Generalized stability of multi-additive mappings

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Abstract

In this paper we unify the system of Cauchy functional equations defining multi-additive mapping to obtain a single equation and prove the generalized Hyers–Ulam stability both of this system and this equation using the so-called direct method.

1. Introduction

Throughout this paper we assume that $V$ is a commutative semigroup, $W$ is a linear space and $n \geq 1$ is an integer. Moreover, $\mathbb{N}$ stands for the set of all positive integers.

Let us recall that a function $f : V^n \rightarrow W$ is called multi-additive if it is additive (satisfies Cauchy’s functional equation) in each variable. Some basic facts on such mappings can be found for instance in [1], where their application to the representation of polynomial functions is also presented (see also [2]).

In this paper we reduce this system of $n$ Cauchy equations to obtain a single functional equation for $f$ and prove the generalized Hyers–Ulam stability both of this system and this equation, using the so-called direct (Hyers) method.

Speaking of the stability of a functional equation we follow the question raised in 1940 by S. M. Ulam: “when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?”. The first answer (in the case of Cauchy’s functional equation in Banach spaces) to Ulam’s question was given by Hyers (see [3]). After his result a great number of papers (see for instance [4–7] and the references given there) on the subject were published, generalizing Ulam’s problem and Hyers’s theorem in various directions and to other functional equations (as the words “differing slightly” and “be close” may have various meanings, different kinds of stability can be dealt with). The classical works on the stability of additive mappings are [3,8–12] and [13], whereas some recent results on this topic can be found in [14–16] and [17]. On the other hand, for some outcomes on Hyers–Ulam stability of multi-additive functions we refer the reader to [18,19] and [20] (see also [1]).

To finish this introductory section let us finally mention some studies of Prager and Schweiger (see [21]) and the author (see [22] and [23]) concerning different kinds of stability of multi-Jensen functions, that is functions satisfying (under some additional assumptions on $V$) Jensen’s functional equation in each variable.

2. Results

First, we prove the stability of the system of equations defining the multi-additive mapping.
Theorem 1. Let $V$ be a commutative semigroup and $W$ be a Banach space. Assume also that $n \in \mathbb{N}$ and for every $i \in \{1, \ldots, n\}$, $\varphi_i : V^{n+1} \rightarrow [0, \infty)$ is a mapping such that for any $(x_1, \ldots, x_{n+1}) \in V^{n+1}$ we have

$$\varphi_i(x_1, \ldots, x_{n+1}) := \sum_{j=0}^{\infty} \frac{1}{2^j}[\varphi_i(2^j x_1, x_2, \ldots, x_{n+1}) + \cdots + \varphi_i(x_1, \ldots, x_{i-2}, 2^j x_{i-1}, x_i, \ldots, x_{n+1})$$

$$+ \frac{1}{2} \varphi_i(x_1, \ldots, x_{i-1}, 2^j x_i, 2^j x_{i+1}, 2^j x_{i+2}, \ldots, x_{n+1}) + \varphi_i(x_1, \ldots, x_{i+1}, 2^j x_{i+2}, x_{i+3}, \ldots, x_{n+1})$$

$$+ \cdots + \varphi_i(x_1, \ldots, x_i, 2^j x_{n+1})] < \infty.$$  \hspace{1cm} (1)

If $f : V^n \rightarrow W$ is a function satisfying

$$\|f(x_1, \ldots, x_{i-1}, x_i + x'_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)\|$$

$$\leq \varphi_i(x_1, \ldots, x_i, x'_i, x_{i+1}, \ldots, x_n), (x_1, \ldots, x_i, x'_i, x_{i+1}, \ldots, x_n) \in V^{i+1}, \quad i \in \{1, \ldots, n\},$$  \hspace{1cm} (2)

then for every $i \in \{1, \ldots, n\}$ there exists a multi-additive mapping $F_i : V^n \rightarrow W$ such that for any $(x_1, \ldots, x_n) \in V^n$ we have

$$\|f(x_1, \ldots, x_n) - F_i(x_1, \ldots, x_n)\| \leq \varphi_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n).$$  \hspace{1cm} (3)

