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Invariant measures and the Kolmogorov equation for the stochastic fast diffusion equation

Viorel Barbu[a], Giuseppe Da Prato[b,*]

[a] *University Al. I. Cuza, 700506, Iasi, Romania*
[b] *Scuola Normale Superiore, 56126, Pisa, Italy*

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Abstract

We prove the existence of an invariant measure μ for the transition semigroup P_t associated with the fast diffusion porous media equation in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$, perturbed by a Gaussian noise. The Kolmogorov infinitesimal generator N of P_t in $L^2(H^{-1}(\mathcal{O}), \mu)$ is characterized as the closure of a second-order elliptic operator in $H^{-1}(\mathcal{O})$. Moreover, we construct the Sobolev space $W^{1,2}(H^{-1}(\mathcal{O}), \mu)$ and prove that $D(N) \subset W^{1,2}(H^{-1}(\mathcal{O}), \mu)$.

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1. Introduction

We are concerned with the following stochastic problem in a bounded domain $\mathscr{O}$ of $\mathbb{R}^d$:

$$\begin{cases} dX(t,\xi) = \Delta\beta(X(t,\xi))dt + \sqrt{Q}dW(t,\xi), & \xi \in \mathscr{O}, \\ X(t,\xi) = 0, \quad \forall\, \xi \in \partial\mathscr{O}, \\ X(0,\xi) = x(\xi), \quad \forall\, \xi \in \mathscr{O}, \end{cases} \tag{1.1}$$

where

$$\beta(r) = a|r|^{\alpha}\operatorname{sgn} r, \quad \alpha \in [0,1],\ a > 0,\ \forall\, r \in \mathbb{R}.$$

* Corresponding author.
E-mail address: daprato@sns.it (G. Da Prato).

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We denote by $\{e_k\}$ and $\{\alpha_k\}$ eigensequences of the Laplace operator in $\mathscr{O}$ endowed with Dirichlet boundary conditions,

$$\Delta e_k = -\alpha_k e_k, \quad \forall\, k \in \mathbb{N}.$$

Let $A = -\Delta$ with $D(A) = H^2(\mathscr{O}) \cap H_0^1(\mathscr{O})$. We shall assume that the boundary $\partial\mathscr{O}$ of $\mathscr{O}$ is sufficiently smooth (of class C^2 for instance) or that $\mathscr{O}$ is convex.

Here W is a cylindrical Wiener process in $L^2(\mathscr{O})$ of the form

$$W(t) = \sum_{k=1}^{\infty} e_k W_k(t), \quad \forall\, t \geq 0,$$

where $\{W_k\}$ is a sequence of independent real Brownian motions on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and

$$\sqrt{Q}\; W(t) = \sum_{k=1}^{\infty} \sqrt{Q}\; e_k W_k(t), \quad \forall\, t \geq 0.$$

We take $Q \in L(L^2(\mathscr{O}))$ such that

$$Qe_k = q_k e_k, \quad \forall\, k \in \mathbb{N},$$

where $\{q_k\}$ is a sequence of nonnegative numbers such that

$$\mathrm{Tr}\,[A^{-1}Q] = \sum_{k=1}^{\infty} \alpha_k^{-1} q_k < \infty.$$

This condition is fulfilled for $q_k \to 0$ sufficiently fast. In fact we shall need a stronger assumption in what follows. (See Proposition 2.6 below.)

For the sake of simplicity we shall take in the following $a = 1$ (here and everywhere in the following we shall denote by Tr the trace in the space $L^2(\mathscr{O})$).

We introduce the nonlinear operator F in $H^{-1}(\mathscr{O})$

$$\begin{cases} F(x) = -\Delta(|x|^{\alpha}\mathrm{sgn}\, x), \quad \forall\, x \in D(F), \\ D(F) = \{x \in L^1(\mathscr{O}) \cap H^{-1}(\mathscr{O}) : |x|^{\alpha}\mathrm{sgn}\, x \in H_0^1(\mathscr{O})\}, \end{cases} \tag{1.2}$$

where $\mathrm{sgn}\, x = \frac{x}{|x|}$ if $x \neq 0$, $\mathrm{sgn}\, 0 = [-1, 1]$. By a classical result (see e.g. [1, Proposition 2.12]) we know that if $\alpha > 0$ then F is maximal monotone on $H^{-1}(\mathscr{O})$. Then we may write (1.1) as

$$\begin{cases} \mathrm{d}X(t) + F(X(t))\mathrm{d}t = \sqrt{Q}\mathrm{d}W(t), \quad \forall\, t \geq 0, \\ X(0) = x \in H^{-1}(\mathscr{O}). \end{cases} \tag{1.3}$$

Definition 1.1. A solution X to (1.1) is an adapted stochastic process $X(t),\ t \geq 0$, with values in $H^{-1}(\mathscr{O})$ such that (see [3,4])

$$X \in L^2_W(\Omega; C[0, T]; H^{-1}(\mathscr{O})) \cap L^{\alpha+1}(\Omega \times [0, T] \times \mathscr{O})$$

and for all $k \in \mathbb{N}$ and all $t \geq 0$ we have

$$\langle X(t), e_k\rangle_2 + \alpha_k \int_0^t \langle |X(s)|^{\alpha}\mathrm{sgn}\, X(s), e_k\rangle_2 \mathrm{d}s = \langle x, e_k\rangle_2 + \sqrt{q_k}W_k(t), \quad \mathbb{P}\text{-a.s.} \tag{1.4}$$

Existence and uniqueness of solutions for Eq. (1.1) for general classes of monotone functions β were established in [3–6,10,11,13]. Moreover in [10] proof was given for the existence of an invariant measure for C^1 functions β satisfying the condition

$$k_0|x|^{m-1} \le \beta'(x) \le k_1|x|^{r-1} + C_2, \quad r > 1,\ k_0, k_1 > 0.$$

In [2] proof was given for the existence of a probability measure μ, infinitesimally invariant, for the Kolmogorov operator N_0 associated with (1.1) and the essential dissipativity of N_0 in the space $L^2(H^{-1}, \nu)$ where ν is an excessive probability measure with respect to N_0. (See also [10,11].)

Here we shall prove the existence of an invariant measure μ for (1.1) (see Theorem 3.2) and we shall describe the corresponding Kolmogorov operator N, i.e., the infinitesimal generator of the transition semigroup P_t associated with (1.1), as the closure in $L^2(H^{-1}(\mathscr{O}), \mu)$ of the elliptic infinite dimensional operator

$$N_0\varphi = \frac{1}{2}\mathrm{Tr}\,[A^{-1}QD^2\varphi] - \langle \beta(x), D^1\varphi\rangle_2.$$

(See Theorem 4.4.) Moreover, the last section is devoted to proving the closability of the gradient in $L^2(H^{-1}, \mu)$ and to defining the Sobolev space $W^{1,2}(H^{-1}, \mu)$. Finally, we show that the domain $D(N)$ of N is included in $W^{1,2}(H^{-1}, \mu)$.

For $\alpha \in (0, 1)$ the equation considered here is known in the literature as the "fast diffusion equation" and it models diffusion in plasma physics (see e.g. [7,8,14]), curvature flows and self-organized criticality in sandpile models. (The case $\alpha = 0$ was recently studied in [4].)

It should be mentioned, however, that the methods used in [2,10–13] for studying the Kolmogorov equation associated with (1.1) are not applicable in the present situation due to the singularity of β' at the origin and so a sharper analysis was necessary.

We shall use the following notation.

- $L^p(\mathscr{O})$ with norm $|\cdot|_p$, $p \ge 1$ and inner product $\langle\cdot,\cdot\rangle_2$ when $p = 2$.
- $H_0^1(\mathscr{O})$ is the standard Sobolev space on $\mathscr{O}$ with norm denoted as $\|\cdot\|_1$.
- $H^{-1}(\mathscr{O})$ is the dual of $H_0^1(\mathscr{O})$ with norm $\|\cdot\|_{-1}$, and inner product $\langle x, y\rangle_{-1} = -\langle \Delta^{-1}x, y\rangle_2,\ x, y \in H^{-1}(\mathscr{O})$. Sometimes we shall write H^{-1} for short, instead of $H^{-1}(\mathscr{O})$ and L^2, instead of $L^2(\mathscr{O})$.

If H is a Hilbert space we shall denote by $\mathrm{D}^1\varphi$ the differential of $\varphi : H \to \mathbb{R}$ and by $D^2\varphi$ the second differential. If $B \in L(H) = L(H, H)$ is a trace class operator we shall denote its trace by $\mathrm{Tr}_H B$. By $C_b^k(H)$, $k = 1, 2$, we shall denote the space of differentiable functions of order k with k derivative continuous and bounded on H. Finally, $B_b(H)$ will represent the space of all Borelian bounded functions on H.

We shall use notation from [4,9] for spaces of adapted processes with values in $H_0^1(\mathscr{O})$, $H^{-1}(\mathscr{O})$ or $L^p(\mathscr{O})$, $p \ge 2$.

