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The Strong Approximation Conjecture holds for amenable groups

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Abstract

Let G be a finitely generated group and $G \rhd G_1 \rhd G_2 \rhd \cdots$ be normal subgroups such that $\bigcap_{k=1}^{\infty} G_k = \{1\}$. Let $A \in \mathrm{Mat}_{d \times d}(\mathbb{C}G)$ and $A_k \in \mathrm{Mat}_{d \times d}(\mathbb{C}(G/G_k))$ be the images of A under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück [W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-theory, Ergeb. Math. Grenzgeb. (3), vol. 44, Springer-Verlag, Berlin, 2002]

$$\dim_G(\ker A) = \lim_{k \to \infty} \dim_{G/G_k}(\ker A_k),$$

where $\dim_G$ denotes the von Neumann dimension. In [J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, Approximating L^2-invariants and the Atiyah conjecture, Comm. Pure Appl. Math. 56 (7) (2003) 839–873] Dodziuk et al. proved the conjecture for torsion free elementary amenable groups. In this paper we extend their result for all amenable groups, using the quasi-tilings of Ornstein and Weiss [D.S. Ornstein, B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Anal. Math. 48 (1987) 1–141].

Keywords: von Neumann dimension; Amenable groups; The approximation conjecture

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1. Introduction

First, let us recall the approximation result of Dodziuk et al. [2]. Let G be a finitely generated group and let $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$.

Let $l^2(G) = \{f : G \to \mathbb{C} \mid \sum_{g\in G} |f(g)|^2 < \infty\}$. By left convolution, A induces a bounded linear operator $A : (l^2(G))^d \to (l^2(G))^d$, which commutes with the right G-action. Let

$$\mathrm{proj}_{\ker A} : \left(l^2(G)\right)^d \to \left(l^2(G)\right)^d$$

be the orthogonal projection onto $\ker A$. Then

$$\dim_G(\ker A) := \mathrm{Tr}_G(\mathrm{proj}_{\ker A}) := \sum_{i=1}^{d} \langle \mathrm{proj}_{\ker A} \mathbf{1}_i, \mathbf{1}_i \rangle_{(l^2(G))^d},$$

where $\mathbf{1}_i \in (l^2(G))^d$ is the function which takes the value e_i on the unit element of G and zero elsewhere ($\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of $\mathbb{C}^d$). $\dim_G(\ker A)$ is called the von Neumann dimension of $\ker A$.

Now let $G \rhd G_1 \rhd G_2 \rhd \cdots$ be normal subgroups such that $\bigcap_{k=1}^{\infty} G_k = \{1\}$.

Let $A_k \in \mathrm{Mat}_{d\times d}(\mathbb{C}(G/G_k))$ be the images of A under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück [5]

$$\dim_G(\ker A) = \lim_{k\to\infty} \dim_{G/G_k}(\ker A_k).$$

In [2] the authors prove the conjecture above in the case when G is a torsion-free elementary amenable group. The goal of this paper is to extend their result to arbitrary amenable groups. If $A \in \mathrm{Mat}_{d\times d}(\mathbb{Z}(G))$ the problem is much easier to handle since one can use the method of Lück [4]. Then the conjecture holds for a large class of groups including amenable and residually finite groups. In the case of complex group algebra the situation seems much more complicated. Dodziuk et al. [2] used noncommutative algebra to prove the conjecture, we shall use the quasi-tilings of Ornstein and Weiss.

2. Preliminaries

Let G be a finitely generated amenable group with a finite symmetric set of generators S. Consider the Cayley-graph C_G, where $V(C_G) = G$ and

$$E(C_G) := \left\{(x, y) \in G \times G \mid y = sx,\ s \in S\right\}.$$

Now we introduce some notation frequently used in the paper later on.

