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On constants in the Füredi–Hajnal and the Stanley–Wilf conjecture

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ABSTRACT

For a given permutation matrix P, let $f_P(n)$ be the maximum number of 1-entries in an $n \times n$ $(0,1)$-matrix avoiding P and let $S_P(n)$ be the set of all $n \times n$ permutation matrices avoiding P. The Füredi–Hajnal conjecture asserts that $c_P := \lim_{n\to\infty} f_P(n)/n$ is finite, while the Stanley–Wilf conjecture asserts that $s_P := \lim_{n\to\infty} \sqrt[n]{|S_P(n)|}$ is finite.

In 2004, Marcus and Tardos proved the Füredi–Hajnal conjecture, which together with the reduction introduced by Klazar in 2000 proves the Stanley–Wilf conjecture.

We focus on the values of the Stanley–Wilf limit (s_P) and the Füredi–Hajnal limit (c_P). We improve the reduction and obtain $s_P \leqslant 2.88c_P^2$ which decreases the general upper bound on s_P from $s_P \leqslant const^{const^{O(k\log(k))}}$ to $s_P \leqslant const^{O(k\log(k))}$ for any $k \times k$ permutation matrix P. In the opposite direction, we show $c_P = O(s_P^{4.5})$.

For a lower bound, we present for each k a $k \times k$ permutation matrix satisfying $c_P = \Omega(k^2)$.

1. Introduction

A $(0,1)$-matrix $A = (a_{i,j})$ is said to be a *permutation matrix*, if each row and each column contains exactly one 1-entry. Each such matrix corresponds to some permutation $\pi = (\pi(1), \pi(2), \dots, \pi(n))$ in such a way, that $a_{i,j} = 1$ exactly if $\pi(i) = j$. We let P_π denote the permutation matrix corresponding to π. An *n-permutation* is a permutation on n elements and its corresponding matrix is an *n-permutation matrix*. We say that B is a *submatrix* of A if it can be obtained from A by removing

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some of its rows and columns. A $(0,1)$-matrix A *contains* a $k \times k$ $(0,1)$-matrix $P = (p_{i,j})$ if A has a $k \times k$ submatrix $B = (b_{i,j})$ such that for all $i, j \in [k]$: $p_{i,j} = 1$ implies $b_{i,j} = 1$. Note that a permutation matrix A contains another permutation matrix P if and only if P is a submatrix of A. A *avoids* B if it does not contain B.

For a $(0,1)$-matrix P let $f_P(n)$ be the maximum number of 1-entries in an $n \times n$ $(0,1)$-matrix avoiding P.

We define the *Füredi–Hajnal limit* of P as follows:

$$c_P = \lim_{n\to\infty} \frac{f_P(n)}{n}.$$

Using the idea of the proof of Theorem 1 from [2], we can prove that c_P always exists and that

$$\forall n \in \mathbb{N}: \quad f_P(n) \leqslant c_P n.$$

In 1992 Füredi and Hajnal [5] conjectured that for any fixed permutation matrix P, $f_P(n) = O(n)$, which is equivalent to asking whether c_P is finite. Marcus and Tardos [9] proved that for any k-permutation matrix P,

$$c_P \leqslant 2k^4 \binom{k^2}{k},$$

which settled the Füredi–Hajnal conjecture (FHC).

Claim 1.

1. *For any k-permutation matrix P and for any $n \geqslant k-1$:*

$$f_P(n) \geqslant (2k-2)n - (k-1)^2 \quad \text{and thus} \quad c_P \geqslant 2k-2.$$

2. *If P is the identity matrix of size $k \times k$, that is $p_{i,j} = 1$ if and only if $i = j$, then*

$$\forall n \geqslant k-1: \quad f_P(n) = (2k-2)n - (k-1)^2.$$

Proof. 1. Take any 1-entry $p_{\alpha,\beta}$ of P. Let A be the $n \times n$ $(0,1)$-matrix with

$$a_{i,j} = \begin{cases} 0 & \text{if } \alpha \leqslant i \leqslant n-k+\alpha \text{ and } \beta \leqslant j \leqslant n-k+\beta, \\ 1 & \text{otherwise.} \end{cases}$$

A has exactly $n^2 - (n-k+1)^2 = (2k-2)n - (k-1)^2$ 1-entries and because $p_{\alpha,\beta}$ cannot be represented by any 1-entry of A, A avoids P.

2. Let P be the $k \times k$ identity matrix and let A be any $n \times n$ $(0,1)$-matrix avoiding P. Then each diagonal of A contains at most $k-1$ 1-entries. Since A has $2n-1$ diagonals and the marginal ones have fewer than $k-1$ elements, we can count that if A avoids P, it has at most $(2k-2)n - (k-1)^2$ 1-entries. □

This has been so far the best known lower bound on c_P. In Section 4 we define a $2k$-permutation matrix *Cross*$(2k)$ and show that $c_{Cross(2k)} \geqslant k^2$.

For a permutation matrix P let $S_P(n)$ be the set of all n-permutation matrices avoiding P, $T_P(n)$ the set of all $n \times n$ $(0,1)$-matrices avoiding P and $T_P(n,m)$ the set of all $n \times n$ $(0,1)$-matrices containing exactly m 1-entries and avoiding P. Obviously

$$T_P(n) \supseteq T_P(n,n) \supseteq S_P(n).$$

The *Stanley–Wilf limit* of a permutation matrix P is defined as

$$s_P = \lim_{n\to\infty} \sqrt[n]{\left|S_P(n)\right|}.$$

The Stanley–Wilf conjecture (SWC) was formulated by Stanley and Wilf around 1992 and asserted that s_P always exists and is finite. A weaker modification claimed that for any given P, $\sqrt[n]{|S_P(n)|}$ is bounded. Arratia [2] showed that both versions are equivalent, that s_P always exists and

$$\forall n \in \mathbb{N}: \quad \left|S_P(n)\right| \leqslant (s_P)^n.$$

Klazar [7] shows that

$$\left|T_P(n)\right| \leqslant 15^{c_P n}.$$

This together with the proof of FHC proves SWC with

$$s_P \leqslant 15^{2k^4\binom{k^2}{k}}.$$

It is known that for every k and every k-permutation matrix P, $s_P \geqslant (k-1)^2/e^3$ [6] and there are infinitely many permutation matrices P with $s_P \geqslant 9.47(k-1)^2/9$ [4].

