Existence, Regularity, and Decay Rate of Solutions of Non-Newtonian Flow

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The existence and the regularity of Young measure-valued solutions to non-Newtonian flows are considered. Furthermore, the uniqueness of solutions and their asymptotic behavior are given.

1. INTRODUCTION AND STATEMENT OF THE RESULT

Fluid dynamics has attracted the attention of many mathematicians and engineers. The Navier–Stokes equations are generally accepted as the correct governing equations for the incompressible motion of viscous fluids. If the relationship between the stress and the rate of strain is linear, then the fluid is called Newtonian. That is, Newtonian fluids satisfy the linear relationship

$$\tau = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where \(\tau\) is the stress and \(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\) is the rate of strain. The coefficient of proportionality \(\mu\) is called the viscosity, and it is a characteristic material quantity for the fluid concerned, which in general depends on temperature and pressure. Air, other gases, water, motor oil, alcohols, and simply hydrocarbon compounds, for example, tend to be Newtonian fluids. Their governing equations of motion will be the Navier–Stokes equations. If the relationship is not linear, the fluid is called non-Newto-

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nian. Examples of non-Newtonian fluids are molten plastics, polymer solutions, dyes, varnishes, suspensions, adhesives, paints, greases, and biological fluids like blood. The simplest model of the stress–strain relationship for such fluids is given by the power law, which states that

\[
\tau = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^q
\]

for \(0 < q < 1\); see Böhme [4].

Ladyzhenskaya [5, 6] proposed a new model to study some kinds of non-Newtonian fluids which is of interest to us. Ladyzhenskaya’s model is

\[
\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \Gamma_{ij}}{\partial x_j} + \rho f_i, \\
\frac{\partial u_i}{\partial x_i} = 0,
\]

(1.1)

in \(Q = \Omega \times (0, \infty)\), the initial condition \(u(x, 0) = u_0\) for \(x \in \Omega\), and with Dirichlet boundary condition, where \(E = (E_{ij})\). Here, \(\Omega \subset \mathbb{R}^n\) is bounded, \(r > -1\), and \(\mu_0, \mu_1 > 0\). These models are called

- Newtonian for \(\mu_0 > 0\) and \(\mu_1 = 0\);
- Rabinowitsch for \(\mu_0, \mu_1 > 0\) and \(r = 2\);
- Ellis for \(\mu_0, \mu_1 > 0\) and \(r > 0\);
- Ostwald–de Wael for \(\mu_0 = 0, \mu_1 > 0\), and \(r > -1\);
- Bingham for \(\mu_0, \mu_1 > 0\) and \(r = -1\).

For \(\mu_0 = 0\), if \(r < 0\), then it is a pseudo-plastic fluid, and if \(r > 0\), then it is a dilatant fluid; see Böhme [4]. The values of the parameters \(\mu_1\) and \(r\) of some of the pseudo-plastic Ostwald–de Wael models are given in Whitaker [11]. For example, for paper pulp, \(\mu_1 = 0.418\) and \(r = -0.425\), and for carboxymethyl cellulose in water, \(\mu_1 = 0.194\) and \(r = -0.434\).

For \(\mu_0, \mu_1 > 0\), Ladyzhenskaya [6] obtained the existence of the weak solutions for \(r \geq 2n/(n + 2) - 1\) and their uniqueness for \(r \geq 0\) \((n = 2)\) and for \(r > \frac{1}{2} (n = 3)\). Bellout, Bloom, and Nečas [2] studied the non-New-
tonian fluids for $\mu_0 = \mu_1 > 0$ for space periodic problems. They showed that there are Young measure-valued solutions if

$$ r \in \begin{cases} (-1, \infty) & \text{for } n = 2, \\ (-\frac{4}{3}, \infty) & \text{for } n = 3, \end{cases} $$

and that Young measure-valued solutions are weak solutions if

$$ r \in \begin{cases} (-\frac{1}{2}, \infty) & \text{for } n = 2, \\ (-\frac{1}{3}, \infty) & \text{for } n = 3, \end{cases} $$

and unique weakly regular solutions if

$$ r \in \begin{cases} (0, \infty) & \text{for } n = 2, \\ (\frac{1}{2}, \infty) & \text{for } n = 3. \end{cases} $$

For Young measure, refer to Pedregal [9]. In Bae and Choe [1], we showed the existence of Young measure-valued solutions for all $r \in (-1, \infty)$ when $\mu_0, \mu_1 > 0$ and for $r \in [0, \infty)$ when $\mu_0 = 0, \mu_1 > 0$. Moreover we have shown the Young measure-valued solutions are weak solutions if certain convexity condition for energy holds.

In this paper, we consider the pseudo-plastic Ostwald-de Wael models $\mu_0 = 0, \mu_1 > 0$, and $-1 < r < 0$ for $n = 2, 3$, and the models such that $\mu_0 > 0$ and $r > -1$. We obtain the existence of Young measure-valued solutions for the Ostwald-de Wael models $\mu_0 = 0$ for all $r$ such that

$$ r \in \begin{cases} (-1, \infty) & \text{when } n = 2, \\ (-\frac{3}{2}, \infty) & \text{when } n = 3. \end{cases} $$

We also showed that for $n = 2$, the Young measure-valued solutions for the periodic problems are regular for all $r \in (-1, 0)$. In a similar way, we can show that for $n = 2$ the Young measure-valued solutions for the periodic problems are weakly regular for all $r \in (-1, \infty)$ for $\mu_0 \geq 0$ and $\mu_1 > 0$ (see the remark following Theorem 4.5). When $n = 2$ and $\mu_0 > 0$, we show the uniqueness of solutions for $r > -1$. Moreover, we estimate the asymptotic behavior of solutions for the Ostwald-de Wael models $\mu_0 = 0$.

In Section 3, we consider (1.1) on a bounded domain for $\mu_0 = 0$. We show that there exist Young measure-valued solutions to (1.1). This existence result works for Dirichlet, Neumann, and the periodic boundary conditions. It turns out that the Galerkin method is suitable to construct the approximate smooth solutions. Then, a compactness lemma for $L^2$
space guarantees that the weak limit of approximate solutions is indeed the Young measure-valued solutions to our problem.

In Section 4 we show that for \(n = 2\) the Young measure-valued solutions for the space periodic problems are weakly regular or regular for all \(r \in (-1, \infty)\) and \(\mu_0 \geq 0, \mu_1 > 0\), and that the solutions are unique for all \(r > -1\) and \(\mu_0 > 0\).

In Section 5, we obtain the decay rates and the asymptotic behavior for solutions.

2. FORMULATION OF NON-NEWTONIAN FLOW

We consider the non-Newtonian flow on a bounded open domain \(\Omega \subset \mathbb{R}^n\),

\[
\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \mu_0 \frac{\partial \Gamma_{0,ij}}{\partial x_j} + \mu_1 \frac{\partial F_{r,ij}}{\partial x_j} + \rho f, \\
\frac{\partial u}{\partial x} = 0, \\
\Gamma_{r,ij} = |E(\nabla u)|^r E_{ij}(\nabla u), \\
\Gamma_{0,ij} = E_{ij}(\nabla u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

in \(Q = \Omega \times (0, \infty)\), and the initial condition \(u(x,0) = u_0\) for \(x \in \Omega\), and Dirichlet boundary data \(u = 0\) on \(\partial \Omega\), where \(r > -1, \mu_0 \geq 0\), and \(\mu_1 > 0\). The constant \(\rho\) is the density; for simplicity, we let \(\rho = 1\).