1. If $g \in G$, then its word-length $w(g)$ is defined as $d_{C_G}(g, 1)$, where d_{C_G} is the shortest path distance on the Cayley-graph.
2. Let $F \subset G$ be a finite set, $k > 0$, then $B_k(F)$ denotes the k-neighborhood of F in the d_{C_G}-metric.
3. We denote by $\Omega_k(F)$ the set of vertices p in F, such that $d_{C_G}(p, F^c) > k$, where F^c is the complement of F.

4. For $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$, its propagation $w(A)$ is just $\sup w(g)$, where g runs through the terms of non-zero coefficients in the entries of A. The propagation of the zero matrix is defined to be 0. Observe that if $f \in (l^2(G))^d$ and $\operatorname{supp} f \subseteq U \subseteq G$, then $\operatorname{supp} A(f) \subseteq B_{w(A)}(U)$, and if $\operatorname{supp} f \subseteq \Omega_{w(A)}(U)$ then $\operatorname{supp} A(f) \subseteq U$. Here, $\operatorname{supp} f := \{g \in G \mid f(g) \neq 0\}$.
5. For a finite set $F \subset G$, ∂F denotes the set of vertices in F such that $d_{C_G}(p, F^c) = 1$. We shall denote the ratio $|\partial F|/|F|$ by $i(F)$.
6. Since G is amenable, it has a *Følner-exhaustion*, that is a sequence of subsets $1 \in F_1 \subset F_2 \subset \cdots$, $\bigcup_{n=1}^{\infty} F_n = G$ such that $i(F_n) \to 0$.

Now we prove some approximation theorems for amenable groups. Let $1 \in F_1 \subseteq F_2 \subseteq \cdots$ be a Følner exhaustion of G and $P_n : (l^2(G))^d \to (l^2(F_n))^d$ be the orthogonal projections. Then by [2, Theorem 3.11] (or [3, Proposition 1]):

$$\dim_G(\ker A) = \lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(\ker P_n A P_n^*)}{|F_n|}.$$

We define the following sequences of vector spaces:

$$Z_n := \left\{ f \in \left(l^2(G)\right)^d \;\middle|\; \operatorname{supp} f \subseteq B_{w(A)}(F_n),\ A(f)|_{F_n} = 0 \right\},$$

$$W_n := \left\{ f \in \left(l^2(G)\right)^d \;\middle|\; \operatorname{supp} f \subseteq \Omega_{w(A)}(F_n),\ A(f) = 0 \right\},$$

$$V_n := \ker(P_n A P_n^*).$$

Proposition 2.1.

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(Z_n)}{|F_n|} = \dim_G(\ker A), \qquad \lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(W_n)}{|F_n|} = \dim_G(\ker A).$$

Proof. It is enough to prove that

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(Z_n)}{\dim_{\mathbb{C}}(V_n)} = 1 \tag{1}$$

and

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(W_n)}{\dim_{\mathbb{C}}(V_n)} = 1. \tag{2}$$

Clearly, $W_n = V_n \cap \{f \in l^2(F_n)^d \mid \operatorname{supp} f \subseteq \Omega_{w(A)}(F_n)\}$. Hence (2) follows from the fact that

$$\lim_{n\to\infty} \frac{|\Omega_{w(A)}(F_n)|}{|F_n|} = 1.$$

Also, $W_n = Z_n \cap \{f \in l^2(F_n)^d \mid \operatorname{supp} f \subseteq \Omega_{w(A)}(F_n)\}$. Therefore (1) follows from the fact that

$$\lim_{n\to\infty} \frac{|\Omega_{w(A)}(F_n)|}{|B_{w(A)}(F_n)|} = 1. \qquad \square$$

Definition 2.1. Let $F \subset G$ be a finite set and $\delta, \epsilon > 0$ be real numbers. We say that F has property $A(\epsilon, \delta, -)$ if for any subset $K \subseteq F$, $|K|/|F| > 1 - \epsilon$, the following holds:

- if $R = \{f \in l^2(F_n)^d \mid \operatorname{supp} f \subseteq \Omega_{w(A)}(K),\ A(f) = 0\}$, then

$$\dim_{\mathbb{C}}(R) \geqslant (1 - \delta)\bigl(\dim_G(\ker A)\bigr).$$

Also, we say that F has property $A(\delta, +)$ if the following holds:

- if Q is the restriction of the space $Z_F := \{f \in (l^2(G))^d \mid \operatorname{supp} f \subseteq B_{w(A)}(F), A(f)|_F = 0\}$ onto F, then

$$\dim_{\mathbb{C}}(Q) \leqslant \bigl(\dim_G(\ker A)\bigr) + \delta.$$

Similarly to Proposition 2.1 one can easily prove the following proposition.

Proposition 2.2. *Let $1 \in F_1 \subseteq F_2 \subseteq \cdots$ be a Følner exhaustion of G as above. Then for any pair of real numbers $\delta, \epsilon > 0$ there exists $n_{\delta,\epsilon}$ such that if $n \geqslant n_{\delta,\epsilon}$ then F_n has both properties $A(\epsilon, \delta, -)$ and $A(\delta, +)$.*

3. Graph convergence and dimension averaging

Let C_G be the Cayley-graph of the previous section. Color the directed edge $(x \to y)$, $x = sy$ by $s \in S$ (hence $(y \to x)$ shall be colored by s^{-1}). Thus we color all edges in both directions with the elements of the set S in such a way that for each $x \in G$ the edges outgoing from x are colored in different ways. The following definition is a variation of the one on random weak convergence in [1].

Let $B_1, B_2, \ldots$ be an infinite sequence of finite graphs. Assume that for any $x \in V(B_n)$: $\deg(x) \leqslant |S|$. We also assume that the directed edges are colored by S in such a way that:

- the color of the edge $(x \to y)$ is the inverse of the color of $(y \to x)$;
- the outgoing edges from any vertex are colored differently.

We say that $p \in V(B_n)$ is k-similar to the identity of G, if its k-neighborhood in B_n is edge-colored isomorphic to the k-neighborhood of the identity in C_G. Let Q_k^B be the set of vertices in B that are k-similar to the identity. Then we say that $\{B_n\}_{n=1}^\infty$ converges to C_G if for any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $n_{\epsilon,k}$ such that if $n \geqslant n_{\epsilon,k}$ then

$$Q_k^{B_n} > (1 - \epsilon)\bigl|V(B_n)\bigr|.$$

Example 1. Let G be a finitely generated group and $\{B_n\}_{n=1}^\infty$ be a sequence of finite induced subgraphs forming a Følner-exhaustion. Then $\{B_n\}_{n=1}^\infty$ converges to C_G.

Example 2. Let G be a finitely generated residually finite group and $G \rhd G_1 \rhd G_2 \rhd \cdots$ be a sequence of finite index normal subgroups such that $\bigcap_{n=1}^\infty G_n = \{1\}$. Let C_n be the Cayley-graph of G/G_n. Then $\{C_n\}_{n=1}^\infty$ converges to C_G.

Now let $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$. One can define the transformation kernel of A, $\widetilde{A}: G\times G \to \mathrm{Mat}_{d\times d}(\mathbb{C})$ in the following way. First write A in the form of $\sum_{\gamma\in G} A_\gamma \cdot \gamma$, where $A_\gamma \in \mathrm{Mat}_{d\times d}(\mathbb{C})$. Then set $\widetilde{A}(\gamma,\delta) := A_{\gamma\delta^{-1}}$. Thus if $f \in l^2(G))^d$, then

$$A(f)(\delta) = \sum_{\gamma\in G} \widetilde{A}(\delta,\gamma) f(\gamma).$$

Now let $\{B_n\}_{n=1}^\infty$ be a sequence of graphs converging to C_G. Then we define the finite-dimensional linear transformations $T_n : (l^2(V(B_n))^d \to (l^2(V(B_n))^d$ approximating A, the following way:

- if $x \in Q^{B_n}_{w(A)}$, $y \in V(B_n)$ and $d_{B_n}(y,x) \leqslant w(A)$, let $\widetilde{T}_n(y,x) := A(\gamma, 1)$, where γ is the element of G satisfying $\phi^x_{w(A)}(\gamma) = y$. Here $\phi^x_{w(A)}$ is the unique colored isomorphism between the $w(A)$-neighborhood of 1 in C_G and the $w(A)$-neighborhood of 1 in B_n;
- if $x \notin Q^{B_n}_{w(A)}$ or $d_{B_n}(y,x) > w(A)$, then let $\widetilde{T}_n(x,y) := 0$.