We show in Section 2 that

$$s_P \leqslant 2.88c_P^2 \quad \left(\leqslant 2.88\left(2k^4\binom{k^2}{k}\right)^2\right)$$

and in Section 3 that

$$c_P \leqslant O\left(s_P^{4.5}\right).$$

These bounds together mean that showing an upper bound polynomial in k on one of the constants c_P, s_P would give an upper bound polynomial in k on the other one.

Originally, the Stanley–Wilf limit was defined for permutations. We only rephrased it in the terms of permutation matrices, so the definitions satisfy $s_\pi = s_{P_\pi}$. To simplify the notation, we will sometimes use s_π instead of s_{P_π}.

Section 5 focuses on similar questions for higher-dimensional permutation matrices. An extension of the Füredi–Hajnal conjecture to higher dimensions was proved by Klazar and Marcus [8]. For any given d-dimensional permutation matrix P, they showed that if a d-dimensional $n \times \cdots \times n$ $(0,1)$-matrix A avoids P, then A has at most $O(n^{d-1})$ 1-entries and there are such matrices A with $\Omega(n^{d-1})$ 1-entries.

It is not known how the Stanley–Wilf conjecture could be extended to higher dimensions. For a d-dimensional permutation matrix P let $S_{P,d}(n)$ be the set of d-dimensional $n \times \cdots \times n$ permutation matrices avoiding P. We provide bounds

$$n^{n(d-2+o(1))} \leqslant \left|S_{P,d}(n)\right| \leqslant n^{n(d(d-2)/(d-1)+o(1))},$$

where the upper bound is obtained by a proof similar to the proof of Theorem 2.

2. FHC to SWC reduction

Theorem 2. *For any permutation matrix P*

$$s_P \leqslant 2.88c_P^2.$$

Thus

$$\forall n \in \mathbb{N}: \quad \left|S_P(n)\right| \leqslant \left(2.88c_P^2\right)^n.$$

Proof. We can assume $c_P \geqslant 1$, since otherwise $s_P = 0$ and the statement is true.

The reduction is based on Klazar's reduction [7]. We start with a 1×1 matrix $A_0 := (1)$. In each step, we transform the matrix A_i of size $2^i \times 2^i$ into A_{i+1} of size $2^{i+1} \times 2^{i+1}$ by replacing each entry ω of A_i by a 2×2 block containing only 0-entries if and only if $\omega = 0$. There is a single possibility how to replace a 0-entry and fifteen possibilities of replacing a 1-entry. The number of 1-entries is non-decreasing, so we are only interested in matrices A_i with at most n 1-entries. Another estimate on the number of 1-entries uses the fact that if A_i contains P, then $A_{i+1}, A_{i+2}, \ldots$ contain P as well. So we consider only matrices A_i that avoid P, thus A_i has at most $f_P(2^i) \leqslant c_P \cdot 2^i$ 1-entries.

- Phase 1: We use the estimate that the number of 1-entries in A_i is at most $c_P \cdot 2^i$ and get

$$\left|T_P(2^i)\right| \leqslant 15^{c_P \cdot 2^{i-1}} \cdot \left|T_P(2^{i-1})\right| \leqslant 15^{c_P \cdot (2^{i-1}+2^{i-2})} \cdot \left|T_P(2^{i-2})\right| \leqslant \cdots \leqslant 15^{c_P \cdot 2^i}. \tag{1}$$

  Klazar continues until $2^i \geqslant n$, but we stop when $i = a$, which will be chosen later.
- Phase 2: This time we use the estimate that the number of 1-entries in A_i is at most n. Using $a = \lfloor \log_2(n/c_P) \rfloor$, we could now easily show $s_P = O(c_P^{\log_2 15})$, but our aim is a better estimate.
  We will count how many transformations of matrices from $T_P(2^{a+i-1})$ give a matrix from $T_P(2^{a+i}, m)$. We define j_1, j_2, j_3, j_4 to be the numbers of 1-entries that were replaced by a block with 1, 2, 3, 4 1-entries, respectively. There are four possible replacements of a 1-entry that do not increase the number of 1-entries, six increase it by one, four by two and one by three. This gives the following recursive formula for the upper bound on $|T_P(2^{a+i}, m)|$:

$$\sum_{\substack{j_1, j_2, j_3, j_4 \geqslant 0 \\ j_1+2j_2+3j_3+4j_4=m}} \binom{m - j_2 - 2j_3 - 3j_4}{j_1, j_2, j_3, j_4} \cdot \left|T_P(2^{a+i-1}, m - j_2 - 2j_3 - 3j_4)\right| 4^{j_1} 6^{j_2} 4^{j_3} 1^{j_4}.$$

  To simplify the computations, we define the function $u : \mathbb{N}_0 \times \mathbb{Z} \to \mathbb{N}_0$:

$$\forall m \geqslant 0: \quad u(0, m) := 1,$$
$$\forall i, \forall m < 0: \quad u(i, m) := 0,$$
$$\forall i > 0, \forall m \geqslant 0: \quad u(i, m) := \sum_{\substack{j_2, j_3, j_4 \geqslant 0 \\ j_2+j_3+j_4 \leqslant m}} \binom{m}{m - j_2 - j_3 - j_4, j_2, j_3, j_4} \times u(i-1, m - j_2 - 2j_3 - 3j_4) 4^{m-2j_2-3j_3-4j_4} 6^{j_2} 4^{j_3} 1^{j_4}.$$