We define the usual Sobolev space \(W^{1,q}(\Omega)\) by the set of all \(L^q(\Omega)\) functions whose \(j\)th order derivatives, \(j \leq 1\), are in \(L^q\). Let

\[
\mathcal{V} = \left\{ v \in C^0_0(\Omega)^n : \text{div} v = 0 \right\}, \\
\mathcal{V}_q = \text{closure of } \mathcal{V} \text{ in } W^{1,q}(\Omega)^n, \\
\mathcal{V}_q = \text{closure of } \mathcal{V} \text{ in } W^{1,q}(\Omega)^n, \\
\mathcal{H} = \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^n.
\]

Notice that the inclusion \(\mathcal{V} \hookrightarrow \mathcal{H}\) is compact. The inner product of \(\mathcal{V}\) is given by \(\langle \nabla u, \nabla v \rangle\), where \(\langle \cdot, \cdot \rangle\) is the usual inner product of \(\mathcal{H}\). Let \(\mathcal{V}\) and \(\mathcal{V}_q\) be the dual spaces of \(\mathcal{V}\) and \(\mathcal{V}_q\), respectively, where \(1/q + 1/q' = 1\).
We will use the notation
\[\|u\|_q = \|u\|_{L^q} = \left(\int_{\Omega} |u|^q \; dx\right)^{1/q} \]  
(2.2)

If \( q \leq 2 \), then \( V \subset V_q \), and if \( q \geq 2 \), then \( V_q \subset V \).

For the convection term we define a bilinear mapping \( B \) such that
\[\langle Bu, v \rangle \overset{\text{def}}{=} b(u, u, v),\]
where
\[b(u, v, w) = \sum_{i,j} \int_{\Omega} u_i \partial_j v_i w_j \; dx.\]

We now present the weak formulation for the problem (2.1): For given \( f, u_0 \) with \( u_0 \in H \) and \( f \in L^q(0,T; L^q(\Omega)^n) \) for some \( q > 1 \), find \( u \) with
\[u \in L^r(0,T; H) \cap L^{r+2}(0,T; V_{r+2}),\]
and satisfying that, for all \( v \in V_{r+2} \),
\[
\frac{d}{dt}\langle u, v \rangle + \mu_0 \Gamma_0(\nabla u, \nabla v) + \mu_1 \Gamma_1(\nabla u, \nabla v) + b(u, u, v) = \langle f, v \rangle, 
\]
(2.4)
\[u(x,0) = u_0(x),\]
where
\[\Gamma_1(\nabla u, \nabla v) = \sum_{i,j} \int_{\Omega} |E(\nabla u)|^r |E_{ij}(\nabla u) E_{ij}(\nabla v) \; dx.\]

Here, Eq. (2.4) is defined in the distribution sense for the scalar functions of time \( t \). The constant \( q \) will be specified later depending on \( \mu_0, \mu_1, r, \) and \( n \).

The subsequent constant \( C \) depends only on the initial data \( u_0 \) and time \( T \) if there is no specific statement.

Remark. If the boundary condition is Neumann or periodic, then we can redefine the solution spaces \( H, V, V_q \) corresponding to the boundary condition.
3. EXISTENCE OF YOUNG MEASURE VALUED SOLUTIONS

For $\mu_0, \mu_1 > 0$, the existence of Young measure-valued solutions is given in Bae and Choe [1] and in Bellout, Bloom, and Nečas [2]. In Bellout, Bloom, and Nečas [2], the existence is given for $r > -1$ when $n = 2$ and for $r > -\frac{3}{2}$ when $n = 3$. In Bae and Choe [1], it is given for $r > -1$ when $n = 3$ and $\mu_0, \mu_1 > 0$, and for $r \geq 0$ when $n = 3$ and $\mu_0 = 0$. In this section, for $\mu_0 = 0$ we find the criterion of $r$ for the existence such that $r > -1$ when $n = 2$, and $r > -\frac{3}{2}$ when $n = 3$.

We use the Galerkin method to get the existence. The energy estimate follows in a standard way. The difficulty lies in showing compactness in $L^2$ space. With the compactness we can prove the strong convergence of approximate convection terms. Consider (2.4) for $\mu_0 = 0$ on $Q_T^{\text{def}} = \Omega \times [0, T]$, where $T > 0$ is fixed. Since $V_{r+2} \subset H$ for $r > -1$ when $n = 2$, and for $r \geq -\frac{3}{2}$ when $n = 3$, we consider $r$ in such ranges. Thus, for such $r$, we have $V_{r+2} \cap H = V_{r+2}$.

Since $V'$ is dense in $V_{r+2}$, and $V_{r+2}$ is separable, there exists an orthonormal subset $(w_i; i = 1, 2, \ldots) \subset V'$ that spans $V_{r+2}$. For each $m$ we define an approximate solution $u^m$ of (2.4),

$$u^m \overset{\text{def}}{=} \sum_{l=1}^{m} g_i^m(t) w_i(x)$$

and

$$\langle u^m_i, w_k \rangle + \mu_1 \langle \nabla u^m, \nabla w_k \rangle + b(u^m, u^m, w_k) = \langle f, w_k \rangle, \quad (3.1)$$

$$u^m(0) = u^m_0, \quad (3.2)$$

where $u^m_0$ is the orthogonal projection in $H$ of $u_0$ onto the space spanned by $(w_1, \ldots, w_m)$. Here the initial data of $g_i^m$ satisfy $g_i^m(0) = \langle u^m_0, w_i \rangle$. For convenience, we ignore the superscript $m$ of $g$. Then we have $u^m_i = \sum_{l=1}^{m} g_l(t) w_i(x)$ and $\partial_t u^m = \sum_{l=1}^{m} g_l(t) \partial_t w_i(x)$. Equation (3.1) becomes

$$\dot{g}_i \langle w_i, w_k \rangle + g_i g_r \langle w_{i,j}, \partial_j w_r, w_k \rangle + \mu_1 \langle \Gamma_{r,i,j}^m, E_{ij}(\nabla w_k) \rangle = \langle f, w_k \rangle, \quad (3.3)$$

where

$$\langle \Gamma_{r,i,j}^m, E_{ij}(\nabla w_k) \rangle = \mu_1 g_i \langle |E_m|', E_{ij}(\nabla w_i), E_{ij}(\nabla w_k) \rangle$$

and $E_m = (E_{ij}(\nabla u^m))$. For summation, we use Einstein notation; that is, the repeated indices mean the summations from $i, j = 1, \ldots, n$ and from
\[ l = 1, \ldots, m. \] Notice that (3.3) is a system of nonlinear ODEs. For the existence and uniqueness of solution of (3.3), we need the Lipschitz property of the nonlinear terms of (3.3), which is easy to check. Thus, there exists a unique solution of (3.3) locally. In other words, for each \( m \geq 1 \), (3.3) has a maximal solution on some interval \([0, t_m] \). If \( t_m < T \), then \( \| u^m(t) \| \to \infty \) as \( t \to t_m \). The following lemma shows that this does not happen; therefore, \( t_m = T \).

