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On the degree and half-degree principle for symmetric polynomials

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ABSTRACT

In this note we aim to give a new, elementary proof of a statement that was first proved by Timofte (2003) [15]. It says that a symmetric real polynomial F of degree d in n variables is positive on $\mathbb{R}^n$ (or on $\mathbb{R}^n_{\geq 0}$) if and only if it is non-negative on the subset of points with at most $\max\{\lfloor d/2 \rfloor, 2\}$ distinct components. We deduce Timofte's original statement as a corollary of a slightly more general statement on symmetric optimization problems. The idea that we are using to prove this statement is that of relating it to a linear optimization problem in the orbit space. The fact that for the case of the symmetric group S_n this can be viewed as a question on normalized univariate real polynomials with only real roots allows us to conclude the theorems in a very elementary way. We hope that the methods presented here will make it possible to derive similar statements also in the case of other groups.

1. Introduction

The problem of certifying that a given polynomial in n real variables is positive has been one of the main motivations for the development of modern real algebraic geometry at the beginning of the 20th century. Besides the general solutions to this question by Hilbert, Artin and Pólya [9,1,11], only little interest has been devoted to the study of the related questions in the case of symmetric polynomials (see [5,13]). However, in [15] Vlad Timofte was able to prove some fundamental properties of the positivity questions for symmetric polynomials with given degree.

For $n \in \mathbb{N}$ the group of all permutations of an n-element set is called the symmetric group S_n. This group acts on $\mathbb{R}^n$ in an obvious way: $\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$ for $\sigma \in S_n$. Let $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$ denote the ring of polynomials in n real variables. A polynomial $F \in \mathbb{R}[X]$ is called *symmetric* if for all $\sigma \in S_n$ we have $F(x) = F(\sigma(x))$. We will write $\mathbb{R}[X]^{S_n}$ for the ring of symmetric polynomials. The essence of the main statements that we present in this paper is that in order to check whether a symmetric polynomial (in)equality holds one only needs to check whether it holds on test sets of (half-)degree dimension of the polynomial. More precisely: let $x \in \mathbb{R}^n$ and let $n(x) = \#\{x_1, \ldots, x_n\}$ denote the number of distinct components of x and $n^*(x) = \#\{x_1, \ldots, x_n \mid x_j \neq 0\}$ denote the number of distinct non-zero components. Then for a given $d \in \mathbb{N}$ we will take a look at sets of the form $A_d := \{x \in \mathbb{R}^n : n(x) \leq d\}$, i.e. the points in $\mathbb{R}^n$ with at most d distinct components and sets $A_d^+ := \{x \in \mathbb{R}^n_{\geq 0} : n^*(x) \leq d\}$, i.e. points with non-negative components such that there are most d distinct positive components. With this setting we aim to prove the following:

Theorem 1.1. *Let $F_0, F_1, \ldots, F_m \in \mathbb{R}[X]^{S_n}$ be symmetric and*

$$K = \left\{x \in \mathbb{R}^n \ : \ F_1(x) \geq 0, \ldots, F_m(x) \geq 0\right\}.$$

If F_0 is of degree d and $k := \max\{2, \lfloor \frac{d}{2} \rfloor, \deg F_1, \ldots, \deg F_m\}$ then

$$\inf_{x\in K} F_0(x) = \inf_{x \in K\cap A_k} F_0(x) \quad \text{and}$$
$$\inf_{x\in K\cap\mathbb{R}^n_+} F_0(x) = \inf_{x\in K\cap A_k^+} F_0(x).$$

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As immediate corollaries we recover the following statements:

Corollary 1.2 (*Degree Principle*)**.** *Let $F_1, \dots, F_m \in \mathbb{R}[X]^{S_n}$ be of degree at most $d \geq 2$. Then the real variety*

$$V_{\mathbb{R}}(F_1, \dots, F_m) := \{x \in \mathbb{R}^n \,:\, F_1(x) = \cdots = F_m(x) = 0\}$$

is empty if and only if $V_{\mathbb{R}}(F_1, \dots, F_m) \cap A_d$ is empty.

