f]) \\
& & \\
& \uparrow & \\
& & \\
A_0 & \longrightarrow & A_0/((J \cap A_0[f]) \cap A_0)
\end{array}
\]

Since $A_0 \cap J = 0$, we must have $((J \cap A_0[f]) \cap A_0) = 0$, so that the lower
horizontal homomorphism is an isomorphism. The corresponding commutative
diagram of affine varieties is as follows:

\[ Y \xrightarrow{\sim} Y_0 \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ V_0 \xleftarrow{\sim} X_0 \]

where \( Y = \text{Max}(A_0[f]) \), \( Y_0 = \text{Max}(A_0[f]/(J \cap A_0[f])) \), and \( X_0 = \text{Max}(A_0/((J \cap A_0[f]) \cap A_0)) \).

Now \( \dim Y = \dim V_0 \) because \( A(Y) = A_0[f] \) is integral over \( A_0 \), and also \( \dim Y_0 = \dim X_0 \) since \( A(Y_0) \) is integral over \( A(X_0) \). But \( \dim V_0 = \dim X_0 \) hence \( \dim Y = \dim V_0 \). Since \( Y \) is irreducible and \( V_0 \) is a subvariety of \( Y \), we must have \( Y = Y_0 \), which means that \( J \cap A_0[f] = 0 \). But \( f \in J \cap A_0[f] \), so \( f = 0 \). Hence \( J = 0 \).

Going back to the first pair of diagrams, we find that \( V = V_1 \), which is not the case. This completes the proof of Proposition (2.7).

**Corollary 2.8.** Let \( V \) be an irreducible almost affine variety over \( k \). Then if \( \dim_k A(V) < \text{card}(k) \), then every separately morphic function \( V \times V \to k \) is a morphic function.

**Proof.** Obvious. (See the proof of (2.3).)

**Remark 2.9.** If we assume that \( k \) is uncountable and the variety is affine in Corollary (2.8) above, then we always have \( \dim_k A(V) < \text{card}(k) \) because \( \dim_k A(V) \) is countable. So every separately polynomial function \( V \times V \to k \) is a polynomial function, which is again [7, (9.3)].

### 3. Bi-Algebraic Groups

Now, we will specialize Corollary (2.8) and Remark (2.9) to the group multiplication of a bi-algebraic group and show that we can be less stringent on cardinality assumptions due to the bi-algebraic group structure.

Let \( G \) be a bi-algebraic group (affine or irreducible almost affine) with the group multiplication \( \mu: G \times G \to G \) and let \( f \) be a morphic function on \( G \). Then we will set \( x \cdot f = f \circ \mu_x \), \( f \cdot x = f \circ \mu_x \) and \( x \cdot f \cdot y = f \circ \mu_y \circ \mu_x \) for \( x, y \in G \), and also we will denote by \( G \cdot f, f \cdot G, \) and \( G \cdot f \cdot G \) (the linear spans (which are subspaces of \( A(G) \)) of \( \{ x \cdot f : x \in G \} \), \( \{ f \cdot x : x \in G \} \), and \( \{ x \cdot f \cdot y : x, y \in G \} \), respectively.

**Proposition 3.1.** If \( k \) is uncountable and \( G \) is a bi-algebraic group (affine or irreducible almost affine), then the following are all equivalent:

1. \( \dim_k f \cdot G < \text{card}(k) \) for every \( f \) in \( A(G) \).
(2) \( \dim_k G \cdot f \) is finite for every \( f \) in \( A(G) \).

(3) \( \dim_k G \cdot f < \text{card}(k) \) for every \( f \) in \( A(G) \).

(4) \( \dim_k f \cdot G \) is finite for every \( f \) in \( A(G) \).

(5) \( \dim_k G \cdot f \cdot G \) is finite for every \( f \) in \( A(G) \).

(6) \( \dim_k G \cdot f \cdot G < \text{card}(k) \) for every \( f \) in \( A(G) \).

