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Resolution of an integral equation with the Thue–Morse sequence

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Abstract

It is a classical fact that the exponential function is a solution of the integral equation $\int_0^X f(x)\,dx + f(0) = f(X)$. If we slightly modify this equation to $\int_0^X f(x)\,dx + f(0) = f(\alpha X)$ with $\alpha \in]0, 1[$, it seems that no classical techniques apply to yield solutions. In this article, we consider the parameter $\alpha = 1/2$. We will show the existence of a solution which takes the values of the Thue–Morse sequence on the odd integers.

Keywords: Integral equation; Thue–Morse sequence

1. Introduction

We consider the functional equation

$$\int_0^X f(x)dx + f(0) = f\left(\frac{X}{2}\right). \tag{1}$$

We can see that the set of continuous solutions is a closed vector space, containing the identically zero function. It is quite clear that any continuous function satisfying Eq. (1) is differentiable infinitely many times. So, Eq. (1) can be rewritten $f(X) = f'(X/2)/2$.

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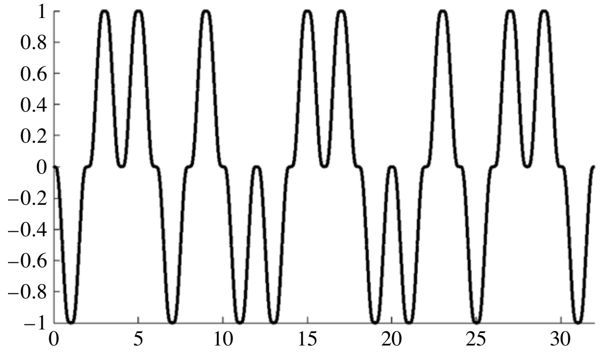

Fig. 1. Representation of the graph of f_∞.

We can easily verify that the nonzero solutions cannot be expanded in a series. In addition, two solutions equal in a neighborhood of 0 are equal everywhere.

We let τ denote the Thue–Morse substitution. It is a morphism of the free monoid generated by -1 and 1, defined by $\tau(-1) = (-1)1$ and $\tau(1) = 1(-1)$ and let $\boldsymbol{u} = (u_n)_{n\geq 0} = (-1)11(-1)1(-1)(-1)1\cdots$ be the Thue–Morse sequence, one of the fixed points of this substitution. See [2,3,5] for details.

The aim of this work is to show the following result:

Theorem 1. *There exists a continuous function f_∞ valued in $[-1, 1]$, solution of Eq. (1), such that (see Fig. 1)*

- *for each integer n, $f_\infty(2n+1) = u_n$ and $f_\infty(2n) = 0$;*
- *for each negative real number x, $f_\infty(x) = 0$;*
- *for each positive real number x, $|f_\infty(x)| = |f_\infty(x+2)|$.*

2. Introduction of some combinatorial objects

For any integers $k \geq 0$ and $n \geq 1$, we define the quantities $(\Sigma_n^k)_{(k,n)\in\mathbb{N}^2}$ by

$$\Sigma_0^k = u_k \quad \text{and} \quad \Sigma_n^0 = 0, \tag{2}$$

and by induction for any integers $k \geq 0$ and $n \geq 0$, by

$$\Sigma_{n+1}^{k+1} = \Sigma_n^k + \Sigma_{n+1}^k. \tag{3}$$

In [7], Prunescu has studied the behavior of certain double sequences, called *recurrent two-dimensional sequences* in a more general context. For example when the initialization of the induction given in Eq. (2) is

$$\Sigma_0^k = v_k \quad \text{and} \quad \Sigma_n^0 = w_n,$$

where $(v_n)_n$ and $(w_n)_n$ are sequences such that $v_0 = w_0$. He is particularly interested in the case where $\boldsymbol{v} = \boldsymbol{w} = \boldsymbol{u}$.