Corollary 1.3 (*Half-Degree Principle*)**.** *Let $F_0 \in \mathbb{R}[X]^{S_n}$ be of degree d and let $k := \max\{2, \lfloor \frac{d}{2} \rfloor\}$. Then the inequality $F_0 \geq 0$ holds on $\mathbb{R}^n$ (resp. on the positive orthant $\mathbb{R}^n_{\geq 0}$) if and only if it holds on A_k (resp. on A_k^+).*
Furthermore there is $x \in \mathbb{R}^n$ with $F_0(x) = 0$ if and only if there is $x \in A_k$ with $F_0(x) = 0$.

Corollary 1.2 and the first part of Corollary 1.3 originate from the work of Timofte [15], but were for special cases already proven by Harris [8]. The second part of the half-degree principle was noted by Grimm [7].

Remark 1.4. In view of the second corollary it seems natural to ask whether the half-degree principle does in general also apply to any system of symmetric equalities. However if one considers the set

$$K := \{x \in \mathbb{R}^n \,:\, x_1 + x_2 + x_3 = 0, x_1^2 + x_2^2 + x_3^2 = 1, x_1^3 + x_2^3 + x_3^3 = 0\}$$

one finds that K is non-empty but $K \cap A_2$ is empty.

The original proofs of these results relied mostly on the existence of a solution to a differential equation and did not fully capture the geometric picture that plays in fact a key role as we intend to show in this article. Therefore we will provide proofs that exploit some underlying geometric properties: we can look at the zeros of a symmetric polynomial F on $\mathbb{R}^n$ as a variety in the orbit space of S_n, which agrees with the coefficient space of monic real univariate polynomials with only real roots. Then to conclude the main theorem we will show that if F has low degree, its minimizers correspond to minimal points of linear functionals over the orbit space.

This article will be structured as follows. In the next section we will give some background from the theory of symmetric polynomials and the geometry of the so called orbit space. In Section 3 we will provide some properties of univariate polynomials of degree n with only real roots. After Section 3 we will be able to give a short and elementary proof of the main theorem using the viewpoint presented in Section 2.

2. Symmetric polynomials and the orbit space of S_n

Among the polynomials that are invariant under the action of the symmetric group the following two families are of special interest:

Definition 2.1. Let $n \in \mathbb{N}$. For each $k \in \{1, \dots, n\}$ let

$$e_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \cdots X_{i_k}$$

denote the kth elementary symmetric polynomial.

One of the things that mark the importance of the elementary symmetric polynomials is that they are algebraically independent generators of the symmetric polynomials over any ring.

Theorem 2.2. *Let $\mathcal{R}$ be any commutative ring. Then the ring of symmetric polynomials $\mathcal{R}[X]^{S_n}$ is a polynomial ring in the n elementary symmetric polynomials $e_1, \dots, e_n$, i.e. every symmetric polynomial F can uniquely be written as $F = G(e_1, \dots, e_n)$ for some polynomial $G \in \mathcal{R}[X]$.*

This statement is classical. Proofs can be found for example in [6,14].

With the above Theorem 2.2 one can deduce the following:

Proposition 2.3. *Let $F \in \mathbb{R}[X]$ be symmetric of degree d. Then there is unique polynomial $G \in \mathbb{R}[Z_1, \dots, Z_d]$ of the form*

$$G = G_0(Z_1, \dots, Z_{\lfloor \frac{d}{2} \rfloor}) + \sum_{i=\lfloor \frac{d}{2} \rfloor + 1}^{d} G_i(Z_1, \dots, Z_{d-i}) Z_i, \tag{2.1}$$

with $G_i \in \mathbb{R}[Z_1, \dots, Z_d]$ such that

$$F = G(e_1, \dots, e_d).$$

Proof. A proof for the announced expression for G in Eq. (2.1) can be given by carefully inspecting a constructive proof of Theorem 2.2 (for example the one presented in [14]). But we will deduce the claim directly from Theorem 2.2.

