Wavelet moment method for the Cauchy problem for the Helmholtz equation

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Received 2 August 2007

Abstract

The paper is concerned with the problem of reconstruction of acoustic or electromagnetic field from inexact data given on an open part of the boundary of a given domain. A regularization concept is presented for the moment problem that is equivalent to a Cauchy problem for the Helmholtz equation. A method of regularization by projection with application of the Meyer wavelet subspaces is introduced and analyzed. The derived formula, describing the projection level in terms of the error bound of the inexact Cauchy data, allows us to prove the convergence and stability of the method.

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MSC: 65M30; 44A60; 65J20; 35J05

Keywords: Cauchy problem; Helmholtz equation; Moment problem; Ill-posed problem; Regularization; Meyer wavelets; Wavelet projection

1. Introduction

Let \( \Omega \) be a simply connected domain in \( \mathbb{R}^d, d = 2, 3 \) with a sufficiently regular boundary \( \partial \Omega \) and, moreover, let \( \Gamma \subset \partial \Omega \) be an open and connected part of the boundary. We consider the problem of the reconstruction of an acoustic or electromagnetic field from inexact data given on \( \Gamma \). Let \( u \) denote a certain component of the considered field. Let us assume further that the field is harmonic with the constant wave number \( k \). In this case, the scalar function \( u \) satisfies in \( \Omega \) the Helmholtz equation

\[
Lu := \Delta u + k^2 u = 0 \quad \text{on } \Omega.
\]  

With respect to applications, we have some freedom in our choice of the domain \( \Omega \): namely, only the part \( \Gamma \) of the boundary \( \partial \Omega \) is given a priori (indicated by measurement possibilities), and in particular we may assume that \( \partial \Omega \in C^{1+\epsilon} \). It means that in the neighborhood of every point \( x \in \partial \Omega \), there exists a normal parametric representation \( \sigma \) and an increasing function \( \epsilon \in C(R_+) \) with \( \int_{R_+} \epsilon(r) \frac{dr}{r} < \infty \) such that for any pair \((\tilde{x}, \hat{x})\) in the neighborhood of \( x \), where the parametric representation is given, \( |\nabla \sigma(\tilde{x}) - \nabla \sigma(\hat{x})| \leq \epsilon(|\tilde{x} - \hat{x}|) \). One can see that Lyapunov surfaces (curves) are in \( C^{1+\epsilon} \).

* The work was supported by EC FP6 MC ToK programmes TODEQ, MTKD-CT-2005-030042 and SPADE2, MTKD-CT-2004-014508.

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Direct problems connected with this equation are typically defined by Dirichlet or Neumann conditions on $\partial \Omega$, or by Dirichlet conditions on one part of boundary (i.e. $\Gamma$) and Neumann conditions on the second one. The inverse problem considered in this paper consists in solving the equation (1) under the both Dirichlet and Neumann conditions posed on the same part $\Gamma$ of the boundary $\partial \Omega$. That means that we deal with the Cauchy problem for the Helmholtz equation

$$
\begin{align*}
Lu &:= \Delta u + k^2 u = 0, & \text{on } \Omega \\
\frac{\partial u}{\partial v} &:= g & \text{on } \Gamma, \tag{2}
\end{align*}
$$

where $\nu$ is the outer unit normal vector to $\partial \Omega$. In all parts of the paper we assume, that $k^2$ is not an eigenvalue of the Neumann problem for $-\Delta$, i.e. that $\nu \equiv 0$ is the unique solution to the following homogeneous boundary-value problem $\Delta v + k^2 v = 0$ in $\Omega$, $\frac{\partial v}{\partial \nu} = 0$ on $\partial \Omega$. We assume that $f \in H^1(\Gamma)$ and $g \in L^2(\Gamma)$ are such that there exists the unique solution $u \in H^{3/2}(\Omega)$. It is known that the Cauchy problem for elliptic equations is ill-posed, which means that the solutions do not depend continuously on the Cauchy data, see e.g. [11,8,10]. This implies serious numerical difficulties in the solving of these problems, especially in the case of perturbed data. However, this case is important from the point of view of real applications for acoustic and electromagnetic fields (cf. [13,10,7,1,15]) where the exact Cauchy data are approximated by their measurements.

For a stable solving of ill-posed problems, regularization techniques are required (cf. [9,16]). Numerical analysis of the Cauchy problem for the Laplace equation is a topic of several papers where different regularization methods were proposed [2,14,3]. Unfortunately, their application to the Helmholtz equation requires some modifications and additional analysis because of essential differences between these two problems.

In this paper is developed the idea of a numerical method based on a transformation of the Cauchy problem to a generalized moment problem: find $\varphi \in L^2(\partial \Omega \setminus \Gamma)$ such that

$$
\int_{\partial \Omega \setminus \Gamma} \varphi \nu d\sigma = \mu(v) \quad \forall v \in V(\Omega), \tag{3}
$$

where $V(\Omega)$ is a certain subspace of $L^2(\Omega)$ and $\mu$ a linear functional on $V(\Omega)$ which will be defined later. This idea was proposed by Cheng et al. in [3] for the Cauchy problem for the Laplace equation.

The paper is organized as follows. In Section 2.1, the equivalence between the Cauchy problem (2) and a moment problem on the boundary $\partial \Omega \setminus \Gamma$ is proved according to an idea in [3]. The rest of the paper is devoted to a regularization method for solving the obtained moment problem in the particular case of the two-dimensional domain $\Omega$. In Section 2.2, a characterization of a dense subspace of the space is given. In conclusion, in Section 2.3, Meyer wavelet projections are chosen for a convergent approximation of the solution in the case of the exact data. In Section 4, the stability of the method is considered with respect to perturbations of the boundary value functions. Finally, the regularization property of the defined wavelet-projection method for the moment problem is established and, as consequence, a stable approximation of a solution to the Cauchy problem (2) is obtained.