(7) The group multiplication \( \mu : G \times G \to G \) is a morphism.

(8) \( G \) is isomorphic to a pro-affine algebraic group \( \text{Alg}_k(B, k) \) for some Hopf algebra \( B \).

(9) \( G \) is a pro-affine algebraic group.

Proof. (1) \( \rightarrow \) (2): Let \( \{f_i : i \in I\} \) be a basis for the span \( f \cdot G \) of \( \{f \cdot x : x \in G\} \), where \( \text{card}(I) < \text{card}(k) \). Then for any \( x \in G \), we can write \( f \cdot x = \sum a_i(x) f_i \) (finite sum), where \( a_i : G \to k \) are some functions. Let \( \Omega = \{F \subseteq I : F \text{ is finite}\} \). Then \( \text{card}(\Omega) = \text{card}(I) \). Define \( V_F = \{x \in G : a_i(x) = 0, i \notin F\} \) for \( F \in \Omega \). Then \( G = \bigcup V_F \) where the union is taken over \( \Omega \). By Proposition (2.7), for some \( F \in \Omega \), \( G = \text{Cl}(V_F) \) because \( \text{card}(\Omega) = \text{card}(I) < \text{card}(k) \).

Let \( x \in V_F \). Then \( f \cdot x = \sum a_i(x) f_i \), where \( i \in F \). We can find a set \( \{g_i \in A(G) : i \in F\} \) of morphic functions in \( A(G) \) such that for all \( x \in V_F \), \( f \cdot x = \sum b_i(x) g_i \), and \( g_i(y_j) = \delta_{ij} \), for some \( y_j \in G, j \in F \) (by Lemma (2.1)).

Now, let \( y \in G \), and consider the function on \( G \), \( y \cdot f = \sum g_i(y)(y_i \cdot f) \), where \( i \in F \). It is in \( A(G) \) since \( y \cdot f, y_i \cdot f \in A(G) \). Evaluating the function at \( x \in V_F \), we get

\[
(y \cdot f - \sum g_i(y)(y_i \cdot f))(x) = f(xy) - \sum g_i(y)f(xy_i)
= (f \cdot x)(y) - \sum (f \cdot x)(y_i)g_i(y)
= (f \cdot x)(y) - \sum b_i(x)g_i(y)
= (f \cdot x - \sum b_i(x)g_i)(y) = 0,
\]

where the sum is taken over \( F \), which means that the morphic function \( y \cdot f - \sum g_i(y)(y_i \cdot f) \) vanishes on a dense subset \( V_F \) of \( G \), so it is an identically zero function on \( G \), i.e., \( y \cdot f = \sum g_i(y)(y_i \cdot f) \), where the sum is taken over \( F \). Hence the span \( G \cdot f \) of the left translates of \( f \), \( \{y \cdot f : y \in G\} \), has finite dimension \( \leq \text{card}(k) \) with generators \( \{y_i \cdot f : i \in F\} \).

(2) \( \rightarrow \) (3): Obvious.

(3) \( \rightarrow \) (4): Modify the proof of the first implication; (1) \( \rightarrow \) (2).

(4) \( \rightarrow \) (1): Obvious.

(4) \( \rightarrow \) (5): Clear because \( y \cdot f \cdot x = y \cdot (f \cdot x) = (y \cdot f) \cdot x \), and (2) and (4) are equivalent.

(5) \( \rightarrow \) (6): Obvious.

(6) \( \rightarrow \) (1): Clear because \( f \cdot x = (1 \cdot f) \cdot x = 1 \cdot f \cdot x \).
(4) → (7): Let \( \{f_1, \ldots, f_n\} \) be a basis of \( f \cdot G \) such that \( f_i(y_j) = \delta_{ij} \) for some \( y_j \in G \) (by Lemma (2.1)). Now for all \( x \in G \) we can write \( f \cdot x = \sum a_i(x) f_i \), where \( a_i: G \to k \) are some functions. Then \( a_i(x) = (f \cdot x)(y_j) = (y_j \cdot f)(x) \), so \( y_j \cdot f = a_j \), which implies that \( a_j \in A(G) \) for all \( j \). But \( (f \cdot x)(y) = \sum a_i(x) f_i(y) \), so \( f(xy) = \sum a_i(x) f_i(y) \), i.e., \( (f \circ \mu)(x, y) = (\sum a_i \otimes f_i)(x, y) \), hence \( f \circ \mu = \sum a_i \otimes f_i \) belongs to \( A(G) \otimes A(G) \), which means that \( \mu \) is a morphism.

