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Entire solutions of certain type of differential equations II ☆

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ABSTRACT

We analyze the transcendental entire solutions of the following type of nonlinear differential equations: $f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$ in the complex plane, where p_1, p_2 and α_1, α_2 are nonzero constants, and $P(f)$ denotes a differential polynomial in f of degree at most $n-1$ with small functions of f as the coefficients.

1. Introduction and results

Let f be a transcendental meromorphic function on the complex plane $\mathbb{C}$ throughout this paper. We assume that the reader is familiar with the standard notations used in the Nevanlinna's value distribution theory such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$, and the counting function $N(r, f)$. We refer the reader to the book [5] for the details of the Nevanlinna's theory and the notations. We use $S(r, f)$ to denote any quantity that satisfies the condition: $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside possibly an exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function of f, if and only if $T(r, a) = S(r, f)$. If $m(r, a) = S(r, f)$, then we say that $a(z)$ is a function of small proximity related to f. In recent years, Nevanlinna's value distribution theory has been used to study solvability and existence of entire or meromorphic solutions of differential equations in complex domains, see, e.g., [3,4,6,7,10, 12–14].

It is straightforward to show that the function $f_1(z) = \sin z$ is a solution of the nonlinear differential equation $4f^3 + 3f'' = -\sin 3z$. It was pointed out in [3] that $f_2(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ is also a solution of this equation. In [14], the authors proved that this equation admits exactly three entire solutions, namely $f_1(z)$, $f_2(z)$ and $f_3(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Note that the function $-\sin 3z$ is a linear combinations of e^{i3z} and e^{-i3z}. So, it is an interesting question to find all entire solutions of the following more general equation:

$$f^n(z) + P(f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1}$$

where p_1, p_2 and λ are nonzero constants, and $P(f)$ denotes a differential polynomial in f of degree at most $n-1$. The following two theorems answered this question partially.

Theorem A. *(See [14].) Let $n \geqslant 3$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-3$, $b(z)$ be a meromorphic function, and λ, p_1, p_2 be three nonzero constants. Then the differential equation:*

$$f^n(z) + P(f) = b(z)\left(p_1 e^{\lambda z} + p_2 e^{-\lambda z}\right)$$

has no transcendental entire solutions $f(z)$ that satisfies $T(r, b) = S(r, f)$.

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Theorem B. *(See [8].) Let $n \geqslant 2$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-2$, and λ, p_1, p_2 be three nonzero constants. If f is an entire solution of Eq.* (1), *then $f(z) = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n}$, where c_1 and c_2 are constants and $c_i^n = p_i$.*

Remark. Theorem B is proved in [8]. From that proof we can see that Theorem B is still true if we suppose that f is a meromorphic function with $N(r, f) = S(r, f)$.

In [9], the authors also discussed the equation similar to the equation in (1) with the right-hand side replaced by a linear combinations of $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ for two nonzero constants α_1 and α_2 with some additional conditions. In the present paper, we weaken the condition on the degree of $P(f)$ in Theorem B and prove the following theorem.

Theorem 1. *Let $n \geqslant 2$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-1$, and λ, p_1, p_2 be three nonzero constants. If f is a meromorphic solution of Eq.* (1) *and $N(r, f) = S(r, f)$, then there exist two nonzero constants c_1, c_2 ($c_j^n = p_j$), and a small function c_0 of f such that*

$$f = c_0 + c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n}. \tag{2}$$

Corollary 1. *Suppose that p_1, p_2, λ are nonzero constants, b_0, b_1, b_2 and c are meromorphic functions. If f is a meromorphic solution of the following nonlinear differential equation*

$$f^2 + c + b_0 f + b_1 f' + b_2 f'' = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{3}$$

such that c, b_0, b_1, b_2 are small function of f, and $N(r, f) = S(r, f)$, then $b_1 = 0$. In particular, if $c = b_0 = 0$, then b_2 is a constant satisfying $b_2^4 \lambda^8 = 2^{14} p_1 p_2$.

For example, equation $f^2 + 8f'' = 16e^{2z} + 4e^{-2z}$ has exactly two entire solutions, namely $f_1(z) = 4e^z - 2e^{-z} - 4$ and $f_2(z) = -4e^z + 2e^{-z} - 4$. In fact, from the proof of Corollary 1, we can see that this equation has no other meromorphic solutions satisfying $N(r, f) = S(r, f)$.

By Theorem 1, we can also prove the following result on linear differential equations.

Corollary 2. *Suppose that $b_1, \dots, b_{n-1}$ are polynomials, p_1, p_1, λ are nonzero constants. Then any non-trivial entire solutions of the linear differential equation*

$$f^{(n)} + b_1 f^{(n-1)} + \cdots + b_{n-1} f' + \left(p_1 e^{\lambda z} + p_2 e^{-\lambda z}\right) f = 0, \tag{4}$$

must have infinitely many zeros.