We shall frequently write $a \lesssim b$ when $a$ is bounded by $b$ multiplied by a positive constant uniformly with respect to all parameters on which $a$ and $b$ depend. Then $a \sim b$ means $a \lesssim b$ and $b \lesssim a$.

2. Generalized moment problem

2.1. General case $\Omega \subset R^3$

Let $v$ be an arbitrary $H^1(\Omega)$ weak solution to the equations

$$
\begin{align*}
Lv &:= \Delta u + k^2 u = 0, & \text{on } \Omega \\
\frac{\partial u}{\partial v} &:= g & \text{on } \partial \Omega \setminus \Gamma. \tag{4}
\end{align*}
$$

Applying Green’s formula

$$
\int_{\Omega} [vLu - uLv] dx = \int_{\partial \Omega} \left[ v \frac{\partial u}{\partial v} - u \frac{\partial v}{\partial v} \right] d\sigma, \tag{5}
$$

where $\sigma$ is the Lebesgue measure on $\partial \Omega$, we get for the solution $u$ of (2) and the solution $v$ of (4)
\[
\int_{\partial \Omega \setminus \Gamma} v \frac{\partial u}{\partial v} d\sigma + \int_{\Gamma} v g d\sigma - \int_{\Gamma} f \frac{\partial v}{\partial v} d\sigma = 0. \tag{6}
\]

Let the test space $V(\Omega)$ be defined as follows:
\[
V(\Omega) := \left\{ v \in H^{3/2}(\Omega) : L v = 0 \text{ on } \Omega \text{ and } \frac{\partial v}{\partial v} = 0 \text{ on } \partial \Omega \setminus \Gamma \right\}. \tag{7}
\]

**Corollary 2.1.** If there exists a solution $u$ of (2) such that
\[
\varphi := \left\{ \frac{\partial u}{\partial v} \right\}_{\partial \Omega \setminus \Gamma} \in L^2(\partial \Omega \setminus \Gamma) \tag{8}
\]
then $\varphi$ is the solution to the following moment problem
\[
\int_{\partial \Omega \setminus \Gamma} \varphi v d\sigma = \int_{\Gamma} \left[ f \frac{\partial v}{\partial v} - g v \right] d\sigma \quad \forall v \in V(\Omega). \tag{9}
\]

Following Cheng et al.’s reasoning [3] we prove

**Theorem 2.2.** Let $g \in L^2(\Gamma)$ and $f \in H^{1/2}(\Gamma)$ and let $k^2$ not be an eigenvalue of the Neumann problem for $-\Delta$. If $\varphi \in L^2(\partial \Omega \setminus \Gamma)$ is a solution to the moment problem (9) and $\partial \Omega \in C^{1+\epsilon}$, then there exists a solution $u$ to the Cauchy problem (2) such that $\frac{\partial u}{\partial v} \in L^2(\partial \Omega \setminus \Gamma)$ and $\frac{\partial u}{\partial v} = \varphi$.

**Proof.** Let us consider the following Neumann problem
\[
\begin{aligned}
\Delta \alpha + k^2 \alpha &= 0, & \text{on } \Omega \\
\frac{\partial \alpha}{\partial v} &= \varphi & \text{on } \partial \Omega \setminus \Gamma, \\
\frac{\partial \alpha}{\partial v} &= g & \text{on } \Gamma.
\end{aligned} \tag{10}
\]

It is known (cf. [6], Section 3 of Chapter XI) that for $\partial \Omega \in C^{1+\epsilon}$, the Neumann problem (10) for $\varphi \in H^{-1/2}(\partial \Omega \setminus \Gamma)$, $g \in H^{-1}(\Gamma)$ admits a unique solution in $H^1(\Omega)$ under Theorem 2.2 assumptions. Classical regularity arguments imply\(^1\), i.e. that there exists a unique $\alpha$ in $H^2(\Omega)$ for $\varphi \in L^2(\partial \Omega \setminus \Gamma)$, $g \in L^2(\Gamma)$.

We are going to prove that
\[
\int_{\Gamma} (\alpha - f)^2 d\sigma = 0. \tag{11}
\]

From (9), it follows that for any $v \in V(\Omega)$
\[
\int_{\partial \Omega \setminus \Gamma} \frac{\partial \alpha}{\partial v} v d\sigma = \int_{\Gamma} \left[ f \frac{\partial v}{\partial v} - \frac{\partial \alpha}{\partial v} \right] d\sigma,
\]
\[
\int_{\partial \Omega} \frac{\partial \alpha}{\partial v} d\sigma = \int_{\Gamma} f \frac{\partial v}{\partial v} d\sigma. \tag{12}
\]

On the other hand, Green’s formula gives
\[
0 = \int_{\Omega} [vL\alpha - \alpha L v] dx = \int_{\partial \Omega} \left[ v \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial v}{\partial v} \right] d\sigma.
\]

\(^1\) A similar result can be obtained for $\partial \Omega$ being Lipschitz continuous under some additional assumptions on the geometry of $\Omega$, (cf. [12]).
which implies
\[ \int_{\partial \Omega} \frac{\partial \alpha}{\partial v} d\sigma = \int_{\partial \Omega} \alpha \frac{\partial v}{\partial v} d\sigma. \]
The formula above and (12) yields
\[ \int_{\Gamma} (\alpha - f) \frac{\partial v}{\partial v} d\sigma = 0. \] (13)
Now, let us take a special element \( \tilde{\nu} \) of \( V(\Omega) \) defined as follows
\[
\begin{aligned}
\Delta \tilde{\nu} + k^2 \tilde{\nu} &= 0, & \text{in } \Omega \\
\frac{\partial \tilde{\nu}}{\partial v} &= 0 & \text{in } \partial \Omega \setminus \Gamma' , \\
\frac{\partial \tilde{\nu}}{\partial v} &= \alpha - f & \text{in } \Gamma ,
\end{aligned}
\] (14)
Since \((\alpha - f) \in L^2(\Gamma')\), we have by the same arguments as used for (10), that there exists a unique solution \( \tilde{\nu} \) to (14) belonging to \( H^{3/2}(\Omega) \). Thus, by (13)
\[ \int_{\Gamma} (\alpha - f)^2 d\sigma = 0, \]
i.e \( \alpha = f \) almost everywhere on \( \Gamma \). It means that \( \alpha \) is a solution to the Cauchy problem (2). \( \square \)