(7) → (8). We divide our demonstration into several steps.

(i) \( A(G) \) is a bi-stable subalgebra of the Hopf algebra of all representative functions on \( G \).

We must show that for any \( f \) in \( A(G) \) the linear span \( f \cdot G \) of the set \( \{f \cdot x; x \in G\} \) is a finite dimensional subspace of \( A(G) \). Since the group multiplication \( \mu: G \times G \to G \) is a morphism, the coordinate ring \( A(G) \) is a bi-stable bi-algebra with the corresponding comorphism as its comultiplication \( \mu^*: A(G) \to A(G) \otimes A(G) \). Let \( \mu^* = \gamma \). If \( f \) is an element of \( A(G) \), let \( \gamma(f) = \sum g_i \otimes h_i \), \( i = 1, \ldots, n \), where \( g_i \) and \( h_i \) are chosen to be linearly independent over \( k \). Then \( f \cdot x = \sum g_i(x) h_i \) and so \( f \cdot G \) is finite dimensional with a basis \( \{h_i\} \).

(ii) \( G \) is isomorphic to the group of all proper algebra automorphisms of \( A(G) \) under the representation of \( G \) in the latter group.

We first recall some facts about proper algebra automorphisms. An algebra automorphism \( \sigma \) of \( A(G) \) is proper if and only if \( \gamma \circ \sigma = (1_A \otimes \sigma) \circ \gamma \). But in our case \( \sigma \) is proper if and only if it commutes with any right translation on \( A(G) \) given by \( g \to g \cdot x \) for \( x \in G \) and \( g \in A(G) \). To prove this, look at the left hand side in the defining equation. If \( f \in A(G) \), and \( x, y \in G \), then \( \gamma(\sigma(f))(x, y) = \sum g_i(x) h_i(y) \) (where we set \( \gamma(\sigma(f)) = \sum g_i \otimes h_i \)). Next consider the right hand side of the equation. \( \gamma((1_A \otimes \sigma) \circ \gamma(f))(x, y) = \sum a_i(y) (\sigma(b_i))(y) \) (where we assume that \( \gamma(f) = \sum a_i \otimes b_i \)). Since \( y \) is arbitrary, this proves our claim.

We now show that the representation of \( G \) in the group of all proper algebra automorphisms of \( A(G) \); \( x \to (1_A \otimes x) \circ \gamma \), is an isomorphism. We note that \( (1_A \otimes x) \circ \gamma \) is nothing but the left translation: \( f \to x \cdot f \), for if \( f \in A(G) \) and \( y \in G \), then \( ((1_A \otimes x) \circ \gamma)(f)(y) = \sum g_i(y) h_i(x) \) (where we let \( \gamma(f) = \sum g_i \otimes h_i \)), hence \( (1_A \otimes x) \circ \gamma(f)(x, y) = (x \cdot f)(y) \). So it is immediate from this fact that \( (1_A \otimes x) \circ \gamma \) is indeed a proper algebra automorphism of \( A(G) \) because it evidently commutes with every right translation given by \( g \to g \cdot y \) for \( y \in G \) and \( g \in A(G) \). The map is clearly injective. To show that it is also surjective, let \( \sigma \) be any proper algebra automorphism of \( A(G) \). Then \( \epsilon \circ \sigma \) is an algebra homomorphism \( A(G) \to k \) (where \( \epsilon: A(G) \to k \) is given by \( f \to f(1) \)), so \( \epsilon \circ \sigma \in G \) (now regarded as \( \text{Alg}_k(A(G), k) \)). If we show that \( (1_A \otimes (\epsilon \circ \sigma)) \circ \gamma = \sigma \) we are done. Now let \( f \in A(G) \) and \( x \in G \). Then \( ((1_A \otimes (\epsilon \circ \sigma)) \circ \gamma(f))(x) = \sum g_i(x)(\sigma(h_i))(1) \) (where we let \( \gamma(f) = \sum g_i \otimes h_i \).
\(\sigma(f \cdot x)(1)\) (by the same argument as we used before) = \((\sigma(f) \cdot x)(1)\) (because \(\sigma\) is proper) = \(\sigma(f)(x)\). Hence we have that \(\mathbb{1}_A \otimes (\varepsilon \circ \sigma) \circ \gamma = \sigma\) as required.