If we cleverly renormalize the lines of the standard Pascal triangle, we can approximate a Gaussian curve. We will renormalize the columns of the Pascal triangle associated to the

	-1	0	0	0	------	0	0	------
	1	-1	0	0	------	0	0	------
	1	0	-1	0	------	0	0	------
	-1	1	-1	-1	------	0	0	------
	1	0	0	-2	------	0	0	------
	⋮					⋮	⋮	
$\Sigma_0^k =$	u_k	Σ_1^k	Σ_2^k	-------------		Σ_n^k	Σ_{n+1}^k	------
$\Sigma_0^{k+1} =$	u_{k+1}	Σ_1^{k+1}	Σ_2^{k+1}	--------------------			Σ_{n+1}^{k+1}	------

Fig. 2. “Pascal’s triangle” associated to the Thue–Morse sequence.

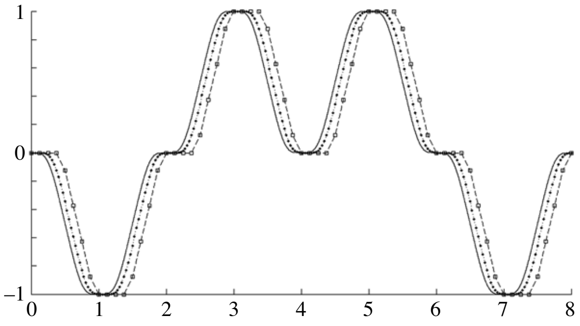

Fig. 3. Representation of graph of f_4, f_6 and f_∞.

Thue–Morse sequence, to approximate the function f_∞ (see Fig. 2). We will see that each column is uniformly bounded. This is a very special property of the Thue–Morse sequence.

This property does not hold for Sturmian words, for which the sequence $(\Sigma_2^k)_k$ is not bounded. More precisely, for each parameter $\alpha \in [0, 1]$, we put $\boldsymbol{v}(\alpha) = \big(v_n(\alpha)\big)_n$ the sequence defined for each integer n by $v_n(\alpha) = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$. We associate to the sequence $\boldsymbol{v}(\alpha)$ the sequence $\boldsymbol{w}(\alpha) = \big(w_n(\alpha)\big)_n$ defined for each integer n by $w_n(\alpha) = \alpha$ if $v_n(\alpha) = 0$, and $w_n(\alpha) = -(1-\alpha)$ otherwise. So, the sequence $\big(\Sigma_1^k\big)_1$ defined in (3) associated to the sequence $\boldsymbol{w}(\alpha)$ is bounded. But the sequence $(\Sigma_2^k)_k$ is not bounded. We refer to [1,4,6].

For all integers n, we define a real function f_n, by $f_n(x) = 0$ if $x \le 0$, and

$$f_n(x) = x_n^k + 2^{n-1}\delta_x (x_n^{k+1} - x_n^k), \quad \text{if } x = \frac{k}{2^{n-1}} + \delta_x \text{ and } 0 \le \delta_x < 2^{1-n}$$

for an integer k, with the notation $x_n^k = 2^{-(n-1)(n-2)/2} \cdot \Sigma_n^k$ (see Fig. 3).

We may also approach this problem from a dynamical point of view. We define T, the application from the set of real sequences into itself by

$$T\left((y_n)_{n\ge 1}\right) = \left(y_1, y_2 + y_1, y_3 + \frac{y_2}{2^1}, \ldots, y_{n+1} + \frac{y_n}{2^{n-1}}, \ldots\right).$$

We must then consider the n-th coordinates of the sequence $(\boldsymbol{y}^k)_{k\ge 0}$ up to renormalization, where $\boldsymbol{y}^k = (y_n^k)_{n\ge 1}$, is defined by induction by $\boldsymbol{y}^0 = \boldsymbol{0} = (0, \ldots, 0, \ldots)$, and for each integer $k \ge 1$,

$$\boldsymbol{y}^{k+1} = T(\boldsymbol{y}^k) - (u_k, 0, \ldots, 0, \ldots).$$

3. Calculation of points x_n^k for the first n

We calculate the initial values of sequences $(x_n^k)_k$. To do this, we note that for each integer k, $u_{2k} = u_k = -u_{2k+1}$.