**Lemma 3.1.** Assume that

\[
\int \left( \int |f| \left( \frac{n(r+1)}{(n(r+1)+r+2)} \right) \, dx \right)^{\frac{n(r+1)+r+2}{(n(r+1)+r+2)}} \, dt
\]  

is bounded. Then we have

\[
\sup_{0 < t < T} \| u^m(t) \|^2 + C \int_0^T \| \nabla u^m \|_{r+\frac{2}{2}}^2 \, dt
\]

\[
\leq C \| u^m(0) \|^2 + C \int_0^T \left( \int |f| \left( \frac{n(r+1)}{(n(r+1)+r+2)} \right) \, dx \right)^{\frac{n(r+1)+r+2}{(n(r+1)+r+2)}} \, dt.
\]

**Proof.** Multiply (3.3) by \( g_k \) and add them for \( k = 1, 2, \ldots, m \). Then

\[
\sum_{l, k = 1}^m g_l g_k \langle w_l, w_k \rangle + \mu_1 \sum_{l, k = 1}^m g_l g_k \langle \nabla u^m, E_{ij}(\nabla w_l), E_{ij}(\nabla w_k) \rangle = \langle f, u^m \rangle,
\]

(3.5)

since \( b(u^m, u^m, u^m) = 0 \). We recall Korn’s inequality given in Nečas and Hlaváček [8] for \( s = 2 \), and in Mosolov and Mjasnikov [7] for \( 1 < s < \infty \):

\[
\left( \int |\nabla E_{ij}(\nabla u^m)|^2 \, dx \right)^{1/4} \geq C \| \nabla u^m \|_s.
\]

(3.6)

Hence, considering the definition of \( u^m \) and Korn’s inequality (3.6), we obtain

\[
\frac{d}{dt} \| u^m \|^2 + C \| \nabla u^m \|_{r+\frac{2}{2}}^2 \leq 2 \langle f, u^m \rangle.
\]

(3.7)
By Hölder’s, Sobolev’s, and Young’s inequalities we get

\[
|\langle f, u^m \rangle| \leq \left( \int |u^m|^{(n(r+2))/((n-r+2))} \, dx \right)^{(n-(r+2))/((n-r+2))} \times \left( \int |f|^{(n(r+2))/((n-r+1)+r+2)} \, dx \right)^{(n(r+1)+r+2))/((n-r+2))}
\]

\[
\leq C \left( \int |\nabla u^m|^{r+2} \, dx \right)^{1/(r+2)} \times \left( \int |f|^{(n(r+2))/((n-r+1)+r+2)} \, dx \right)^{(n(r+1)+r+2)/((n-r+2))}
\]

\[
\leq \varepsilon C \int |\nabla u^m|^{r+2} \, dx + \varepsilon^{-1} \times \left( \int |f|^{(n(r+2))/((n-r+1)+r+2)} \, dx \right)^{(n(r+1)+r+2)/((n-r+2))}
\]

We have from (3.7), by taking \( \varepsilon \) small, that

\[
\frac{d}{dt} \|u^m\|^2 + C \|\nabla u^m\|_{r+2}^2 \leq C \left( \int |f|^{(n(r+2))/((n-r+1)+r+2)} \, dx \right)^{(n(r+1)+r+2)/((n-r+2))}
\]

By integrating the preceding equation with respect to \( t \), we complete the proof.

From Lemma 3.1, we have

\[ u^m \in L^r(0, T; H) \cap L^{r+2}(0, T; V_{r+2}). \]

Thus, we have a subsequence, still denoted by \( u^m \), that converges to \( u \) in

\[ L^{r+2}(0, T; V_{r+2}) \]

weakly, and in \( L^r(0, T; H) \) is weak-star, as \( m \to \infty \). For such a subsequence we need to take the limit to (3.1) in order to show the limit \( u \) is a Young measure-valued solution. Owing to the foregoing statement, we can take the limit to the infinity for the first term of (3.1). For the nonlinear viscosity term \( \Gamma \), we need to find the corresponding Young measure to the weak limit of \( \nabla u^m \). For Young measure, we refer to Pedregal [9]. For the nonlinear convection term of (3.1) we need a compactness lemma. We restate the compact embedding lemma from Theorem 2.1 of Temam [10, Chap. III].

**Lemma 3.2.** Assume that \( X_0, X, \) and \( X_1 \) are Banach spaces with

\[ X_0 \subset X \subset X_1, \]
where the injection is continuous and the injection of $X_0$ into $X$ is compact, and $X_0$ and $X_1$ are reflexive. Let $T > 0$ be finite and let $\alpha_0, \alpha_1 > 1$ be finite numbers. Define the space

$$
\mathcal{Y} = \mathcal{Y}(0, T; \alpha_0, \alpha_1; X_0, X_1)
$$

where

$$
\mathcal{Y} = \left\{ v \in L^{\alpha_0}(0, T; X_0), \dot{v} \text{ def } = \frac{d}{dt} v \in L^{\alpha_1}(0, T; X_1) \right\},
$$

which is a Banach space for the norm

$$
\|v\|_{\mathcal{Y}} \text{ def } = \|v\|_{L^{\alpha_0}(0, T; X_0)} + \|\dot{v}\|_{L^{\alpha_1}(0, T; X_1)}.
$$

Then the injection of $\mathcal{Y}$ into $L^{\alpha_0}(0, T; X)$ is compact.

We define by $\mathcal{V}_{r+2}$ the intersection of $\mathcal{V}_{r+2}$ and $W^{2, r+2}(\Omega)^n$, and by $\mathcal{V}^2$ the dual space of $\mathcal{V}_{r+2}$. We take $X_0 = W^{1, r+2}$, $X = L^2$, and $X_1 = \mathcal{V}^2$. Then the embedding $X_0 \hookrightarrow X$ is compact for

$$
r > \begin{cases} 
-1 & \text{for } n = 2, \\
-\frac{4}{5} & \text{for } n = 3.
\end{cases}
$$

By Lemma 3.1, we have $L^{r+2}(0, T; W^{1, r+2}(\Omega)^n)$ and $L^r(0, T; L^2(\Omega)^n)$ for all $s \geq 1$.