(iii) \(G\) is isomorphic to some pro-affine algebraic group.

Let \(B\) be the smallest Hopf algebra containing the bi-algebra \(A(G)\). Then \(\text{Alg}_k(B, k)\) (the set of algebra homomorphisms from \(B\) onto \(k\)) is a pro-affine algebraic group.

Hochschild and Mostow proved that \(\text{Alg}_k(B, k)\) is isomorphic with the group of all proper algebra automorphisms of \(A(G)\) [2, p. 1129]: They showed that every proper algebra automorphism \(\sigma\) of \(A(G)\) can be mapped to a unique extension of \(\varepsilon \circ \sigma\) (in \(\text{Alg}_k(A(G), k) = G\)) to \(\text{Alg}_k(B, k)\). Hence every element of \(G\) can be regarded also as an element of \(\text{Alg}_k(B, k)\), so we must have \(G = \text{Alg}_k(A(G), k) = \text{Alg}_k(B, k)\) as we wished. (We remark that they were interested to show only that the group of all proper algebra automorphisms of \(A(G)\) is isomorphic with \(\text{Alg}_k(B, k)\). In fact, in their context, \(G\) is not isomorphic with the group of all proper automorphisms of \(A(G)\) because they did not assume that the pair \((G, A(G))\) is a pro-affine algebraic variety but only that \(A(G)\) is a bi-stable algebra of representative functions on \(G\).

\((8) \rightarrow (9)\): Nothing to prove.

\((9) \rightarrow (4)\): Obvious; it is (i) in the implication; \((7) \rightarrow (8)\). This completes the proof of Proposition (3.1).

We should remark that the crucial point in the above discussion is the first implication: \(\dim_k f \cdot G < \text{card}(k) \rightarrow \dim_k G \cdot f\) is finite, in whose proof we used Proposition (2.7) under the assumption \(k\) is uncountable.

We established by Proposition (3.1) that if the ground field is sufficiently large every bi-algebraic group (affine or irreducible almost affine) is an algebraic group.

**Corollary 3.2.** Every affine bi-algebraic group over an uncountable field is an algebraic group.

**Proof.** Clear because \(\dim_k G \cdot f \cdot G \leq \dim_k A(G)\) which is always countable.

In conclusion, we summarize what is obtained: First, the answers to (1) and (2) of the problems of Palais stated at the outset of Section 1 are both in the affirmative if and only if \(\dim_k G \cdot f\) is finite (or \(\dim_k G \cdot f < \text{card}(k)\) when \(\text{card}(k)\) is uncountable) for every morphic function \(f\) on \(G\); and secondly, (2) always follows from (1). However, there still remains the question whether or not \(\dim_k G \cdot f\) is finite (or \(\dim_k G \cdot f < \text{card}(k)\) when \(\text{card}(k)\) is uncountable) for any irreducible almost affine bi-algebraic group, although it is always the case for every affine bi-algebraic group due to the result in [4].
ACKNOWLEDGMENT

The author is indebted to the referee for numerous comments and suggestions for improving the exposition.

REFERENCES