If λ and $-\lambda$ are replaced by two constants α_1 and α_2, respectively, then we have the following result.

Theorem 2. *Let $n \geqslant 2$ be an integer, $P(f)$ be a differential polynomial in f of degree at most $n-2$, and p_1, p_2, α_1, α_2 be nonzero constants and $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the following equation*

$$f^n + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{5}$$

and satisfying $N(r, f) = S(r, f)$, then one of the following holds:

(i) $f(z) = c_0 + c_1 e^{\alpha_1 z/n}$;
(ii) $f(z) = c_0 + c_2 e^{\alpha_2 z/n}$;
(iii) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$, *and* $\alpha_1 + \alpha_2 = 0$,

where c_0 is a small function of $f(z)$ and c_1, c_2 are constants satisfying $c_1^n = p_1$, $c_2^n = p_2$.

Remark. From the proof of Theorem 2, we can deduce that α_1/α_2 must be a rational number under the assumption of Theorem 2.

For further study, we propose the following question.

Question. How to find the solutions of Eq. (5) under the condition $\deg P(f) = n - 1$?

2. Some lemmas

The following lemmas will be used in the proofs of the theorems.

Lemma 1 *(Clunie's lemma). (See [1,2].) Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n(z)P(f) = Q(f)$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m\big(r, P(f)\big) = S(r, f).$$

Lemma 2. *(See [5].) Suppose that f is a nonconstant meromorphic function and $F = f^n + Q(f)$, where $Q(f)$ is a differential polynomial in f with degree $\leqslant n-1$. If $N(r, f) + N(r, 1/F) = S(r, f)$, then $F = (f - c_0)^n$, where c_0 is meromorphic and $T(r, c_0) = S(r, f)$.*

Lemma 3. *(See [14].) Let n be a positive integer, $a, b_0, b_1, \ldots, b_{n-1}$ be polynomials, and b_n be a nonzero constant. Let $L(f) = \sum_{k=0}^{n} b_k f^{(k)}$. If $a(z) \not\equiv 0$, then the transcendental meromorphic solution of the following equation:*

$$f^2 + \big(L(f)\big)^2 = a,$$

must have the form $f(z) = \frac{1}{2}(P(z)e^{R(z)} + Q(z)e^{-R(z)})$, where P, Q, R are polynomials, and $PQ = a$. If furthermore all b_k are constants, then $\deg P + \deg Q \leqslant n - 1$. Moreover, $R(z) = \lambda z$, where λ is a nonzero constant satisfying the following equations:

$$\sum_{k=0}^{n} b_k \lambda^k = \frac{1}{i}, \qquad \sum_{k=j}^{n} b_k \binom{k}{j} \lambda^{k-j} = 0, \quad j = 1, \ldots, \deg P,$$
$$\sum_{k=0}^{n} b_k (-\lambda)^k = -\frac{1}{i}, \qquad \sum_{k=j}^{n} b_k \binom{k}{j} (-\lambda)^{k-j} = 0, \quad j = 1, \ldots, \deg Q.$$

Lemma 4. *(See [11].) Let n, m be positive integers satisfying $1/n + 1/m < 1$. Then there exist no transcendental entire solutions f and g that satisfy the equation $af^n + bg^m = 1$, with a, b being small functions of f and g, respectively.*

Lemma 5. *Let $n \geqslant 2$ be an integer, $P(f)$ be a differential polynomial in f of degree $\leqslant n-1$, and λ, p_1, p_2 be three nonzero constants. If f is a meromorphic solution of Eq. (1) and $N(r, f) = S(r, f)$, then the function $\varphi = \lambda^2 f - n^2 f''$ is a small function of f. Furthermore,*

$$\lambda^{2k} f^n - n^{2k} f^{n-2k} \big(f'\big)^{2k} \in \mathcal{D}_{n-1}, \quad n \geqslant 2k, \tag{6}$$
$$\lambda^{2k} f^{n-1} f' - n^{2k} f^{n-2k-1} \big(f'\big)^{2k+1} \in \mathcal{D}_{n-1}, \quad n \geqslant 2k+1, \tag{7}$$

where and in the sequel $\mathcal{D}_{n-1}$ denotes the family of all differential polynomials in f of degree at most $n-1$ with coefficients being small functions of f.