**Lemma 3.3.** Let $s_n$ be numbers such that for $n = 2$,

$$
s_2 = r + 2 \quad \text{for each } -1 < r < 0,
$$

and for $n = 3$, $s_3$ is given by

$$
s_3 = \begin{cases} 
\frac{r + 2}{r + 1} & \text{for } -\frac{4 + \sqrt{6}}{5} \leq r < 0, \\
5r + 4 & \text{for } -\frac{3}{5} < r \leq -\frac{4 + \sqrt{6}}{5}.
\end{cases}
$$

Assume that for $n = 2$, there is a number $\alpha \geq 2$ such that

$$
\int \left( \int |f|^{\alpha/(\alpha - 1)} dx \right)^{(\alpha/(\alpha - 1))s_2} dt
$$

(3.8)
is bounded, and for \( n = 3 \),

\[
\int \left( \int |f|^{6/5} \, dx \right)^{(5/6)s_3} \, dt
\]

is bounded. Then we have

\[ \dot{u}^m(t) \in L^\infty(0,T;\mathcal{V}^2). \]

**Proof.** For all \( \nu \in \mathcal{V}_2^2 \),

\[
\frac{d}{dt} \langle u^m, \nu \rangle = -\mu_2 \langle |E(\nabla u^m)|^{\nu} E_{ij}(\nabla u^m), E_{ij}(\nabla \nu) \rangle
\]

\[ - b(u^m, u^m, \nu) + \langle f, \nu \rangle. \]

Observe that

\[
\|\nabla \cdot \left( |E(\nabla u^m)|^{\nu} E_{ij}(\nabla u^m) \right)\|_{\mathcal{V}^2}
\]

\[
= \sup_{\nu \in \mathcal{V}_2^2} \frac{|\langle \nabla \cdot \left( |E(\nabla u^m)|^{\nu} E_{ij}(\nabla u^m) \right), \nu \rangle|}{\|\nu\|_{\mathcal{V}_2^2}}
\]

\[
= \sup_{\nu \in \mathcal{V}_2^2} \frac{|\langle |E(\nabla u^m)|^{\nu} E_{ij}(\nabla u^m) \rangle, E_{ij}(\nabla \nu) \rangle|}{\|\nu\|_{\mathcal{V}_2^2}}
\]

\[
\leq \| |E(\nabla u^m)|^{\nu + 1}\|.
\]

By Hölder’s inequality, we have

\[
\| |E(\nabla u^m)|^{\nu + 1}\| \leq C \left( \int |\nabla u^m|^{2(\nu + 1)} \, dx \right)^{1/2}
\]

\[
\leq C \left( \int |\nabla u^m|^{(\nu + 1)/(\nu + 2)} \, dx \right)^{(\nu + 1)/(\nu + 2)}.
\]

Thus we have

\[
\int_0^T \|\nabla \cdot \left( |E(\nabla u^m)|^{\nu} E_{ij}(\nabla u^m) \right)\|_{\mathcal{V}_2^2/(\nu + 1)} \, dt \leq C \int \|\nabla u^m\|^{\nu + 2} \, dx \, dt \leq C.
\]
By Sobolev’s inequality, we have that for $n = 2$ and for all $\alpha \geq 2$,

$$
\|Bu^m\|_{V^2} = \sup_{v \in V^2} \frac{\langle Bu^m, v \rangle}{\|v\|_{V^2}} = \sup_{v \in V^2} \frac{\langle (u^m \cdot \nabla) v, u^m \rangle}{\|v\|_{V^2}}
\leq \sup_{v \in V^2} \left( \frac{\int |u^m|^{2a/(a-1)} \, dx}{\|v\|_{V^2}^{(a-1)/a}} \right)^{(a-1)/a}
\leq C \sup_{v \in V^2} \left( \frac{\int |u^m|^{2a/(a-1)} \, dx}{\|v\|_{V^2}^{(a-1)/a}} \right)^{(a-1)/a}
\leq \left( \int |u^m|^{2a/(a-1)} \, dx \right)^{(a-1)/a}.
$$

In addition, for each $\alpha \geq 2$ we have that for $r < 0$ with $r > -2(\alpha - 1)/(2\alpha - 1),

$$
\left( \int |u^m|^{2a/(a-1)} \, dx \right)^{(a-1)/a}
= \left( \int |u^m|^{(r+2)/(r+1)(a-1)} + (2a(r+1) - (r+2))/(r+1)(a-1) \, dx \right)^{(a-1)/a}
\leq \left( \int |u^m|^{2(r+2)/(r+1)} \, dx \right)^{-r/(2a(r+1))}
\left( \int |u^m|^2 \, dx \right)^{(2a(r+1) - (r+2))/(2a(r+1))}
\leq C \left( \int |\nabla u^m|^2 \, dx \right)^{1/(a(r+1))}
$$
since $u^m \in L^2(0, T; L^2(\Omega)).$ Thus we have that for all $\alpha \geq 2$ and for $r < 0$ with $r > -2(\alpha - 1)/(2\alpha - 1),

$$
\int_0^T \|Bu^m\|_{V^2}^{(r+1)} \, dt \leq C. \tag{3.9}
$$

Notice that

$$
\alpha(r + 1) > 1 \quad \text{if and only if} \quad r > \frac{1}{\alpha} - 1.
$$

We take

$$
\alpha = \frac{r + 2}{r + 1}.
$$

Thus, for $n = 2$, for $0 > r > -1$, setting $s_2 = r + 2$, we have that

$$
\int_0^T \|Bu^m\|_{V^2}^{s_2} \, dt \leq C.
$$
For $n = 3$, by Sobolev’s inequality we have that for $0 > r > -\frac{2}{5}$,

\[
\|Bu^m\|_{V^2} = \sup_{v \in V^2} \frac{|\langle Bu^m, v \rangle|}{\|v\|_{V^2}} \\
\leq \sup_{v \in V^2} \left( \frac{\int |u^m|^{12/5} \, dx}{\|v\|_{V^2}} \right)^{5/6} (\int |\nabla v|^6 \, dx)^{1/6} \leq \left( \int |u^m|^{12/5} \, dx \right)^{5/6} \\
= \left( \int |u^m|^{(18(3r+2))/(5(5r+4))} |u^m|^{(6(r+2))/(5(5r+4))} \, dx \right)^{5/6} \\
\leq \left( \int |u^m|^2 \, dx \right)^{(3(3r+2))/(2(5r+4))} \left( \int |u^m|^{3(r+2)/(1-r)} \, dx \right)^{(1-r)/(3(5r+4))} \\
\leq C \left( \int |\nabla u^m|^r \, dx \right)^{1/(5r+4)}
\]

since $u^m \in L^\infty(0, T; L^2(\Omega))$. Thus, for $0 > r > -\frac{3}{5}$ we have

\[
\int_0^T \|Bu^m\|_{V^2}^2 \, dt \leq C. \quad (3.10)
\]

Define $s_3 > 1$ by

\[
s_3 = \begin{cases} 
\frac{r + 2}{r + 1} & \text{for } -4 + \frac{\sqrt{6}}{5} \leq r < 0, \\
5r + 4 & \text{for } -\frac{3}{5} < r \leq -4 + \frac{\sqrt{6}}{5}.
\end{cases}
\]

Then, we have

\[
\int_0^T \|Bu^m\|_{V^2}^{s_3} \, dt \leq C.
\]