2. The approximating problem

Note that β is m-accretive in $\mathbb{R}$ and denote by β_ϵ, $\epsilon > 0$, its Yosida approximation, i.e.,

$$\beta_\epsilon(r) = \frac{1}{\epsilon}\,(r - J_\epsilon(r)), \quad \forall\, r \in \mathbb{R},$$

where

$$J_\epsilon = (1+\epsilon\beta)^{-1}, \quad \forall\, \epsilon > 0.$$

We set

$$\widetilde{\beta}_\epsilon(r) = \beta_\epsilon(r) + \epsilon r, \quad \forall\, r \in \mathbb{R}.$$

Since $\widetilde{\beta}_\epsilon$ is Lipschitz continuous and strongly monotone the stochastic equation

$$\begin{cases} dX_\epsilon(t) - \Delta\widetilde{\beta}_\epsilon(X_\epsilon(t))dt = \sqrt{Q}dW(t), \\ X_\epsilon(0) = x \in H^{-1}(\mathscr{O}), \end{cases} \tag{2.1}$$

has a unique solution

$$X_\epsilon \in L^2_W(\Omega; C([0,T]; H^{-1}\mathscr{O})) \cap L^2(\Omega \times [0,T]; H^1_0(\mathscr{O})).$$

(See [4, Proposition 3.4].)

We set

$$j_\epsilon(r) := \int_0^r \beta_\epsilon(r)dr. \tag{2.2}$$

Lemma 2.1. *We have*

$$r\beta_\epsilon(r) \geq j_\epsilon(r), \quad \forall\, r \in \mathbb{R}. \tag{2.3}$$

Moreover for any $\alpha \in (0,1)$ *we have*

$$\beta'_\epsilon(r) = \frac{\alpha|J_\epsilon(r)|^{\alpha-1}}{1+\alpha\epsilon|J_\epsilon(r)|^{\alpha-1}}, \quad \forall\, r \in \mathbb{R} \tag{2.4}$$

and for $\alpha = 0$

$$\beta'_\epsilon(r) = \begin{cases} 0 & \text{for } |r| > \epsilon \\ 1 & \text{for } |r| < \epsilon. \end{cases}$$

Proof. Since j_ϵ is convex we have

$$j_\epsilon(s) - j_\epsilon(r) \geq \beta_\epsilon(r)(s-r), \quad \forall\, r, s \in \mathbb{R}.$$

Setting $s = 0$ yields (2.3).

Let us show (2.4). Set $s = J_\epsilon(r) = (1+\epsilon\beta)^{-1}(r)$, so $r = s + \epsilon\beta(s)$. If $s \neq 0$ we have

$$1 = \frac{ds}{dr} + \alpha\epsilon|r|^{\alpha-1}\frac{ds}{dr} = \frac{ds}{dr}(1+\alpha\epsilon|s|^{\alpha-1}).$$

Hence

$$J'_\epsilon(r) = \frac{ds}{dr} = \frac{1}{1+\alpha\epsilon|J_\epsilon(r)|^{\alpha-1}}.$$

Therefore

$$\beta'_\epsilon(r) = \frac{\alpha|J_\epsilon(r)|^{\alpha-1}}{1+\alpha\epsilon|J_\epsilon(r)|^{\alpha-1}}. \tag{2.5}$$

In the case $\alpha = 0$ we have

$$\beta_\epsilon(r) = \begin{cases} \text{sign}\, r & \text{for } |r| > \epsilon \\ r & \text{for } |r| < \epsilon \end{cases}$$

and this implies the desired formula for β'_ϵ. □

2.1. A few estimates for the approximating equation

We shall give below a few estimates for the solution X_ϵ to (2.1). We begin with an estimate for $\mathbb{E}\|X_\epsilon(t)\|_{-1}^2$.

Proposition 2.2. *If $x \in H^{-1}$ we have*

$$\mathbb{E}\|X_\epsilon(t)\|_{-1}^2 + 2\mathbb{E}\int_0^t \int_{\mathscr{O}} \beta_\epsilon(X_\epsilon(s))X_\epsilon(s)\mathrm{d}s\mathrm{d}\xi = \|x\|_{-1}^2 + t\mathrm{Tr}\,[QA^{-1}]. \tag{2.6}$$

Proof. By Itô's formula applied to $\|X_\epsilon(t)\|_{-1}^2$, we have

$$\begin{aligned} \mathrm{d}\|X_\epsilon\|_{-1}^2 &= 2\langle \Delta\beta_\epsilon(X_\epsilon)\mathrm{d}t + \sqrt{Q}\mathrm{d}W(t), X_\epsilon\rangle_{-1} + \mathrm{Tr}\,[QA^{-1}]\mathrm{d}t \\ &= -2\langle \beta_\epsilon(X_\epsilon), X_\epsilon\rangle_2 \mathrm{d}t + \left\langle \sqrt{Q}\mathrm{d}W(t), X_\epsilon \right\rangle_{-1} + \mathrm{Tr}\,[QA^{-1}]\mathrm{d}t. \end{aligned}$$

Taking the expectation yields (2.6). □

Let us now estimate $\mathbb{E}|X_\epsilon(t)|_2^2$ (we refer the reader to [3] and [3] for a justification of Itô's formula for Eq. (2.8) below).

Proposition 2.3. *Assume that* $\mathrm{Tr}Q < \infty$. *Then, if $x \in L^2(\mathscr{O})$ we have*

$$\mathbb{E}|X_\epsilon(t)|_2^2 + 2\mathbb{E}\int_0^t \int_{\mathscr{O}} \beta'_\epsilon(X_\epsilon(s))|\nabla X_\epsilon(s)|^2 \mathrm{d}\xi\mathrm{d}s \le |x|_2^2 + t\mathrm{Tr}\, Q. \tag{2.7}$$

Proof. The proof is exactly the same as that of [4, Proposition 3.4], so it will only be outlined. Namely, we consider the equation

$$\begin{cases} \mathrm{d}X_\epsilon^\lambda(t) + (F_\epsilon)_\lambda(X_\epsilon^\lambda(t))\mathrm{d}t = \sqrt{Q}\mathrm{d}W_t, \\ X_\epsilon^\lambda(0) = x, \end{cases} \tag{2.8}$$

where $(F_\epsilon)_\lambda$ is the Yosida approximation of $F_\epsilon = -\Delta\widetilde{\beta}_\epsilon$ with the domain $\{x \in L^1(\mathscr{O}) \cap H^{-1}(\mathscr{O}) : \widetilde{\beta}_\epsilon(x) \in H_0^1(\mathscr{O})\}$. Then applying Itô's formula with $\varphi(x) = \frac{1}{2}\,|x|_2^2$ and taking into account that

$$(F_\epsilon)_\lambda(X_\epsilon^\lambda) = -\Delta\widetilde{\beta}_\epsilon(Y_\epsilon^\lambda), \qquad Y_\epsilon^\lambda = (1 + \lambda F_\epsilon)^{-1}(X_\epsilon^\lambda),$$

we get

$$\mathbb{E}|X_\epsilon^\lambda(t)|_2^2 + 2\mathbb{E}\int_0^t \langle (F_\epsilon)_\lambda(X_\epsilon^\lambda(s)), X_\epsilon^\lambda(s)\rangle_2 \mathrm{d}s = |x|_2^2 + t\mathrm{Tr}\, Q. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} \langle (F_\epsilon)_\lambda(X_\epsilon^\lambda), X_\epsilon^\lambda\rangle_2 &= \langle F_\epsilon(Y_\epsilon^\lambda), Y_\epsilon^\lambda\rangle_2 + \lambda|(F_\epsilon)_\lambda(X_\epsilon^\lambda)|^2 \\ &= -\langle \Delta\widetilde{\beta}_\epsilon(Y_\epsilon^\lambda), Y_\epsilon^\lambda(x)\rangle_2 + \lambda|(F_\epsilon)_\lambda(X_\epsilon^\lambda)|_2^2. \end{aligned} \tag{2.10}$$

So, substituting in (2.9) and using Green's formula yields

$$\mathbb{E}|X_\epsilon^\lambda(t)|_2^2 + 2\mathbb{E}\int_0^t \int_{\mathscr{O}} \widetilde{\beta}'_\epsilon(Y_\epsilon^\lambda(s))|\nabla Y_\epsilon^\lambda(s)|_2^2 \mathrm{d}s\mathrm{d}\xi \le |x|_2^2 + t\mathrm{Tr}\, Q. \tag{2.11}$$

Recalling that by [4, Proposition 3.4] we have

$$\lim_{\lambda\to 0} X_\epsilon^\lambda = X_\epsilon \quad \text{strongly in } L^2(\Omega \times [0,T] \times \mathscr{O}),$$

we obtain that

$$\lim_{\lambda\to 0} Y_\epsilon^\lambda = X_\epsilon \quad \text{strongly in } L^2(\Omega \times [0,T] \times \mathscr{O}). \tag{2.12}$$

Moreover, since $\widetilde{\beta}'_\epsilon \ge \epsilon$ it follows by (2.10) and (2.11) that $\{Y_\epsilon^\lambda\}$ is bounded in $L^2(0,T; L^2(\Omega, H_0^1(\mathscr{O})))$ and so

$$\lim_{\lambda\to 0} Y_\epsilon^\lambda = X_\epsilon \quad \text{weakly in } L^2(0,T; L^2(\Omega, H_0^1(\mathscr{O}))),$$

i.e.,

$$\lim_{\lambda\to 0} \nabla Y_\epsilon^\lambda = \nabla X_\epsilon \quad \text{weakly in } L^2(\Omega \times [0,T] \times \mathscr{O}). \tag{2.13}$$