Proof. Set $P = P(f)$. Suppose that f is a meromorphic solution of Eq. (1) and $N(r, f) = S(r, f)$. By differentiating (1), we get

$$nf^{n-1} f' + P' = \lambda p_1 e^{\lambda z} - \lambda p_2 e^{-\lambda z}. \tag{8}$$

Eliminating $e^{-\lambda z}$ from (1) and (8) yields

$$\lambda f^n + nf^{n-1} f' + \lambda P + P' = 2\lambda p_1 e^{\lambda z}. \tag{9}$$

By taking the derivative of the above equation, we get

$$n\lambda f^{n-1} f' + n(n-1) f^{n-2} \big(f'\big)^2 + nf^{n-1} f'' + \lambda P' + P'' = 2\lambda^2 p_1 e^{\lambda z}. \tag{10}$$

Then eliminating $e^{\lambda z}$ from (9) and (10) gives

$$\lambda^2 f^n - n(n-1) f^{n-2} \big(f'\big)^2 - nf^{n-1} f'' + \lambda^2 P - P'' = 0. \tag{11}$$

By eliminating $e^{\lambda z}$ from (1) and (8), we have

$$\lambda f^n - nf^{n-1} f' + \lambda P - P' = 2\lambda p_2 e^{-\lambda z}. \tag{12}$$

It follows from (9) and (12) that

$$\lambda^2 f^{2n} - n^2 f^{2n-2}\big(f'\big)^2 + 2\lambda^2 f^n P - 2nf^{n-1} f' P' + \lambda^2 P^2 - \big(P'\big)^2 = 4\lambda^2 p_1 p_2. \tag{13}$$

Eliminating $(f')^2$ from (11) and (13) yields

$$\varphi f^{2n-1} = (n-2)\lambda^2 f^n P - 2n(n-1) f^{n-1} f' P' + nf^n P'' + (n-1)\lambda^2 P^2 - (n-1)\big(P'\big)^2 - 4(n-1)\lambda^2 p_1 p_2, \tag{14}$$

where $\varphi = \lambda^2 f - n^2 f''$. Since the right-hand side of the above equation is a differential polynomial in f of degree at most $2n-1$, by Lemma 1, we get $m(r,\varphi) = S(r,f)$. By the assumption, we have $N(r,\varphi) = S(r,f)$ and thus $T(r,\varphi) = S(r,f)$, which means that φ is a small function of f. By substituting $f'' = (\lambda^2 f - \varphi)/n^2$ into (11), we get

$$\lambda^2 f^n - n^2 f^{n-2}\big(f'\big)^2 + \frac{\varphi}{n-1} f^{n-1} + \frac{n}{n-1}\lambda^2 P - \frac{n}{n-1} P'' = 0, \tag{15}$$

which implies that

$$\lambda^2 f^n - n^2 f^{n-2}\big(f'\big)^2 \in \mathcal{D}_{n-1}. \tag{16}$$

Differentiating the left-hand side of (16), and then replacing f'' by $(\lambda^2 f - \varphi)/n^2$ in the result, we get

$$\lambda^2 f^{n-1} f' - n^2 f^{n-3}\big(f'\big)^3 \in \mathcal{D}_{n-1}, \quad n \geqslant 3. \tag{17}$$

Taking the derivative and then replacing f'' by $(\lambda^2 f - \varphi)/n^2$ in the result, and combining (16), we derive

$$\lambda^4 f^n - n^4 f^{n-4}\big(f'\big)^4 \in \mathcal{D}_{n-1}, \quad n \geqslant 4. \tag{18}$$

Formulas (6) and (7) can be derived by using mathematical induction. □

Lemma 6. *Suppose that $f(z)$ is a transcendental meromorphic function, $a(z)$, $b(z)$, $c(z)$ and $d(z)$ are small functions of $f(z)$, and $acd \not\equiv 0$. If*

$$af^2 + bff' + c\big(f'\big)^2 = d, \tag{19}$$

then

$$c\big(b^2 - 4ac\big)\frac{d'}{d} + b\big(b^2 - 4ac\big) - c\big(b^2 - 4ac\big)' + \big(b^2 - 4ac\big)c' = 0. \tag{20}$$

In particular, if a, b, c, d are constants and $b^2 - 4ac \neq 0$, then $b = 0$, and

$$f(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z},$$

where c_1, c_2 and λ are nonzero constants.

Proof. It is seen from (19) that the poles of f must be the poles of d if they are not the zeros or poles of a, b and c. Therefore, $N(r,f) = S(r,f)$. Eq. (19) can be written as

$$\frac{1}{f^2} = \frac{a}{d} + \frac{b}{d}\frac{f'}{f} + \frac{c}{d}\left(\frac{f'}{f}\right)^2.$$

By the lemma of logarithmic derivative, we get $m(r,1/f) = S(r,f)$, and thus $T(r,f) = N(r,1/f) + S(r,f)$. Also we can see from (19) that the multiple zeros of f must be the zeros of d if they are not the poles of a, b and c. Hence $N(r,1/f) = \overline{N}(r,1/f) + S(r,f)$. Differentiating (19) yields

$$a' f^2 + \big(2a + b'\big) ff' + \big(b + c'\big)\big(f'\big)^2 + bff'' + 2cf'f'' = d'. \tag{21}$$