For every $i \in \{1, \ldots, n\}$ the function $F_i$ is given by

$$F_i(x_1, \ldots, x_n) := \lim_{\delta \to -\infty} \frac{1}{2^i} f(x_1, \ldots, x_{i-1}, 2^i x_i, x_{i+1}, \ldots, x_n), \quad (x_1, \ldots, x_n) \in V^n.$$  \hspace{1cm} (4)

Proof. Fix $x_1, \ldots, x_n \in V, j \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \ldots, n\}$. Putting $x'_i := x_i$ in (2) we get

$$\left\| \frac{1}{2^j} f(x_1, \ldots, x_{i-1}, 2^j x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n) \right\| \leq \frac{1}{2} \varphi_i(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_n).$$

Dividing both sides of the above inequality by $2^j$ and replacing $x_i$ by $2^i x_i$ we see that

$$\left\| \frac{1}{2^i} f(x_1, \ldots, x_{i-1}, 2^i x_i, x_{i+1}, \ldots, x_n) - \frac{1}{2^{i+1}} f(x_1, \ldots, x_{i-1}, 2^{i+1} x_i, x_{i+1}, \ldots, x_n) \right\|$$

$$\leq \frac{1}{2^{i+1}} \varphi_i(x_1, \ldots, x_i, x_i, 2^i x_i, x_{i+1}, \ldots, x_n),$$

and consequently for any non-negative integers $l$ and $m$ with $l < m$ we obtain

$$\left\| \frac{1}{2^l} f(x_1, \ldots, x_{i-1}, 2^l x_i, x_{i+1}, \ldots, x_n) - \frac{1}{2^m} f(x_1, \ldots, x_{i-1}, 2^m x_i, x_{i+1}, \ldots, x_n) \right\|$$

$$\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi_i(x_1, \ldots, x_i, x_i, 2^j x_i, x_{i+1}, \ldots, x_n).$$  \hspace{1cm} (5)

Therefore from (1) it follows that $(\frac{1}{2^i} f(x_1, \ldots, x_{i-1}, 2^i x_i, x_{i+1}, \ldots, x_n))_{i \in \mathbb{N}}$ is a Cauchy sequence. Since the space $W$ is complete, this sequence is convergent and we define $F_i : V^n \rightarrow W$ by (4). Putting $l = 0$, letting $m \to \infty$ in (5) and using (1) we see that (3) holds.

Finally, fix also $x'_i \in V$ and note that according to (2) we have

$$\left\| \frac{1}{2^l} f(x_1, \ldots, x_{i-1}, 2^l x_i + x'_i, x_{i+1}, \ldots, x_n) - \frac{1}{2^l} f(x_1, \ldots, x_{i-1}, 2^l x_i, x_{i+1}, \ldots, x_n) \right\|$$

$$+ \frac{1}{2} \varphi_i(x_1, \ldots, x_i, x_i, 2^i x_i, x_{i+1}, \ldots, x_n).$$

Next, fix $k \in \{1, \ldots, n\} \setminus \{i\}$. $x'_k \in V$ and assume that $k < i$ (the same arguments apply to the case where $k > i$). From (2) it follows that

$$\left\| \frac{1}{2^l} f(x_1, \ldots, x_{k-1}, x_k + x'_k, x_{k+1}, \ldots, x_{i-1}, 2^l x_i, x_{i+1}, \ldots, x_n) - \frac{1}{2^l} f(x_1, \ldots, x_{k-1}, 2^l x_i, x_{i+1}, \ldots, x_n) \right\|$$

$$+ \frac{1}{2} \varphi_k(x_1, \ldots, x_k, x'_k, x_{k+1}, \ldots, x_{i-1}, 2^i x_i, x_{i+1}, \ldots, x_n).$$

Letting $j \to \infty$ in the above two inequalities and using (1) we see that the mapping $F_i$ is multi-additive. \qed
Next, we reduce the system of $n$ Cauchy equations to obtain a single functional equation for $f$.