  We have $|T_P(2^{a+i}, m)| \leqslant u(i, m)|T_P(2^a)|$ because it is true for $i = 0$ and the differences between the recursive formulas are that the one for $u(i, m)$ adds several nonnegative summands and changes the multinomial coefficient. But, as one can check, the value of the multinomial coefficient never decreases.
  For each nonnegative i, we will find some positive d_i such that for all integers m we will have $u(i, m) \leqslant (4^i d_i)^m$. First, $d_0 := 1$ satisfies the inequality for $i = 0$. For $i > 0$, if m is negative, the inequality is trivial, otherwise

$$\begin{aligned} u(i, m) &\leqslant \sum_{\substack{j_2, j_3, j_4 \geqslant 0 \\ j_2+j_3+j_4 \leqslant m}} \binom{m}{m - j_2 - j_3 - j_4, j_2, j_3, j_4} \\ &\quad \times \left(4^{i-1} d_{i-1}\right)^{m-j_2-2j_3-3j_4} \cdot 4^{m-2j_2-3j_3-4j_4} \cdot 6^{j_2} 4^{j_3} 1^{j_4} \\ &= \left(4^i d_{i-1}\right)^m \sum_{\substack{j_1', j_2, j_3, j_4 \geqslant 0 \\ j_1'+j_2+j_3+j_4=m}} \binom{m}{j_1', j_2, j_3, j_4} \\ &\quad \times \left(\frac{6}{d_{i-1} 4^{i+1}}\right)^{j_2} \left(\frac{4}{d_{i-1}^2 4^{2i+1}}\right)^{j_3} \left(\frac{1}{d_{i-1}^3 4^{3i+1}}\right)^{j_4} \\ &= \left(4^i d_{i-1}\right)^m \left(1 + \frac{6}{d_{i-1} 4^{i+1}} + \frac{4}{d_{i-1}^2 4^{2i+1}} + \frac{1}{d_{i-1}^3 4^{3i+1}}\right)^m. \end{aligned}$$

  Thus we can set d_i to $d_{i-1} \cdot (1 + 6/(d_{i-1} 4^{i+1}) + 4/(d_{i-1}^2 4^{2i+1}) + 1/(d_{i-1}^3 4^{3i+1}))$ or anything larger. Then $d_i \geqslant d_{i-1}$. We will count d_1 and d_2 exactly and then the rest.

For $i = 1$, the expression above becomes 1.44140625, so we can set $d_1 := 1.4415$. Then we get $d_2 \geqslant 1.537989\ldots$, so let $d_2 := 1.538$. For $i \geqslant 3$ let

$$\begin{aligned}
d_i &= d_{i-1} \cdot \left(1 + \frac{6}{d_{i-1}4^{i+1}} + \frac{4}{d_{i-1}^2 4^{2i+1}} + \frac{1}{d_{i-1}^3 4^{3i+1}}\right) \\
&\leqslant d_{i-1} \cdot \left(1 + \frac{6}{d_2 4^{i+1}} + \frac{4}{d_2^2 4^{2i+1}} + \frac{1}{d_2^3 4^{3i+1}}\right) \\
&\leqslant d_{i-1} \exp\left(\frac{6}{d_2 4^{i+1}} + \frac{4}{d_2^2 4^{2i+1}} + \frac{1}{d_2^3 4^{3i+1}}\right) \\
&\leqslant d_2 \prod_{j=3}^{i} \left(\exp\left(\frac{6}{d_2 4^{j+1}} + \frac{4}{d_2^2 4^{2j+1}} + \frac{1}{d_2^3 4^{3j+1}}\right)\right) \\
&= d_2 \exp\left(\sum_{j=3}^{i} \frac{6}{d_2 4^{j+1}} + \sum_{j=3}^{i} \frac{4}{d_2^2 4^{2j+1}} + \sum_{j=3}^{i} \frac{1}{d_2^3 4^{3j+1}}\right) \\
&\leqslant d_2 \exp\left(\frac{4}{3}\frac{6}{d_2 4^4} + \frac{16}{15}\frac{4}{d_2^2 4^7} + \frac{64}{63}\frac{1}{d_2^3 4^{10}}\right) \\
&\leqslant 1.57.
\end{aligned}$$

Let $d_\infty := 1.57$. All in all, we have just proven that for any i and m:

$$\left|T_P\left(2^{a+i}, m\right)\right| \leqslant 4^{im} d_\infty^m \left|T_P\left(2^a\right)\right| \leqslant 4^{im} d_\infty^m \cdot 15^{c_P \cdot 2^a},$$

where the last inequality follows from Eq. (1). We could finish when $2^{a+i} \geqslant n$ for the first time, which would already result in $s_P = O(c_P^2)$, but to achieve a better multiplication constant, we continue until $a + i$ equals some b such that $2^b \geqslant 2n^2$.

Every n-permutation matrix avoiding P can be expanded by adding empty rows and columns to form a matrix from $T_P(2^b, n)$. This can be done in $\binom{2^b}{n}^2$ ways while the reverse process is unique—we just delete all empty rows and columns and see what remains. Therefore $|T_P(2^b, n)| \geqslant |S_P(n)|\binom{2^b}{n}^2$.

Since $2^b \geqslant 2n^2$, we can estimate:

$$\binom{2^b}{n} \geqslant \frac{(2^b - n)^n}{n!} \geqslant \frac{2^{b\cdot n}(1 - \frac{1}{2n})^n}{en(\frac{n}{e})^n} \geqslant \frac{2^{b\cdot n} \cdot e^{-1}}{en(\frac{n}{e})^n}.$$

We now have

$$\begin{aligned}
\left|S_P(n)\right| &\leqslant \left|T_P\left(2^b, n\right)\right| \cdot \binom{2^b}{n}^{-2} \\
&\leqslant 4^{n\cdot(b-a)} \cdot d_\infty^n \cdot 15^{c_P \cdot 2^a} \cdot \left(en\left(\frac{n}{e}\right)^n \cdot e \cdot 2^{-b\cdot n}\right)^2 \\
&= e^4 n^2 \left(4^{b-a} d_\infty \frac{n^2}{e^2} 4^{-b}\right)^n 15^{c_P \cdot 2^a}
\end{aligned}$$

and so

$$\sqrt[n]{\left|S_P(n)\right|} \leqslant \sqrt[n]{e^4 n^2}\frac{d_\infty}{e^2} n^2 4^{-a} 15^{c_P 2^a/n} = \sqrt[n]{e^4 n^2}\frac{d_\infty}{e^2} 4^{-a} \exp\left(2\ln(n) + \frac{\ln(15) c_P 2^a}{n}\right).$$

Let $g_a(n) := 2\ln(n) + \ln(15)c_P 2^a/n$. A simple calculation shows that for any given $a > 0$, $g_a(n)$ has its minimum at $n = \ln(15)c_P 2^{a-1}$ and is decreasing on the interval $(0, \ln(15)c_P 2^{a-1})$. So we will set

$$n(a) := \left\lfloor \ln(15) c_P 2^{a-1} \right\rfloor$$

and estimate

$$g_a\big(n(a)\big) \leqslant g_a\big(\ln(15)c_P 2^{a-1} - 1\big) \leqslant g_a\left(\ln(15)c_P 2^{a-1}\left(1 - \frac{1}{2^a}\right)\right).$$