Due to Corollary 2.1 and Theorem 2.2, the problem (2) can be equivalently formulated as the following moment problem
\[ \int_{\partial \Omega \setminus \Gamma'} \varphi v d\sigma = \mu(v) \quad \forall v \in V(\Omega). \] (15)
where
\[ \mu(v) = \int_{\Gamma} \left[ f \frac{\partial v}{\partial v} - g v \right] d\sigma, \] (16)
which has at most one solution. In the Eq. (15) we can replace the space \( V \) by any dense subset of \( V \), for instance by a dense sequence.

2.2. Model problem in \( R^2 \)

Now, our consideration is restricted to the case \( \Omega \subset R^2 \). Let us assume that
\[ \partial \Omega \setminus \Gamma' = \{(x_1, x_2) : x_1 \in [0, 1], x_2 = 0\} \] (17)
and \( \Gamma' \subset R \times R^+ \) is such that \( \partial \Omega \in C^{1+\epsilon} \). Let
\[ \gamma := \max\{x_2 : (x_1, x_2) \in \Gamma' \}. \] (18)

**Theorem 2.3.** Let \( U \) denote the set of functions
\[ U := \left\{ \beta \in L^2(R) : \hat{\beta}(\xi) \xi^2 \cosh(\gamma \sqrt{\xi^2 - k^2}) \in L^2(R) \right\} \] (19)
where \( \hat{\beta} \) denotes the Fourier transform of \( \beta \), and \( \forall \beta \in U \) let
\[ w_\beta(x_1, x_2) := \frac{1}{\sqrt{2\pi}} \int_R \hat{\beta}(\xi) \cosh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} d\xi. \] (20)
The set of functions \( \{w_\beta : \beta \in U\} \) is dense in \( V(\Omega) \).
Proof. Let \( \Omega_1 := R \times (0, \gamma) \) and
\[
W := \left\{ w \in H^2(\Omega_1) : Lw = 0 \text{ on } \Omega_1 \text{ and } \frac{\partial w}{\partial x_2}(x, 0) = 0 \text{ for } x \in R \right\}. \tag{21}
\]
If \( w \in W \), then \( w|_\Omega \in V(\Omega) \). Moreover, the set \( W(\Omega) := \{ w|_\Omega : w \in W \} \) is dense in \( V(\Omega) \). So, in the formulation of the moment problem (15), the space \( V(\Omega) \) can be replaced by \( W(\Omega) \).

Now, we are going to prove that \( w \) belongs to \( W \) if and only if \( \exists \beta \in U \) such that \( \forall (x_1, x_2) \in \Omega_1 \ w(x_1, x_2) = w_\beta(x_1, x_2) \).

Applying a Fourier transform to \( w \in W \) with respect to the variable \( x_1 \in R \) we easily see that \( \hat{w}(\xi, x_2) \) is a solution to the following problem:
\[
\begin{aligned}
\frac{\partial^2 \hat{w}}{\partial x_1^2}(\xi, x_2) &= (\xi^2 - k^2) \hat{w}(\xi, x_2) \quad \text{for } \xi \in R, \ x_2 \in (0, \gamma), \\
\frac{\partial \hat{w}}{\partial x_2}(\xi, 0) &= 0 \quad \text{for } \xi \in R.
\end{aligned} \tag{22}
\]
The general solution of this problem has the form
\[
\hat{w}(\xi, x_2) = h(\xi) \cosh(x_2 \sqrt{\xi^2 - k^2}).
\]
If \( w \in W \), then
\[
\frac{\partial^2 w}{\partial x_1^2}(\cdot, x_2) = -\xi^2 \hat{w}(\cdot, x_2) \in L^2(R),
\]
and thus, the function \( h \) appearing in the formula above has to be such that \( \xi^2 \hat{w}(\xi, x_2) \) belongs to \( L^2(R) \) as a function of \( \xi \), i.e. \( \tilde{w}_{x_2} \) and \( \hat{w}_{x_2x_2} \in L^2(R) \). So, \( h = \tilde{\beta} \) where \( \beta \in U \) and \( w \) is the inverse Fourier transform of \( \hat{\beta} \cosh(x_2 \sqrt{\xi^2 - k^2}) \). Inversely, if \( \hat{w} \) is given by (20), where \( h = \tilde{\beta} \) and \( \beta \in U \), then \( \hat{w} \) satisfies the equations (22).
Since \( \xi^2 \hat{w}(\xi, x_2) \in L^2(R) \), \( w(\cdot, x_2) \in H^2(R) \) and \( Lw = 0 \) on \( R \times (0, \gamma) \), \( \frac{\partial w}{\partial x_2}(x, 0) = 0 \) for \( x \in R \). This ends the proof. \( \square \)

Remark 2.4. If \( w(x_1, x_2) = w_\beta(x_1, x_2) \), then \( w(x_1, 0) = \beta(x_1) \) for almost all \( x_1 \in R \). Thus the moment problem (15) can be now formulated as follows: find \( \varphi \in L^2(0, 1) \) such that
\[
\int_0^1 \varphi(x) \beta(x) \, dx = \eta(\beta), \quad \forall \beta \in U, \tag{23}
\]
where
\[
\eta(\beta) := \mu(w_\beta) = \int_I \left[ \int \frac{\partial w_\beta}{\partial \nu} - gw_\beta \right] \, d\sigma \tag{24}
\]
and \( w_\beta \) is given by (20).