**Theorem 2.** Let $V$ be a commutative semigroup with the identity element $0$ and $W$ be a linear space. A mapping $f : V^n \rightarrow W$ is multi-additive if and only if

$$f(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) = \sum_{i_1, \ldots, i_n \in \{1, 2\}} f(x_{i_1}, \ldots, x_{i_n}), \quad (x_{11}, \ldots, x_{n1}), (x_{12}, \ldots, x_{n2}) \in V^n. \quad (6)$$

**Proof.** Assume that $f : V^n \rightarrow W$ satisfies Eq. (6). Putting

$$(x_{11}, \ldots, x_{n1}) = (x_{12}, \ldots, x_{n2}) = (0, \ldots, 0)$$

in (6) we get $f(0, \ldots, 0) = 2^n f(0, \ldots, 0)$, and consequently $f(0, \ldots, 0) = 0$. Next, fix $j \in \{1, \ldots, n\}$, $x_{j1} \in V$ and put $x_{j2} = x_{k3} = 0$, where $k \in \{1, \ldots, n\} \setminus \{j\}$. Then, by (6),

$$f(0, \ldots, 0, x_{j1}, 0, \ldots, 0) = 2^{n-1} f(0, \ldots, 0, x_{j1}, 0, \ldots, 0),$$

and thus $f(0, \ldots, 0, x_{j1}, 0, \ldots, 0) = 0$.

We continue in this fashion obtaining $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero. Finally, fix $j \in \{1, \ldots, n\}$, $x_{11}, \ldots, x_{n1}$, $x_{j2} \in V$ and put $x_{i2} = 0$ for $k \in \{1, \ldots, n\} \setminus \{j\}$ in (6). Then

$$f(x_{11}, \ldots, x_{j-1}, x_{j1} + x_{j2}, x_{j+1}, \ldots, x_{n1}) = f(x_{11}, \ldots, x_{n1}) = f(x_{11}, \ldots, x_{j-1}, x_{j2}, x_{j+1}, \ldots, x_{n1}),$$

which proves that $f$ is multi-additive. The rest of the proof is clear. \(\square\)

Finally, we prove the stability of Eq. (6). The below theorem generalizes Găvruta’s result from \cite{13} (where the case when $n = 1$ and $V$ is a commutative group was considered).

**Theorem 3.** Let $V$ be a commutative semigroup with an identity element and $W$ be a Banach space. Assume also that $\varphi : V^{2n} \rightarrow [0, \infty)$ is a mapping such that

$$\varphi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) := \sum_{j=0}^{\infty} \frac{1}{2^{2j+1}} \varphi(2^j x_{11}, 2^j x_{12}, \ldots, 2^j x_{n1}, 2^j x_{n2}) < \infty, \quad (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in V^{2n}. \quad (7)$$

If $f : V^n \rightarrow W$ is a function satisfying

$$\left\| f(x_{11} + x_{12}, \ldots, x_{n1} + x_{n2}) - \sum_{i_1, \ldots, i_n \in \{1, 2\}} f(x_{i_1}, \ldots, x_{i_n}) \right\| \leq \varphi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}), \quad (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in V^{2n}, \quad (8)$$

then there exists a unique multi-additive mapping $F : V^n \rightarrow W$ for which

$$\|f(x_{11}, \ldots, x_{n1}) - F(x_{11}, \ldots, x_{n1})\| \leq \varphi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n1}), \quad (x_{11}, \ldots, x_{n1}) \in V^n. \quad (9)$$

The function $F$ is given by

$$F(x_{11}, \ldots, x_{n1}) := \lim_{j \to \infty} \frac{1}{2^j} f(2^j x_{11}, \ldots, 2^j x_{n1}), \quad (x_{11}, \ldots, x_{n1}) \in V^n. \quad (10)$$

**Proof.** Fix $(x_{11}, \ldots, x_{n1}) \in V^n$ and $j \in N \cup \{0\}$. Putting $x_{i2} := x_{i1}$ for $i \in \{1, \ldots, n\}$ in (8) we get

$$\|f(2x_{11}, \ldots, 2x_{n1}) - 2^n f(x_{11}, \ldots, x_{n1})\| \leq \varphi(x_{11}, x_{12}, \ldots, x_{n1}, x_{n1}).$$