Since $\lim_{a\to\infty} n(a) = \infty$ and from [2] $\lim_{n\to\infty} \sqrt[n]{|S_P(n)|}$ exists, we obtain

$$\begin{aligned}
s_P &= \lim_{n\to\infty} \sqrt[n]{\big|S_P(n)\big|} = \lim_{a\to\infty} \sqrt[n(a)]{\big|S_P\big(n(a)\big)\big|} \\
&\leqslant \lim_{a\to\infty} \big(\sqrt[n(a)]{e^4 n(a)^2}\big) \frac{d_\infty}{e^2} \lim_{a\to\infty} \big(4^{-a} \exp\big(g_a\big(n(a)\big)\big)\big) \\
&\leqslant 1 \cdot \frac{d_\infty}{e^2} \lim_{a\to\infty} 4^{-a} \exp\left(g_a\left(\ln(15)c_P 2^{a-1}\left(1 - \frac{1}{2^a}\right)\right)\right) \\
&= \frac{d_\infty}{e^2} \lim_{a\to\infty} 4^{-a} \exp\left(2\ln\left(\ln(15)c_P 2^{a-1}\left(1 - \frac{1}{2^a}\right)\right) + \frac{\ln(15)c_P 2^a}{\ln(15)c_P 2^{a-1}(1 - \frac{1}{2^a})}\right) \\
&\leqslant \frac{d_\infty}{e^2} \lim_{a\to\infty} 4^{-a}\big(\ln(15)c_P 2^{a-1}\big)^2 \left(1 - \frac{1}{2^a}\right)^2 \exp\left(\frac{2}{1 - \frac{1}{2^a}}\right) \\
&= \frac{d_\infty}{e^2} \cdot \frac{\ln^2(15)}{4} c_P^2 \lim_{a\to\infty} 4^{-a} 4^a \left(1 - \frac{1}{2^a}\right)^2 \exp\left(\frac{2}{1 - \frac{1}{2^a}}\right) \\
&= d_\infty \cdot \frac{\ln^2(15)}{4} c_P^2 \\
&\leqslant 2.88 c_P^2.
\end{aligned}$$

Theorem 1 from [2] now gives

$$\forall n \geqslant 1: \quad \big|S_P(n)\big| \leqslant \big(2.88 c_P^2\big)^n. \qquad \square$$

Notice that a similar proof can be used to show $\sqrt[n]{|T_P(n,n)|} \leqslant O(c_P^2)$. However, $\lim_{n\to\infty} \sqrt[n]{|T_P(n)|} \geqslant 2^{c_P}$. To show this we will take an $n \times n$ (0, 1)-matrix A with $f_P(n)$ 1-entries that avoids P. The matrix A contains $2^{f_P(n)}$ different $n \times n$ matrices and all such matrices avoid P.

3. SWC to FHC reduction

Lemma 3. *Let P be any permutation matrix and let B be a matrix of size $b \times c$ containing at least b 1-entries in each row. If B avoids P, then*

$$\big|S_P(b)\big| \geqslant \left(\frac{b^2}{e^2 c}\right)^b.$$

Proof. We take the rows of B one by one from top to bottom and from each of them, we select some 1-entry in a column that was not used previously. This way, we constructed a b-permutation matrix contained in B, thus avoiding P. This construction gives us at least $b!$ occurrences of b-permutation matrices, but some can be different occurrences of the same matrix. To count the largest possible number of occurrences of a given b-permutation matrix, we observe that the rows are given but we can select any b-tuple out of the c columns. All in all, the number of different b-permutation matrices avoiding P is at least

$$\frac{b!}{\binom{c}{b}} \geqslant \frac{(\frac{b}{e})^b}{(\frac{ce}{b})^b} = \left(\frac{b^2}{e^2 c}\right)^b. \qquad \square$$

Lemma 4. *For a given permutation matrix P take any $l \in \mathbb{N}$ such that $\sqrt[7]{l}$ is an integer and*

$$\left|S_P\left(l^{10/7}\right)\right| < \left(\frac{l^{6/7}}{2e^2}\right)^{l^{10/7}}.$$

Then

$$\forall n \in \mathbb{N}: \quad f_P(n) \leqslant \left(2l^{27/7} + 10l^{24/7} + 8l^2\right)n.$$

Proof. First observe that if P has size 1×1, then the lemma holds.

By a theorem of Arratia [2], if P has size at least 2×2, then for every $i, j \geqslant 1$ we have $|S_P(i+j)| \geqslant |S_P(i)| \cdot |S_P(j)|$. Extending this, we have $|S_p(\alpha \cdot i)| \geqslant |S_p(i)|^\alpha$, and so the conditions of the lemma also imply

$$\left|S_P(l)\right| < \left(\frac{l^{6/7}}{e^2}\right)^{l} \quad \text{and} \quad \left|S_P\left(l^{8/7}\right)\right| < \left(\frac{l^{6/7}}{e^2}\right)^{l^{8/7}}.$$

Let $A = (a_{i,j})$ be any $n \times n$ permutation matrix avoiding P. We start similarly to the proof of FHC [9]—we cut the matrix A by horizontal and vertical cuts into a grid of blocks $K_{i,j}$ of sizes $2l^2 \times 2l^2$ and discard the incomplete blocks on the right and at the bottom. That is, $K_{i,j} := \{a_{i',j'}\colon i' \in \{2l^2 i + 1, \ldots, 2l^2(i+1)\},\ j' \in \{2l^2 j + 1, \ldots, 2l^2(j+1)\}\}$. The *$j$th column of blocks* is $\mathcal{C}_j := \{K_{i,j}\colon i \in \{0, 1, \ldots, \lfloor n/(2l^2)\rfloor - 1\}\}$ and the *ith row of blocks* is $\mathcal{R}_i := \{K_{i,j}\colon j \in \{0, 1, \ldots, \lfloor n/(2l^2)\rfloor - 1\}\}$. We say that a block is *wide* if it contains more than l nonzero columns, *very wide* if it contains more than $l_1 = l^{8/7}$ nonzero columns and *ultrawide* if it contains more than $l_2 = l^{10/7}$ nonzero columns. Similarly, a block is *tall, very tall, ultratall* if it has more than l, l_1, l_2 nonzero rows, respectively. Throughout the proof we will use the following observation:

Observation 5. *We take b blocks from the same column of blocks and separately contract the columns of each of them. This way we obtain a $b \times 2l^2$ matrix $B = (b_{i,j})$ with one row for each block, such that $b_{i,j} = 0$ if and only if the ith selected block contains no 1-entry in its jth column. If B contains P, then A contains P as well.*

Proof. For each 1-entry in the occurrence of P in the contracted matrix B, we take any 1-entry from the column from which it was contracted. Because P is a permutation matrix, the relative positions of these 1-entries do not change and they form an occurrence of P in the original matrix A. □

Now, we return to the proof of Lemma 4. If $n \leqslant 2l^{27/7}$, the claim is trivial, otherwise we count the maximal number of 1-entries in a matrix A that avoids P:

- The discarded blocks have together at most $2 \cdot 2l^2 n$ 1-entries.
- Each nonzero block which is neither wide nor tall, has at most l^2 1-entries. As was shown in [9], if we contract each block of A into a single element (whose value is 1 exactly if the block is nonzero), we obtain a matrix that avoids P. So the number of nonzero blocks is at most $f_P(\lfloor n/2l^2 \rfloor)$ and this value can be estimated from the induction hypothesis.
- Each ultrawide or ultratall block has at most $4l^4$ 1-entries. We will show that there are fewer than $l_2 = l^{10/7}$ ultrawide blocks in one column of blocks and fewer than l_2 ultratall blocks in one row of blocks. It is enough to prove this only for ultrawide blocks; the proof for ultratall blocks is the same. For contradiction, suppose there are at least l_2 ultrawide blocks in the same column of blocks. We contract the columns of each of them as in Observation 5 and obtain a $l_2 \times 2l^2$ matrix B with l_2 rows each of which has at least l_2 1-entries. Lemma 3 then gives

  $$\left|S_P\left(l^{10/7}\right)\right| \geqslant \left(\frac{l_2^2}{2e^2 l^2}\right)^{l_2} = \left(\frac{l^{6/7}}{2e^2}\right)^{l^{10/7}},$$

  which contradicts the conditions of Lemma 4.

- Each very wide or very tall block which is neither ultrawide nor ultratall has at most $l_2^2 = l^{20/7}$ 1-entries. To count the maximal number of very wide blocks in one column of blocks we first contract each such block to a row with at least $l_1 = l^{8/7}$ 1-entries. If some l_1 consecutive rows have all their 1-entries in at most l_2 columns of A, we will remove all the other columns and obtain an $l_1 \times l_2$ matrix with at least l_1 1-entries in each row and consequently

$$\left|S_P\left(l^{8/7}\right)\right| \geqslant \left(\frac{l_1^2}{e^2 l_2}\right)^{l_1} = \left(\frac{l^{6/7}}{e^2}\right)^{l^{8/7}},$$

which is not possible and so there are at least l_2 nonzero columns in each group of l_1 consecutive rows. Contracting this group gives a row with at least l_2 1-entries and as was previously shown, there are fewer than l_2 such rows. We conclude that there are fewer than $l_2 l_1 = l^{18/7}$ very wide blocks in one column of blocks.
- In each wide or tall block which is neither very wide nor very tall, there are at most $l_1^2 = l^{16/7}$ 1-entries. We divide the wide blocks into groups of l consecutive blocks. If all the 1-entries in the blocks of one group lied in only l_1 columns, there would be at least

$$\left|S_P(l)\right| \geqslant \left(\frac{l^2}{e^2 l_1}\right)^{l} = \left(\frac{l^{6/7}}{e^2}\right)^{l}$$

l-permutation matrices avoiding P. So each group can be contracted into a row with at least l_1 1-entries. But as we have shown, there are fewer than $l_2 l_1$ such rows and therefore there are at most $l_2 l_1 l = l^{25/7}$ wide blocks in one column of blocks.

The overall number of 1-entries is at most

$$\begin{aligned} f_P(n) &\leqslant 2 \cdot 2l^2 n + l^2 f_P\left(\left\lfloor \frac{n}{2l^2} \right\rfloor\right) + 2\left(4l^4 l^{10/7} + l^{20/7} l^{18/7} + l^{16/7} l^{25/7}\right)\frac{n}{2l^2} \\ &\leqslant \left(4l^2 + l^{27/7} + 5l^{24/7} + 4l^2 + 4l^{24/7} + l^{24/7} + l^{27/7}\right)n \\ &\leqslant \left(2l^{27/7} + 10l^{24/7} + 8l^2\right)n. \qquad \square \end{aligned}$$

Theorem 6. *For any permutation matrix P*

$$c_P \leqslant \left(2^{32.5} e^9 s_P^{4.5} + 5 \cdot 2^{29} e^8 s_P^4 + 2^{58/3} e^{14/3} s_P^{7/3}\right) = O\left(s_P^{4.5}\right).$$

Proof. We take the smallest $l > (2e^2 s_P)^{7/6}$ that is a seventh power of an integer. Because $(2e^2 s_P)^{7/6} \geqslant 1$, we will find a suitable l not larger than $2^7 (2e^2 s_P)^{7/6}$. For every integer i, the number of i-permutation matrices avoiding P is at most s_P^i and from the choice of l, $s_P^i < (\frac{l^{6/7}}{2e^2})^i$. Thus we can use Lemma 4 and substituting $l \leqslant 2^7 (2e^2 s_P)^{7/6}$ into its result gives the claim that was to be proven. □

Lemma 4 might be useful even if we do not know the Stanley–Wilf limit, for if we manage to count $|S_P(n)|$ for several small n, we might be able to find some l that would satisfy the conditions of the lemma.