Notice that for $n = 2$, for all $\alpha \geq 2$,

\[
\|f\|_{V^2} = \sup_{v \in V^2} \frac{|\langle f, v \rangle|}{\|v\|_{V^2}} \\
\leq \sup_{v \in V^2} \left( \frac{\int |f|^{\alpha/(\alpha - 1)} \, dx}{\|v\|_{V^2}} \right)^{\alpha - 1/\alpha} (\int |v|^\alpha \, dx)^{1/\alpha} \\
\leq C \sup_{v \in V^2} \left( \frac{\int |f|^{\alpha/(\alpha - 1)} \, dx}{\|v\|_{V^2}} \right)^{\alpha - 1/\alpha} (\int |\nabla v|^2 \, dx)^{1/2} \\
\leq C \left( \int |f|^{\alpha/(\alpha - 1)} \, dx \right)^{(\alpha - 1)/\alpha}.
\]
Thus if there is a number $\alpha \geq 2$ such that
\[
\int \left( \int |f|^{a/(a-1)} dx \right)^{(a-1)/a(r+2)} dt
\]
is finite. Then we have
\[
\dot{u}^m \in L^{s_3}(0, T; \mathbb{V}^2).
\]
Notice that for $n = 3$,
\[
\|f\|_{\mathbb{V}^2} = \sup_{v \in \mathbb{V}^2} \frac{|\langle f, v \rangle|}{\|v\|_{\mathbb{V}^2}^{5/6} (\int |v|^6 dx)^{1/6}}
\begin{align*}
&\leq \sup_{v \in \mathbb{V}^2} \frac{(\int |f|^{5/5} dx)^{5/6} (\int |v|^6 dx)^{1/6}}{\|v\|_{\mathbb{V}^2}^{5/6}} \\
&\leq C \sup_{v \in \mathbb{V}^2} \frac{(\int |f|^{5/5} dx)^{5/6} (\int |\nabla v|^2 dx)^{1/2}}{\|v\|_{\mathbb{V}^2}^{5/6}} \\
&\leq C \left( \int |f|^{5/5} dx \right)^{5/6}.
\end{align*}
\]
Thus if
\[
\int \left( \int |f|^{5/5} dx \right)^{(5/6)s_3} dt
\]
is finite, then we have
\[
\dot{u}^m \in L^{s_3}(0, T; \mathbb{V}^2),
\]
where $s_3$ is given depending on $r$.  

Now we are ready to show our existence theorem for Young measure-valued solutions.

**Theorem 3.4 (Existence of Young measure-valued solutions).** Let $\mu_0 = 0$, $\mu_1 > 0$, and $n = 2$ or 3. Let $r$ be a number in Lemma 3.3. If $u_0 \in H$ and $f$ satisfies the conditions in Lemmas 3.1 and 3.3, then there is a measure-valued solution $(u, \nu_{x,i})$ of (2.4) satisfying
\[
- \int_{Q_T} u_i \partial_i \phi dx dt - \int_{Q_T} u_i u_j \nabla_j \phi_i dx dt \\
+ \mu_1 \int_{Q_T} \int_{\mathbb{R}^n} |E(\lambda)| \cdot E(\lambda) d\nu_{x,i}(\lambda) \cdot E(\nabla \phi) dx dt = \int_{\mathbb{R}^n} \phi_i dx dt
\]
for all \( \phi \in C_0^\infty(Q_T) \) with \( \nabla \cdot \phi = 0 \), where
\[
E_{ij}(\lambda) = \frac{1}{2}(\lambda_{ij} + \lambda_{ji})
\]
and \( Q_T \overset{\text{def}}{=} \Omega \times [0, T] \).

**Proof.** In order to use Lemma 3.2, we need the fact that the injection \( X_0 \hookrightarrow X \) is compact. If we take \( X_0 = V_{r+2} \) and \( X = H \), then the injection is compact. Thus from Lemma 3.3, we have that the injection of \( \mathcal{Y} \) into \( L^{r+2}(0, T; H) \),
\[
\mathcal{Y} = (0, T; r + 2, s_n; V_{r+2}, \mathcal{V}^2) \hookrightarrow L^{r+2}(0, T; H),
\]
is compact.

As in the paragraph before Lemma 3.2, we have a subsequence, still denoted by \( u^m \), that converges to \( u \) in \( L^{r+2}(0, T; V_{r+2}) \) weakly and in \( L^r(0, T; H) \) is weak-star, as \( m \to \infty \). For such a subsequence we need to take the limit to (3.1) in order to show the limit \( u \) is a weak solution. Owing to the preceding statement, we can take the limit to infinity for the first term of (3.1). By Lemma 3.3, we have the strong convergence of the subsequence \( u^m \) to \( u \) in \( L^{r+2}(0, T; H) \); in other words, \( u^m \to u \) in \( L^{r+2}(0, T; H) \) strongly, and this suffices to pass to the limit in (3.3). Let \( \psi \) be a continuously differentiable function on \( [0, T] \) with \( \psi(T) = 0 \). We multiply (3.1) by \( \psi(t) \) and then integrate by parts. This leads to the equation
\[
- \int_0^T \langle u^m(t), \psi'(t)w_k \rangle \, dt + \mu \int_0^T \Gamma_\gamma(\nabla u^m(t), \nabla w_k \psi(t)) \, dt \\
+ \int_0^T b(u^m(t), u^m(t), w_k \psi(t)) \, dt \\
= -\langle u_0^m, w_k \rangle \psi(0) + \int_0^T \langle f(t), w_k \psi(t) \rangle \, dt.
\]
The first term can pass to the limit. Consider the limit of the third term. Notice that
\[
\int_0^T b(u^m(t), u^m(t), w_k \psi(t)) \, dt = -\int_0^T \int_\Omega u^m_j \partial w_{k,i} \psi u^m_i \, dx \, dt,
\]
which converge to
\[
-\int_0^T \int_\Omega u_j \partial w_{k,i} \psi u_i \, dx \, dt = \int b(u, u, w_k \psi) \, dt
\]
due to the $L^{r+2}(0,T;V_{r+2})$ weak and $L^{r+2}(0,T;L^2(\Omega))$ strong convergence of $u^m$. Since $\Gamma^r_r$ is nonlinear with respect to $E$, we need to introduce a Young measure $\nu_{x,t}$ corresponding to the weak limit $(\nabla u^m)$. From the lower semicontinuity of $L^{r+2}$ norm and Korn's inequality (3.6), we have a uniform bound such that
\[ \int_0^T \int_{\Omega} |\nabla u^m|^{r+2} \, dx \, dt \leq C \]
for some $C$ independent of $m$. Thus, as in the case of Bellout, Bloom, and Nečas [2], we can find a Young measure $\nu_{x,t}$ for the weak limit of $(\nabla u^m)$ since $|\Gamma^r_r(u,u)| \leq C(1 + |\nabla u|^{r+2})$. For the existence of the Young measure $\nu_{x,t}$, refer to Theorem 3.2 in Bellout, Bloom, and Nečas [2] or to Pedregal [9]. Therefore, we conclude that
\[ \int_{\Gamma^r_r} (\nabla u^m(t), \nabla w_k \psi(t)) \, dt \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} |E(\lambda)|^r E(\lambda) \, d\nu_{x,t} \cdot E(\nabla w_k \phi_t) \, dx \, dt \]
as $m$ tends to the infinity.

4. REGULARITY FOR $n = 2$

In this section we consider the periodic problem on $\Omega = [0,1]^2$. When $r > - \frac{1}{2}$, Bellout, Bloom, and Nečas [2] have shown that there is a weak solution. We prove that the Young measure-valued solution $(u, \nu_{x,t})$ is regular for all $r > -1$ for $n = 2$.