$$\begin{cases} \Sigma_1^{2k} = 0, \\ \Sigma_1^{2k+1} = u_{2k} = u_k, \end{cases} \Longleftrightarrow \begin{cases} x_1^{2k} = 0, \\ x_1^{2k+1} = u_k. \end{cases}$$

$$\begin{cases} \Sigma_2^{4k} = 0, \\ \Sigma_2^{4k+1} = \Sigma_2^{4k} + \Sigma_1^{4k} = 0, \\ \Sigma_2^{4k+2} = \Sigma_2^{4k+1} + \Sigma_1^{4k+1} = u_{2k} = u_k, \\ \Sigma_2^{4k+3} = \Sigma_2^{4k+2} + \Sigma_1^{4k+2} = u_{2k} = u_k, \end{cases} \Longleftrightarrow \begin{cases} x_2^{4k} = 0, \\ x_2^{4k+1} = 0, \\ x_2^{4k+2} = u_k, \\ x_2^{4k+3} = u_k. \end{cases}$$

$$\begin{cases} \Sigma_3^{8k} = 0, \\ \Sigma_3^{8k+1} = \Sigma_3^{8k} + \Sigma_2^{8k} = 0, \\ \Sigma_3^{8k+2} = \Sigma_3^{8k+1} + \Sigma_2^{8k+1} = 0, \\ \Sigma_3^{8k+3} = \Sigma_3^{8k+2} + \Sigma_2^{8k+2} = u_{4k} = u_k, \\ \Sigma_3^{8k+4} = \Sigma_3^{8k+3} + \Sigma_2^{8k+3} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+5} = \Sigma_3^{8k+4} + \Sigma_2^{8k+4} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+6} = \Sigma_3^{8k+5} + \Sigma_2^{8k+5} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+7} = \Sigma_3^{8k+6} + \Sigma_2^{8k+6} = u_k, \end{cases} \Longleftrightarrow \begin{cases} x_3^{8k} = 0, \\ x_3^{8k+1} = 0, \\ x_3^{8k+2} = 0, \\ x_3^{8k+3} = u_k/2, \\ x_3^{8k+4} = u_k, \\ x_3^{8k+5} = u_k, \\ x_3^{8k+6} = u_k, \\ x_3^{8k+7} = u_k/2. \end{cases}$$

$$\begin{cases} \Sigma_4^{16k} = 0, \\ \Sigma_4^{16k+1} = \Sigma_4^{16k} + \Sigma_3^{16k} = 0, \\ \Sigma_4^{16k+2} = \Sigma_4^{16k+1} + \Sigma_3^{16k+1} = 0, \\ \Sigma_4^{16k+3} = \Sigma_4^{16k+2} + \Sigma_3^{16k+2} = 0, \\ \Sigma_4^{16k+4} = \Sigma_4^{16k+3} + \Sigma_3^{16k+3} = u_k, \\ \Sigma_4^{16k+5} = \Sigma_4^{16k+4} + \Sigma_3^{16k+4} = 3u_k, \\ \Sigma_4^{16k+6} = \Sigma_4^{16k+5} + \Sigma_3^{16k+5} = 5u_k, \\ \Sigma_4^{16k+7} = \Sigma_4^{16k+6} + \Sigma_3^{16k+6} = 7u_k, \\ \Sigma_4^{16k+8} = \Sigma_4^{16k+7} + \Sigma_3^{16k+7} = 8u_k, \\ \Sigma_4^{16k+9} = \Sigma_4^{16k+8} + \Sigma_3^{16k+8} = 8u_k, \\ \Sigma_4^{16k+10} = \Sigma_4^{16k+9} + \Sigma_3^{16k+9} = 8u_k, \\ \Sigma_4^{16k+11} = \Sigma_4^{16k+10} + \Sigma_3^{16k+10} = 8u_k, \\ \Sigma_4^{16k+12} = \Sigma_4^{16k+11} + \Sigma_3^{16k+11} = 7u_k, \\ \Sigma_4^{16k+13} = \Sigma_4^{16k+12} + \Sigma_3^{16k+12} = 5u_k, \\ \Sigma_4^{16k+14} = \Sigma_4^{16k+13} + \Sigma_3^{16k+13} = 3u_k, \\ \Sigma_4^{16k+15} = \Sigma_4^{16k+14} + \Sigma_3^{16k+14} = u_k, \end{cases} \Longleftrightarrow \begin{cases} x_4^{16k} = 0, \\ x_4^{16k+1} = 0, \\ x_4^{16k+2} = 0, \\ x_4^{16k+3} = 0, \\ x_4^{16k+4} = u_k/8, \\ x_4^{16k+5} = 3u_k/8, \\ x_4^{16k+6} = 5u_k/8, \\ x_4^{16k+7} = 7u_k/8, \\ x_4^{16k+8} = u_k, \\ x_4^{16k+9} = u_k, \\ x_4^{16k+10} = u_k, \\ x_4^{16k+11} = u_k, \\ x_4^{16k+12} = 7u_k/8, \\ x_4^{16k+13} = 5u_k/8, \\ x_4^{16k+14} = 3u_k/8, \\ x_4^{16k+15} = u_k/8. \end{cases}$$