4. Quadratic lower bound in FHC

In this section we will construct an $n \times n$ matrix $A(k,n)$ that avoids the matrix $Cross(2k)$ and has $\Omega(nk^2)$ 1-entries. $Cross(2k)$ will be a permutation matrix. Let cr_i denote the unique 1-entry in ith column of $Cross(2k)$ and let rcr_i be the row containing cr_i. $Cross(2k)$ is defined as follows:

$$rcr_i := \begin{cases} i & \forall i \leqslant k,\ i \text{ even}, \\ 2k+1-i & \forall i \leqslant k,\ i \text{ odd}, \\ i & \forall i > k,\ i \text{ odd}, \\ 2k+1-i & \forall i > k,\ i \text{ even}. \end{cases}$$

For example,

$$Cross(8) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A *diagonal* of an $n \times n$ matrix $A = (a_{i,j})$ is the set of elements $a_{i,j}$ satisfying $i - j = d$ for d fixed from $\{-(n-1), -(n-2), \dots, n-1\}$. The elements of a *skew diagonal* are $a_{i,j}$ with $i + j = d$, where $d \in \{2, 3, \dots, 2n\}$. The *main (skew) diagonal* is the longest one. The *diagonal distance* between a_{i_1,j_1} and a_{i_2,j_2} is $|(i_1 - j_1) - (i_2 - j_2)|$ and their *skew diagonal distance* is $|(i_1 + j_1) - (i_2 + j_2)|$.

Let $n' := n^{(k-1)/k}$. To simplify the proof, we consider only such n that are the kth power of an integer.

Let $A(1,n)$ be the $n \times n$ matrix with only 0-entries. Thus $A(1,n)$ avoids $Cross(2)$.

For $k \geqslant 2$, $A(k,n)$ will contain several copies of $A(k-1,n')$ rotated by 90°. The 1-entries not lying in any of the copies will be called *proper* 1*-entries*. We will show that, if $A(k,n)$ were to contain $Cross(2k)$, then (without loss of generality) the 1-entries corresponding to cr_1 and cr_{2k} would be proper, and that the 1-entries corresponding to all the other cr_i would be in a single copy of $A(k-1,n')$. This will contradict the induction hypothesis.

The proper 1-entries are all the entries of $A(k,n)$ such that their diagonal distance from the main diagonal is in the set $\{n', n'+1, \dots, n'+2k-2\}$. Thus the proper 1-entries form two groups of $2k-1$ consecutive diagonals, one group to the left and the other to the right of the main diagonal. These diagonals will be called the *proper diagonals*. The matrix $A(k,n)$ has no 1-entry at diagonal distance larger than $n'+2k-2$ from the main diagonal. The element cr_1 is the leftmost and lowermost 1-entry of $Cross(2k)$, and thus if $A(k,n)$ contains $Cross(2k)$ so that cr_1 is represented by $a_{i,j}$, then cr_1 can also be represented by any 1-entry $a_{i',j'}$ with $i' \geqslant i$ and $j' \leqslant j$. If $a_{i,j}$ is not in the leftmost proper diagonal, then we can find such a 1-entry $a_{i',j'}$ in the leftmost proper diagonal. Thus we can without loss of generality assume that cr_1 occurs in the leftmost and similarly cr_{2k} occurs in the rightmost proper diagonal.

The rest of $Cross(2k)$ must appear inside the axis-parallel rectangle which has cr_1 and cr_{2k} in its corners. We must place cr_2 at least $2k-2$ rows above and at least 1 column to the right from the occurrence of cr_1. But no such entry lies inside the left $2k-1$ proper diagonals. Using a similar reasoning with cr_{2k}, cr_2 can neither lie in the right $2k-1$ proper diagonals. Similarly, cr_{2k-1} cannot be represented by any proper 1-entry of $A(k,n)$ and later, we will show that only cr_1 and cr_{2k} can be represented by a proper 1-entry.

Now we take a number of copies (to be determined later) of $A(k-1,n')$, rotate them by 90° and place them between the two groups of proper diagonals. We leave $2n'+2(2k-2)+1$ skew diagonals between the rightmost nonzero skew diagonal of a copy of $A(k-1,n')$ and the leftmost nonzero skew diagonal of the nearest copy to the right. We also leave n' skew diagonals to the left from the leftmost nonzero skew diagonal of the leftmost copy and to the right from the rightmost nonzero skew diagonal of the rightmost copy. Because all 1-entries of $A(k-1,n')$ lie in only $2(n')^{(k-2)/(k-1)} + 2(2k-4)+1$ skew diagonals around the main skew diagonal, the number of copies of $A(k-1,n')$ that we can place is

$$\left\lfloor \frac{2n-1}{2(n')^{\frac{k-2}{k-1}} + 2(2k-4) + 2n' + 2(2k-2) + 2} \right\rfloor = \left\lfloor \frac{n - \frac{1}{2}}{n^{\frac{k-1}{k}} + n^{\frac{k-2}{k}} + 2(2k-2) - 1} \right\rfloor$$
$$\geqslant \left\lfloor \frac{n}{(1 + \frac{1}{2(k-1)})n^{\frac{k-1}{k}}} \right\rfloor \geqslant \frac{k-1}{k} n^{\frac{1}{k}}.$$

The last two inequalities are true for $k \geqslant 2$ and n large enough.

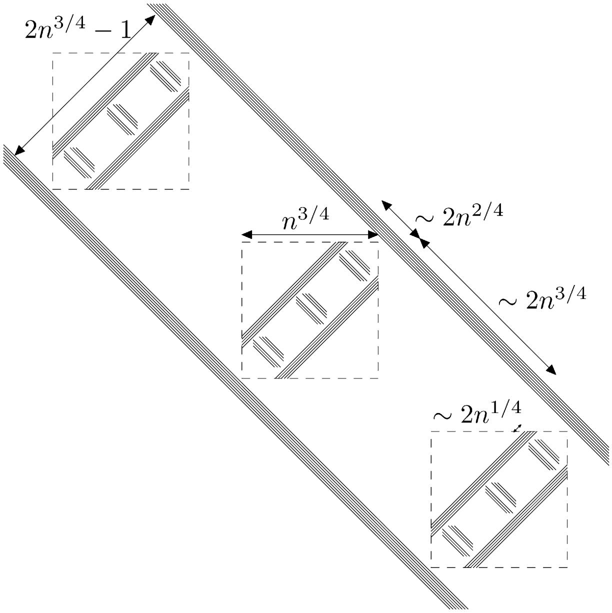

Fig. 1. Schematic figure of $A(4,n)$ which avoids $Cross(8)$. Full lines represent diagonals with 1-entries.