Suppose z_0 is a simple zero of f that is not the pole of a and b. Then from (19) and (21), we get $c(f')^2(z_0) = d(z_0)$ and $(b + c')(f')^2(z_0) + 2cf'f''(z_0) = d'(z_0)$, which implies that z_0 is a zero of $(cd' - bd - dc')f' - 2cdf''$. Let

$$\alpha = \frac{(cd' - bd - dc')f' - 2cdf''}{f}. \tag{22}$$

Then we have $T(r,\alpha) = S(r,f)$, i.e., α is a small function of f. It follows that

$$f'' = \frac{cd' - bd - dc'}{2cd} f' - \frac{\alpha}{2cd} f. \tag{23}$$

By substituting the above equation into (21), we get

$$\left(a' - \frac{b\alpha}{2cd}\right) f^2 + \left(2a + b' + \frac{b(cd' - bd - dc')}{2cd} - \frac{\alpha}{d}\right) ff' + c\frac{d'}{d}(f')^2 = d'. \tag{24}$$

From this and (19) we get

$$\beta f + \gamma f' = 0, \tag{25}$$

where

$$\beta = a' - \frac{b\alpha}{2cd} - a\frac{d'}{d}, \tag{26}$$

and

$$\gamma = 2a + b' - \frac{bd'}{2d} - \frac{b^2 + bc'}{2c} - \frac{\alpha}{d}. \tag{27}$$

Note that β and γ are small functions of f. If $\gamma \not\equiv 0$, then it follows from (25) that $\overline{N}(r, f) = S(r, f)$, which is impossible. Hence $\gamma \equiv 0$, and thus $\beta \equiv 0$. By eliminating α from the above two equations, we can derive (20). In particular, if a, b, c, d are constants and $b^2 - 4ac \neq 0$, then we get $b = 0$. By Lemma 3, we see that there exist nonzero constants c_1, c_2 and λ such that $f(z) = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$. This completes the proof of Lemma 6. □

3. Proof of Theorem 1

First of all, we prove Theorem 1 in the special case that $P(z) = cf^{n-2} f' + Q(f)$ where c is a small function of f and $Q(f) \in \mathcal{D}_{n-2}$. Set $P = P(f)$, $Q = Q(f)$. By Lemma 5, we see that $\varphi = \lambda^2 f - n^2 f''$ is a small function of f. By taking the derivatives of P and substituting $f'' = (\lambda^2 f - \varphi)/n^2$ into the results, we get

$$P' = c' f^{n-2} f' + c(n-2) f^{n-3} (f')^2 + \frac{c\lambda^2}{n^2} f^{n-1} + Q_1,$$

$$P'' = \left(c'' + \frac{3n-5}{n^2} c\lambda^2\right) f^{n-2} f' + 2(n-2)c' f^{n-3} (f')^2 + \frac{2c'\lambda^2}{n^2} f^{n-1} + c(n-2)(n-3) f^{n-4} (f')^3 + Q_2,$$

where $Q_1 = Q' - \frac{c\varphi}{n^2} f^{n-2} \in \mathcal{D}_{n-2}$, and $Q_2 = Q_1' - \frac{c'\varphi\lambda^2}{n^2} f^{n-2} - \frac{2c(n-2)\varphi}{n^2} f^{n-3} f' \in \mathcal{D}_{n-2}$. It is obviously that

$$fP = cf^{n-1} f' + R_1, \tag{28}$$

where $R_1 = fQ \in \mathcal{D}_{n-1}$. By (16) and (17), we have

$$fP'' = \left(c'' + \frac{n^2 - 3n + 1}{n^2} c\lambda^2\right) f^{n-1} f' + \frac{2(n-1)}{n^2} c' \lambda^2 f^n + R_2 \tag{29}$$

where R_2 is a function in $\mathcal{D}_{n-1}$. Multiplying (15) by f and then substituting (28), (29) into the result, we get

$$f^{n-1} \psi = \frac{n}{n-1} R_2 - \frac{n}{n-1} \lambda^2 R_1, \tag{30}$$

where

$$\psi = \lambda^2 f^2 - n^2 (f')^2 + \left(\frac{\varphi}{n-1} - \frac{2c'}{n}\lambda^2\right) f + \left(\frac{3n-1}{n(n-1)} c\lambda^2 - \frac{n}{n-1} c''\right) f'. \tag{31}$$

Since the right-hand side of (30) is a function in $\mathcal{D}_{n-1}$, by Lemma 1, we get $m(r, \psi) = S(r, f)$. And thus $T(r, \psi) = S(r, f)$, i.e., ψ is a small function of f. Let

$$\alpha = \frac{\varphi}{n-1} - \frac{2c'}{n}\lambda^2, \tag{32}$$

$$\beta = \frac{3n-1}{n(n-1)} c\lambda^2 - \frac{n}{n-1} c''. \tag{33}$$

We can write (31) as

$$\psi = \lambda^2 f^2 - n^2 (f')^2 + \alpha f + \beta f'. \tag{34}$$

Taking the derivative of this equation and substituting $f'' = (\lambda^2 f - \varphi)/n^2$ into the result, we get

$$(2\varphi + \alpha + \beta')f' + \left(\alpha' + \frac{\beta\lambda^2}{n^2}\right)f = \psi' + \frac{\beta\varphi}{n^2}. \tag{35}$$