Now by (2.11) and (2.5) we see that

$$\lim_{\lambda\to 0} |\widetilde{\beta}'_\epsilon(Y_\epsilon^\lambda) - \widetilde{\beta}'_\epsilon(X_\epsilon)| = 0, \quad \text{a.e. in } \Omega \times (0,T) \times \mathscr{O}$$

and

$$|\widetilde{\beta}'_\epsilon(Y_\epsilon^\lambda) - \widetilde{\beta}'_\epsilon(X_\epsilon)| \le C, \quad \text{a.e. in } \Omega \times (0,T) \times \mathscr{O}.$$

This implies that

$$\lim_{\lambda\to 0} \nabla Y_\epsilon^\lambda [\widetilde{\beta}'_\epsilon(Y_\epsilon^\lambda)]^{1/2} = \nabla X_\epsilon [\widetilde{\beta}'_\epsilon(X_\epsilon)]^{1/2} \quad \text{weakly in } L^2(\Omega \times [0,T] \times \mathscr{O})$$

and therefore by weak lower semicontinuity of the integral, we have

$$\liminf_{\lambda\to 0} \mathbb{E}\int_0^t \int_{\mathscr{O}} \widetilde{\beta}'_\epsilon(Y_\epsilon^\lambda(s))|\nabla Y_\epsilon^\lambda(s)|_2^2 \mathrm{d}s\mathrm{d}\xi \ge \mathbb{E}\int_0^t \int_{\mathscr{O}} \widetilde{\beta}'_\epsilon(X_\epsilon(s))|\nabla X_\epsilon(s)|_2^2 \mathrm{d}s\mathrm{d}\xi.$$

Then by (2.9) and (2.11) the conclusion (2.7) follows. □

We have a similar estimate for $\mathbb{E}|X_\epsilon(t)|_{2m}^{2m}$.

Proposition 2.4. *Assume that $\sum_{k=1}^\infty q_k \|e_k\|_\infty^2 < \infty$. Then, if $x \in L^{2m}(\mathscr{O})$ and $m > \frac{1}{2}$ we have*

$$\begin{aligned} &\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + 2m(2m-1)\mathbb{E}\int_0^t \int_{\mathscr{O}} \beta'_\epsilon(X_\epsilon(s))|X_\epsilon(s)|^{2m-2}|\nabla X_\epsilon(s)|^2 \mathrm{d}\xi\mathrm{d}s \\ &\quad \le |x|_{2m}^{2m} + m(2m-1)\sum_{k=1}^\infty q_k \mathbb{E}\int_0^t \int_{\mathscr{O}} |X_\epsilon(s)|^{2m-2} e_k^2 \mathrm{d}\xi\mathrm{d}s. \end{aligned} \tag{2.14}$$

Proof. One proceeds as in the proof of Proposition 2.3, i.e. one applies Itô's formula to the function $\varphi(x) = \frac{1}{2}\,|x|_{2m}^{2m}$ and to Eq. (2.8). One gets as above

$$\begin{aligned}&\mathbb{E}|X_\epsilon^\lambda|_{2m}^{2m} + 2m \int_{\mathscr{O}} (\widetilde{\beta}_\epsilon)(Y_\epsilon^\lambda(s))|Y_\epsilon^\lambda(s)|^{2m-2}|\nabla Y_\epsilon^\lambda(s)|^2 \mathrm{d}\xi\,\mathrm{d}s\\ &\quad = |x|_{2m}^{2m} + m(2m-1)\sum_{k=1}^\infty q_k \mathbb{E}\int_0^t\int_{\mathscr{O}} |Y_\epsilon^\lambda(s)|^{2m-2} e_k^2 \mathrm{d}\xi\,\mathrm{d}s\end{aligned}$$

and let $\lambda \to 0$ to get by (2.12), (2.13) estimate (2.14). □

By (2.14) we have:

Corollary 2.5. *Assume that* $\sum_{k=1}^\infty q_k\|e_k\|_\infty^2 < \infty$. *Then, if* $x \in L^{2m}(\mathscr{O})$, $m \in \mathbb{N}$, *there exists* $C_T > 0$ *independent of* ϵ *such that*

$$\begin{aligned}&\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + 2m(2m-1)\mathbb{E}\int_0^t\int_{\mathscr{O}} \beta'_\epsilon(X_\epsilon(s))|X_\epsilon(s)|^{2m-2}|\nabla X_\epsilon(s)|^2 \mathrm{d}\xi\,\mathrm{d}s\\ &\quad \le C_T(1+|x|_{2m}^{2m}), \quad \forall\, t\in[0,T].\end{aligned} \tag{2.15}$$

2.2. *Existence for equation* (1.1)

Proposition 2.6. *Assume that* $\sum_{k=1}^\infty q_k\|e_k\|_\infty^2 < \infty$. *Then if* $x \in H^{-1}(\mathscr{O})$ *and* $\int_{\mathscr{O}} |x|^{2\alpha}\mathrm{d}\xi < \infty$, $\alpha \ge 0$ *problem* (1.1) *has a unique solution.*

Proof. The existence and uniqueness of a solution to (1.1) was proved earlier in [2,3,5] for more general β. Here we give, however, for later use a direct proof relying on Proposition 2.4.

Let us show that sequence $\{X_\epsilon\}$ is Cauchy in $L_W^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$, where X_ϵ is the solution to (2.1). In fact, for $\epsilon, \eta > 0$ we have by (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}(X_\epsilon - X_\eta) = \Delta\widetilde{\beta}_\epsilon(X_\epsilon) - \widetilde{\Delta\beta_\eta}(X_\eta), \quad \mathbb{P}\text{-a.s.}$$

Multiplying both sides, in scalar fashion, in $H^{-1}(\mathscr{O})$ by $X_\epsilon - X_\eta$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X_\epsilon - X_\eta|_{-1}^2 - \langle \Delta\beta_\epsilon(X_\epsilon) - \Delta\beta_\eta(X_\eta), X_\epsilon - X_\eta\rangle_{-1} + \epsilon|\nabla X_\epsilon - \nabla X_\eta|_2^2 = 0, \quad \mathbb{P}\text{-a.s.},$$

and therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X_\epsilon - X_\eta|_{-1}^2 + \langle \beta_\epsilon(X_\epsilon) - \beta_\eta(X_\eta), X_\epsilon - X_\eta\rangle_2 \le 0, \quad \mathbb{P}\text{-a.s.}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|X_\epsilon - X_\eta|_{-1}^2 + \langle \beta(J_\epsilon(X_\epsilon)) - \beta(J_\eta(X_\eta)), J_\epsilon(X_\epsilon) - J_\eta(X_\eta)\rangle_2\\ &\quad + \langle \beta_\epsilon(X_\epsilon) - \beta_\eta(X_\eta), \epsilon\beta_\epsilon(X_\epsilon) - \eta\beta_\eta(X_\eta)\rangle_2 \le 0.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|X_\epsilon - X_\eta|_{-1}^2 &\le 2(\epsilon+\eta)\left(|\beta_\epsilon(X_\epsilon)|_{2\alpha}^{2\alpha} + |\beta_\eta(X_\eta)|_{2\alpha}^{2\alpha}\right)\\ &\le 2C(\epsilon+\eta)\left(1 + |X_\epsilon|_{2\alpha}^{2\alpha} + |X_\eta|_{2\alpha}^{2\alpha}\right).\end{aligned}$$

Consequently,

$$\mathbb{E} \sup_{t\in[0,T]} |X_\epsilon - X_\eta|_{-1}^2 \leq 2C(\epsilon+\eta)\int_0^T (1+|X_\epsilon(s)|_{2\alpha}^{2\alpha} + |X_\eta(s)|_{2\alpha}^{2\alpha})\mathrm{d}s,$$

and by Proposition 2.3 it follows that $\{X_\epsilon\}$ is Cauchy in $L^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$ and so, it is convergent to an element $X \in L^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$.

It is easily seen that X is a solution of (1.1). □

3. The invariant measure

We shall assume here that $\sum_{k=1}^\infty q_k \|e_k\|_\infty^2 < \infty$. We denote by P_t the transition semigroup associated with Eq. (1.1), i.e.,

$$P_t\varphi(x) := \mathbb{E}[\varphi(X(t,x))], \quad \forall\, \varphi \in B_b(H^{-1}),\ x \in H^{-1},\ t \geq 0$$

where $X(t,x)$ is the solution to (1.1).

We notice that P_t is *Feller*, that is $P_t\varphi \in C_b(H^{-1})$ for all $\varphi \in C_b(H^{-1})$ and all $t \geq 0$. This an easy consequence on the fact that $X(t,x)$ depends continuously on x in the topology of H^{-1}.

Lemma 3.1. *If* $m \in [1,2]$ *then the following estimate holds:*

$$\frac{1}{T}\mathbb{E}\int_0^T |X(s,x)|_{p^*(2m+\alpha-1)/2}^{2m+\alpha-1}\mathrm{d}s \leq \frac{C_1}{T}|x|_{2m}^{2m} + C_2, \quad \forall\, T>0,\ x \in L^{2m}(\mathscr{O}), \tag{3.1}$$

where $p^* = \frac{2d}{d-2}$ *if* $d>2$, $p^* \in [2,\infty)$ *if* $d = 1,2$.