Then if $f \in l^2(V(B_n))^d$ and $p \in V(B_n)$

$$T_n(f)(p) = \sum_{q\in V(B_n)} \widetilde{T}_n(p,q) f(q).$$

The main goal of our paper is to prove the following theorem.

Theorem 1. *If G is a finitely generated amenable group and $\{B_n\}_{n=1}^\infty$, $\{T_n\}_{n=1}^\infty$ are as above, then*

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}} \ker T_n}{|V(B_n)|} = \dim_G(\ker A).$$

The Strong Approximation Conjecture for amenable groups follows from the theorem:

Corollary 3.1. *If G is a finitely generated amenable group and $G \triangleright G_1 \triangleright G_2 \cdots$ are normal subgroups of G such that $\bigcap_{n=1}^\infty G_n = \{1\}$, then*

$$\lim_{n\to\infty} \dim_{G/G_n}(\ker A_n) = \dim_G(\ker A),$$

where $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$ and $A_n \in \mathrm{Mat}_{d\times d}(\mathbb{C}(G/G_n))$ are the images of A under the maps induced by the epimorphisms $G \to G/G_n$.

Proof.

Case 1. Suppose that all G_n has finite index. Note that in this case $T_n = A_n$ if n is large enough, hence the corollary immediately follows.

Case 2. Assume that for large enough n, the amenable group G/G_n is infinite. Let $1 \in F_1^n \subset F_2^n \subset \cdots$ be a Følner-exhaustion of the Cayley graph C_{G/G_n} (using the image of the generator system S). Then

$$\dim_{G/G_n}(A_n) = \lim_{k\to\infty} \frac{\dim_{\mathbb{C}}(\ker P_k^n A_n (P_k^n)^*)}{|F_k^n|},$$

where $P_k^n : (l^2(G/G_n))^d \to (l^2(F_k^n))^d$ is the orthogonal projection.

Pick a sequence $F_{m_1}^1, F_{m_2}^2, \ldots$ such that

- $i(F_{m_j}^j) \to 0$;
- $(\dim_{G/G_n}(A_n) - \dim_{\mathbb{C}}(\ker P_{m_n}^n A_n (P_{m_n}^n)^*)/|F_{m_n}^n|) \to 0$.

Now let $B_{m_n}^n$ be the graph induced by $F_{m_n}^n$.

Lemma 3.1. *$\{B_{m_n}^n\}_{n=1}^\infty$ converges to C_G.*

Proof. Since $\bigcap_{k=1}^\infty G_k = \{1\}$, for any $d \in \mathbb{N}$ there exists $n_d > 0$ such that if $n \geqslant n_d$ then the d-balls in C_{G/G_n} are colored-isomorphic to the d-ball of C_G. Let $H_{m_n}^n = \Omega_d(F_{m_n}^n)$. Clearly $H_{m_n}^n \subseteq Q_d^{F_{m_n}^n}$. Since the vertex degrees of G/G_n are at most S, $|H_{m_n}^n| \geqslant |F_{m_n}^n| - |S|^d |\partial F_{m_n}^n|$. Now our lemma easily follows. □

Lemma 3.2.

$$\lim_{n\to\infty} \frac{\dim_{\mathbb{C}}(\ker P_k^n A_n (P_k^n)^*)}{\dim_{\mathbb{C}} \ker T_n} = 0.$$

Here T_n is the linear operator associated to $B_{m_n}^n$.