If $2\varphi + \alpha + \beta' \not\equiv 0$, then $f' = \gamma_1 f + \gamma_2$, where γ_1 and γ_2 are two small functions of f. Hence $P = c\gamma_1 f^{n-1} + c\gamma_2 f^{n-2} + Q$. Let $f_1 = f + c\gamma_1/n$. Then Eq. (1) can be written as $f_1^n + \tilde{P} = p_1 e^{\lambda z} + p_2 e^{-\lambda z}$, where $\tilde{P}$ is a differential polynomial in f of degree at most $n-2$. By Theorem B, there exist two nonzero constants c_1 and c_2 ($c_j^n = p_j$) such that $f_1 = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n}$. Therefore, $f = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n} - c\gamma_1/n$.

If $2\varphi + \alpha + \beta' \equiv 0$, then from (35) we get $\alpha' + \beta\lambda^2/n^2 = 0$ and $\psi' + \beta\varphi/n^2 = 0$. It follows that $\beta^2 - 4n^2\psi - n^2\alpha^2/\lambda^2 := d$ is a constant. Eq. (34) can be written as

$$\left(f' - \frac{\beta}{2n^2}\right)^2 - \left(\frac{\lambda}{n} f + \frac{\alpha}{2n\lambda}\right)^2 = \frac{d}{4n^4}. \tag{36}$$

Let $h = \lambda f/n + \alpha/(2n\lambda)$. By $\alpha' + \beta\lambda^2/n^2 = 0$, we get $f' - \beta/2n^2 = nh'/\lambda$. Therefore, $h^2 - (nh'/\lambda)^2 = -d/(4n^4)$. By Lemma 3, there exist two nonzero constants d_1 and d_2 such that $h(z) \equiv d_1 e^{\lambda z/n} + d_2 e^{-\lambda z/n}$. Hence there exist constants c_1, c_2 and a small function c_0 of f such that $f(z) \equiv c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n} + c_0$, which means that the conclusion of Theorem 1 is true in the special case.

Now we prove Theorem 1 in the general case. Since $P(z)$ is a differential polynomial in f of degree at most $n-1$, by using $f'' = (\lambda^2 f - \varphi)/n^2$, we see that $P(z)$ can be expressed as a polynomial in f and f' with total degree at most $n-1$. Therefore,

$$P = \sum_{k=0}^{n-1} b_k f^{n-1-k} (f')^k + P_1, \tag{37}$$

where $P_1 \in \mathcal{D}_{n-2}$, and b_k ($k = 0, 1, \ldots, n-1$) are small functions of f. Squaring both sides of (1), we get

$$f^{2n} + 2f^n P + P^2 - 2p_1 p_2 = p_1^2 e^{2\lambda z} + p_2^2 e^{-2\lambda z}.$$

That is

$$f^{2n} + \sum_{k=0}^{n-1} 2b_k f^{2n-1-k} (f')^k + Q_1 = p_1^2 e^{2\lambda z} + p_2^2 e^{-2\lambda z},$$

where Q_1 is a function in $\mathcal{D}_{2n-2}$. By (6) and (7), the above equation can be expressed as

$$f^{2n} + \alpha_1 f^{2n-1} + \alpha_2 f^{2n-2} f' + Q_2 = p_1^2 e^{2\lambda z} + p_2^2 e^{-2\lambda z},$$

where α_1, α_2 are small functions of f and $Q_2 \in \mathcal{D}_{2n-2}$. Let $g = f + \alpha_1/(2n-1)$. It follows that

$$g^{2n} + cg^{2n-2} g' + Q_3 = p_1^2 e^{2\lambda z} + p_2^2 e^{-2\lambda z},$$

where c is small function of g, and Q_3 is a differential polynomial in g with degree at most $2n-2$. By the result of Theorem 1 in the special case, we conclude that Theorem 1 is still true in the general case.

4. Proof of Corollary 1

Suppose that f is a meromorphic solution of Eq. (3) and $N(r, f) = S(r, f)$. By Theorem 1, we have

$$f(z) = c_0(z) + c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2},$$

where c_1 and c_2 are constants satisfying $c_j^2 = p_j$, and $c_0(z)$ is a small function of f. By substituting the above equation into (3) and noting that the coefficients of $e^{\lambda z/2}$ and $e^{-\lambda z/2}$ must vanish, we get

$$2c_0 + b_0 + \frac{\lambda}{2} b_1 + \frac{\lambda^2}{4} b_2 = 0, \tag{38}$$

$$2c_0 + b_0 - \frac{\lambda}{2} b_1 + \frac{\lambda^2}{4} b_2 = 0, \tag{39}$$

$$c_0^2 + 2c_1 c_2 + c + b_0 c_0 + b_1 c_0' + b_2 c_0'' = 0. \tag{40}$$

From (38) and (39), we get $b_1 = 0$ and $2c_0 + b_0 + \frac{\lambda^2}{4} b_2 = 0$. In particular, if $c = b_0 = 0$, then $b_2 = -\frac{8}{\lambda^2} c_0$. It follows from (40) that