4. First combinatorial results

Lemma 1. *For any integers $n \geq 1$, $k \geq 0$ and $l \in \{0, \ldots, 2^n - 1\}$, there exists $a(n,l)$, which does not depend on k, such that $\Sigma_n^{2^n k+l} = a(n,l)u_k$. In particular, $\Sigma_n^{2^n k} = a(n,0) = 0$. For any integer $n \geq 1$ and $l \in \{0, \ldots, 2^n - 1\}$, the coefficients $a(n,l)$ satisfy the following relation:*

$$\begin{aligned} &a(n+1, l+1) = a(n+1, l) + a(n,l) \\ &\text{and} \quad a(n+1, l+2^n+1) = a(n+1, l+2^n) - a(n,l). \end{aligned} \tag{4}$$

We conclude that $a(n+1, l+2^n) = a(n+1, 2^n) - a(n+1, l)$.

Proof. We have seen in Section 3, that this result is true for the first values of the integer n. We suppose that the result is true up to a rank $n-1$ and we will show that it is still true up to order n. We start by verifying that $\Sigma_{n+1}^{2^n k}$ is zero for each integer k:

$$\begin{aligned} \Sigma_n^{2^n k} &= \sum_{l=0}^{2^n k-1} \Sigma_{n-1}^l + \Sigma_{n+1}^0 = \sum_{j=0}^{k-1} \sum_{l=0}^{2^n-1} \Sigma_{n-1}^{2^n j+l} \\ &= \sum_{j=0}^{k-1} \left(\sum_{l=0}^{2^{n-1}-1} \Sigma_{n-1}^{2^{n-1}(2j)+l} + \sum_{l=0}^{2^{n-1}-1} \Sigma_{n-1}^{2^{n-1}(2j+1)+l} \right) \\ &= \sum_{j=0}^{k-1} \left(u_{2j} \sum_{l=0}^{2^{n-1}-1} a(n-1,l) + u_{2j+1} \sum_{l=0}^{2^{n-1}-1} a(n-1,l) \right) \\ &= \left(\sum_{l=0}^{2^{n-1}-1} a(n-1,l) \right) \cdot \left(\sum_{j=0}^{k-1} u_{2j} + u_{2j+1} \right) \\ &= 0. \end{aligned}$$

Now, we focus on the recurrence relations verified by the coefficients $a(n,k)$. The integer n is already fixed, we show this result by induction on l and k. For $l = 0$, we have seen that this result was true for all integers k. Suppose Eq. (4) holds for all k up to a rank l and show that it is still true for all k the rank $l+1$.