2.3. Moment problem with Meyer wavelets

The Meyer wavelet \( \psi \) is a function from \( C^\infty(R) \) defined by its Fourier transform as follows:
\[
\hat{\psi}(\xi) = e^{i\xi b(\xi)},
\]
where
\[
b(\xi) = \begin{cases}
\frac{1}{\sqrt{2\pi}} \sin \left[ \frac{\pi}{2} \left( \frac{3}{2\pi} |\xi| - 1 \right) \right], & \text{for } \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3}, \\
\frac{1}{\sqrt{2\pi}} \cos \left[ \frac{\pi}{2} \left( \frac{3}{4\pi} |\xi| - 1 \right) \right], & \text{for } \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}, \\
0, & \text{otherwise}
\end{cases}
\]
and the function \( \zeta \) equals 0 if \( x \leq 0 \), equals 1 if \( x \geq 1 \), and \( \zeta(x) \) equals 1 if \( x = 0 \). Then

\[
\psi_{jl}(x) := 2^j \psi(2^j x - l) \quad j, l \in \mathbb{Z}
\]

form the orthonormal basis of \( L^2(R) \) (cf. [5]). We have

\[
\hat{\psi}_{jl} = 2^{j/2} e^{-i \xi 2^{-j}} \hat{\psi}(2^{-j} \xi).
\]

and

\[
\supp(\hat{\psi}_{jl}) = \left\{ \xi; \frac{2}{3} \pi 2^j \leq |\xi| \leq \frac{8}{3} \pi 2^j \right\},
\]

for any \( j, l \in \mathbb{Z} \). Thus \( \psi_{jl} \in U \). Moreover, the set \( \{\psi_{j,k}\}_{j,k \in \mathbb{Z}} \) is dense in \( U \), because it is dense in \( L^2(R) \).

**Remark 2.5.** From the above conclusion and from Theorem 2.3, the moment problem (15), (16) can be equivalently formulated as follows: find \( \varphi \in L^2(0, 1) \) such that

\[
\int_0^1 \varphi(x) \psi_{j,k}(x) dx = \eta_{j,k}, \quad \forall j, k \in \mathbb{Z},
\]

where

\[
\eta_{j,k} := \eta(\psi_{j,k}),
\]

and \( \eta(\psi_{j,k}) \) is given by (24), (20).

### 2.4. Approximation for the exact data

Let

\[
\tilde{\varphi}(x) := \begin{cases} \varphi(x) & x \in (0, 1), \\ 0 & x \in R \setminus (0, 1) \end{cases}
\]

where \( \varphi \) is the exact solution of (27). Since \( \tilde{\varphi} \in L^2(R) \) and \( \{\psi_{j,k}\}_{j,k \in \mathbb{Z}} \) is the basis of \( L^2(R) \), from (27) it follows that \( \tilde{\varphi} \) has the following wavelet representation

\[
\tilde{\varphi}(x) = \sum_{j,k \in \mathbb{Z}} \eta_{j,k} \psi_{j,k}.
\]

**Definition 2.6.** As an approximate solution, let us take the orthogonal projection of \( \tilde{\varphi} \) onto \( V_J = \text{span} \{\psi_{j,k}\}_{j < J, k \in \mathbb{Z}} \), i.e.:

\[
\varphi_J(x) = \sum_{j < J} \sum_{k \in \mathbb{Z}} \eta_{j,k} \psi_{j,k}(x).
\]

It is clear that

\[
\|\varphi_J - \varphi\|_{L^2(0, 1)} \leq \|\varphi_J - \tilde{\varphi}\|_{L^2(R)} \longrightarrow 0 \quad \text{as} \quad J \longrightarrow \infty.
\]

A rate of convergence can be obtained under an additional assumption on the smoothness. Let \( B^s_{p,q} \), for \( p, q \geq 1, 0 < s < 1 \) denote the Besov space, i.e.

\[
B^s_{p,q}(R) = \{ f \in L^p(R) : \{2^j \omega_{n,p}(f, 2^{-j})\}_{j \geq 0} \in l_q \},
\]

where \( n > s \) and

\[
\omega_{n,p}(f, t) = \sup_{|\xi| \leq t} \|\Delta^r_{\xi} f\|_{L^p(R)}
\]
where \( \Delta_h f(x) := f(x + h) - f(x) \) is the \( n \)-th order \( L^p \) modulus of smoothness of \( f \) (cf. [4]). A natural norm is then given by

\[
\|f\|_{B_p^q} := \|f\|_{L^p} + |f|_{B_p^q},
\]

(34)

where

\[
|f|_{B_p^q} := \|2^{sj}\omega_{n,p}(f,2^{-j})\|_{L^q}.
\]

(35)

For \( q < \infty \), the Besov semi-norm has equivalent integral form

\[
|f|_{B_p^q} = \left( \int (t^{-s}\omega_{n,p}(f,t))^{q} \frac{dt}{t} \right)^{1/q} .
\]

(36)

**Lemma 2.7.** If \( \varphi \in H^1(0,1) \), then \( \varphi \in B_{2,q}^s \) for \( s \in (0, \frac{1}{2}) \) and \( 0 < q < \infty \).