Proof. By Proposition 2.4 and by (2.5) we see that

$$\frac{1}{2m}\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + (2m-1)\mathbb{E}\int_0^t\int_{\mathscr{O}}\left(\frac{\alpha|J_\epsilon(X_\epsilon(s))|^{\alpha-1}}{1+\alpha\epsilon|J_\epsilon(X_\epsilon(s))|^{\alpha-1}}+\epsilon\right)|X_\epsilon(s)|^{2m-2}|$$
$$\times\,\nabla X_\epsilon(s)|^2\mathrm{d}s\mathrm{d}\xi \leq |x|_{2m}^{2m} + (m-1/2)\sum_{k=1}^\infty q_k\mathbb{E}\int_0^t\int_{\mathscr{O}}|X_\epsilon(s)|^{2m-2}e_k^2\mathrm{d}\xi\mathrm{d}s.$$

Taking into account that J_ϵ is non-expansive, that is

$$|J_\epsilon(\xi) - J_\epsilon(\eta)| \leq |\xi-\eta|, \quad \forall\, \xi,\eta \in \mathbb{R},$$

we have

$$|J_\epsilon(x)| \leq |x|, \qquad |\nabla J_\epsilon(x)| \leq |\nabla x| \quad \text{a.e. in } \mathscr{O},\ \forall\, x \in H_0^1(\mathscr{O}).$$

Then we obtain that

$$\frac{1}{2m}\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + \alpha(2m-1)\mathbb{E}\int_0^t\int_{\mathscr{O}}\left(\frac{\alpha|Y_\epsilon(s)|^{2m-3+\alpha}}{1+\alpha\epsilon|Y_\epsilon(s)|^{\alpha-1}}\right)|\nabla Y_\epsilon(s)|^2\mathrm{d}s\mathrm{d}\xi$$
$$\leq |x|_{2m}^{2m} + C\mathbb{E}\int_0^t\int_{\mathscr{O}}|X_\epsilon(s)|^{2m-2}\mathrm{d}\xi\mathrm{d}s, \quad \forall\, \epsilon>0,$$

where $Y_\epsilon(s) = J_\epsilon(X_\epsilon(s))$. Equivalently,

$$\frac{1}{2m}\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + C\alpha\mathbb{E}\int_0^t\int_{\mathscr{O}}|\nabla g_\epsilon(Y_\epsilon(s))|^2\mathrm{d}s\mathrm{d}\xi$$
$$\leq |x|_{2m}^{2m} + C\mathbb{E}\int_0^t\int_{\mathscr{O}}|X_\epsilon(s)|^{2m-2}\mathrm{d}\xi\mathrm{d}s, \quad \forall\, \epsilon>0. \tag{3.2}$$

Here $g_\epsilon : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_\epsilon(r) := \int_0^r \left(\frac{z^{2m-3+\alpha}}{1 + \alpha\epsilon z^{\alpha-1}} \right)^{1/2} \mathrm{d}z, \quad \forall\, r \geq 0.$$

Next by the Sobolev embedding theorem we have for $p^* = \frac{2d}{d-2}$ if $d > 2$, $p^* \in [2, \infty)$ if $d = 1, 2$,

$$\mathbb{E} \int_0^t \int_{\mathscr{O}} |\nabla g_\epsilon(Y_\epsilon(s))|^2 \mathrm{d}s\mathrm{d}\xi \geq C\mathbb{E} \int_0^t |g_\epsilon(Y_\epsilon(s))|^2_{p^*} \mathrm{d}s. \tag{3.3}$$

(Here and everywhere in the following we shall denote by C several positive constants independent of ϵ.)

On the other hand, by the integral mean theorem we have

$$\begin{aligned} g_\epsilon(r) &\geq \int_0^r \frac{z^{m-3/2+\alpha/2}}{1 + (\alpha\epsilon)^{1/2} z^{(\alpha-1)/2}} \mathrm{d}z = \int_0^r \frac{z^{m-1}}{z^{(1-\alpha)/2} + (\alpha\epsilon)^{1/2}} \mathrm{d}z \\ &= \frac{2}{1-\alpha} \int_0^{r^{\frac{1-\alpha}{2}}} \frac{u^{\frac{2m-1+\alpha}{1-\alpha}}}{u + \sqrt{\alpha\epsilon}} \mathrm{d}u = \frac{\tilde{g}}{m}\, r^m \end{aligned}$$

where

$$\tilde{g} \in \left[\frac{1}{r^{\frac{1-\alpha}{2}} + (\alpha\epsilon)^{1/2}}, \frac{1}{(\alpha\epsilon)^{1/2}} \right].$$

Hence

$$g_\epsilon(r) \geq \frac{r^m}{m\left(r^{\frac{1-\alpha}{2}} + (\alpha\epsilon)^{1/2}\right)}, \quad \forall\, r \geq 0.$$

This yields

$$g_\epsilon(r) \geq \frac{r^m}{m\left(r^{\frac{1-\alpha}{2}} + r^{\frac{1-\alpha}{2}}\right)} = \frac{1}{2m}\, r^{m-\frac{1-\alpha}{2}} \quad \text{if } r \geq (\alpha\epsilon)^{\frac{1}{1-\alpha}}.$$

For $0 < r < (\alpha\epsilon)^{\frac{1}{1-\alpha}}$ we have (because $g_\epsilon(r) \leq \frac{1}{2m-1+\alpha} r^{\frac{2m-1+\alpha}{1-\alpha}}$)

$$g_\epsilon(r) \geq -\frac{1}{2m-1+\alpha} (\alpha\epsilon)^{\frac{2m+\alpha-1}{2(1-\alpha)}}.$$

Hence for all $r > 0$ we have

$$g_\epsilon(r) \geq C\left(r^{\frac{2m-1+\alpha}{2}} - C\epsilon^{\frac{2m-1+\alpha}{2(1-\alpha)}} \right). \tag{3.4}$$

By (3.2), (3.3) and (3.4) we obtain that

$$\begin{aligned} &\frac{1}{2m} \mathbb{E}|X_\epsilon(t)|^{2m}_{2m} + C\mathbb{E} \int_0^t |Y_\epsilon(s)|^{2m+\alpha-1}_{p^*(2m+\alpha-1)/2} \mathrm{d}s \\ &\quad \leq |x|^{2m}_{2m} + C\mathbb{E} \int_0^t |X_\epsilon(s)|^{2m-2}_{2m-2} \mathrm{d}s + C\epsilon^{\frac{m-1+\alpha}{2(1-\alpha)}}, \quad \forall\, \epsilon > 0. \end{aligned} \tag{3.5}$$

On the other hand, we have by (2.3) and by estimate (2.6) that

$$\mathbb{E}\int_0^t \int_{\mathscr{O}} j_\epsilon(X_\epsilon(s))\mathrm{d}s\mathrm{d}\xi \le C, \quad \forall\, \epsilon > 0.$$

This yields

$$\mathbb{E}\int_0^t \int_{\mathscr{O}} |X_\epsilon(s) - Y_\epsilon(s)|^2 \mathrm{d}s\mathrm{d}\xi \le C\epsilon, \quad \forall\, \epsilon > 0,$$

because

$$j_\epsilon(r) = \frac{1}{2\epsilon}\,|r - J_\epsilon(r)|^2 + j(J_\epsilon(r)), \quad \forall\, r \in \mathbb{R}.$$

Then for $2m - 2 \le 2$ (i.e., $1 \le m \le 2$ as we assumed) we have for ϵ small enough

$$\mathbb{E}\int_0^t |X_\epsilon(s)|_{2m-2}^{2m-2}\,\mathrm{d}s \le \mathbb{E}\int_0^t |Y_\epsilon(s)|_{2m-2}^{2m-2}\,\mathrm{d}s + C\epsilon$$

and substituting in (3.5) we obtain that

$$\frac{1}{2m}\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + C\mathbb{E}\int_0^t |Y_\epsilon(s)|_{p^*(2m+\alpha-1)/2}^{2m+\alpha-1}\mathrm{d}s$$
$$\le |x|_{2m}^{2m} + C\mathbb{E}\int_0^t |X_\epsilon(s)|_{2m-2}^{2m-2}\mathrm{d}s + C\epsilon, \quad \forall\, \epsilon > 0. \tag{3.6}$$

Now taking into account that for any $\delta > 0$ there exists $C_\delta > 0$ independent of ϵ such that

$$|r|^{2(m-1)} \le \delta|r|^{2m+\alpha-1} + C_\delta, \quad \forall\, r \in \mathbb{R},$$

we obtain by (3.6) that

$$\frac{1}{2m}\mathbb{E}|X_\epsilon(t)|_{2m}^{2m} + \frac{1}{2}\,C\mathbb{E}\int_0^t \int_{\mathscr{O}} |Y_\epsilon(s)|_{p^*(2m+\alpha-1)/2}^{2m+\alpha-1}\mathrm{d}s\mathrm{d}\xi \le |x|_{2m}^{2m} + C\epsilon t,$$
$$\forall\, \epsilon > 0,\ t > 0. \tag{3.7}$$

Now letting $\epsilon \to 0$ in (3.7) and recalling that $\lim_{\epsilon\to 0}(X_\epsilon - Y_\epsilon) = 0$ in $L^2(\Omega \times [0,T] \times \mathscr{O})$ and $X_\epsilon \to X$ in $L^2(\Omega; C([0,T]; H^{-1}(\mathscr{O})))$, we obtain by the weak lower semicontinuity of the norm that X satisfies (3.1) as claimed. □

Theorem 3.2 below is the main result of this section.