**Lemma 4.1.**
\[ b(u^m, u^m, \Delta u^m) = 0. \]

**Proof.** Notice that
\[ b(u^m, u^m, \Delta u^m) = \sum_{i,j,k=1}^{2} \int_{\Omega} u_j^m \partial_i u_i^m \partial_k^2 u_i^m \, dx \]
\[ = - \sum_{i,j,k=1}^{2} \int_{\Omega} u_j^m \partial_k \partial_j u_i^m \partial_k u_i^m \, dx \]
\[ - \sum_{i,j,k=1}^{2} \int_{\Omega} \partial_k u_j^m \partial_j u_i^m \partial_k u_i^m \, dx. \]
The first part is zero due to the divergence-free condition and integration by parts; the second part is zero by direct calculation.
We recall the generalized form of Korn’s inequality in Bellout, Bloom, and Nečas [2, 3]:
\[
\int_{\Omega} \left| \frac{\partial E_{ij}(\nabla u)}{\partial x_k} - \frac{\partial E_{ij}(\nabla u)}{\partial x_k} \right|^2 dx \geq C\|u\|_2^2, \quad \text{for} \ 1 < s < \infty. \tag{4.1}
\]

The following lemma will be useful for the proof of the weak regularity for \( \mu_0 > 0 \).

**Lemma 4.2.** For all \( q \geq 2 \),
\[
\int_{\Omega} \partial_k \left( |E(\nabla u)|^q E_{ij}(\nabla u) \right) \partial_k E_{ij}(\nabla u) dx \geq C\|u\|_{1,q(1+1/2)}^2.
\]

**Proof.** Observe that
\[
\partial_k \left( |E|^q E_{ij} \right) \partial_k E_{ij} = \partial_k \left( (E_{ij} E_{ij})^{q/2} E_{ij} \right) \partial_k E_{ij}
\]
\[
= |E|^q \partial_k E_{ij} \partial_k E_{ij} + r |E|^{-2} E_{ij} \partial_k E_{ij} \partial_k E_{ij}.
\]

We note that
\[
\partial_k \left( |E|^q E_{ij} \right) \partial_k \left( |E|^q E_{ij} \right)
\]
\[
= \partial_k \left( (E_{ij} E_{ij})^{q/4} E_{ij} \right) \partial_k \left( (E_{ij} E_{ij})^{q/4} E_{ij} \right)
\]
\[
= \left( \frac{r}{2} |E|^{-2} E_{ij} \partial_k E_{ij} E_{ij} + |E|^{q/2} \partial_k E_{ij} \right)^2
\]
\[
= \frac{r^2}{4} |E|^{-2} E_{ij} \partial_k E_{ij} E_{ij} \partial_e E_{ij} \partial_k E_{ij}
\]
\[
+ r |E|^{-2} E_{ij} \partial_k E_{ij} \partial_k E_{ij} \partial_k E_{ij} + |E|^{q} \partial_k E_{ij} \partial_k E_{ij}
\]
\[
= \frac{r(r + 4)}{4} |E|^{-2} E_{ij} \partial_k E_{ij} E_{ij} \partial_k E_{ij} + |E|^{q} \partial_k E_{ij} \partial_k E_{ij}.
\]

Thus we have the equivalence of \( \partial_k (|E|^q E_{ij}) \partial_k E_{ij} \) and \( \partial_k (|E|^q E_{ij})^2 \). In particular, we have
\[
\frac{r + 4}{4} \partial_k (|E|^q E_{ij}) \partial_k E_{ij} \geq \left( \partial_k (|E|^q E_{ij}) \right)^2.
\]
Thus we have by Sobolev’s and Korn’s inequalities that
\[
\frac{r + 4}{4} \int_{\Omega} \partial_k (|E'| E_{ij}) \partial_k E_{ij} \geq \int_{\Omega} |\partial_k (|E|^{r/2} E_{ij})|^2 \\
\geq C \left( \int_{\Omega} |E|^{r/2} E_{ij}|^q \, dx \right)^{2/q} \\
\geq C \left( \int_{\Omega} |\nabla u|^{q(1+r/2)} \, dx \right)^{2/q}
\]
for all \( q \geq 2 \). \( \blacksquare \)

The following lemma will be useful for the proof of the regularity for \( \mu_0 = 0, \mu_1 > 0 \).

**Lemma 4.3.**

\[
\int |\nabla^2 u|^s \, dx \leq \frac{s}{2} \int \partial_k (|E(\nabla u)' E_{ij}(\nabla u)|) \partial_k E_{ij}(\nabla u) \, dx \\
+ C \int |\nabla u|^{-r/(2-s)} \, dx
\]
for \( 1 < s < 2 \).

**Proof.** Notice that for \( 0 < s < 2 \),

\[
\int |\nabla E(\nabla u)|^r \, dx \\
= \int |E(\nabla u)|^{-r/2} |E(\nabla u)|^{r/2} |\nabla E(\nabla u)|^2 \, dx \\
\leq \left( \int |E(\nabla u)|^{-r/2} |\nabla E(\nabla u)|^2 \, dx \right)^{r/2} \left( \int |E(\nabla u)|^{-r/(2-s)} \, dx \right)^{(2-s)/2} \\
\leq \frac{s}{2} \int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 \, dx + \frac{2 - s}{2} \int |E(\nabla u)|^{-r/(2-s)} \, dx.
\]

By Korn’s inequality (4.1) for \( 1 < s < 2 \), we have

\[
\int |\nabla^2 u|^s \, dx \leq \frac{s}{2} \int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 \, dx + C \int |\nabla u|^{-r/(2-s)} \, dx.
\]

For \( s = r + 2 \), we have

\[
\int |\nabla^2 u|^{r+2} \, dx \leq \frac{r + 2}{2} \int |E(\nabla u)|^r |\nabla E(\nabla u)|^2 \, dx + C \int |\nabla u| \, dx.
\]
We have from (3.5) with $\Delta w_k$ in the place of $w_k$ that

$$\frac{d}{dt}\|u^m\|^2 + \mu_2 \langle |E^m|^r, E_{ij}(\nabla u^m), E_{ij}(\nabla u^m) \rangle = -\langle f, \Delta u^m \rangle.$$  

Thus, we have

$$\frac{d}{dt}\|u^m\|^2 + C \int |\nabla^2 u^m|^{r+2} \, dx$$

$$\leq C \int |\nabla u^m| \, dx + |\langle f, \Delta u^m \rangle|$$

$$\leq C \int |\nabla u^m| \, dx + \varepsilon^{-1} \int |f|^{(r+2)/(r+1)} \, dx + \varepsilon \int |\Delta u^m|^{r+2} \, dx.$$  

Taking $\varepsilon$ small, we have

$$\frac{d}{dt}\|u^m\|^2 + C \int |\nabla^2 u^m|^{r+2} \, dx \leq C \int |\nabla u^m| \, dx + C \int |f|^{(r+2)/(r+1)} \, dx.$$  

