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An algorithm for feedback linearization*

Robert B. Gardner[1]
Department of Mathematics, University of North Carolina, Chapel Hill, N.C. 27599-3250, U.S.A.

William F. Shadwick[2]
Pure Mathematics Department, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

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Abstract: Previous methods for exact linearization by feedback have relied on solving Frobenius systems of partial differential equations of dimensions equal to the Kronecker indices. We will describe an algorithm whereby one may find the linearizing feedback for any controlable linearizable system having distinct Kronecker indices with p-controls by purely algebraic calculations and integration of at most p one-dimensional Frobenius systems. The paper concludes with a concrete example considered by Hunt-Su-Meyer in their paper [3].

Keywords: Exact linearization, Brunovský normal form, symmetry pseudo-group, Kronecker indices.

MS classification: 53C10, 93B27.

1. Introduction

The basic idea is to use the symmetries of a system which can be brought into a Brunovský normal form to establish a standard presentation and then use the symmetries to uncouple the various blocks. The analysis proceeds by utilizing the method of equivalence [1] to compute the structure equations of the symmetry pseudo-group of a controlable linearizable system. Off of rest points this symmetry pseudo-group is transitive on the space of states and controls and the structure equations coincide with those of the original control system. Finally in the case that the Kronecker indices are distinct each block can be put into linearized normal form by computations involving at most integration of one one-dimensional Frobenius system.

2. The Algorithm

The details of the computation and characterization of the structure equations of the

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symmetry pseudo-group of a controlable linearizable system have already been given in [2]. The structure equations become more difficult to describe if there are repeated Kronecker indices and hence in this short report we will assume that all Kronecker indices $\{\kappa_i\}$ are distinct and are ordered $\kappa_1 > \ldots > \kappa_p$. In this case we will have as part of the structure equations, those of the general group on the line prolonged (i.e. differentiated) κ_1 times. These will appear as the equations generated by one of the form

$$d\eta_1^1 = \eta_1^1 \wedge \eta_2^1$$

and all the equations one obtains by taking exterior derivatives. One derivative yields

$$d\eta_2^1 = \eta_1^1 \wedge \eta_3^1$$

and its derivative yields

$$d\eta_3^1 = \eta_1^1 \wedge \eta_4^1 + \eta_2^1 \wedge \eta_3^1,$$

and so on with all of the equations taking the form

$$d\eta_i^1 \equiv \eta_1^1 \wedge \eta_{i+1}^1 \mod \eta_2^1, \ldots, \eta_{i-1}^1.$$

The coefficients of the terms which are congruent modulo $\eta_2^1, \ldots, \eta_{i-1}^1$ are obtained by applying the identity $d^2 = 0$ to the previous line so these may all be computed by symbolic manipulation if a concrete value of κ_1 is given.

In order to obtain the linearizing coordinates one proceeds as follows. The 1-form η_1^1 satisfies $d\eta_1^1 \neq 0$, but $d\eta_1^1 \wedge \eta_1^1 = 0$, so η_1^1 is exact up to a factor. Hence we obtain linearly independent functions x_1^1, x_2^1 of the original variables such that

$$\eta_1^1 = \frac{dx_1^1}{x_2^1}.$$

We also have the structure equation

$$d\eta_1^1 = \eta_1^1 \wedge \eta_2^1$$

which tells us that $d\eta_1^1$ determines η_2^1 up to a multiple of η_1^1. In fact that multiple determines x_3^1 from

$$\eta_2^1 = \frac{dx_2^1}{x_2^1} - \frac{x_3^1}{x_2^1}\eta_1^1.$$

Likewise the remaining state variables $\{x_i^1\}$, $4 \leqslant i \leqslant \kappa_1$ corresponding to this longest Brunovský block are obtained by purely algebraic calculations.

Next we proceed to calculate the state variables corresponding to the other Brunovský blocks. The structure equations will have the form

$$d\eta_1^2 = \eta_1^1 \wedge \eta_2^2 + \beta \wedge \eta_1^2.$$

This means that modulo η_1^1, η_1^2 is exact up to a factor, thus we may solve

$$\eta_1^2 = \lambda(dx_1^2 - x_2^2\eta_1^1),$$

where x_1^2, x_2^2 are linearly independent functions of the original variables, linearly independent from $x_1^1, \dots, x_{\kappa_1}^1$. As before, we obtain all the successive state variables $x_1^2, \dots, x_{\kappa_2}^2$ by purely algebraic calculations. All of the calculations of x_i^A for $2 < A \leqslant p$ are identical with the one for x_i^2.

3. Example of the linearization algorithm for (3,2) systems

We illustrate our method for the control system

$$\frac{dx^1}{dt} = \sin x^2, \quad \frac{dx^2}{dt} = \sin x^3, \quad \frac{dx^3}{dt} = (x^4)^3 + v^1,$$
$$\frac{dx^4}{dt} = x^5 + (x^4)^3 - (x^1)^{10}, \quad \frac{dx^5}{dt} = v^2,$$

an example of Hunt-Su-Meyer [3].