Theorem 3.2. *For each $\alpha \in [0,1]$ there exists at least one invariant measure μ for P_t, i.e.,*

$$\int_{H^{-1}} P_t\varphi(x)\mu(\mathrm{d}x) = \int_{H^{-1}} \varphi(x)\mu(\mathrm{d}x), \quad \forall\, \varphi \in B_b(H^{-1}).$$

Moreover, one has

$$\text{support}\ \mu \subset L^{p^*(3+\alpha)/2}(\mathscr{O}), \tag{3.8}$$

$$\int_{H^{-1}} |x|_{p^*(3+\alpha)/2}^{3+\alpha}\mu(\mathrm{d}x) < \infty, \tag{3.9}$$

where p^* was defined in Lemma 3.1.

Proof. We shall use estimate (3.1) for $m = 2$. We set

$$\mu_T = \frac{1}{T} \int_0^T \pi_{t,x} \mathrm{d}t \quad \forall\, T > 0,$$

where $\pi_{t,x}$ is the law of $X(t, x)$. Let $x \in L^4(\mathscr{O})$. Since by the Sobolev theorem the embedding of $L^{p^*(3+\alpha)/2}(\mathscr{O})$ into $H^{-1}(\mathscr{O})$ is compact we see that the set

$$\mathscr{M}_R := \{x \in L^{p^*(3+\alpha)/2}(\mathscr{O}) : \ |x|_{p*(3+\alpha)/2} \le R\}$$

is compact in $H^{-1}(\mathscr{O})$. Also by Lemma 3.1 we have that

$$\mu_T(\mathscr{M}_R^c) \le C_1 \frac{|x|_4^4}{T R^{3+\alpha}} + \frac{C_2}{R^{3+\alpha}}.$$

Hence $\{\mu_T\}_{T>1}$ is tight and so, by the Prokhorov theorem, it is weakly convergent (on a subsequence $T_n \uparrow \infty$) to an invariant measure μ of P_t. Clearly we have

$$\mu_T(\mathscr{M}_R^c) \le \frac{C_2}{R^{3+\alpha}}, \quad \forall\, R > 0,$$

and this implies that also $\mu(L^{p^*(3+\alpha)/2}(\mathscr{O})) = 1$ as claimed.

It remains to prove (3.9), i.e. that

$$|x|_{p*(3+\alpha)/2} \in L^{3+\alpha}(H^{-1}, \mu).$$

With this purpose we shall use estimate (3.1) where x is replaced by $\frac{x}{1+\epsilon|x|_{2m}}$.

If $X^\epsilon = X^\epsilon(t, x)$ is the corresponding solution to (1.1) we have by (3.1) that

$$\frac{1}{T}\mathbb{E} \int_0^T |X^\epsilon(t, x)|^{2m+\alpha-1}_{p^*(2m+\alpha-1)/2} \mathrm{d}t \le \frac{C_1}{T} \frac{|x|_{2m}^{2m}}{(1+\epsilon|x|_{2m})^{2m}} + C_2, \quad \forall\, T > 0,\ x \in L^{2m}(\mathscr{O})$$

and by the invariance of measure μ,

$$\frac{1}{T} \int_{H^{-1}} \mu(\mathrm{d}x) \mathbb{E} \int_0^T |X^\epsilon(t, x)|^{2m+\alpha-1}_{p^*(2m+\alpha-1)/2} \mathrm{d}t = \int_{H^{-1}} \frac{|x|^{2m+\alpha-1}_{\frac{1}{2}p^*(2m+\alpha-1)}}{(1+\epsilon|x|_{2m})^{2m+\alpha-1}} \mu(\mathrm{d}x), \quad \forall\, T > 0.$$

Hence

$$\int_{H^{-1}} \frac{|x|^{2m+\alpha-1}_{\frac{1}{2}p^*(2m+\alpha-1)}}{(1+\epsilon|x|_{2m})^{2m+\alpha-1}} \mu(\mathrm{d}x) \le \frac{C_1}{T} \int_{H^{-1}} \frac{|x|_{2m}^{2m}}{(1+\epsilon|x|_{2m})^{2m}} \mu(\mathrm{d}x) + C_2, \quad \forall\, T > 0,\ \epsilon > 0.$$

Then letting $\epsilon \to 0$ it follows by Fatou's lemma that (3.9) holds. This completes the proof. □

Remark 3.3. The problem of uniqueness of invariant measure μ remains open.

4. The Kolmogorov equation

Everywhere in this section we shall assume that $\sum_{k=1}^\infty q_k \|e_k\|_\infty^2 < \infty$.

We denote by $X(t, x)$ the solution of (1.1) and by P_t the corresponding transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \forall\, t \ge 0,\ x \in H^{-1}(\mathscr{O}),\ \varphi \in B_b(H^{-1}).$$

It is well known that P_t has a unique extension to a contraction C_0-semigroup in $L^2(H^{-1}, \mu)$, still denoted P_t (see e.g. [9]). The corresponding infinitesimal generator will be denoted by N. The Kolmogorov equation associated with Eq. (1.1) is the infinite dimensional elliptic equation in H^{-1}

$$\lambda\varphi - N\varphi = f.$$

Set

$$\begin{aligned} N_0\varphi : &= \frac{1}{2}\,\mathrm{Tr}_{H^{-1}}\,[QD^2\varphi] + \langle F(x), D^1\varphi\rangle_{-1} \\ &= \frac{1}{2}\,\mathrm{Tr}_{L^2}\,[A^{-1}QD^2\varphi] - \langle \beta(x), D^1\varphi\rangle_2, \quad \forall\, \varphi \in D(N_0), \end{aligned} \tag{4.1}$$

where

$$D(N_0) := \{\varphi \in C_b^2(H^{-1}) \cap C_b^1(H_0^1) :\ D^2\varphi A \in C_b(L(L^2))\}.$$

(Here $A = -\Delta$ with $D(A) = H^2((\mathscr{O})) \cap H_0^1((\mathscr{O}))$.) Notice that $D(N_0)$ is dense in $L^2(H^{-1}, \mu)$. Our aim is to prove that N_0 is a core for N. We shall prove first:

Proposition 4.1. *N is an extension of N_0.*

Proof. Let $\varphi \in D(N_0)$. By Taylor's formula we have

$$\begin{aligned} \varphi(X(t,x)) - \varphi(x) = \langle D^1\varphi(x), X(t,x) - x\rangle_{-1} + \frac{1}{2}\langle D^2\varphi(x)(X(t,x) - x),& \\ X(t,x) - x\rangle_{-1} + R(t,x),& \end{aligned}$$

where $\|R(t,x)\|_{-1} = o(t)$ as $t \to 0$. We obtain that

$$\begin{aligned} \frac{1}{t}\,(P_t\varphi(x) - \varphi(x)) = &-\mathbb{E}\frac{1}{t}\int_0^t \langle D^1\varphi(x), \beta(X(s,x))\rangle_2 \mathrm{d}s \\ &+ \frac{1}{2t}\mathbb{E}\int_0^t \mathrm{d}s \int_0^t \langle D^2\varphi(x)\Delta\beta(X(s,x)), \beta(X(r,x))\rangle_2 \mathrm{d}r \\ &+ \frac{1}{t}\mathbb{E}\int_0^t \mathrm{d}s \langle D^2\varphi(x)\Delta\beta(X(s,x)), \sqrt{Q}\,W(t)\rangle_{-1} \\ &+ \frac{1}{2t}\,\mathbb{E}\langle D^2\varphi(x)\sqrt{Q}\,W(t), \sqrt{Q}\,W(t)\rangle_{-1} + R(t,x) \\ =:&\ I_1(t,x) + I_2(t,x) + I_3(t,x) + I_4(t,x) + R(t,x). \end{aligned} \tag{4.2}$$

We notice that (4.2) makes sense because, since X is the solution to (1.1), we have (Definition 1.1) $\beta(X) \in L^{\frac{1+\alpha}{\alpha}}(\Omega \times (0,T) \times \mathscr{O}) \subset L^2(\Omega \times (0,T) \times \mathscr{O})$ and $D^2\varphi(x)\Delta\beta(X) \in L^2(\Omega \times (0,T) \times \mathscr{O})$.

As regards $I_1(t,x)$ we have

$$I_1(t,x) = -\frac{1}{t}\int_0^t \langle D^1\varphi(x), P_s(\beta)(x)\rangle_2 \mathrm{d}s$$

and so, taking into account the invariance of μ, we obtain that

$$\begin{aligned}\int_{H^{-1}} |I_1(t,x)|^2 \mu(\mathrm{d}x) &\le \|\varphi\|_{C_b^1(H_0^1)} \frac{1}{t}\int_0^t \mathrm{d}s \int_{H^{-1}} P_s(\|\beta\|_{-1})(x)\mu(\mathrm{d}x)\\ &= \|\varphi\|_{C_b^1(H_0^1)} \int_{H^{-1}} \|\beta(x)\|_{-1}\mu(\mathrm{d}x) \le C\int_{H^{-1}} |\beta(x)|_{\frac{p^*}{p^*-1}}\mu(\mathrm{d}x)\\ &\le C\int_{H^{-1}} |x|^{\alpha}_{\frac{\alpha p^*}{p^*-1}}\mu(\mathrm{d}x) < +\infty. \end{aligned} \tag{4.3}$$

Here we have used once again Sobolev's embedding theorem, $L^{\frac{p^*}{p^*-1}}(\mathscr{O}) \subset H^{-1}(\mathscr{O})$, and Theorem 3.2 part (3.9).