If cr_2 and cr_{2k-1} lied in different copies of $A(k-1,n')$, their skew diagonal distance would be at least $2n' + 2(2k-1)$. But obviously, this distance must be smaller than the diagonal distance between cr_1 and cr_{2k} which is only $2n' + 2(2k-1) - 1$. So cr_2 and cr_{2k-1} lie in the same copy. Because $cr_3, cr_4, \ldots, cr_{2k-2}$ must lie in the rectangle with cr_2 and cr_{2k-1} in its corners, all $cr_2, cr_3, \ldots, cr_{2k-1}$ lie in the same copy. From the definition follows that $cr_2, cr_3, \ldots, cr_{2k-1}$ form an occurrence of $Cross(2(k-1))$ rotated by $90°$, which is, by the induction hypothesis, avoided by the rotated copy of $A(k-1,n')$.

See Fig. 1 for an example of $A(k,n)$.

Lemma 7. *Let $k \geqslant 2$. If n is large enough and a kth power of an integer, then*:

1. *$A(k,n)$ avoids $Cross(2k)$.*
2. *$A(k,n)$ contains at least $k^2 n$ 1-entries.*

Proof. 1. This has already been proven in preceding paragraphs.

2. Let $h(k,n)$ denote the number of 1-entries in $A(k,n)$. $A(k,n)$ has $2(2k-1)(n - n^{(k-1)/k}) - 2(2k-1)(2k-2)$ proper 1-entries. For $k=2$ and n large enough, this is at least $4n$, which was to be proven. Otherwise, from the previous calculations, $A(k,n)$ has at least $h(k-1,n')\frac{k-1}{k}n^{1/k}$ 1-entries in the copies of $A(k-1,n')$. Since $n' = n^{(k-1)/k}$ is a $(k-1)$st power of an integer and large enough, we can use the induction hypothesis. All in all, for $k \geqslant 3$,

$$h(k,n) \geqslant 2(2k-1)\bigl(n - n^{\frac{k-1}{k}}\bigr) - 2(2k-1)(2k-2) + h\bigl(k-1, n^{\frac{k-1}{k}}\bigr)\frac{k-1}{k}n^{1/k}$$

$$\geqslant 3kn + (k-1)^2 n^{\frac{k-1}{k}}\frac{k-1}{k}n^{1/k} \geqslant 3kn + \frac{k^3 - 3k^2}{k}n = n\bigl(3k + k^2 - 3k\bigr) = k^2 n. \qquad \square$$

Theorem 8. *For every $k \geqslant 2$, there exists a $k \times k$ matrix B such that*

$$c_B \geqslant \frac{(k-1)^2}{4}.$$

Proof. From the previous lemma, if k is even, we have $c_{Cross(k)} \geqslant k^2/4$. Otherwise we take any matrix B containing $Cross(k-1)$ and obtain $c_B \geqslant c_{Cross(k-1)} \geqslant (k-1)^2/4$. $\square$

5. Higher-dimensional matrices

We will call $M \in \{0,1\}^{[n_1]\times\cdots\times[n_d]}$ a *d-dimensional* (0, 1)*-matrix of size* $n_1 \times \cdots \times n_d$.

A d-dimensional (0, 1)-matrix P of size $k \times \cdots \times k$ is a *d-dimensional k-permutation matrix* if P contains k 1-entries and the positions of each two 1-entries of P differ in all coordinates.

We say that a d-dimensional (0, 1)-matrix $P = (p_{i_1,\dots,i_d})$ of size $k_1 \times \cdots \times k_d$ is *contained* in a d-dimensional (0, 1)-matrix $A = (a_{i_1,\dots,i_d})$ of size $n_1 \times \cdots \times n_d$ if there exist d increasing injections $f_i : [k_i] \to [n_i]$, $i = 1, 2, \dots, d$, such that for all $i_1, i_2, \dots, i_d \in [k]$: $p_{i_1,\dots,i_d} = 1$ implies $a_{f_1(i_1),\dots,f_d(i_d)} = 1$. If P is not contained in A, we say that A *avoids* P.

For a d-dimensional k-permutation matrix P and $a, b \in [d]$, let the (a,b)*-projection* of P, $\mathrm{proj}_{a,b}(P)$, be the (2-dimensional) k-permutation matrix P' with $p'_{i,j} = 1$ exactly if P has a 1-entry whose ath coordinate has value i and bth coordinate has value j.

Klazar and Marcus [8] proved that for a fixed d-dimensional k-permutation matrix P, the maximum number of 1-entries in a d-dimensional matrix A of size $n \times \cdots \times n$ that avoids P is $f_{P,d}(n) = \Theta(n^{d-1})$. This generalizes the Füredi–Hajnal conjecture.

Let P be a given d-dimensional k-permutation matrix P. Define $S_{P,d}(n)$ to be the set of all d-dimensional n-permutation matrices avoiding P and $T_{P,d}(n,m)$ to be the set of all d-dimensional matrices of size $n \times \cdots \times n$ that avoid P and have at most m 1-entries. Obviously $T_{P,d}(n,n) \supseteq S_{P,d}(n)$.

Theorem 9. *For a fixed d-dimensional k-permutation matrix P*

$$(n!)^{d-2} = n^{n(d-2+o(1))} \leqslant \left|S_{P,d}(n)\right| \leqslant n^{n(\frac{d(d-2)}{d-1}+o(1))} < (n!)^{d-1}.$$

Proof. The rightmost expression is the number of d-dimensional n-permutation matrices which is a trivial upper bound on $|S_{P,d}(n)|$.

The lower bound will follow from the following observation. Let P and A be permutation matrices and let $P' := \mathrm{proj}_{1,2}(P)$ and $A' := \mathrm{proj}_{1,2}(A)$. If A' avoids P', then A avoids P.

For the given matrix P, we take a matrix A' that avoids P'. For any 2-dimensional n-permutation matrix A', there are $(n!)^{d-2} = n^{n(d-2+o(1))}$ d-dimensional n-permutation matrices A such that $A' = \mathrm{proj}_{1,2}(A)$. Because A' avoids P', all such matrices A are in $S_{P,d}(n)$.