**Proof.** Let us observe that if \( \varphi \in H^1(0,1) \), then for \( \Delta_h \varphi(x) = \varphi(x + h) - \varphi(x) \) and \( h \in R \), we have

\[
\|\Delta_h \varphi\|_{L^2(R)} = \begin{cases} \sqrt{|h|}C_\varphi & \text{when } |h| < 1, \\ \sqrt{3}\|\varphi\|_{L^2(0,1)} & \text{when } |h| \geq 1, \end{cases}
\]

(37)

where

\[
0 < C_\varphi \leq \sqrt{3}(\varphi^2(0) + \varphi^2(1) + \|\varphi\|^2_{L^2(0,1)})^{\frac{1}{2}}.
\]

Since for \( |h| \geq 1 \) it is obvious, and for \( h < 0 \) \( \|\Delta_h \varphi\|_{L^2(R)} = \|\Delta|\varphi\|_{L^2(R)} \), we have to consider the case \( h \in (0,1) \) only. In this case

\[
\Delta_h \varphi(x) = \begin{cases} \int_{x-h}^{x+h} \varphi'(\xi)d\xi + \varphi(0) & \text{for } x \in (-h,0), \\ \int_{x}^{x+h} \varphi'(\xi)d\xi & \text{for } x \in (0,1-h), \\ \int_{x}^{1} \varphi'(\xi)d\xi - \varphi(1) & \text{for } x \in (1-h,1), \end{cases}
\]

and (37) follows from the following inequalities

\[
\int_{-h}^{0} \left[ \int_{0}^{x+h} \varphi'(\xi)d\xi \right]^2 dx \leq \|\varphi'\|^2_{L^2(0,h)} \int_{-h}^{0} (x+h)dx = \frac{h^2}{2} \|\varphi'\|^2_{L^2(0,h)},
\]

\[
\int_{0}^{1-h} \left[ \int_{x}^{x+h} \varphi'(\xi)d\xi \right]^2 dx \leq h \int_{0}^{1-h} \int_{x}^{x+h} (\varphi'(\xi))^2 d\xi dx = h^2 \|\varphi'\|^2_{L^2(0,1)},
\]

\[
\int_{1-h}^{1} \left[ \int_{x}^{x+h} \varphi'(\xi)d\xi \right]^2 dx \leq \int_{1-h}^{1} (1-x) \int_{x}^{1} (\varphi'(\xi))^2 d\xi dx \leq \frac{h^2}{2} \|\varphi'\|^2_{L^2(1-h,1)}.
\]

Using the definition (36) and denoting \( B_\varphi := \sqrt{3}\|\varphi\|_{L^2(0,1)} \), we get

\[
|\varphi|_{B_{2,q}^s} = \int_{0}^{\infty} \left[ \sup_{|h| \leq t} \|\Delta_h \varphi\|_{L}^2 \right]^q \frac{dt}{t}
\]

\[
= C_\varphi \int_{0}^{1} (t^{-s+\frac{1}{2}})^q \frac{dt}{t} + B_\varphi \int_{1}^{\infty} t^{-s-q-1}dt.
\]

These integrals are convergent for \( s \in (0, \frac{1}{2}) \) and \( q > 0 \), which ends the proof. 

\[ \square \]
Remark 2.8. If $\varphi \in H^r(0, 1)$, $r \in (1, \infty)$, then by Lemma 2.7 $\widetilde{\varphi} \in B^{2}_{2,q}(R)$ for $s \in (0, 1)$, $q \in (0, \infty)$. This result cannot be improved, since for $\chi$ being the characteristic function of the interval $(0, 1)$

$$|\chi|_{B^{2}_{2,q}}^q = \int_{R} \left[ t^{-s} \sup_{|t| \leq t} \|\Delta_h^r \chi\|_{L^2(R)} \right]^q \frac{dt}{t} \geq \int_{0}^{1} \left[ t^{-s} \sup_{0 < h < t} \|\Delta_h^r \chi\|_{L^2((-r h, (1-r)h))} \right]^q \frac{dt}{t} = \int_{0}^{1} t^{s(1-2q)} \frac{dt}{t}. $$

Lemma 2.9. If $\varphi \in H^1(0, 1)$, then for $s \in (0, \frac{1}{2})$

$$\|\varphi_J - \widetilde{\varphi}\|_{L^2(R)} \lesssim 2^{-sJ}. $$

Proof. Since the Meyer scaling function $\phi$ as well as the Meyer wavelet $\psi$ satisfy the assumptions $|\phi(x)|, |\phi'(x)|, |\psi(x)| \leq C (1 + |x|)^{-n}$ for $n > 3$, Theorem 10.9 from [17] holds for $0 < s < 1$, $1 \leq p, q \leq \infty$. Taking into account that $\widetilde{\varphi} \in B^{2}_{2,q}$ (Lemma 2.7), from this theorem it follows that

$$\left( \sum_{j=0}^{\infty} \left[ 2^{js} \|\varphi - \varphi_j\|_{L^2(R)} \right]^q \right)^{\frac{1}{q}} < \infty $$

and (38).

3. Perturbed data

Let $f_\delta$ and $g_\delta$ be perturbed boundary value functions on $\Gamma$ such that

$$\|f_\delta - f\|_{L^2(\Gamma)} + \|g_\delta - g\|_{L^2(\Gamma)} \leq \delta < 1. $$

Let

$$\eta_{j,k}^\delta := \int_{\Gamma} \left[ f_\delta \frac{\partial w_{j,k}}{\partial v} - g_\delta w_{j,k} \right] d\sigma, $$

where $w_{j,k} := w_{\psi_{j,k}}$, cf. (24), (20). The approximate solution for the perturbed data can be defined as follows:

$$\varphi_J^\delta(x) := \sum_{j < J} \sum_{k \in \mathbb{Z}} \eta_{j,k}^\delta \psi_{j,k}(x). $$

We are going to show that for any $\delta$, it is possible to choose a positive integer $J = J(\delta)$ in such a way that $\|\varphi - \varphi_J^\delta\|_{L^2(0,1)}$ tends to $0$ as $\delta \to 0$.