$$\begin{aligned} \Sigma_n^{2^n k+l+1} &= \Sigma_n^{2^n k+l} + \Sigma_{n-1}^{2^{n-1}(2k)+l} = a(n,l)u_k + a(n-1,l)u_{2k} \\ &= a(n,l)u_k + a(n-1,l)u_k = (a(n,l) + a(n-1,l))\, u_k. \end{aligned}$$

$$\begin{aligned} \Sigma_n^{2^n k+2^{n-1}+l+1} &= \Sigma_n^{2^n k+2^{n-1}+l} + \Sigma_{n-1}^{2^{n-1}(2k+1)+l} \\ &= a(n, l+2^{n-1})u_k + a(n-1,l)u_{2k+1} \\ &= a(n,l)u_k - a(n-1,l)u_k \\ &= \left(a(n+2^{n-1}, l) - a(n-1,l) \right) u_k. \end{aligned}$$

Then, we verify the last relation of the lemma:

$$\begin{aligned} a(n+1, l+2^n) &= a(n+1, l+2^n-1) - a(n, l-1) \\ &= a(n+1, l+2^n-2) - a(n, l-2) - a(n, l-1), \\ &= a(n+1, 2^n) - \sum_{j=0}^{l-1} a(n,j). \end{aligned}$$

We get $a(n+1, l+2^n) = a(n+1, 2^n) - a(n+1, l)$. $\square$

Lemma 2. *For any integer n,* $a(n, 2^{n-1}) = 2^{(n-1)(n-2)/2}$.

Proof. Since $a(1, 1) = 1$, this result is immediate by induction from the relation:

$$\begin{aligned} a(n+1, 2^n) &= \sum_{l=0}^{2^n-1} a(n,l) = \sum_{l=0}^{2^{n-1}-1} a(n,l) + \sum_{l=0}^{2^{n-1}-1} a(n, l+2^{n-1}) \\ &= \sum_{l=0}^{2^{n-1}-1} a(n,l) + \sum_{l=0}^{2^{n-1}-1} \Big(a(n, 2^{n-1}) - a(n,l)\Big) \\ &= \sum_{l=0}^{2^{n-1}-1} a(n, 2^{n-1}) = 2^{n-1} a(n, 2^{n-1}). \end{aligned}$$

So, $a(n+1, 2^n) = 2^{n-1} \cdot 2^{(n-1)(n-2)/2} = 2^{(n+1-1)(n+1-2)/2}$. □

Lemma 3. *For every integer n, and* $l \in \{0, \ldots, 2^n - 1\}$,

$$0 \le a(n,l) \le 2^{(n-1)(n-2)/2}. \tag{5}$$

Proof. We will show this by induction on the integer n. We initialized the recurrence. We suppose that the result is true up to the rank n and show that it is still true to the rank $n + 1$.

Suppose then that for each integer $l \in \{0, \ldots, 2^n - 1\}$, Eq. (5) holds. Since for every $l \in \{0, \ldots, 2^n - 1\}$,

$$a(n+1, l+1) = a(n,l) + a(n+1, l) \ge 0,$$

the sequence $(a(n+1,l))_{l \in \{0,\ldots,2^n\}}$ increases from 0 to $2^{(n-1)(n-2)/2}$ for $l = 2^n$. We can then conclude because if $l \in \{0, \ldots, 2^n - 1\}$,

$$0 \le a(n+1, l+2^n) = 2^{(n-1)(n-2)/2} - a(n+1, l) \le 2^{(n-1)(n-2)/2}. \quad \square$$

Lemma 4. *For every integer n, and* $l \in \{0, \ldots, 2^{n-2} - 1\}$,

$$a(n, 2l+1) \ge a(n, 2l) \ge 2^{n-2} a(n,l).$$

Proof. We prove this lemma by induction on n. For $n = 1$, the result is immediate. We show that if the result is true up to the rank n, it is still true to the rank $n + 1$. We show this by induction on l. From Lemma 1, this is true for $l = 0$ and $l = 1$. We suppose that the result is true for $2l$ and $2l + 1$, and we show that it is still true for $2l + 2$ and $2l + 3$.