The proof of the upper bound is similar to the proof of Theorem 2. We start with A_0, the $1 \times \cdots \times 1$ matrix containing one 1-entry. In each step, we transform the matrix A_i of size $2^i \times \cdots \times 2^i$ into A_{i+1} of size $2^{i+1} \times \cdots \times 2^{i+1}$ by replacing each 0-entry of A_i by a $2 \times \cdots \times 2$ block containing only 0-entries and each 1-entry of A_i by a $2 \times \cdots \times 2$ block containing at least one 1-entry. There is a single possibility how to replace a 0-entry and $2^{2^d} - 1$ possibilities of replacing a 1-entry. However only 2^d of the possible replacements of the 1-entry do not increase the number of 1-entries.

In the first phase we use the above mentioned estimate $f_{P,d}(2^i) = \Theta(2^{i(d-1)})$ from [8]. Thus $f_{P,d}(2^i) \leqslant c_{P,d} 2^{i(d-1)}$ for some constant $c_{P,d}$ and

$$\left|T_{P,d}\left(2^i, n\right)\right| \leqslant 2^{2^d \cdot c_{P,d} \cdot 2^{(i-1)(d-1)}} \cdot \left|T_{P,d}\left(2^{i-1}, n\right)\right| \leqslant \cdots \leqslant 2^{2^d \cdot c_{P,d} \cdot 2^{i(d-1)}}.$$

We stop when $i = a$, where $a := \lceil \log_2(n^{1/(d-1)}) \rceil$. Then $|T_{P,d}(2^a, n)| \leqslant 2^{O(n)}$.

In the second phase which consists of $b := \lceil \log_2(n) \rceil - a \leqslant \log_2(n)(d-2)/(d-1) + 1$ steps, we use the fact that all matrices A_i have at most n 1-entries. During this phase, we will do at most bn replacements of a 1-entry, but only at most n of them will increase the number of 1-entries.

For each matrix A_{a+b} that was created from A_a, we order the replacements of 1-entries during the second phase primarily by the step in which the replacement occurred and secondarily by the lexicographic order of the position of the 1-entry being replaced. The $2^{2^d} - 1$ types of replacements of a 1-entry are assigned numbers from $[2^{2^d} - 1]$ so that the types of replacements that do not increase the number of 1-entries get numbers from $[2^d]$. Each matrix A_{a+b} that was created from A_a, is assigned a vector whose elements are from $[2^{2^d}-1]$. The entries of the vector represent the replacements of 1-entries in the above defined order, the value of each entry is the type of the replacement. The vector has length at most bn and at most n its entries are from $[2^{2^d}-1]\setminus[2^d]$. Then we append several entries to the vector to obtain a vector of length $(b+1)n$ with exactly n entries from $[2^{2^d}-1]\setminus[d]$. Different matrices A_{a+b} created from the same matrix A_a get different vectors and thus

$$\begin{aligned}\left|T_{P,d}\left(2^{a+b},n\right)\right| &\leqslant 2^{2^d n}2^{dbn}\binom{(b+1)n}{n}\left|T_{P,d}\left(2^a,n\right)\right|\\ &\leqslant 2^{O(n)}2^{\log_2(n)\frac{d(d-2)}{d-1}n}\left(\left(\log_2(n)+2\right)e\right)^n\\ &\leqslant n^{n(\frac{d(d-2)}{d-1}+o(1))}.\end{aligned}$$

We have $|T_{P,d}(n,n)|\leqslant|T_{P,d}(2^{a+b},n)|$, because $2^{a+b}\geqslant n$ and thus all matrices from $T_{P,d}(n,n)$ are in a one-to-one correspondence with matrices from $T_{P,d}(2^{a+b},n)$ that have 0-entries in all places with some coordinate larger than n. $\square$

6. Conclusions and open problems

We have shown that the values s_P and c_P are closely related. On the other hand, it is impossible to find an increasing function $g:\mathbb{R}\to\mathbb{R}$ such that $s_P=g(c_P)$. This can be seen on the permutation matrices I and F corresponding to the permutations 1234 and 1342 respectively. First, from Claim 1, $c_I=6$ and $c_F\geqslant 6$. But for any identity matrix P, Regev [10] showed an asymptotic formula for $|S_P(n)|$ giving $s_{1,\ldots,k}=(k-1)^2$, so in our case $s_I=9$. Bóna [3] found an exact formula for $|S_F(n)|$, from which $s_F=8$. Thus $c_I\leqslant c_F$ while $s_I>s_F$. But the following still might be true:

Problem 1. Does there exist a constant r_1, such that for all permutation matrices P

$$r_1c_P^2\leqslant s_P\leqslant r_2c_P^2?$$

What are the best possible values of r_1 (if it exists) and r_2?

From Theorem 2, r_2 exists and $r_2\leqslant 2.88$.

Another interesting open problem is to find the highest possible value of s_P for P of a given size.

Problem 2. Does there exist a constant q, such that for any k and all k-permutation matrices P

$$s_P\leqslant qk^2?$$

This is a weaker form of the conjecture [2] asserting that $s_P\leqslant(k-1)^2$, which was disproved in [1], where it was shown that $s_{1324}\geqslant 9.47$. A generalization of this result [4] implies $q\geqslant s_{1324}/9$, which is the highest known lower bound on q. In Section 4, we have shown that a related question for the Füredi–Hajnal limit has a negative answer, because there is a sequence of permutation matrices whose Füredi–Hajnal limit grows faster than linearly in their size. Moreover, due to this fact, at least one of the two above-mentioned problems has negative answer.

The following even weaker version already seems to be true.

Conjecture 1. *There exist constants s and t, such that for any k and every k-permutation matrix P*

$$s_P\leqslant sk^t.$$

Equivalently, we could have conjectured that $c_P\leqslant s'k^{t'}$ for some constants s' and t'.

Although Marcus and Klazar [8] proved an extension of the Füredi–Hajnal conjecture for higher-dimensional matrices, no such extension exists for the Stanley–Wilf conjecture. One possible extension could be the following

Problem 3. Given any d-dimensional permutation matrix P, does there exist a constant $s_{P,d}$ such that

$$\left|S_{P,d}(n)\right| \leqslant s_{P,d}^n \cdot n^{(d-2)n}?$$

Section 5 contains an improvement of the trivial upper bound on $|S_{P,d}(n)|$, but the gap between the bounds still remains very large.

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