The first step is to show that for fixed $J$, the approximate solution is stable with respect to perturbations of $f$ and $g$ and to derive an error bound. We have

$$\|\varphi_J - \varphi_J^\delta\|_{L^2(\Gamma)} = \sum_{j < J} \sum_{k \in \mathbb{Z}} |\eta_{j,k} - \eta_{j,k}^\delta|^2. $$

Let us introduce two auxiliary functions

$$Q_1^\delta(\xi) := \int_{\Gamma} (f - f_\delta) \frac{\partial}{\partial v} \left( \cosh(x_2 \sqrt{\xi^2 - k^2})e^{i\xi x_1} \right) d\sigma, $$

$$Q_2^\delta(\xi) := \int_{\Gamma} (g - g_\delta) \cosh(x_2 \sqrt{\xi^2 - k^2})e^{i\xi x_1} d\sigma. $$
Lemma 3.1.

\[ \| \varphi - \varphi^j \|^2_{L^2(R)} \leq \sum_{j = -\infty}^{j-1} 2^j \sup_{|\xi| \in (\frac{\pi}{2}, \frac{3\pi}{2})} \left| Q_1^j(2^j \xi) - Q_2^j(2^j \xi) \right|^2. \]

**Proof.** Applying definitions of \( \eta_{j,k} \) and \( \eta_{j,k}^j \) (cf. Remark 2.5 and (41)), and the formula (20) we can write (43) in the terms of functions \( Q_i^j, i = 1, 2 \):

\[ \| \varphi - \varphi^j \|^2_{L^2(R)} = \sum_{j = -\infty}^{j-1} 2^j \sum_{l \in \mathbb{Z}} \left| \frac{1}{\sqrt{2\pi}} \int_R \hat{\psi}_{j,l}(\xi) (Q_1^j(\xi) - Q_2^j(\xi)) d\xi \right|^2. \]

Taking into account the compact support of \( \hat{\psi} \) and the formula (25), by change the variable \( \zeta = 2^{-j} \xi \) we get

\[ \| \varphi - \varphi^j \|^2_{L^2(R)} = \sum_{j = -\infty}^{j-1} 2^j \sum_{l \in \mathbb{Z}} \left| \frac{1}{\sqrt{2\pi}} \int_{|\xi| \in (\frac{\pi}{2}, \frac{3\pi}{2})} \hat{\psi}(\xi) (Q_1^j(\xi) - Q_2^j(\xi)) e^{-i\xi l} d\xi \right|^2. \]

The integral under the sum is the sum of two integrals: one is over the interval \((-\frac{\pi}{2}, -\frac{\pi}{2})\), and the second one over \((\frac{\pi}{2}, \frac{3\pi}{2})\). So, each of two integrals can be considered as the \( l \)-th Fourier coefficient of the function \( \hat{\psi}(\xi) (Q_1^j(2^j \xi) - Q_2^j(2^j \xi)) \) with respect to orthogonal set \( \{e^{-i\xi l}\} \) of 2\( \pi \) periodic functions in \( L^2(-\frac{\pi}{2}, -\frac{\pi}{2}) \) and \( L^2(\frac{\pi}{2}, \frac{3\pi}{2}) \), respectively. Thus

\[ \| \varphi - \varphi^j \|^2_{L^2(R)} \leq \sum_{j = -\infty}^{j-1} 2^j \| \hat{\psi}(\cdot) (Q_1^j(2^j \cdot) - Q_2^j(2^j \cdot)) \|^2_{L^2(R)}. \]

But

\[ \| \hat{\psi}(\cdot) (Q_1^j(2^j \cdot) - Q_2^j(2^j \cdot)) \|_{L^2(R)} \leq \| \hat{\psi} \|_{L^2(R)} \sup_{|\xi| \in (\frac{\pi}{2}, \frac{3\pi}{2})} \left| Q_1^j(2^j \xi) - Q_2^j(2^j \xi) \right| \]

and \( \| \hat{\psi} \|_{L^2(R)} = 1 \), which ends the proof. \( \Box \)

Let \( \eta > 0 \) be the wave number in the Helmholtz equation (2). Let

\[ c_j := k \frac{3}{8\pi} 2^{-j}; \] \hspace{1cm} (46)

then \( c_j < 1 \) if and only if \( k < \frac{8}{3} 2^j \). Let

\[ j_0 := j_0(k) = \max\{j : c_j \geq 1\}. \] \hspace{1cm} (47)

**Lemma 3.2.** If \( |\xi| \in (\frac{\pi}{2}, \frac{3\pi}{2}) \) and the error bound (40) holds, then

\[ |Q_1^j(2^j \xi)| \leq \begin{cases} \frac{\sqrt{|T|}k}{\sqrt{|T|}} \left( \frac{k}{c_j} \sqrt{2 - c_j^2} \cosh \left( \frac{k}{c_j} \sqrt{1 - c_j^2} \right) \right) & \text{for } j \leq j_0, \\
\frac{\sqrt{|T|}}{\sqrt{|T|}} \left( \frac{k}{c_j} \sqrt{2 - c_j^2} \cosh \left( \frac{k}{c_j} \sqrt{1 - c_j^2} \right) \right) & \text{for } j > j_0 \end{cases} \]

\[ |Q_2^j(2^j \xi)| \leq \begin{cases} \frac{\sqrt{|T|}}{\sqrt{|T|}} \left( \frac{k}{c_j} \sqrt{2 - c_j^2} \cosh \left( \frac{k}{c_j} \sqrt{1 - c_j^2} \right) \right) & \text{for } j \leq j_0, \\
\frac{\sqrt{|T|}}{\sqrt{|T|}} \left( \frac{k}{c_j} \sqrt{2 - c_j^2} \cosh \left( \frac{k}{c_j} \sqrt{1 - c_j^2} \right) \right) & \text{for } j > j_0, \end{cases} \]

where \( \gamma \) and \( c_j \) are given by (18) and (46), respectively, and \( |T| \) denotes length of the curve \( \Gamma \).