If $l \in \{0, \ldots, 2^{n-2} - 1, \}$, then $a(n, 2l) \le a(n, 2l + 1)$ and

$$\begin{aligned} a(n+1, 2(l+1)+1) &\ge a(n+1, 2(l+1)) \\ &\ge a(n+1, 2l) + a(n, 2l) + a(n, 2l+1) \\ &\ge 2^{n-1} a(n,l) + a(n, 2l) + a(n, 2l+1) \\ &\ge 2^{n-1} a(n,l) + 2a(n, 2l) \\ &\ge 2^{n-1} a(n,l) + 22^{n-2} a(n-1, l) \\ &\ge 2^{n-1} \left(a(n,l) + a(n-1,l)\right) \\ &\ge 2^{n-1} a(n, l+1). \end{aligned}$$

If $l \in \{2^{n-2}, \dots, 2^{n-1}-1, \}$, then $a(n,2l) \geq a(n,2l+1)$ and

$$\begin{aligned} a(n+1, 2(l+1)+1) &\geq a(n+1, 2(l+1)) \\ &\geq a(n+1,2l) + a(n,2l) + a(n,2l+1) \\ &\geq 2^{n-1}a(n,l) + a(n,2l) + a(n,2l+1) \\ &\geq 2^{n-1}a(n,l) + 2a(n,2l+1) \\ &\geq 2^{n-1}a(n,l) + 22^{n-2}a(n-1,l) \\ &\geq 2^{n-1}\left(a(n,l) + a(n-1,l)\right) \\ &\geq 2^{n-1}a(n,l+1). \quad \square \end{aligned}$$

5. Proof of Theorem 1

Let us start by proving the following lemma.

Lemma 5. *Let n be an integer greater than or equal to* 1.

1. *For each integer m,* $f_n(2m+1) = u_m$ *and* $f_n(2m) = 0$.
2. *For each real x,* $f_n(x) \in [-1,1]$.
3. *For each integer m and for each* $x \in [0,2]$,

$$f_n(x+2m) = -f_n(x)u_m. \tag{6}$$

4. *For each integer m, if* $u_m = -1$, f_n *increases on* $[m, m+1]$, *and if* $u_m = 1$, f_n *decreases on* $[m, m+1]$. *In particular,* f_n *and* u_m *have the same sign on* $[2m, 2m+2]$.
5. *For each couple of reals* $(x,y) \in [0,2]^2$*:* $|f_n(x) - f_n(y)| \leq |x-y|$.
6. *For each real* $x \in [0,1]$, *the sequence* $(f_n(x))_n$ *decreases.*
7. *For each real* $x \in [2m, 2m+1]$, $(f_n(x))_n$ *is decreasing if* $u_m = -1$, *and increasing otherwise. And for each real* $x \in [2m+1, 2m+2]$, $(f_n(x))_n$ *is increasing if* $u_m = -1$, *and decreasing otherwise.*

Proof of point 1 of Lemma 5. We fix an integer $n \geq 1$, and an integer m. By the definition of functions f_n, $f_n(2m) = \Sigma_n^{2^n m} 2^{-(n-1)(n-2)/2} = 0$ and

$$f_n(2m) = \Sigma_n^{2^n m + 2^{n-1}} 2^{-(n-1)(n-2)/2} = 2^{(n-1)(n-2)/2} u_m 2^{-(n-1)(n-2)/2} = u_m. \quad \square$$

Proof of point 2 of Lemma 5. From Lemma 3, for each positive real x:

$$\begin{aligned} |f_n(x)| &\leq \sup\left\{|x_n^{2^n k + l}|; k \in \mathbb{N} \text{ and } l \in \{0, \dots, 2^n\}\right\} \\ &\leq \sup\left\{|\Sigma_n^{2^n k + l} 2^{-(n-1)(n-2)/2}|; k \in \mathbb{N} \text{ and } l \in \{0, \dots, 2^n\}\right\} \\ &\leq \sup\left\{a(n,l)2^{-(n-1)(n-2)/2}; l \in \{0, \dots, 2^n\}\right\}. \end{aligned}$$

From Lemma 3, $|a(n,l)2^{-(n-1)(n-2)/2}| \leq 1$ and $|f_n(x)| \leq 1$. $\square$

Proof of point 3 of Lemma 5. We fix a real $x = \frac{k}{2^{n-1}} + \delta_x \in [0,2)$, and an integer m.