$$c_0^2 + 2c_1c_2 - \frac{8}{\lambda^2} c_0 c_0'' = 0, \tag{41}$$

which implies that c_0 has no zeros and poles. Therefore, $c_0 = e^h$ for an entire function h. From the above equation, we have $(1 - \frac{8}{\lambda^2}(h'' + h'^2))e^{2h} = -2c_1c_2$. It follows that h, and thus c_0, is a constant. Hence $2c_1c_2 = -c_0^2$. Note that $c_j^2 = p_j$ and $c_0 = -\frac{\lambda^2}{8} b_2$. We can derive $\lambda^8 b_2^4 = 2^{14} p_1 p_2$ easily.

5. Proof of Corollary 2

If Eq. (4) has a non-trivial entire solution f with finitely many zeros, then $f = pe^{\alpha}$, where p is a polynomial and α is an entire function. Let $g = p'/p + \alpha'$. By a simple computation, we get $f' = gf$ and

$$f^{(k)} = \left(g^k + \frac{k(k+1)}{2} g^{k-2} g' + P_{k-2}(g)\right) f, \quad k \geqslant 2, \tag{42}$$

where $P_{k-2}(g)$ is a differential polynomial in g of degree $k-2$. From (4) and the above equation, we get the following equation:

$$g^n + \frac{n(n+1)}{2} g^{n-2} g' + b_1 g^{n-1} + Q_{n-2}(g) = -p_1 e^{\lambda z} - p_2 e^{-\lambda z}, \tag{43}$$

where $Q_{n-2}(g)$ is a differential polynomial in g of degree $n-2$ with coefficients being polynomials. Since the right-hand side of the above equation is transcendental, we see that g must be transcendental. It follows from $g = p'/p + \alpha'$ that $N(r, g) = S(r, g)$. By Theorem 1, there exist two nonzero constants c_1, c_2 and a small function c_0 such that $g = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n} + c_0$. Substitute this into (43) and compare the coefficients of $e^{\lambda z}$, $e^{-\lambda z}$, $e^{\frac{n-1}{n}\lambda z}$ and $e^{-\frac{n-1}{n}\lambda z}$ in the resulting equation, we have

$$\begin{aligned}
&c_1^n = -p_1, \qquad c_2^n = -p_2,\\
&nc_0c_1^{n-1} + \frac{n(n+1)}{2}\frac{\lambda}{n} c_1^{n-1} + b_1 c_1^{n-1} = 0,\\
&nc_0c_2^{n-1} - \frac{n(n+1)}{2}\frac{\lambda}{n} c_2^{n-1} + b_1 c_2^{n-1} = 0.
\end{aligned}$$

From these equations, we get $\lambda = 0$, a contradiction. This also completes the proof of Corollary 2.

6. Proof of Theorem 2

Suppose that $f(z)$ is a transcendental meromorphic solution of Eq. (5) and satisfies $N(r, f) = S(r, f)$. By differentiating (5), we get

$$nf^{n-1} f' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{44}$$

Eliminating $e^{\alpha_1 z}$ and $e^{\alpha_2 z}$ from (5) and (44), respectively, we get

$$\alpha_1 f^n - nf^{n-1} f' + \alpha_1 P - P' = (\alpha_1 - \alpha_2) p_2 e^{\alpha_2 z}, \tag{45}$$

$$\alpha_2 f^n - nf^{n-1} f' + \alpha_2 P - P' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{46}$$

Differentiating (46) yields

$$n\alpha_2 f^{n-1} f' - n(n-1) f^{n-2} (f')^2 - nf^{n-1} f'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{47}$$

It follows from (46) and (47) that

$$f^{n-2} \varphi = -Q, \tag{48}$$

where

$$\varphi = \alpha_1\alpha_2 f^2 - n(\alpha_1 + \alpha_2) ff' + n(n-1)(f')^2 + nff'', \tag{49}$$

and

$$Q = \alpha_1\alpha_2 P - (\alpha_1 + \alpha_2) P' + P''. \tag{50}$$

Since Q is a differential polynomial in f of degree $\leqslant n-2$, from (48) and by Lemma 1, we have $m(r, \varphi) = S(r, f)$. Therefore, $T(r, \varphi) = S(r, f)$. We distinguish two cases below.