Let $G \in \mathbb{R}[Z_1, \dots, Z_n]$ be the unique polynomial with $F = G(e_1, \dots, e_n)$. Take a finite set $I \subset \mathbb{Z}^n_{\geq 0}$ such that $G = \sum_{i \in I} g_i Z_1^{i_1} \cdots Z_n^{i_n}$, with $g_i \in \mathbb{R}$. As $e_1^{i_1} \cdots e_n^{i_n}$ is a homogeneous polynomial of degree $i_1 + 2i_2 + \cdots + ni_n$ and $\deg F = d$, we infer that $g_i = 0$ for all i with $\sum_l l i_l > d$. Now assume that $j \geq \lfloor \frac{d}{2} \rfloor + 1$ and $i_j \geq 1$. Then the condition $\sum_l l i_l \leq d$ implies first that $i_j = 1$ and further that $i_k = 0$ for all $k > d - j$. This means that the sum of terms of G that contain Z_j with $j \geq \lfloor \frac{d}{2} \rfloor + 1$ can be written in the form

$$\sum_{j=\lfloor \frac{d}{2} \rfloor}^{d} G_j(Z_1, \dots, Z_{d-j}) Z_j$$

for certain polynomials G_j. Finally combining all the other terms of G into a polynomial $G_1(Z_1, \dots, Z_{\lfloor \frac{d}{2} \rfloor})$ we arrive at the above representation. $\square$

Given $x \in \mathbb{C}^n$ we can view x as the n roots of the univariate monic polynomial

$$f = \prod_{i=1}^{n} (T - x_i).$$

The classical Vieta formula implies that f can also be written as

$$f = T^n - e_1(x) T^{n-1} + \cdots \pm e_n(x).$$

The identification of the n roots with the n coefficients is realized by the map

$$\begin{aligned} \pi \ : \ \mathbb{C}^n/S_n &\longrightarrow \mathbb{C}^n \\ x := (x_1, \dots, x_n) &\longmapsto \pi(x) := (e_1(x), \dots, e_n(x)). \end{aligned}$$

By the Fundamental Theorem of Algebra, π is a bijective map. Moreover, π is a homeomorphism (see [2] Proposition 1.1.5). In particular, π^{-1} is continuous, i.e., the roots of a univariate polynomial depend continuously on its coefficients.

However, as we are concerned only with real points we will have to restrict π to $\mathbb{R}^n$. In this case the restriction maps into $\mathbb{R}^n$ but it fails to be surjective: already the easy example $X^2 + 1$ shows that we can find n real coefficients that define a polynomial with strictly less than n real zeros. Polynomials with real coefficients that only have real roots are sometimes called *hyperbolic*.

Now the strategy used in order to prove Theorem 1.1 is to take the viewpoint of the real orbit space. Instead of F on $\mathbb{R}^n$, we will have to examine G over the set

$$\mathcal{H} := \{z \in \mathbb{R}^n : T^n - z_1 T^{n-1} + \cdots \pm z_n \text{ has only real roots}\}$$

and the sets

$$\mathcal{H}^k := \{z \in \mathcal{H} : T^n - z_1 T^{n-1} + \cdots \pm z_n \text{ has at most } k \text{ distinct zeros}\}.$$

So Theorem 1.1 follows directly from the following:

Theorem 2.4. *Let $F \in \mathbb{R}[X]$ of degree d and $G \in \mathbb{R}[Z_1, \dots, Z_n]$ such that $F = G(e_1, \dots, e_n)$. Then we have:*

(1) $G(\mathcal{H}) = G(\mathcal{H}^d)$,
(2) $\inf_{z \in \mathcal{H}} G(z) = \inf_{z \in \mathcal{H}^k} G(z)$ *for all k with* $\max\{2, \lfloor d/2 \rfloor\} \leq k$,
(3) $\inf_{z \in \mathcal{H} \cap \mathbb{R}^n_+} G(z) = \inf_{z \in \mathcal{H}^k \cap \mathbb{R}^n_+} G(z)$ *for all k with* $\max\{2, \lfloor d/2 \rfloor\} \leq k$.