Moreover, we have also that for each $\psi \in L^2(H^{-1},\mu)$

$$\begin{aligned}\lim_{t\to 0}\int_{H^{-1}} I_1(t,x)\psi(x)\mu(\mathrm{d}x) &= \int_{H^{-1}} \left\langle D^1\varphi(x), \lim_{t\to 0}\frac{1}{t}\int_0^t P_s(\beta)(x)\mathrm{d}s\right\rangle_2 \psi(x)\mu(\mathrm{d}x)\\ &= \int_{H^{-1}} \langle D^1\varphi(x),\beta(x)\rangle_2\psi(x)\mu(\mathrm{d}x),\end{aligned} \tag{4.4}$$

because, as seen earlier by (3.8), $\beta \in L^2(H^{-1},\mu)$, $D^1\varphi(x) \in H_0^1(\mathscr{O})$ and P_t is a C_0-semigroup on $L^2(H^{-1},\mu)$.

We shall estimate now the integral

$$I_2(t,x) = \frac{1}{2t}\mathbb{E}\int_0^t \mathrm{d}s\int_0^t \langle D^2\varphi(x)\Delta\beta(X(s,x)),\beta(X(r,x))\rangle_2 \mathrm{d}r.$$

We have

$$|D^2\varphi(x)\Delta y|_2 \le C|y|_2, \quad \forall\, y \in L^2,$$

because $\varphi \in D(N_0)$. This yields

$$|\langle D^2\varphi(x)\Delta\beta(X(s,x)),\beta(X(r,x))\rangle_2| \le C_1|\beta(X(s,x))|_2|\beta(X(r,x))|_2, \quad \forall\, s,r \in [0,t].$$

We have, therefore,

$$|I_2(t,x)| \le C\frac{1}{t}\mathbb{E}\left(\int_0^t |\beta(X(r,x))|_2\mathrm{d}r\right)^2 \le C\mathbb{E}\int_0^t |\beta(X(r,x))|_2^2\mathrm{d}r,$$
$$\forall\, t \ge 0,\ x \in H^{-1}.$$

This yields

$$\begin{aligned}\int_{H^{-1}} |I_2(t,x)|^2\mu(\mathrm{d}x) &\le Ct\int_{H^{-1}} |\beta(x)|_2^4\mu(\mathrm{d}x) = Ct\int_{H^{-1}} |x|_{2\alpha}^{4\alpha}\mu(\mathrm{d}x)\\ &\le Ct\int_{H^{-1}} |x|_{4\alpha}^{\frac{(3+\alpha)p^*}{2}}\mu(\mathrm{d}x),\end{aligned}$$

and so by (3.9) we have that

$$\int_{H^{-1}} |I_2(t,x)|^2\mu(\mathrm{d}x) \le C_2 t, \quad \forall\, t \ge 0. \tag{4.5}$$

Similarly,

$$\begin{aligned}|I_3(t,x)| &\le \frac{C}{t}\mathbb{E}\left(\int_0^t |\beta(X(s,x))|_2 \mathrm{d}s\, |\sqrt{Q}\, W(t)|_2\right)\\ &\le C\left(\frac{1}{t}\mathbb{E}\int_0^t |\beta(X(s,x))|_2^2 \mathrm{d}s\right)^{1/2}\left(\mathbb{E}|\sqrt{Q}\, W(t)|_2^2\right)^{1/2}\\ &= C\sqrt{t}\,(\mathrm{Tr}\, Q)^{1/2}\left(\frac{1}{t}\int_0^t P_s(|\beta(x)|_2^2)\mathrm{d}s\right)^{1/2}.\end{aligned}$$

This yields

$$\int_{H^{-1}} |I_3(t,x)|^2 \mu(\mathrm{d}x) \le C\sqrt{t}\int_{H^{-1}} |\beta(x)|_2^2 \mu(\mathrm{d}x) \to 0 \quad \text{as } t \to 0. \tag{4.6}$$

Finally, we have for each $\psi \in L^2(H^{-1},\mu)$

$$\begin{aligned}\int_{H^{-1}} I_3(t,x)\psi(x)\mu(\mathrm{d}x) &= \frac{1}{2t}\int_{H^{-1}} \psi(x)\mathbb{E}\left\langle D^2\varphi(x)\sqrt{Q}\, W(t), \sqrt{Q}\, W(t)\right\rangle_{-1} \mu(\mathrm{d}x)\\ &= \frac{1}{2t}\int_{H^{-1}} \psi(x)\mathbb{E}\left(\sum_{k=1}^{\infty}\langle D^2\varphi(x)e_k, e_k\rangle_{-1} q_k W_k^2(t)\right)\mu(\mathrm{d}x)\\ &= \int_{H^{-1}} \psi(x)\, \mathrm{Tr}_{H^{-1}}\, [QD^2\varphi(x)]\mu(\mathrm{d}x).\end{aligned} \tag{4.7}$$

By (4.2)–(4.7) it follows that $\varphi \in D(N)$ and

$$\lim_{t\to 0}\frac{1}{t}\,(P_t\varphi(x) - \varphi(x)) = N\varphi = N_0\varphi,$$

weakly and therefore strongly in $L^2(H^{-1},\mu)$, as claimed. □

Remark 4.2. Since N is dissipative it follows that N_0 is dissipative as well.

Now we are going to show that N is the closure of N_0 in $L^2(H^{-1},\mu)$. To this end it is convenient to approximate the Kolmogorov equation $\lambda\varphi - N_0\varphi = f$ by one with smooth drift term F.

More precisely we consider the Yosida approximation of F (defined by (1.2)),

$$F_\epsilon(x) = \frac{1}{\epsilon}\,(x - (1+\epsilon F)^{-1}(x)) = -\Delta\beta((1+\epsilon F)^{-1}(x)).$$

It is well known that F_ϵ is Lipschitz in H^{-1}. Next, we introduce a further regularization of F_ϵ defining

$$F_{\epsilon,\rho}(x) = \int_{H^{-1}} F_\epsilon(\mathrm{e}^{-\rho A}x + y)N_{\frac{1}{2A}\,(1-\mathrm{e}^{-2\rho A})}(\mathrm{d}y), \quad x \in H^{-1}, \tag{4.8}$$

for any $\rho > 0$ and $\epsilon > 0$, where $N_{\frac{1}{2A}\,(1-\mathrm{e}^{-2\rho A})}$ is the Gaussian measure centered at 0 and having covariance $\frac{1}{2A}\,(1-\mathrm{e}^{-2\rho A})$. $F_{\epsilon,\rho}$ is monotone increasing, Lipschitz and has bounded derivatives of any order.

Finally, we consider the equation

$$\lambda\varphi_{\epsilon,\rho} - \frac{1}{2}\,\mathrm{Tr}_{H^{-1}}\,[QD^2\varphi_{\epsilon,\rho}] + \langle F_{\epsilon,\rho}(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1} = f, \tag{4.9}$$

where $\lambda > 0$ and $f \in L^2(H^{-1}, \mu)$.

Lemma 4.3. *The following statements hold.*

(i) *For any $f \in C_b^2(H^{-1}(\mathscr{O}))$ and any $\lambda > 0$ there is a unique solution $\varphi_{\epsilon,\rho} \in C_b^2(H^{-1}(\mathscr{O}))$ of* (4.9) *given by*

$$\varphi_{\epsilon,\rho}(x) = \int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}[f(X_{\epsilon,\rho}(t,x))]\mathrm{d}t, \quad \forall\, x \in H^{-1}, \tag{4.10}$$

where $X_{\epsilon,\rho}$ is the solution to

$$\begin{cases} \mathrm{d}X_{\epsilon,\rho}(t,x) + F_{\epsilon,\rho}(X_{\epsilon,\rho}(t,x))\mathrm{d}t + \sqrt{Q}\mathrm{d}W(t), \\ X_{\epsilon,\rho}(0,x) = x. \end{cases} \tag{4.11}$$

(ii) *If in addition $f \in C_b^1(H_0^1(\mathscr{O}))$ then $\varphi_{\epsilon,\rho}$ possesses the Gateaux derivative at each point $x \in L^2(\mathscr{O})$ and the following estimate holds:*

$$|\langle D^1\varphi_{\epsilon,\rho}(x), h\rangle_2| \le C|h|_{\frac{2d}{d+2}}, \quad \forall\, h \in L^{\frac{2d}{d+2}}(\mathscr{O}),\ x \in L^2(\mathscr{O}). \tag{4.12}$$