**Lemma 4.4.** Assume that

$$\int_{Q_T} |f|^{(r+2)/(r+1)} \, dx \, dt$$

is bounded. Then we have

$$\sup_{0 < t < T} \|\nabla u^m(t)\|^2 + \int_{Q_T} |\nabla^2 u^m|^{r+2} \, dx \, dt$$

$$\leq C\|u^m(0)\|^2 + C \int_{Q_T} |f|^{(r+2)/(r+1)} \, dx \, dt.$$  

**Remark.** Condition (4.2) implies (3.4) and (3.8) for $n = 2$. 

From Lemmas 3.1 and 4.4, we have that for $n = 2$,

$$u^m \in L^r(0, T; H) \cap L^{r+2}(0, T; V_{r+2}) \cap L^r(0, T; V)$$

$$\cap L^{r+2}(0, T; W^{2, r+2}).$$  

Thus, we have a subsequence, still denoted by $u^m$, that converges to $u$ in $L^{r+2}(0, T; V_{r+2})$ weakly, and in $L^r(0, T; H)$ is weak-star, and in $L^{r+2}(0, T; W^{1, r+2})$ converges strongly as $m \to \infty$. For such a subsequence we can take the limit to (3.1) in order to show the limit $u$ is a regular solution.
THEOREM 4.5. For $n = 2$ and for each $r \in (-1, 0)$, if $(4.2)$ is bounded, then the Young measure-valued solution is regular and the Young measure is Dirac; that is, $\nu_{\lambda}(\lambda) = \delta(\lambda - \nabla u(x,t))$ a.e. in $\Omega \times (0,T)$. Furthermore, the solution $u$ satisfies

$$\sup_{0 < t < T} \|\nabla u(t)\|^2 + \int_{Q_T} |\nabla^2 u|^{r+2} \, dx \, dt \leq C\|u(0)\|^2 + C + \int_{Q_T} |f|^{(r+2)/(r+1)} \, dx \, dt.$$ 

Remarks. 1. For $r \geq 0$, our Young measure-valued solution is weakly regular such that for all $q \geq 2$,

$$\sup_{0 < t < T} \|\nabla u(t)\|^2 + \mu_1 \int \left( \int_{\Omega} |\nabla u|^{q(1+r/2)} \, dx \right)^{2/q} \, dt \leq C,$$

where $C$ depends on $u_0$ and $f$.

2. For $\mu_0 \geq 0$, in Bae and Choe [1] the existence of Young measure-valued solutions is given. In a similar way to Theorem 4.5, using Lemma 4.1 and Lemma 4.2 or Lemma 4.3, we can show that the Young measure-valued solutions are regular for $\mu_0 > 0$ such that for $-1 < r < 0$,

$$\sup_{0 < t < T} \|\nabla u(t)\|^2 + \mu_0 \int_{Q_T} |\nabla^2 u|^2 \, dx \, dt + \mu_1 \int_{Q_T} |\nabla^2 u|^{r+2} \, dx \, dt \leq C,$$

and weakly regular for $r \geq 0$ and for all $q \geq 2$,

$$\sup_{0 < t < T} \|\nabla u(t)\|^2 + \mu_0 \int_{Q_T} |\nabla^2 u|^2 \, dx \, dt + \mu_1 \int \left( \int_{\Omega} |\nabla u|^{q(1+r/2)} \, dx \right)^{2/q} \, dt \leq C,$$

where $C$ depends on $u_0$ and $f$.

We now consider the uniqueness of solutions for $\mu_0 > 0$ when $n = 2$. Before showing the uniqueness, we observe that since

$$\int_{Q_T} |u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \, dx \, dt \leq C,$$
by Sobolev’s inequality we have
\[
\int_\Omega \sup |u|^2 \, dx \leq C \int_Q |u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \, dx \, dt \leq C. \tag{4.3}
\]

Let \( u \) and \( v \) be two solutions of (2.4) for \( \mu_0 > 0 \). If we take the difference of (2.4) for \( u \) and \( v \), then we have, setting \( w = u - v \),
\[
\frac{d}{dt} \|w\|^2 + \mu_0 \int_\Omega |\nabla w|^2 \, dx + b(w, u, w)
+ \mu_1 \int_\Omega \left( |E(\nabla u)|^r \ E_{ij}(\nabla u) - |E(\nabla v)|^r \ E_{ij}(\nabla v) \right) E_{ij}(\nabla w) \, dx = 0.
\tag{4.4}
\]

Observe that
\[
|b(w, u, w)| \leq \int |w| |\nabla w| |u| \, dx \leq \left( \sup_\Omega |u| \right) \int |w| |\nabla w| \, dx
\]
and that
\[
|b(w, u, w)| \leq \int |w|^2 |\nabla u| \, dx \leq \left( \sup_\Omega |\nabla u| \right) \int |w|^2 \, dx.
\]

Since for \( r > -1 \),
\[
\int_\Omega \left( |E(\nabla u)|^r \ E_{ij}(\nabla u) - |E(\nabla v)|^r \ E_{ij}(\nabla v) \right) E_{ij}(\nabla w) \, dx \geq 0,
\]
we have
\[
\frac{d}{dt} \|w\|^2 + \mu_0 \int_\Omega |\nabla w|^2 \, dx \leq \left( \sup_\Omega |u|^2 \right) \int |w| |\nabla w| \, dx
\leq \left( \sup_\Omega |u|^2 \right) \int |w|^2 \, dx + \varepsilon \int |\nabla w|^2 \, dx.
\]

Taking \( \varepsilon \) small, we have
\[
\frac{d}{dt} \|w\|^2 \leq \left( \sup_\Omega |u|^2 \right) \int |w|^2 \, dx.
\]

Setting \( Y \overset{\text{def}}{=} \|w\|^2 \), we have
\[
Y' \leq \left( \sup_\Omega |u|^2 \right) Y.
\]
Then we have

\[ Y(t) \leq \exp\left( \int_0^t \sup_{\Omega} |u|^2 \, dt \right) Y(0) \leq \exp\left( \int_0^t \|u\|^2_{\mathcal{H}(\Omega)} \right) Y(0). \]

Owing to (4.3) and \( Y(0) = 0 \), we have

\[ \int |u(t)|^2 \, dx = 0 \quad \text{for} \quad t \geq 0. \]

Thus we have the following uniqueness theorem.