I) Set up the associated exterior differential system Σ_0 on the space of states and controls as in [2]. Thus given

$$\frac{dx^1}{dt} = f^1, \quad \frac{dx^2}{dt} = f^2, \quad \frac{dx^3}{dt} = f^3, \quad \frac{dx^4}{dt} = f^4, \quad \frac{dx^5}{dt} = f^5,$$

take the matrix

$$A_0 = \begin{pmatrix} 1/f^1 & 0 & 0 & 0 & 0 \\ -f^2 & f^1 & 0 & 0 & 0 \\ -f^3 & 0 & f^1 & 0 & 0 \\ -f^4 & 0 & 0 & f^1 & 0 \\ -f^5 & 0 & 0 & 0 & f^1 \end{pmatrix}$$

and set $\eta_0 = A_0 dx$, and $\mu_0 = dv$. This results in

$$\Sigma_0 = \begin{pmatrix} \eta_0^2 \\ \eta_0^3 \\ \eta_0^4 \\ \eta_0^5 \end{pmatrix} = \begin{pmatrix} -\sin x^3 dx^1 + \sin x^2 dx^2 \\ -((x^4)^3 + v^1)dx^1 + \sin x^2 dx^3 \\ -(x^5 + (x^4)^3 - (x^1)^{10})dx^1 + \sin x^2 dx^4 \\ -v^2 dx^1 + \sin x^2 dx^5 \end{pmatrix}$$

with independence condition $\eta_0^1 = dx^1/\sin x^2$.

II) Compute the derived flag, which amounts to putting K_0, the 5×2 matrix defined by

$$d\eta_0^1 \equiv \eta_0^1 K_0 \mu_0 \mod \Sigma_0$$

into normal form. This results in

$$K_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sin x^2 & 0 \\ 0 & 0 \\ 0 & \sin x^2 \end{pmatrix} \xrightarrow{\text{normalizes to}} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by the change of generators

$$\bar{\eta}_0^3 = \eta_0^4, \qquad \bar{\eta}_0^4 = \eta_0^3, \qquad \bar{\mu}^1 = \sin x^2 \mu^1, \qquad \bar{\mu}^2 = \sin x^2 \mu^2.$$

Now modulo Σ_0

$$d\eta_0^2 \equiv 0, \qquad d\bar{\eta}_0^3 \equiv 0, \qquad d\bar{\eta}_0^4 \equiv \eta_0^1 \wedge \mu^1, \qquad d\eta_0^5 \equiv \eta_0^1 \wedge \mu^2$$

and we see the derived flag is $\Sigma_0^{(1)} = \{\eta_0^2, \bar{\eta}_0^3\}$, and $\Sigma_0^{(2)} = \{0\}$. In the inductive notation of the general theory we now set the complement of $\Sigma_0^{(1)}$ in Σ_0 equal to

$$\mu_0^{(1)} = \{\bar{\eta}_0^4, \eta_0^5\}$$

indicating that these forms act as the new controls for $\Sigma_0^{(1)}$.

III) Normalize the exterior derivatives of $\Sigma_0^{(1)}$, which amounts to putting $K_0^{(1)}$, the 2×2 matrix defined by

$$d\Sigma_0^{(1)} = \eta_0^1 K_0^{(1)} \wedge \mu_0^{(1)} \mod \Sigma_0^{(1)}$$

into normal form.

In our example

$$d\eta_0^2 \equiv (dx^1/\sin x^2)\cos x^3(\sin x^2 dx^3 - ((x^4)^3 + v^1)dx^1) = \eta_0^1 \cos x^3 \wedge \bar{\eta}_0^4,$$
$$d\bar{\eta}_0^3 \equiv (dx^1/\sin x^2) \wedge (\sin x^2 dx^5 - v^2 dx^1),$$

hence

$$K_0^{(1)} = \begin{pmatrix} \cos x^3 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{normalizes to}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

by the change of generators $\bar{\bar{\eta}}_0^4 = \cos x^3 \bar{\eta}_0^4$.

IV) Normalize $d\eta_0^1$ with respect to $\Sigma_0^{(1)}$, which amounts to putting $\kappa^{(2)}$ defined by

$$d\eta_0^1 = \eta_0^1 \kappa^{(2)} \wedge \Sigma_0^{(2)}$$

into normal form.