Case 1. $\varphi \equiv 0$. In this case, we have $Q \equiv 0$, i.e.,

$$\alpha_1\alpha_2 P - (\alpha_1+\alpha_2)P' + P'' \equiv 0. \tag{51}$$

From (5) and by Lemma 4, we see that $P \not\equiv 0$. Therefore, $\alpha_1 P - P' \equiv 0$ and $\alpha_2 P - P' \equiv 0$ cannot hold simultaneously. Suppose $\alpha_2 P - P' \not\equiv 0$. By (51), we deduce that

$$\alpha_2 P - P' = Ae^{\alpha_1 z}, \tag{52}$$

where A is a nonzero constant. Combining this and (46), we get

$$f^{n-1}\bigl(\alpha_2 f - nf'\bigr) = \frac{\alpha_2(\alpha_2-\alpha_1-A)}{A}P + (1-\alpha_2+\alpha_1)P'. \tag{53}$$

Note that the right-hand side of the above equation is a differential polynomial in f of degree $\leqslant n-2$. By Lemma 1, we see that $\alpha_2 f - nf'$ and $f(\alpha_2 f - nf')$ are small functions of f. Therefore, $\alpha_2 f - nf' = 0$, which yields

$$f^n = \tilde{p}_2 e^{\alpha_2 z}, \tag{54}$$

where $\tilde{p}_2$ is a nonzero constant. By this and (5), (52), we get

$$\left(1-\frac{p_2}{\tilde{p}_2}\right)f^n = \frac{-\alpha_1}{\alpha_1-\alpha_2}P + \frac{1}{\alpha_1-\alpha_2}P'. \tag{55}$$

If $\tilde{p}_2 \neq p_2$, then by the above equation and Lemma 1 we get $T(r,f) = S(r,f)$, which is impossible. Therefore, $\tilde{p}_2 = p_2$, and thus $f = c_2 e^{\alpha_2 z/n}$, where c_2 is a nonzero constant satisfying $c_2^n = p_2$.

If $\alpha_1 P - P' \not\equiv 0$, then by a similar method we can deduce that $f = c_1 e^{\alpha_1 z/n}$, where c_1 is a nonzero constant satisfying $c_1^n = p_1$.

Case 2. $\varphi \not\equiv 0$. It follows from (49) that the multiple zero of f must be the zero of φ. Therefore, $N_{(2}(r, 1/f) = S(r,f)$. By differentiating (49) we get

$$\varphi' = 2\alpha_1\alpha_2 ff' - n(\alpha_1+\alpha_2)\bigl(f'\bigr)^2 - n(\alpha_1+\alpha_2)ff'' + n(2n-1)f'f'' + nff'''. \tag{56}$$

If z_0 is a simple zero of f, then it follows from (49) and (56) that z_0 is a zero of $(2n-1)\varphi f'' - ((n-1)\varphi' + (\alpha_1+\alpha_2)\varphi)f'$. Define

$$\psi := \frac{(2n-1)\varphi f'' - ((n-1)\varphi' + (\alpha_1+\alpha_2)\varphi)f'}{f}. \tag{57}$$

Then we have $T(r,\psi) = S(r,f)$. It follows that

$$f'' = \gamma_1 f' + \gamma_0 f, \tag{58}$$

where

$$\gamma_1 = \frac{n-1}{2n-1}\frac{\varphi'}{\varphi} + \frac{\alpha_1+\alpha_2}{2n-1}, \qquad \gamma_0 = \frac{\psi}{(2n-1)\varphi}. \tag{59}$$

By substituting (58) into (49), we have

$$af^2 + bff' + c\bigl(f'\bigr)^2 = \varphi, \tag{60}$$

where $a = \alpha_1\alpha_2 + n\gamma_0$, $b = n\gamma_1 - n(\alpha_1-\alpha_2)$, and $c = n(n-1)$. By Lemma 6, we have

$$c\bigl(4ac-b^2\bigr)\frac{\varphi'}{\varphi} = c\bigl(4ac-b^2\bigr)' - b\bigl(4ac-b^2\bigr). \tag{61}$$

Now we distinguish two subcases below.

Subcase 2.1. Suppose $4ac - b^2 = 0$. It follows from (60) that $c(f' - \frac{b}{2c}f)^2 = \varphi$, which implies that $\beta = f' + \frac{b}{2c}f$ is a small function of f. By substituting $f' = -\frac{b}{2c}f + \beta$ into (45) and (46), respectively, we get

$$\left(\alpha_1 + \frac{nb}{2c}\right)f^n - n\beta f^{n-1} + \alpha_1 P - P' = (\alpha_1-\alpha_2)p_2 e^{\alpha_2 z}, \tag{62}$$

$$\left(\alpha_2 + \frac{nb}{2c}\right)f^n - n\beta f^{n-1} + \alpha_2 P - P' = (\alpha_2-\alpha_1)p_2 e^{\alpha_1 z}, \tag{63}$$

and the left-hand sides of the above two equations are polynomials in f with coefficients being small functions of f. Since $\alpha_1 \neq \alpha_2$, one of $\alpha_1 + \frac{nb}{2c}$ and $\alpha_2 + \frac{nb}{2c}$ is not zero.