**Theorem 4.6.** Let \( n = 2 \) and \( \mu_0 > 0 \). Then the solution of (2.4) is unique for all \( r > -1 \).

### 5. Asymptotic Behavior

In this section we estimate the asymptotic behavior of solutions. Observe that

\[ \left( \int |u|^2 \, dx \right)^{(r+2)/2} \leq \left( \int |u|^{(n(2+r)/(n-2-r))} \, dx \right)^{(n-2-r)/n} \leq C \int |\nabla u|^{r+2} \, dx. \]

First consider the case \( f = 0 \). From

\[ \frac{d}{dt} \int |u|^2 \, dx + C \int |\nabla u|^{r+2} \, dx \leq 0, \]

we have

\[ \frac{d}{dt} \int |u|^2 \, dx + C \left( \int |u|^2 \, dx \right)^{(r+2)/2} \leq 0. \]

Let \( Y = \|u\|^2 \). Then

\[ Y' + CY^{(r+2)/2} \leq 0, \]

\[ \frac{Y'}{Y^{(r+2)/2}} \leq -C, \]

\[ -\frac{r}{2} Y^{-r/2} \leq -Ct - \frac{r}{2} Y(0)^{-r/2}. \]
From (5.1), we have that for $r > 0$,
\[
Y^{-r/2} \geq \frac{C}{r} t + Y(0)^{-r/2},
\]
\[
Y \leq \left( \frac{C}{r} t + Y(0)^{-r/2} \right)^{-2/r},
\]
which means that $\|u(t)\|$ is decreasing with rate $t^{-1/r}$ as $t$ tends to infinity. Again from (5.1), we have that for $r < 0$,
\[
Y^{-r/2} \leq \frac{C}{r} t + Y(0)^{-r/2},
\]
\[
Y \leq \left( \frac{C}{r} t + Y(0)^{-r/2} \right)^{-2/r},
\]
which means that $\|u(t)\|$ vanishes in a finite time, $t_0 = -(r/C)Y(0)^{-r/2}$.
If $r = 0$, then the decreasing rate of $Y(t)$ is exponential.

Now we consider with $f$. Observe that
\[
\left\langle f, u \right\rangle \leq \|f\| \|u\| \leq C \left( \int |f|^2 \, dx \right)^{(r+2)/(2(r+1))} + \varepsilon \left( \int |u|^2 \, dx \right)^{(r+2)/2}.
\]
Taking $\varepsilon$ small,
\[
\frac{d}{dt} \int |u|^2 \, dx + C \left( \int |u|^2 \, dx \right)^{(r+2)/2} \leq C \left( \int |f|^2 \, dx \right)^{(r+2)/(r+1)}.
\]
Setting
\[
C_f \overset{\text{def}}{=} C \left( \int |f|^2 \, dx \right)^{(r+2)/(2(r+1))},
\]
we have
\[
Y' + CY^{(r+2)/2} \leq C_f.
\]
Consider
\[
W' + CW^{(r+2)/2} = C_f. \tag{5.2}
\]
Since $W \geq 0$ and
\[
W' = -CW^{(r+2)/2} + C_f,
\]
if \(-CW(t)^{(r+2)/2} + C_f \leq 0\) for a time \(t\), then \(W(t)\) is decreasing at such a time \(t\), and if \(-CW(t)^{(r+2)/2} + C_f \geq 0\), \(W(t)\) is increasing at such a time \(t\).

Setting

\[
W_* \equiv \left( \frac{C_f}{C} \right)^{2/(r+2)},
\]

we observe that \(W_*\) is the asymptotically stable equilibrium of (5.2). If there is a \(t_0 \geq 0\) such that \(W(t_0) < W_*\), then \(0 \leq W(t) < W_* \) for all time \(t > t_0\) and \(W(t)\) converges to \(W_*\) monotonically as \(t\) tends to infinity. Let \(t_0\) be a time such that \(W_0 \equiv W(t_0) > W_*\). Then \(W(t) > W_*\) for all time \(t > t_0\) and \(W(t)\) converges to \(W_*\) monotonically as \(t\) tends to infinity.

Observe that

\[
W' + \left( C W^{(r+2)/2} - C_f \right) = 0,
\]

\[
(W - W_*)' + \left( W - W_* \right) \left( \frac{C W^{(r+2)/2} - C_f}{W - W_*} \right) = 0.
\]

As \(t \to \infty\),

\[
\frac{C W^{(r+2)/2} - C_f}{W - W_*} = C \frac{W^{(r+2)/2} - W_*^{(r+2)/2}}{W - W_*} \to \left( 1 + \frac{r}{2} \right) C^{2/(2+r)} C_f^{r/(2+r)}.
\]

For \(W > W_*\), we have that for \(r < 0\),

\[
C \frac{W^{(r+2)/2} - W_*^{(r+2)/2}}{W - W_*} < \left( 1 + \frac{r}{2} \right) C^{2/(2+r)} C_f^{r/(2+r)},
\]

and that for \(r > 0\),

\[
C \frac{W^{(r+2)/2} - W_*^{(r+2)/2}}{W - W_*} > \left( 1 + \frac{r}{2} \right) C^{2/(2+r)} C_f^{r/(2+r)}.
\]

For \(W \in (W_*, W_0]\) and for \(r < 0\),

\[
C \frac{W^{(r+2)/2} - W_*^{(r+2)/2}}{W - W_*} > C \frac{W_0^{(r+2)/2} - W_*^{(r+2)/2}}{W_0 - W_*}.
\]

Setting \(Z = W - W_*\) and

\[
Z_0 = \left( \frac{C W_0^{(r+2)/2} - C_f}{W_0 - W_*} \right),
\]

(5.3)
we have that for $Z > 0$,

$$Z' + Z_0Z \leq 0.$$

Then we have

$$Z(t) \leq \exp(-Z_0(t - t_0))Z(t_0) \quad \text{for all } t > t_0.$$ 

That is, for $W_0 = W(t_0)$,

$$W(t) \leq \exp(-Z_0(t - t_0))(W_0 - W_\ast) + W_\ast \quad \text{for all } t > t_0, \quad (5.4)$$

where $Z_0$ is given in (5.3). In a similar way, we have the asymptotic decay rate (5.4) for $r > 0$, where

$$Z_0 = \left(1 + \frac{r}{2}\right)C^{2/(2+r)}C^{r/(2+r)}.$$

Thus we have the asymptotic behavior of $Y$:

$$Y(t) \leq \exp(-Z_0(t - t_0))(W_0 - W_\ast) + W_\ast \quad \text{for all } t > t_0.$$

**Theorem 5.1.** For $f = 0$ we have that for $r < 0$,

$$\int_\Omega |u(t)|^2 \, dx \leq \left(\frac{C}{r} t + \left(\int_\Omega |u(0)|^2 \, dx\right)^{-r/2}\right)^{-2/r},$$

which means that $\|u(t)\|$ vanishes in a finite time $t_0$,

$$t_0 = -\frac{r}{C} \|u(0)\|^{-r}.$$

For $r > 0$ and $f = 0$,

$$\int_\Omega |u(t)|^2 \, dx \leq \left(\frac{C}{r} t + \left(\int_\Omega |u(0)|^2 \, dx\right)^{-r/2}\right)^{-2/r},$$

which means that $\|u(t)\|$ is decreasing with rate $t^{-1/-r}$ as $t$ tends to the infinity.

Let

$$C_r^\ast = C\left(\int |f|^2 \, dx\right)^{(r+2)/(2(r+1))}.$$ 

Then for $r > -1$, we have that,

$$\int_\Omega |u(t)|^2 \, dx \leq \exp(-Z_0(t - t_0)) \int_\Omega |u(t_0)|^2 \, dx + W_\ast \quad \text{for all } t > t_0,$$
where
\[ W_*= \left( \frac{C_f}{C} \right)^{2/(r+2)}, \]

and \( Z_0 \) is given in (5.3) or in (5.5) depending on \( r \).

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