Dividing both sides of the above inequality by $2^{n(j+1)}$ and replacing $x_{11}$ by $2^j x_{11}$ for $i \in \{1, \ldots, n\}$ we see that

$$\left\| \frac{1}{2^{n(j+1)}} f(2^{j+1} x_{11}, \ldots, 2^{j+1} x_{n1}) - \frac{1}{2^n} f(2^j x_{11}, \ldots, 2^j x_{n1}) \right\| \leq \frac{1}{2^{n(j+1)}} \varphi(2^j x_{11}, 2^j x_{12}, \ldots, 2^j x_{n1}, 2^j x_{n1}).$$

and consequently for any non-negative integers $l$ and $m$ with $l < m$ we obtain

$$\left\| \frac{1}{2^{ml}} f(2^m x_{11}, \ldots, 2^m x_{n1}) - \frac{1}{2^{nl}} f(2^n x_{11}, \ldots, 2^n x_{n1}) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{n(j+1)}} \varphi(2^j x_{11}, 2^j x_{12}, \ldots, 2^j x_{n1}, 2^j x_{n1}). \quad (11)$$

Therefore from (7) it follows that $(\frac{1}{2^{nl}} f(2^n x_{11}, \ldots, 2^n x_{n1}))_{j \in \mathbb{N}}$ is a Cauchy sequence. Since the space $W$ is complete, this sequence is convergent and we define $F : V^n \rightarrow W$ by (10). Putting $l = 0$, letting $m \to \infty$ in (11) and using (7) we see that (9) holds.
Next, fix also \((x_{12}, \ldots, x_{n2}) \in V^n\) and note that according to (8) we have
\[
\left\| \frac{1}{2^{n_2}} f(2^2(x_{11} + x_{12}), \ldots, 2^2(x_{n1} + x_{n2})) - \sum_{i_1, \ldots, i_n \in \{1, 2\}} \frac{1}{2^{n_2}} f(2^i x_{i1}, \ldots, 2^i x_{in}) \right\| 
\leq \frac{1}{2^{n_2}} \omega(2^2 x_{11}, 2^2 x_{12}, \ldots, 2^2 x_{n1}, 2^2 x_{n2}).
\]

Letting \(j \to \infty\) in the above inequality and using (7) we see that the mapping \(F\) satisfies Eq. (6), and Theorem 2 now shows that \(F\) is multi-additive.

Finally, assume that \(F' : V^n \to W\) is another multi-additive mapping satisfying (9) and fix \(k \in \mathbb{N} \cup \{0\}\). Then, using the multi-additivity of \(F\) and \(F', (7)\) and (9), we have
\[
\|F(x_{11}, \ldots, x_{n1}) - F'(x_{11}, \ldots, x_{n1})\| = \left\| \frac{1}{2^{nk}} F(2^k x_{11}, \ldots, 2^k x_{n1}) - \frac{1}{2^{nk}} F'(2^k x_{11}, \ldots, 2^k x_{n1}) \right\|
\leq \left\| \frac{1}{2^{nk}} F(2^k x_{11}, \ldots, 2^k x_{n1}) - \frac{1}{2^{nk}} f(2^k x_{11}, \ldots, 2^k x_{n1}) \right\|
+ \left\| \frac{1}{2^{nk}} f(2^k x_{11}, \ldots, 2^k x_{n1}) - \frac{1}{2^{nk}} F'(2^k x_{11}, \ldots, 2^k x_{n1}) \right\|
\leq \frac{2}{2^{nk}} \omega(2^k x_{11}, 2^k x_{12}, \ldots, 2^k x_{n1}, 2^k x_{n1})
\leq 2 \sum_{j=k}^{\infty} \frac{1}{2^{n(j+1)}} \omega(2^j x_{11}, 2^j x_{11}, \ldots, 2^j x_{n1}, 2^j x_{n1}),
\]
whence letting \(k \to \infty\) we obtain \(F = F'\). \(\square\)

References