Suppose $\alpha_1 + \frac{nb}{2c} \neq 0$. By Lemma 2, there exists a small function c_0 of f such that

$$\left(\alpha_1 + \frac{nb}{2c}\right)(f - c_0)^n = (\alpha_1 - \alpha_2)p_2 e^{\alpha_2 z}, \tag{64}$$

which implies that $f = c_0 + c_2 e^{\alpha_2 z/n}$, and $c_2^n = \frac{(\alpha_1-\alpha_2)p_2}{\alpha_1+\frac{nb}{2c}}$. Similarly, if $\alpha_2 + \frac{nb}{2c} \neq 0$, then we have $f = \tilde{c}_0 + \tilde{c}_2 e^{\alpha_1 z/n}$. This cannot hold in such case. Therefore, $\alpha_2 + \frac{nb}{2c} = 0$. Thus $c_2^n = p_2$.

Suppose $\alpha_2 + \frac{nb}{2c} \neq 0$. We can deduce that $f = c_0 + c_1 e^{\alpha_1 z/n}$, and $c_1^n = p_1$, by a similar argument.

Subcase 2.2. Suppose $4ac - b^2 \neq 0$. From (61) and the definitions of γ_1 and b, we get

$$\frac{2n^2(n-1)}{2n-1}\frac{\varphi'}{\varphi} = \frac{2n(n-1)}{2n-1}(\alpha_1+\alpha_2) + \frac{(4ac-b^2)'}{4ac-b^2}. \tag{65}$$

By integration, we see that there exists a nonzero constant B such that

$$\varphi^{2n^2(n-1)} = B\left(4ac-b^2\right)^{2n-1} e^{2n(n-1)(\alpha_1+\alpha_2)z}, \tag{66}$$

which implies that $e^{2n(n-1)(\alpha_1+\alpha_2)z}$ is small function of f. But from (5) we have $nT(r,f) \leqslant T(r, e^{\alpha_1 z}) + T(r, e^{\alpha_2 z}) + S(r,f)$. Therefore, $\alpha_1 + \alpha_2 = 0$. It follows from (45) and (46) that

$$f^{2n-2}\varphi_1 + R = -(\alpha_2 - \alpha_1)^2 p_1 p_2, \tag{67}$$

where R is a differential polynomial in f of degree $\leqslant 2n-2$, and $\varphi_1 = \alpha_1\alpha_2 f^2 + n^2(f')^2$. By Lemma 1 we see that φ_1 is small function of f. Combining (60), we get $\varphi_1 = \frac{n}{n-1}\varphi$. Finally, by Lemma 6, we can deduce that $f = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$, where c_1 and c_2 are nonzero constants satisfying $c_i^n = p_i$. This also completes the proof of Theorem 2.

7. Concluding remark

By slightly modifying the proof of Theorem 1, we can prove the following result.

Theorem 3. *Let $n \geqslant 2$ be an integer, and α a nonconstant entire function. Let $P(f)$ be a differential polynomial in f of degree at most $n-1$, and p_1, p_2 be two nonzero constants. If f is a meromorphic solution of the equation*

$$f^n + P(f) = p_1 e^{\alpha} + p_2 e^{-\alpha}, \tag{68}$$

and $N(r,f) = S(r,f)$, then

$$f = c_0 + c_1 e^{\alpha/n} + c_2 e^{-\alpha/n}, \tag{69}$$

where c_0 is a small function of f, and c_1, c_2 are nonzero constants satisfying $c_i^n = p_i$.

Furthermore, if we suppose that the degree of $P(f)$ is at most $n-2$ in Theorem 3, then we can show $c_0 = 0$ in the following way. Let $g = c_1 e^{\alpha/n} + c_2 e^{-\alpha/n}$. We have

$$e^{\alpha/n} = \frac{1}{2c_1}g + \frac{n}{2c_1\alpha'}g', \qquad e^{-\alpha/n} = \frac{1}{2c_2}g - \frac{n}{2c_c\alpha'}g',$$

and $f = c_0 + g$. Hence $f^n = g^n + nc_0 g^{n-1} + P_1(g)$, where $P_1(g)$ is a polynomial in g of degree at most $n-2$. Note that

$$g^n = p_1 e^{\alpha} + p_2 e^{-\alpha} + \sum_{k=1}^{n-1}\binom{n}{k}\left(c_1 e^{\alpha/n}\right)^k \left(c_2 e^{-\alpha/n}\right)^{n-k}.$$

And $(c_1 e^{\alpha/n})^k (c_2 e^{-\alpha/n})^{n-k}$ is a polynomial in $e^{\alpha/n}$ or in $e^{-\alpha/n}$ of degree at most $n-2$. Therefore, the last summation in the above equation is a differential polynomial in g of degree at most $n-2$. It follows from (68) that

$$nc_0 g^{n-1} + P_2(g) = 0,$$

where $P_2(g)$ is a differential polynomial in g of degree at most $n-2$. Note that $N(r,g) = S(r,g)$. The above equation implies $c_0 = 0$.

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