Before we give the proof of the above theorem and therefore the proof of Theorem 1.1, we will need some very elementary facts about polynomials with only real roots. We will show these facts concerning hyperbolic polynomials in the next section.

3. Hyperbolic polynomials

The main problem that we will have to deal with in order to prove the main theorem is the question of which changes of the coefficients of a hyperbolic polynomial will result in polynomials that are still hyperbolic. This question is in fact very old and has already been studied by Pólya and Schur (see [12,3,10]). Despite the complexity of this question in full generality, we will argue that the statements that we need mainly follow directly from classical facts on the relation between the zeros of a polynomial and the zeros of its derivative.

If $f \in \mathbb{R}[T]$ factors as

$$f = \prod_{i=1}^{k} (T - x_i)^{m_i},$$

then the numbers m_i are called the orders of the corresponding roots. Then from Rolle's theorem (see [4] Proposition 2.22) one can directly infer the following:

Proposition 3.1. *Let $f = T^n + a_1T^{n-1} + \cdots + a_n$ be hyperbolic. Then the following hold:*

(1) *Let $a, b \in \mathbb{R}$ with $a \leq b$. If f has k roots (counted with multiplicities) in $[a, b]$ then f' has at least $k - 1$ roots in $[a, b]$ (and exactly $k - 1$ if $f(a) = f(b) = 0$).*
(2) *All derivatives of f are also hyperbolic.*
(3) *If $a \in \mathbb{R}$ is a multiple root of order $k > 1$ of f' then a is also a root of order $k + 1$ of f.*
(4) *If $a_i = a_{i+1} = 0$ then $a_j = 0$ for all $j \geq i$.*

We need to specify small perturbations of the coefficients such that all real roots of a hyperbolic polynomial stay real. The following proposition is in fact sufficient for our reasoning.

Proposition 3.2. *Let $f \in \mathbb{R}[T]$ be a hyperbolic polynomial of degree n with exactly k distinct roots.*

(a) *If $k = n$ then for any non-zero polynomial g of degree at most n there exists $\delta_n > 0$ such that for $0 < \varepsilon < \delta_n$ the polynomials $f \pm g$ are also hyperbolic with n distinct roots.*
(b) *If $k < n$ then for each $1 \leq s \leq k$ there is a polynomial g_s of degree $n - s$ and a $\delta_s > 0$ such that for all $0 < \varepsilon < \delta_s$ the polynomials $f \pm \varepsilon g_s$ are also hyperbolic and have strictly more distinct zeros.*

Proof. (a) This follows directly by the fact that the roots of f depend continuously on the coefficients and come as complex conjugated pairs.

(b) Let $x_1, \ldots, x_k$ be the distinct roots of f. We can factor as follows:

$$f = \underbrace{\prod_{i=1}^{s}(T - x_i)}_{:=p} \cdot g_1,$$

where the set of zeros of g_1 contains only elements from $\{x_1, \ldots, x_k\}$ and g_1 is of degree $n - s$. Now we can apply (a) to see that $p \pm \varepsilon_s$ is hyperbolic for all $\varepsilon_s < \delta_s$. Furthermore we see that $p \pm \varepsilon_s$ has none of its roots in the set $\{x_1, \ldots, x_k\}$. Hence $(p \pm \varepsilon_s) \cdot g_1 = f \pm \varepsilon_s g_1$ is hyperbolic and has more than k distinct roots. □

We also want restrict to $\mathbb{R}^n_+$. Thus in the following proposition we note what happens in this case.

Proposition 3.3. *The map π maps $\mathbb{R}^n_+$ onto $\mathcal{H}_+ := \mathbb{R}^n_{\geq 0} \cap \mathcal{H}$.*

By definition of the set $\mathcal{H}_+$ it could be possible that there are all sorts of polynomials with zero coefficients. But to conclude the main theorem also in the version on $\mathcal{H}_+$ we will need the following proposition which follows from Proposition 3.1.