In our example

$$d\eta_0^1 = \frac{(\sin x^2)^2}{\cos x^2} dx^2 \wedge dx^1$$
$$= \frac{dx^1}{\sin x^2} \frac{\cos x^2}{\sin x^2}(\sin x^2 dx^2 - \sin x^3 dx^1) = \eta_0^1 \frac{\cos x^2}{\sin x^2} \eta_0^2.$$

Hence

$$\kappa^{(2)} = (\cos x^2/\sin x^2, 0) \xrightarrow{\text{normalizes to}} (1,0)$$

by the change of generators $\bar{\eta}_0^2 = \cos x^2/\sin x^2 \ \eta_0^2$ and results in

$$d\eta_0^1 = \eta_0^1 \wedge \bar{\eta}_0^2.$$

If we follow the inductive notation, we set

$$\eta_1^1 = dx^1/\sin x^2 \quad \text{and} \quad \eta_2^1 = \frac{\cos x^2}{\sin x^2}\ (\sin x^2 dx^2 - \sin x^3 dx^1),$$

and from the general theory we know there are forms $\eta_3^1, \eta_1^2, \eta_2^2, \mu^1, \mu^2$ such that

$$\begin{aligned}
d\eta_1^1 &= \eta_1^1 \wedge \eta_2^1, \\
d\eta_2^1 &= \eta_1^1 \wedge \eta_3^1, \\
d\eta_3^1 &= \eta_1^1 \wedge \mu^1 + \eta_2^1 \wedge \eta_3^1, \\
d\eta_1^2 &\equiv \eta_1^1 \wedge \eta_2^2 \mod \eta_1^2, \eta_2^1, \\
d\eta_2^2 &\equiv \eta_1^1 \wedge \mu^2 \mod \eta_2^2, \eta_3^1, \eta_1^2, \eta_2^1.
\end{aligned}$$

V) Compute representative forms for the eta's satisfying the previous structure equations. We already have η_1^1 and η_2^1. Next

$$d\eta_2^1 = -d\left(\frac{\cos x^2}{(\sin x^2)^2} \sin x^3\right) \wedge dx^1$$
$$= dx^1/(\sin x^2) \wedge d\left(\frac{\cos x^2}{(\sin x^2)^2} \sin x^3\right).$$

Hence we may take

$$\eta_3^1 = d\left(\frac{\cos x^2}{(\sin x^2)^2} \sin x^3\right).$$

Similarly we may take

$$\eta_1^2 = -(x^5 + (x^4)^3 - (x^1)^{10}) dx^1 + \sin x^2 dx^3,$$
$$\eta_2^2 = -v^2 dx^1 + \sin x^2 dx^5.$$

VI) Use the coordinate algorithm and solve for $x_1^1, x_2^1, x_3^1, u^1, x_1^2, x_2^2, u^2$ where

$$\begin{aligned}
\eta_1^1 &= dx_1^1/x_2^1, \\
\eta_2^1 &= dx_1^1/x_1^2 - (x_3^1/x_1^2)\eta_1^1, \\
\eta_3^1 &= dx_3^1/x_2^1 - 2(x_3^1/x_2^1)\eta_2^1 - (u^1/x_2^1)\eta_1^1, \\
\eta_1^2 &\equiv \lambda(dx_1^2 - x_2^2\eta_1^1) \mod \eta_1^2, \eta_2^1, \\
\eta_2^2 &\equiv \zeta(dx_2^2 - u^2\eta_1^1) \mod \eta_2^2, \eta_3^1, \eta_1^2, \eta_2^1.
\end{aligned}$$

Since $\eta_1^1 = dx^1/\sin x^2$ we may take $x_1^1 = x^1$ and $x_2^1 = \sin x^2$. We may rewrite η_2^1 in the form

$$\eta_2^1 = d(\sin x^2)/\sin x^2 - \frac{\sin x^3 \cos x^2}{\sin x^2} dx^1/\sin x^2$$

and hence read off $x_3^1 = \sin x^3 \cos x^2$. The form η_3^1 can be rewritten

$$\begin{aligned}
\eta_3^1 = &\frac{1}{\sin x^2}(d(\cos x^2 \sin x^3) - (v^1 + (x^4)^3)\cos x^2 \cos x^3 \\
&- \sin^2\ x^3 \sin x^2))\eta_1^1 - 2\cos x^2 \sin x^3 \eta_2^1
\end{aligned}$$

from which we see

$$u^1 = \cos x^3 \sin x^2 (v^1 + (x^4)^3)\cos x^2 \cos x^3 - \sin^2\ x^3 \sin x^2.$$

Since

$$\begin{aligned}
\eta_1^2 &= -(x^5 + (x^4)^3 - (x^1)^{10})dx^1 + \sin x^2 dx^4 \\
&= \sin x^2(dx^4 - (x^5 + (x^4)^3 - (x^1)^{10})\eta_1^1,
\end{aligned}$$

we do not have to use the congruence and may take $x_1^2 = x^4$, and $x_2^2 = x^5 + (x^4)^3 - (x^1)^{10}$. Similarly

$$\begin{aligned}
\eta_2^2 &= -v^2 dx^1 + \sin x^2 dx^5 \\
&= \sin x^2 d(x^5 + (x^4)^3 - (x^1)^{10}) - 3(x^4)^2 dx^4 + 10(x^1)^9 dx^1 - v^2 dx^1,
\end{aligned}$$

and using the congruence to replace $\sin x^2 dx^4$ by $(x^5 + (x^4)^3 - (x^1)^{10}$ we see that we may take

$$u^2 = v^2 + 3(x^4)^2 dx^4 + 10(x^1)^9 - 10\sin x^2 (x^1)^9.$$

This completes the construction of the linearizing coordinate transformation.

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