Proof. Part (i) is standard; let us prove (ii). Fix $x, h \in L^2(\mathscr{O})$. Then by (4.10) we have

$$\langle D^1\varphi_{\epsilon,\rho}(x), h\rangle_2 = \int_0^\infty \mathrm{e}^{-\lambda t}\mathbb{E}[\langle D^1 f(X_{\epsilon,\rho}(t,x)), Z_{\epsilon,\rho}(t,x)\rangle_2]\mathrm{d}t, \tag{4.13}$$

where $Z_{\epsilon,\rho}$ is the solution to the linear equation

$$\begin{cases} \dfrac{\partial}{\partial t} Z_{\epsilon,\rho}(t,x) = -D^1F_{\epsilon,\rho}(X_{\epsilon,\rho}(t,x))(Z_{\epsilon,\rho}(t,x)), \quad \xi \in \mathscr{O}, \\ Z_{\epsilon,\rho}(t,x) = 0, \quad \forall\, \xi \in \partial\mathscr{O}, \\ Z_{\epsilon,\rho}(0,x) = h. \end{cases} \tag{4.14}$$

By the monotonicity of $F_{\epsilon,\rho}$ it follows that

$$\|Z_{\epsilon,\rho}(t,x)\|_{-1} \le \|h\|_{-1}, \quad \forall\, h \in L^{\frac{2d}{d+2}}(\mathscr{O}).$$

So, by the Sobolev embedding theorem ($L^{\frac{p^*}{p^*-1}}(\mathscr{O}) \subset H^{-1}(\mathscr{O})$) we have

$$\|Z_{\epsilon,\rho}(t,x)\|_{-1} \le C|h|_{\frac{2d}{d+2}}, \quad \forall\, h \in L^{\frac{2d}{d+2}}(\mathscr{O}).$$

Now by (2.7) we deduce that if $x \in L^2(\mathscr{O})$ then $X_{\epsilon,\rho}(t,x) \in H_0^1(\mathscr{O})$ and so by (4.13) we get (4.12) as claimed. □

We are now in position to prove the main result of this section.

Theorem 4.4. *Let $\alpha \in (0,1]$. Then N_0 is essentially m-dissipative in $L^1(H^{-1}(\mathscr{O}), \mu)$ and its closure $\overline{N_0}$ coincides with N.*

Proof. We first notice that N_0 is dissipative in $L^1(H^{-1}(\mathscr{O}), \mu)$ by Proposition 4.1. To prove the essential m-dissipativity of N_0 we fix $\lambda > 0$ and $f \in C_b^2(H^{-1}(\mathscr{O})) \cap C_b^1(H_0^1(\mathscr{O}))$. Then we consider the solution $\varphi_{\epsilon,\rho}$ of (4.9) expressed by (4.10). We have therefore

$$\lambda\varphi_{\epsilon,\rho} - L_0\varphi_{\epsilon,\rho} = f + \langle F(x) - F_{\epsilon,\rho}(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1}. \quad \square \tag{4.15}$$

Claim. *We have*

$$\lim_{\epsilon\to 0}\lim_{\rho\to 0}\langle F(x) - F_{\epsilon,\rho}(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1} = 0 \quad \text{in } L^2(H^{-1}, \mu).$$

The latter implies that the range of $\lambda I - N_0$ includes $C_b^2(H^{-1}(\mathscr{O})) \cap C_b^1(H_0^1(\mathscr{O}))$ so it is dense in $L^2(H^{-1}, \mu)$ and, therefore, the conclusion of Theorem 4.4 follows by the Lumer–Phillips theorem.

Proof of the Claim. We set

$$\langle F(x) - F_{\epsilon,\rho}(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1} = I_\epsilon^1 + I_{\epsilon,\rho}^2,$$

where

$$I_\epsilon^1 := \langle F(x) - F_\epsilon(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1}$$
$$I_{\epsilon,\rho}^2 := \langle F_\epsilon(x) - F_{\epsilon,\rho}(x), D^1\varphi_{\epsilon,\rho}\rangle_{-1}.$$

As regards $I_{\epsilon,\rho}^2$, we have by (4.8) that

$$F_\epsilon(x) - F_{\epsilon,\rho}(x) = \int_{H^{-1}} (F_\epsilon(x) - F_\epsilon(\mathrm{e}^{-\rho A}x + y))N_{\frac{1}{2A}(1-\mathrm{e}^{-2\rho A})}(\mathrm{d}y).$$

Since F_ϵ is Lipschitz, there exists $C_\epsilon > 0$ such that

$$\|F_\epsilon(x)\|_{-1} \le C_\epsilon(1 + \|x\|_{-1}), \quad \forall x \in H^{-1}.$$

Therefore

$$\begin{aligned}\|F_{\epsilon,\alpha}(x)\|_{-1}^2 &\le 2C_\epsilon^2 \int_{H^{-1}} (1 + \|y\|_{-1}^2)N_{\frac{1}{2A}(1-\mathrm{e}^{-2\rho A})}(\mathrm{d}y)\\ &\le 2C_\epsilon^2\left(1 + \|\mathrm{e}^{-\rho A}x\|_{-1}^2 + \frac{1}{2}\operatorname{Tr} A^{-1}\right),\end{aligned} \tag{4.16}$$

and so, by the Lebesgue dominated convergence theorem, we have that

$$\lim_{\rho\to 0}\int_{H^{-1}} I_{\epsilon,\rho}^2(x)\mu(\mathrm{d}x) = 0.$$

As regards $I_{\epsilon,\rho}^1$, by Lemma 4.3-(ii) we have

$$\begin{aligned}|\langle F(x) - F_\epsilon(x), D^1\varphi_\epsilon\rangle_{-1}| &\le |\langle \beta(x) - \beta((I + \epsilon F)^{-1}(x)), D^1\varphi_\epsilon\rangle_2|\\ &\le C|\beta(x) - \beta((I + \epsilon F)^{-1}(x))|_{\frac{2d}{d+2}}.\end{aligned}$$

This yields

$$\begin{aligned}&\int_{H^{-1}} |\langle F(x) - F_\epsilon(x), D^1\varphi_\epsilon\rangle_{-1}|^2\mu(\mathrm{d}x)\\ &\quad\le C\int_{H^{-1}} |\beta(x) - \beta((I + \epsilon F)^{-1}(x))|_{\frac{2d}{d+2}}^2\mu(\mathrm{d}x).\end{aligned} \tag{4.17}$$

We are going to apply the Lebesgue dominated convergence theorem in $L^1(H^{-1},\mu)$ to pass to the limit in (4.17). To this end further estimates are necessary. We recall that $y_\epsilon := (I+\epsilon F)^{-1}(x)$ is the solution to the equation

$$y_\epsilon - \epsilon\Delta\beta(y_\epsilon) = x \quad \text{in } \mathscr{O},\ y_\epsilon = 0 \text{ on } \partial\mathscr{O}. \tag{4.18}$$

If we multiply (4.18) by $|y_\epsilon|^{p-1}\text{sign } y_\epsilon$ and integrate over $\mathscr{O}$ we get that

$$\begin{aligned}\int_{\mathscr{O}} |y_\epsilon|^p d\xi &\le \int_{\mathscr{O}} |y_\epsilon|^{p-1} \text{ sign } y_\epsilon \ x d\xi \\ &\le \left(\int_{\mathscr{O}} |y_\epsilon|^p d\xi\right)^{\frac{p-1}{p}} \left(\int_{\mathscr{O}} |x|^p d\xi\right)^{\frac{1}{p}}.\end{aligned}$$

This yields

$$|y_\epsilon|_p \le |x|_p, \quad \forall\, \epsilon > 0,\ x \in L^p(\mathscr{O}),\ p \ge 1. \tag{4.19}$$

Since $y_\epsilon \to y$ in $H^{-1}(\mathscr{O})$ as $\epsilon \to 0$ we have by (4.19) that $y_\epsilon \to x$ weakly and then strongly in $L^p(\mathscr{O})$ for each $x \in L^p(\mathscr{O})$ and $p > 1$. (The space $L^p(\mathscr{O})$ is uniformly convex.)

Similarly by the inequality

$$|\beta(y_\epsilon)|_p \le |\beta(x)|_p, \quad \forall\, \epsilon > 0,\ x \in L^{2p}(\mathscr{O}),\ p \ge 1/2, \tag{4.20}$$

we conclude that for $\epsilon \to 0$, $\beta(y_\epsilon) \to \beta(x)$ strongly in $L^p(\mathscr{O})$ for each $x \in L^{2p}(\mathscr{O})$ and $p > 1/2$.

This implies therefore that

$$\lim_{\epsilon\to 0} |((I+\epsilon F)^{-1}x) - \beta(x)|_{\frac{2d}{d+2}} = 0, \quad \forall\, x \in L^{\frac{2d}{d+2}}(\mathscr{O}). \tag{4.21}$$

On the other hand, by (4.20) and Theorem 3.2 ((3.8) and (3.9)) we have that

$$|\beta((I+\epsilon F)^{-1}x) - \beta(x)|^2_{\frac{2d}{d+2}} \le 2|\beta(x)|^2_{\frac{2\alpha d}{d+2}}$$

and

$$\int_{H^{-1}} |\beta(x)|^2_{\frac{2d}{d+2}} \mu(dx) = \int_{H^{-1}} |x|^{2\alpha}_{\frac{2d}{d+2}} \mu(dx) < \infty.$$

Thus by (4.17) and (4.21) we infer via Lebesgue's dominated convergence theorem that

$$\lim_{\epsilon\to 0} \int_{H^{-1}} I^1_\epsilon(x)\mu(dx) = 0,$$

as claimed. This completes the proof. □

Remark 4.5. Theorem 4.4 remains true for $\alpha = 0$ if in the definition of N_0 (see (4.1)) we take $\beta(x) = \frac{x}{|x|}$ if $x \neq 0$, $\beta(0) = 0$. In this case the operator F is no longer maximal monotone, but replacing in the proof of Theorem 4.4 F by $F^\epsilon(x) = -\Delta$ (sign $x + \epsilon x$) the previous argument works in this case too. The details are omitted.