$$\begin{aligned} f_n(x+2m) &= f_n\left(\frac{k+m2^n}{2^{n-1}} + \delta_x\right) = x_n^{k+m2^n} + \delta_x(x_n^{k+m2^n+1} - x_n^{k+m2^n}) \\ &= -u_m\left(x_n^{k+m2^n} + \delta_x(x_n^{k+m2^n+1} - x_n^{k+m2^n})\right) = -u_m f_n(x). \end{aligned}$$

We treat now the case where $x = 2m$. From Point 1, $f_n(2m) = f_n(0) = 0$ and this point is demonstrated. □

Proof of point 4 of Lemma 5. We verify this result by induction on n. For $n = 1$, the result is true. Now, we verify that if it is true up to the rank $n - 1$, it will be still true to the rank n. For any integer k,

$$\begin{aligned} f_n\left(\frac{k}{2^{n-1}}\right) - f_n\left(\frac{k+1}{2^{n-1}}\right) &= x_n^k - x_n^{k+1} = (\Sigma_n^k - \Sigma_n^{k+1})2^{-(n-1)(n-2)/2} \\ &= \Sigma_{n-1}^k 2^{-(n-1)(n-2)/2} = \Sigma_{n-1}^k 2^{-(n-1-1)(n-1-2)/2} 2^{-n+1} \\ &= x_{n-1}^k 2^{-n+1} = f_{n-1}\left(\frac{k}{2^{n-2}}\right) 2^{-n+1}. \end{aligned}$$

So, if $k = m2^{n-1} + l$, with $0 \le l \le 2^{n-1} - 1$, then $\frac{k}{2^{n-2}} = \frac{l}{2^{n-2}} + 2m$, and $f_{n-1}(\frac{k}{2^{n-2}})$ and u_m have the same sign. Then, f_n is decreasing on $[m, m + 1]$ if $u_m = -1$, and increasing otherwise.

And if $k = (m + 1)2^{n-1} + l$, with $0 \le l \le 2^{n-1} - 1$, then $\frac{k}{2^{n-2}} = \frac{l}{2^{n-2}} + 2(m + 1)$, and $f_{n-1}(\frac{k}{2^{n-2}})$ have the same sign as u_{m+1}, and so f_n is decreasing on $[m, m + 1]$ if $u_{m+1} = -1$, and increasing otherwise.

We suppose now that $u_m = u_{2m} = -1$. The function f_n decreases from 0 to -1 on $[2m, 2m + 1]$, and increases from -1 to 0 on $[2m + 1, 2m + 2]$. So, the function is negative on $[2m, 2m + 2]$. We can then use the same argument if $u_m = 1$ to complete the proof of this point. □

Proof of point 5 of Lemma 5. We fix two reals x and y, such that $x \le y$, verifying:

$$x = \frac{k}{2^{n-1}} + \delta_x \quad \text{and} \quad y = \frac{l}{2^{n-1}} + \delta_y,$$

where δ_x and δ_y are less than $1/2^{n-1}$.

$$\begin{aligned} f_n(x) - f_n(y) &= x_n^k + 2^{n-1}\delta_x(x_n^{k+1} - x_n^k) - x_n^l - 2^{n-1}\delta_y(x_n^{l+1} - x_n^l) \\ &= x_n^k - x_n^{k+1} + \cdots + x_n^{l-1} + x_n^l + 2^{n-1}\delta_x(x_n^{k+1} - x_n^k) \\ &\quad - 2^{n-1}\delta_y(x_n^{l+1} - x_n^l) \\ &= \left(f_{n-1}\left(\frac{k}{2^{n-2}}\right) + \cdots + f_{n-1}\left(\frac{l-1}{2^{n-1}}\right)\right)2^{1-n} - \delta_x f_{n-1}\left(\frac{k}{2^{n-2}}\right) \\ &\quad + \delta_y f_{n-1}\left(\frac{l}{2^{n-2}}\right). \end{aligned}$$

Since f_{n-1} is negative on $[0, 1]$:

$$|f_n(x) - f_n(y)| \le \frac{l-k}{2^{n-1}} + \delta_y - \delta_x = |x - y|. \quad \square$$