**Proof.** From (44) it follows that

\[ |Q_1^j(\xi)| \leq \| f - f_3 \|_{L^2(|T|)} \sqrt{|T|} q_1(\xi). \] \hspace{1cm} (50)
where \( q_1^2(\xi) \) is equal to
\[
\sup_{(x_1, x_2) \in \Gamma} \left\{ \left| \xi \cosh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right|^2 + \left| \sqrt{\xi^2 - k^2} \sinh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right|^2 \right\}.
\]
Similarly, from (45) it follows that
\[
|Q_2^\delta(\xi)| \leq \|g - g_0\|_{L^2(\Gamma)} \sqrt{\|\Gamma\|} \sup_{(x_1, x_2) \in \Gamma} \left| \cosh(x_2 \sqrt{\xi^2 - k^2}) e^{i\xi x_1} \right|.
\]
If \( j \leq j_0 \), then for \( \xi = 2^j \gamma \sqrt{\xi^2 - k^2} \) is an imaginary number and thus \( |\cosh(x_2 \sqrt{\xi^2 - k^2})| \) as well as \( |\sinh(x_2 \sqrt{\xi^2 - k^2})| \) are less or equal to 1. Hence
\[
|q_1(\xi)| \leq (\xi^2 + (\sqrt{k^2 - \xi^2})^2)^{1/2},
\]
which proves Lemma 3.2 for \( j \leq j_0 \).
Since \( k = c_j \sqrt{\xi^2 - k^2} \) and \( c_j < 1 \) for \( j > j_0 \), we have for \( \xi = 2^j \gamma \sqrt{\xi^2 - k^2} \)
\[
\cosh(x_2 \sqrt{\xi^2 - k^2}) \leq \cosh \left( \frac{8}{3} \pi 2^j \gamma \sqrt{1 - c_j^2} \right) = \cosh \left( \frac{c_j}{c_j} \gamma \sqrt{1 - c_j^2} \right),
\]
which gives (49). Taking into account that for \( \alpha > 0 \), \( \sinh^2 \alpha = \cosh^2 \alpha - 1 \), we obtain
\[
q_1(\xi) \leq \frac{k}{c_j} \sqrt{2 - c_j^2} \cosh \left( \frac{k}{c_j} \gamma \sqrt{1 - c_j^2} \right).
\]

**Theorem 3.3.** If (40) holds, then
\[
\|\varphi_J - \varphi_J^\delta\| \lesssim d(J),
\]
where
\[
d(J) := \begin{cases} 
2^{j-1} & \text{as } J \leq j_0 \\
2^j e^{\frac{1}{2} \pi 2^j} & \text{as } J > j_0.
\end{cases}
\]

**Proof.** Let \( b_{k,j} := 1 + \frac{k}{c_j} \sqrt{2 - c_j^2} \). Taking into account (48), (49), we get for \( j \leq j_0 \)
\[
\sup_{|\xi| \in \left( \frac{3}{2} \pi, \frac{7}{2} \pi \right)} \left| Q_1^\delta(2^j \xi) - Q_2^\delta(2^j \xi) \right| \leq \sup_{|\xi| \in \left( \frac{3}{2} \pi, \frac{7}{2} \pi \right)} \left( |Q_1^\delta(2^j \xi)| + |Q_2^\delta(2^j \xi)| \right) \leq \delta \sqrt{|\Gamma|} (k + 1).
\]
Similarly, for \( j > j_0 \)
\[
\sup_{|\xi| \in \left( \frac{3}{2} \pi, \frac{7}{2} \pi \right)} \left| Q_1^\delta(2^j \xi) - Q_2^\delta(2^j \xi) \right| \leq \delta \sqrt{|\Gamma|} b_{k,j} \cosh \left( \frac{k}{c_j} \gamma \sqrt{1 - c_j^2} \right).
\]
Since \( \sum_{l=-\infty}^{j} 2^l = 2^{j+1} \), from Lemma 3.1 it follows that
\[
\|\varphi_J - \varphi_J^\delta\|^2 \leq \delta^2 |\Gamma| (k + 1)^2 \frac{2^j}{2^j} \quad \text{as } J \leq j_0
\]
\[
\|\varphi_J - \varphi_J^\delta\|^2 \leq \delta^2 |\Gamma| \left\{ (k + 1)^2 \frac{2^{j_0}}{2^{j_0} + \sum_{j=j_0+1}^{j-1} 2^j b_{k,j}^2 \cosh^2 \left( \frac{k}{c_j} \gamma \sqrt{1 - c_j^2} \right)} \right\} \quad \text{as } J > j_0.
\]
According to (47), \( c_j \in (0, 1) \) for \( j > j_0 \), and thus \( 1 < \sqrt{2 - c_j} < \sqrt{2} \). Moreover, by (46), \( \frac{k}{c_j} = \frac{8 \pi}{3} 2^j \). Thus
\[
b_{k,j}^2 < \left( 1 + \frac{8 \pi}{3} 2^j \sqrt{2} \right)^2 \leq \delta^2 2^{2j},
\]
where \( \vartheta = (2^{-j_0} + \frac{8\pi}{3}\sqrt{2}) \). Taking into account that

\[
\cosh\left(\frac{k}{c_j} \sqrt{1 - c_j^2}\right) < e^{\frac{8\pi}{3} \gamma 2^j},
\]

we get

\[
\sum_{j=j_0+1}^{J-1} 2^j b_{k_j,j} \cosh\left(\frac{k}{c_j} \sqrt{1 - c_j^2}\right) \leq \vartheta^2 e^{\frac{8\pi}{3} \gamma 2^j} J - 1 \sum_{j=j_0+1}^{J-1} 2^j.
\]