5. The Sobolev space $W^1(H^{-1},\mu)$

Everywhere in this section we shall assume that $\sum_{k=1}^\infty q_k \|e_k\|^2_\infty < \infty$ and that $q_k > 0$ for all $k \in \mathbb{N}$, or in other words that Ker $Q = \{0\}$. We denote by μ an invariant measure of P_t and by

N its infinitesimal generator in $L^2(H,\mu)$. We know by Theorem 4.4 that N is the closure of the Kolmogorov operator N_0 defined by (4.1).

For any $\varphi \in D(N_0)$ we have, as is easily checked,

$$N_0(\varphi^2) = 2\varphi N_0\varphi + |Q^{1/2}D^1\varphi|_2^2.$$

Integrating this identity with respect to μ and taking into account that, by the invariance of μ, $\int_{H^{-1}} N_0(\varphi^2)\mathrm{d}\mu = 0$, yields

$$\int_{H^{-1}} N_0\varphi\,\varphi\,\mathrm{d}\mu = -\frac{1}{2}\int_{H^{-1}} |Q^{1/2}D^1\varphi|_2^2\mathrm{d}\mu, \quad \forall\,\varphi \in D(N_0). \tag{5.1}$$

Proposition 5.1. *The operator $Q^{1/2}D^1$ can be uniquely extended to a bounded operator, still denoted by $Q^{1/2}D^1$, from $D(N)$ (endowed with the graph norm) into $L^2(H^{-1},\mu;H^{-1})$. Moreover the following identity holds:*

$$\int_{H^{-1}} N\varphi\,\varphi\,\mathrm{d}\mu = -\frac{1}{2}\int_{H^{-1}} |Q^{1/2}D^1\varphi|_2^2\mathrm{d}\mu, \quad \forall\,\varphi \in D(N). \tag{5.2}$$

Proof. Let $\varphi \in D(N)$. Then there exists a sequence $\{\varphi_n\} \subset D(N_0)$ such that

$$\varphi_n \to \varphi, \qquad N_0\varphi_n \to N\varphi \quad \text{in } L^2(H^{-1},\mu).$$

By (5.1) it follows that

$$\int_{H^{-1}} |Q^{1/2}D^1(\varphi_n-\varphi_m)|_2^2\mathrm{d}\mu \le 2\int_{H^{-1}} |N(\varphi_n-\varphi_m)|\,|\varphi_n-\varphi_m|\,\mathrm{d}\mu.$$

Therefore the sequence $\{Q^{1/2}D^1\varphi_n\}$ is Cauchy in $L^2(H^{-1},\mu;H^{-1})$ and the conclusion follows. □

Corollary 5.2. *Let $\varphi \in D(N_0)$ and let $t \ge 0$. Then the following identity holds:*

$$\int_{H^{-1}} (P_t\varphi)^2\,\mathrm{d}\mu + \frac{1}{2}\int_0^t \mathrm{d}s\int_{H^{-1}} |Q^{1/2}D^1P_s\varphi|_2^2\mathrm{d}\mu = \int_{H^{-1}} \varphi^2\,\mathrm{d}\mu. \tag{5.3}$$

Proof. Let $\varphi \in D(N_0)$. Then from the Hille–Yosida theorem one has $P_t\varphi \in D(N)$ for any $t \ge 0$ and moreover

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t\varphi = NP_t\varphi.$$

Multiplying both sides of this identity by φ and integrating with respect to x, and taking into account (5.2), yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{H^{-1}} (P_t\varphi)^2\mathrm{d}\mu = \int_{H^{-1}} N\varphi\,\varphi\,\mathrm{d}\mu = -\frac{1}{2}\int_{H^{-1}} |Q^{1/2}D^1P_s\varphi|_2^2\mathrm{d}s.$$

Integrating finally with respect to t yields (5.2). □

As a consequence of identity (5.2), we are going to prove now that the mapping

$$D^1 : D(N_0) \subset L^2(H^{-1},\mu) \to L^2(H^{-1},\mu;H^{-1}), \qquad \varphi \to D^1\varphi,$$

is closable. First we shall prove the following lemma.

Lemma 5.3. *Let $\{\varphi_n\} \subset D(N_0)$ and $G \in L^2(H^{-1}, \mu; H^{-1})$ be such that $D^1\varphi_n \to G$ in $L^2(H^{-1}, \mu; H^{-1})$. Then we have*

$$\lim_{n\to\infty} D^1 P_t\varphi_n = \mathbb{E}[X_x^*(t,x)G(X(t,x))] \quad \text{in } L^2(H^{-1}, \mu; H^{-1}) \text{ uniformly in } t.$$

Here $X_x^*(t,x)$ is the Gâteaux differential of the map $x \to X(t,x)$.

Proof. Taking into account that $\|X_x^*(t,x)\|_{L(H^{-1})} \leq 1$ and μ is invariant we have that

$$\begin{aligned}
&\int_{H^{-1}} \left\| D^1 P_t\varphi_n(x) - \mathbb{E}[X_x^*(t,x)G(X(t,x))]\right\|_{-1}^2 \mu(\mathrm{d}x) \\
&\quad = \int_{H^{-1}} \left\|\mathbb{E}[X_x^*(t,x)(D^1\varphi_n(X(t,x)) - G(X(t,x)))]\right\|_{-1}^2 \mu(\mathrm{d}x) \\
&\quad \leq \int_{H^{-1}} \mathbb{E}[\|D^1\varphi_n(X(t,x)) - G(X(t,x))\|_{-1}^2]\mu(\mathrm{d}x) \\
&\quad = \int_{H^{-1}} \|D^1\varphi_n(x) - G(x)\|_{-1}^2 \mu(\mathrm{d}x),
\end{aligned}$$

and the conclusion follows. □

We can now prove the announced result.

Theorem 5.4. *D^1 is closable.*

Proof. Let $\{\varphi_n\} \subset D(N_0)$ and $G \in L^2(H^{-1}, \mu; H^{-1})$ be such that

$$\varphi_n \to 0 \quad \text{in } L^2(H^{-1}, \mu), \qquad D^1\varphi_n \to G \quad \text{in } L^2(H^{-1}, \mu; H^{-1}).$$

By (5.3) we have

$$\int_{H^{-1}} (P_t\varphi_n)^2 \, \mathrm{d}\mu + \frac{1}{2}\int_0^t \mathrm{d}s \int_{H^{-1}} |Q^{1/2}D^1 P_s\varphi_n|_2^2 \mathrm{d}\mu = \int_{H^{-1}} \varphi_n^2 \, \mathrm{d}\mu.$$

Letting $n \to \infty$ yields

$$\lim_{n\to\infty} \int_0^t \mathrm{d}s \int_{H^{-1}} \|A^{-1}D^1 Q^{1/2} P_s\varphi_n\|_{-1}^2 \mathrm{d}\mu = 0.$$

Consequently, by Lemma 5.3, it follows that

$$\int_0^t \mathrm{d}s \int_{H^{-1}} \|Q^{1/2}X^*(t,x)G(X(t,x))\|_{-1}^2 \mu(\mathrm{d}x) = 0,$$

which yields

$$X^*(t,x)G(X(t,x)) = 0, \quad \forall\, t \geq 0, \ \mu\text{-a.s.},$$

because $A^{-1}Q$ is one-to-one. Therefore

$$G(X(t,x)) = 0, \quad \forall\, t \geq 0, \ \mu\text{-a.s.},$$

and so

$$P_t(\|G(\cdot)\|_{-1}^2)(x) = 0, \quad \forall\, t \geq 0, \ \mu\text{-a.s.}$$

Integrating with respect to μ over H^{-1} and taking into account the invariance of μ yields

$$\int_{H^{-1}} \|G(x)\|_{-1}^2 \mu(\mathrm{d}x) = 0,$$

so $G = 0$ as required. □

We shall define the Sobolev space $W^{1,2}(H^{-1}, \mu)$ as the domain of the closure of D^1.

We conclude this section with a regularity property of elements of $D(N)$.

Proposition 5.5. *We have* $D(N) \subset W^{1,2}(H^{-1}, \mu)$.

Proof. Let $\varphi \in D(N)$ and let $\{\varphi_n\} \subset D(N_0)$ be a sequence such that

$$\varphi_n \to \varphi, \qquad N_0\varphi_n \to N\varphi \quad \text{in } L^2(H^{-1}, \mu).$$

By (5.2) it follows that

$$\int_{H^{-1}} |D^1(\varphi_n - \varphi_m)|_2^2 \mathrm{d}\mu \le 2 \int_{H^{-1}} |N(\varphi_n - \varphi_m)|\, |\varphi_n - \varphi_m|\, \mathrm{d}\mu.$$

Therefore the sequence $\{D^1\varphi_n\}$ is Cauchy in $L^2(H^{-1}, \mu; H^{-1})$. Since D^1 is closed it follows that $\varphi \in W^{1,2}(H^{-1}, \mu)$ as required. □

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