Proposition 3.4. *Let $f := T^n + a_1T^{n-1} + \cdots + a_n$ be a hyperbolic polynomial with only non-negative roots. If $a_{n-i} = 0$ for one i then $a_{n-j} = 0$ for all $j \leq i$.*

Proof. First observe that if f has only positive roots then by Rolle's theorem all its derivatives share this property. If $a_{n-i} = 0$ we know that the ith derivative $f^{(i)}$ of f has a root at $t = 0$. But as $f^{(i-1)}$ has also only positive roots, also $f^{(i-1)}(0) = 0$. Now the statement follows since Proposition 3.1(3) now implies that f has a multiple root of order i at $t = 0$. □

To study the polynomials on the boundary of $\mathcal{H}_+$ the following consequence of Proposition 3.2 will be helpful:

Proposition 3.5. *Let $f \in \mathbb{R}[T]$ be a hyperbolic polynomial of degree n with $k < n$ distinct roots and assume that f has a root of order $m < k$ at 0. Then for each $1 \leq s \leq k$ there is a polynomial g_s of degree $n - s$ with an m-fold root at 0 and $\delta_s > 0$ such that for all $0 < \varepsilon < \delta_s$ the polynomials $f \pm \varepsilon g$ are also hyperbolic and have strictly more distinct roots.*

Proof. Just consider the hyperbolic polynomial $\tilde{f} := \frac{f}{T^m}$ of degree $n - m$ with $k - m$ distinct zeros. Applying 3.2 to $\tilde{f}$ we construct a polynomial $\tilde{g}_s$ of degree $n - m - s$ but then obviously $g_s := \tilde{g}_s T^m$ meets the announced stated requirements. □

4. Proof of the main theorem

This last section uses the statements about univariate polynomials given in the previous section to prove the main statements. The proofs will be based on an optimization problem. In order to introduce this problem we will first give some notation:

Recall that with each S_n-orbit of any $x \in \mathbb{R}^n$ we associate the polynomial

$$f = \prod_{i=1}^{n}(T - x_i) = T^n + \sum_{i=1}^{n}(-1)^i a_i T^{n-i}.$$

We will consider optimization problems over sets of the form

$$\mathcal{H}(a_1, \dots, a_s) := \{z \in \mathbb{R}^n \,:\, z_1 = a_1, \dots, z_s = a_s,\ T^n - z_1 T^{n-1} + \cdots \pm z_n \text{ is hyperbolic}\},$$

i.e. over the set of all monic hyperbolic polynomials of degree n that agree with f on the leading $s + 1$ coefficients.

Now for the proof of the main theorem will take a look at linear optimization problems of the form

$$\begin{aligned} &\min c^t z \\ &z \in \mathcal{H}(a_1, \dots, a_s), \end{aligned}$$

where $c \in \mathbb{R}^n$ defines any linear function and $a_1, \dots, a_s$ are fixed. To make the later argumentation easier, we set the minimum of any function over the empty set to be infinity.

A priori it may not be obvious that such problems have an optimal solution. But, this is a consequence of the following lemma:

Lemma 4.1. *For any $s \geq 2$ every set $\mathcal{H}(a_1, \dots, a_s)$ is compact.*

Proof. As the empty set is compact we can assume that there is $z \in \mathcal{H}(a_1, a_2)$. Let $x_1, \dots, x_n$ be the roots of $f_z := T^n - z_1 T^{n-1} + \cdots \pm z_n$. Then we have $e_1(x) = -a_1$ and $e_2(x) = a_2$. Hence we have $\sum_{i=1}^n x_i^2 = (e_1(x))^2 - 2e_2(x) = a_1^2 - 2a_2$. This shows that x is contained in a ball; thus $\mathcal{H}(a_1, a_2)$ is bounded, and hence so is $\mathcal{H}(a) \subseteq \mathcal{H}(a_1, a_2)$. Furthermore as the roots of a polynomial depend continuously on the coefficients it is clear that $\mathcal{H}(a)$ is closed and therefore compact. □

We will use $\mathcal{H}^k(a_1, \dots, a_s)$ to refer to the points in $\mathcal{H}(a_1, \dots, a_s) \cap \mathcal{H}^k$, i.e. to those monic hyperbolic polynomials which have at most k distinct zeros and prescribed coefficients $a_1, \dots, a_s$.