Proof of point 6 of Lemma 5. We show that for each integer N and each integer $l \in \{0, \ldots, 2^{N-1} - 1\}$, the sequence $\left(f_{n+N+1}\left(\frac{l}{2^{N-1}}\right)\right)_n$ is decreasing:

$$\begin{aligned} f_{n+N+1}(l/2^{N-1}) &= f_{n+N+1}\left(\frac{l2^n}{2^{N+n-1}}\right) = f_{n+N+1}\left(\frac{l2^n - 1}{2^{N+n-1}}\right) = x_{N+n+1}^{l2^n} \\ &= \Sigma_{N+n+1}^{l2^n} 2^{(N+n)(N+n-2)} = a(N + n + 1, l2^n)2^{(N+n)(N+n-2)}u_0 \end{aligned}$$

$$= -a(N+n+1, l2^n)2^{(N+n)(N+n-2)}$$
$$\le -2^{N+n-1}a(N+n, l2^{n-1})2^{(N+n)(N+n-1)/2}.$$

This result is then proved because

$$\begin{aligned} -2^{N+n-1}a(N+n, l2^{n-1})2^{(N+n)(N+n-1)/2} &= -a(N+n, l2^{n-1})2^{(N+n-1)(N+n-2)/2} \\ &= a(N+n, l2^{n-1})2^{(N+n-1)(N+n-2)/2}u_0 \\ &= f_{n+N}(l/2^{N-1}). \end{aligned}$$

Then, $f_{N+n+1}(l/2^{N-1}) \le f_{N+n}(l/2^{N-1})$. □

Proof of point 7 of Lemma 5. Let $x \in [0, 1]$ and m be an integer. We deduce the proof of this point from the following remarks:

$$f_n(x+1) = -1 - f_n(x) \quad \text{and} \quad f_n(x+2m) = -u_m \cdot f_n(x). \quad \square$$

Proof of Theorem 1. For each real number x, the sequence $(f_n(x))_n$ is monotone and bounded, so it converges. Let $f_\infty(x)$ denote the limit. It is clear that the function f_∞ is 1-Lipschitz. The third point is proved from Eq. (6). With the previous lemma, we can deduce that the range of the function f_∞ is included in $[-1, 1]$ and that for each positive integer m:

$$f_\infty(2m) = 0 \quad \text{and} \quad f_\infty(2m+1) = u_m.$$

We need to verify that it is a solution of Eq. (1). We fix a positive real $X \in [2m, 2m+2[$ and an integer n such that $[X, X + 1/2^{n-1}] \subset]2m, 2m+2[$. We fix l, the integer such that $0 \le \delta_X = X - l/2^{n-1} \le 1/2^{n-1}$.

Then for any integer m sufficiently large:

$$\begin{aligned} f_{n+m+1}(X) &= f_{n+m+1}\left(\frac{l2^m}{2^{n+m-1}}\right) + f_{n+m+1}(X) - f_{n+m+1}\left(\frac{l}{2^{n-1}}\right) \\ &= 2^{1-n-m}\sum_{j=0}^{l2^m-1} f_{n+m}\left(\frac{j}{2^{n+m}}\right) + f_{n+m+1}(X) - f_{n+m+1}\left(\frac{l}{2^{n-1}}\right). \end{aligned}$$

Since for each real x, the sequence $(f_n(x))_n$ is monotone, we let m tend to infinity to find:

$$f_\infty(X) = \int_0^{2\frac{l}{2^{n-1}}} f_\infty(x)dx + f_\infty(X) - f_\infty\left(\frac{l}{2^{n-1}}\right).$$

We deduce therefore that

$$\left|f_\infty(X) - \int_0^{2X} f_\infty(x)dx\right| \le 2\left|f_\infty(X) - f_\infty\left(\frac{l}{2^{n-1}}\right)\right| \le \frac{1}{2^{n-2}}.$$

Then, when n goes to infinity, $f_\infty(X) = \int_0^{2X} f_\infty(x)dx$. □

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