Let \( \eta \) be a constant such that \( \sum_{j=j_0+1}^{J-1} 2^j = \frac{1}{7} (2^{3j} - 2^{3(j_0+1)}) \leq \eta 2^J \). Thus, for \( J > j_0 \)

\[
\|\varphi - \varphi_\delta J\| \leq \delta^2 |\Gamma| \left\{ (k + 1)^2 2^j_0 + \vartheta^2 \eta 2^J e^{\frac{8\pi}{3} \gamma 2^j} \right\} \leq \left( \eta 2^J e^{\frac{4}{3} \pi 2^J} \right)^2,
\]

which ends the proof. \( \Box \)

### 4. Regularization by wavelet-projection

Thus, from Lemma 2.9 and Theorem 3.3, it follows that

\[
\|\varphi - \varphi_\delta J\|_{L^2(0,1)} \lesssim 2^{-sJ} + \delta d(J).
\]

Looking for a proper value of the parameter of regularization \( J \) as a function of \( \delta \) (such that it implies the convergence when \( \delta \to 0 \)), we get the following result with an a priori choice of \( J \).

**Theorem 4.1.** Let \( \alpha \) be a fixed constant such that \( 0 < \alpha < \frac{1}{\gamma} \) for \( \gamma \) given by (18), and let \( c_0 = \frac{3\alpha}{4\pi} \). If the assumptions of Lemma 2.9 and Theorem 3.3 are satisfied and

\[
J(\delta) := E\left[ \log_2 \left( c_0 \ln \frac{1}{\delta} \right) \right],
\]

then for \( \delta \leq \left( \frac{8\pi}{3\kappa} \right)^{1/2+s} \)

\[
\|\varphi - \varphi_\delta J(\delta)\|_{L^2(0,1)} \lesssim (-\ln \delta)^{-s}.
\]

**Proof.** If \( J(\delta) \) is given by (55), then \( J(\delta) \to \infty \) as \( \delta \to 0 \). Moreover, denoting \( \tau := \frac{4\pi}{3} \), we have

\[
e^{2^j} \sim \delta^{-\alpha} \quad \text{and} \quad 2^j \sim \frac{1}{\tau} \ln \left( \frac{1}{\delta} \right)^{\alpha}.
\]

For sufficiently small \( \delta \), \( J(\delta) > j_0 \), and then, according to (53),

\[
\frac{\delta d(J(\delta))}{\delta^{1-\gamma \alpha}} \lesssim \delta^{1-\gamma \alpha} \left( \alpha \ln \left( \frac{1}{\delta} \right) \right)^{\frac{3}{2}}.
\]

Since \( s + \frac{3}{2} < 2 \), from (54) it follows that

\[
\|\varphi - \varphi_\delta J\|_{L^2(0,1)} \lesssim \left( \frac{1}{\ln \delta} \right)^{s} \left[ c_0^{-s} + c_0^{\frac{3}{2}} (\ln \delta)^{2 \delta^{1-\gamma \alpha}} \right].
\]

Using twice the rule of l’Hospital, we see that

\[
\lim_\delta \delta^{1-\gamma \alpha} (\ln(\delta))^2 = \frac{2}{(\gamma \alpha - 1)^2} \lim_\delta \delta^{1-\gamma \alpha} = 0.
\]

Therefore, the expression in the square brackets is bounded, which proves the theorem. \( \Box \)
Now, we can formulate the projection-regularized wavelet moment method for the problem (2) in the domain $\Omega \subset \mathbb{R}^2$ with the boundary described in Section 2.2 (cf. (17)). Let $\varphi_j^\delta$ be the approximate solution of moment problem given by (42) with coefficients (41). For given perturbed boundary value functions $f_\delta, g_\delta$ we define the regularized solution $u_j^\delta$ as a solution of the well posed Neumann problem (for $\partial \Omega$ sufficiently smooth)

$$
\begin{align*}
\Delta u_j^\delta + k^2 u_j^\delta &= 0, \quad \text{on } \Omega \\
\frac{\partial u_j^\delta}{\partial v} &= \varphi_j^\delta, \quad \text{on } \partial \Omega \setminus \Gamma, \\
\frac{\partial u_j^\delta}{\partial v} &= g_\delta, \quad \text{on } \Gamma.
\end{align*}
$$

(56)

Finally, from Theorem 4.1 and the continuous dependence of the solution of the problem (56) on the boundary conditions, we get an asymptotic convergence of projection-regularized wavelet moment method:

**Remark 4.2.** If the assumptions of Theorem 4.1 are satisfied, then

$$
\|u - u_{(\delta)}\|_{H^1(\Omega)} \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.
$$

5. Conclusion

A new approach to the stable solving of the Cauchy problem for the Helmholtz equation is investigated. The generalized moment problem (9), equivalent to the Cauchy problem for the Helmholtz equation (2), is found. Next, it is proved that in the two dimensional case, the Meyer wavelets generate a dense subset of the space appearing in the moment problem formulation.

Based on these facts, the wavelet projection method is applied and its self-regularization property is proved. Specifically, for the exact data an approximation error bound is found in the Besov spaces $B^{s}_{2,q}$, ($s < \frac{1}{2}$). For noisy data, a rate of convergence is evaluated under an a priori choice of projection level depending on a data error bound.

However, the order of convergence is rather small. An additional regularization procedure for the moment problem on $V_J$ could be a way to improve presented method. It will be discussed in a future work.

References