The crucial observation which will be the core of the theorems that we want to prove lies in the geometry of the optimal points of the above optimization problems. This is noted in the following:

Theorem 4.2. *Let $n > 2$, $s \in \{2, \dots, n\}$, $c \in \mathbb{R}^n$ with $c_j \neq 0$ for at least one $j \in \{s + 1, \dots, n\}$ and $a \in \mathbb{R}^s$ such that $\mathcal{H}(a) \neq \emptyset$. We consider the optimization problem*

$$\min_{z \in H(a)} c^t z.$$

Let M denote the set of minimizers of this problem. Then we have $\emptyset \neq M \subseteq \mathcal{H}^s(a)$.

Proof. Since by the above lemma $\mathcal{H}(a)$ is compact, there is at least one minimizer z, showing the non-emptyness of M. Now in order to prove $M \subseteq H^s(a)$ take $z \in M$ such that the number k of distinct roots of

$$f_z := T^n - z_1 T^{n-1} + \cdots \pm z_n$$

is maximal. We have to show that $s < k \leq n$ is impossible.
Assume that $k = n$. Then we can choose $y \in \mathbb{R}^n$ such that $c^t y < 0$. By Proposition 3.2 (a) we deduce that there is a $\delta_n > 0$ such that for all $0 < \varepsilon < \delta_n$ we find that the polynomial $f_z + \varepsilon(y_1 T^{n-1} + \cdots \pm y_n)$ is still hyperbolic. Thus by the choice of y we have $z + \varepsilon y \in \mathcal{H}(a)$ but by construction we have $c^t(z + \varepsilon y) < c^t z$ for all $0 < \varepsilon < \delta_n$ which clearly implies $z \notin M$. If on the other hand we assume $s < k < n$ then by Proposition 3.2 we find $y \in \{0\}^k \times \mathbb{R}^{n-k}$ and $\delta_k > 0$ such that for

$$g := T^{n-k} - y_{k+1} T^{n-k-1} + \cdots \pm y_n$$

we have that $f \pm \varepsilon g$ is hyperbolic for all $0 < \varepsilon < \delta_k$. Thus by the choice of y the point $z \pm \varepsilon y$ will be in $\mathcal{H}(a)$ for all $0 < \varepsilon < \delta_k$. Without loss of generality we may assume that $c^t y \leq 0$. This in turn implies

$$c^t(z + \varepsilon y) \leq c^t z \leq z - \varepsilon y,$$

and since z is supposed to be a minimizer we must have that also $(z + \varepsilon y)$ is a minimizer. However, by Proposition 3.2 $f + \varepsilon g$ has strictly more distinct components, which clearly contradicts our choice of z and we can conclude. □

From the above lemma we can conclude the following important corollary:

Corollary 4.3. *Every set $\mathcal{H}(a_1, \dots, a_s) \neq \emptyset$ with $s \geq 2$ contains a point $\tilde{z}$ with $\tilde{z} \in \mathcal{H}^s(a_1, \dots, a_s)$.*

Proof. If $n \in \{1, 2\}$ the statement is clear. So we can assume $n > 2$ and the statement follows directly from Theorem 4.2. □

To transfer the half-degree principle to $\mathbb{R}^n_{\geq 0}$ we will also need to know what happens to the minima when we intersect a set $\mathcal{H}(a_1, \dots, a_s)$ with $\mathbb{R}^n_+$. We denote this intersection with $\mathcal{H}^+(a_1, \dots, a_s)$ and define

$$\mathcal{H}^{(s,+)}(a_1, \dots, a_s) := \mathcal{H}^+(a_1, \dots, a_s) \cap \mathcal{H}^s(a_1, \dots, a_s) \cup \mathcal{H}(a_1, \dots, a_s, 0, 0, \dots, 0).$$

With this appropriate notation we have a same type of argument as in Theorem 4.2:

Lemma 4.4. *Let $n > 2$, $s \in \{2, \dots, n\}$, $c \in \mathbb{R}^n$ with $c_j \neq 0$ for at least one $j \in \{s+1, \dots, n\}$ and $a \in \mathbb{R}^s$ such that $\mathcal{H}^+(a) \neq \emptyset$. Consider the optimization problem*

$$\min_{z \in H^+(a)} c^t z.$$

Let M denote the set of minimizers of this problem. Then we have $\emptyset \neq M \subseteq \mathcal{H}^{(s,+)}(a)$.

Proof. The argument works out almost the same way as in Theorem 4.2: indeed if $z \in \mathcal{H}^+(a_1, \dots, a_s)$ has strictly positive components, small perturbations of these will not change the positivity and the same arguments can be used. So just the cases of $z \in \mathcal{H}^+(a_1, \dots, a_s)$ with zero components need special consideration. So assume that we have a $\tilde{z} \in \mathcal{H}(a_1, \dots, a_s)$ with zero components such that

$$c^t \tilde{z} = \min_{z \in \mathcal{H}^+(a_1, \dots, a_s)} c^t z.$$

But with Proposition 3.4 we see that there is $i \in \{1, \dots, n\}$ such that $\tilde{z}_j = 0$ for all $j \geq i$. If $i \leq s+1$ we have already that $\tilde{z} \in \mathcal{H}^{(s,+)}(a_1, \dots, a_s)$. But if $s+1 < i$ we can see from Proposition 3.5 that there is $0 \neq \tilde{y} \in \{0\}^s \times \mathbb{R}^{i-s} \times \{0\}^{n-i}$ such that $\tilde{z}_1 \pm \varepsilon \tilde{y} \in \mathcal{H}(a_1, \dots, a_s) \cap \mathbb{R}^N_+$ for small positive ε and argue as in the previous lemma. □

Now to we are able to deduce the proof of Theorem 2.4:

Proof of Theorem 2.4. (1) We know from Proposition 2.3 that

$$G = G_0(Z_1, \dots, Z_{\lfloor \frac{d}{2} \rfloor}) + \sum_{i=\lfloor \frac{d}{2} \rfloor + 1}^{d} G_i(Z_1, \dots, Z_{d-i}) Z_i.$$

So G is constant on any set $\mathcal{H}(a_1, \dots, a_d)$. As we have

$$\bigcup_{(a_1, \dots, a_d) \in \mathbb{R}^d} \mathcal{H}(a_1, \dots, a_d) = \mathcal{H},$$

the first statement in Theorem 2.4 follows now directly from Corollary 4.3.

(2) We will have to see that

$$\min_{z \in \mathcal{H} \subset \mathbb{R}^n} G(z) = \min_{z \in \mathcal{H}^k} G(z).$$

Again we decompose the space in the form

$$\bigcup_{(a_1, \dots, a_k) \in \mathbb{R}^k} \mathcal{H}(a_1, \dots, a_k) = \mathcal{H}.$$

Therefore

$$\min_{z \in \mathcal{H}} G(z) = \min_{a_1, \dots, a_k} \min_{z \in \mathcal{H}(a_1, \dots, a_k)} G(z).$$

But for fixed $z_1 = a_1, \dots, z_k = a_k$ the function $G(z)$ is just linear and now we can apply Theorem 4.2 and see that

$$\min_{z \in \mathcal{H}(a_1, \dots, a_k)} G(z) = \min_{z \in \mathcal{H}^k(a_1, \dots, a_k)} G(z).$$

and we get the second statement in Theorem 2.4.

(3) Again the function G is linear over the sets $\mathcal{H}^+(a_1, \dots, a_k)$ and we can argue as above by using Lemma 4.4. □

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