Proof. If $\operatorname{supp} f \subset F_{m_n}^n \backslash B_{w(A)}(\partial F_{m_n}^n)$ then $T_n(f) = P_k^n A_n (P_k^n)^*(f)$. Since

$$\frac{|F_{m_n}^n \setminus B_{w(A)}(\partial F_{m_n}^n)|}{|F_{m_n}^n|} \to 1$$

our lemma follows. □

Obviously, Lemmas 3.1 and 3.2 imply the corollary. □

4. Quasi-tilings

Let us recall the notion of quasi-tilings from [6]. Let X be a finite set and $\{A_i\}_{i=1}^n$ be finite subsets of X. Then we say that $\{A_1, A_2, \ldots, A_n\}$ are ϵ-disjoint if there exist subsets $\overline{A_i} \subseteq A_i$ such that:

- for any $1 \leqslant i \leqslant n$, $|\overline{A_i}|/|A_i| \geqslant 1 - \epsilon$;
- If $i \neq j$ then $\overline{A_i} \cap \overline{A_j} = \emptyset$.

On the other hand, if $\{H_j\}_{j=1}^m$ are finite subsets of X, then we say that they α-cover X if

$$\frac{|X \cap (\bigcup_{j=1}^m H_j)|}{|X|} \geqslant \alpha.$$

Finally, we say that the collection $\{H_1, H_2, \dots, H_m\}$ δ-evenly covers X if there exists some $M \in \mathbb{R}^+$ such that:

- for any $p \in X$, $\sum_{j:\ p \in H_j}^m 1 \leqslant M$;
- $\sum_{j=1}^m |H_j| \geqslant (1-\delta)M|X|$.

According to [6, Lemma 4], if $\{H_1, H_2, \dots, H_m\}$ form a δ-even covering of X, then for any $0 < \epsilon < 1$ there exists an ϵ-disjoint subcollection of the H_j's that $\epsilon(1-\delta)$-covers X.

Now we define *tiles* for our S-edge colored graphs. Let G be a finitely generated group with a symmetric generator set S and let $1 \in F_1 \subseteq F_2 \subseteq \cdots$, $\bigcup_{n=1}^\infty F_n = G$ be a Følner-exhaustion. Let B be a finite graph as in the previous section with edge-colorings by the elements of S. Also, let L be a natural number. Let $\{F_{\alpha_1}, F_{\alpha_2}, \dots, F_{\alpha_n}\}$ be a finite collection of the Følner sets above such that for any $1 \leqslant i \leqslant n$, $F_{\alpha_i} \subset B_{1/2L}(1)$. Then for any $x \in Q_L^B$ and $1 \leqslant i \leqslant n$, $T_x(F_{\alpha_i})$ is the image of F_{α_i} under the unique colored isomorphism $\phi_L^x : B_L(1) \to B_L(x)$ mapping 1 to x. We call such a subset a tile of type F_{α_i} and say that x is the center of $T_x(F_{\alpha_i})$. A system of tiles ϵ-quasi-tile $V(B)$ if they are ϵ-disjoint and form an $(1-\epsilon)$-cover. The following theorem is a version of [6, Theorem 6].

Theorem 2. *For any $\epsilon > 0$, $n > 0$, there exist $L > 0$, $\delta > 0$ and a finite collection $\{F_{n_1}, F_{n_2}, \dots, F_{n_s}\} \subset B_L(1)$ of the Følner sets, such that $n_i > n$ and if*

$$\frac{|Q_L^B|}{|V(B)|} > 1 - \delta$$

then $V(B)$ can be ϵ-quasi-tiled by tiles of the form $T_x(F_{n_i})$, $x \in Q_L^B$, $1 \leqslant i \leqslant s$.

5. The inductional step

First of all fix a constant $\epsilon_1 < \epsilon/100$. Let us call a finite set $H \subset G$ a set of type (K, α), $K \in \mathbb{N}$, $\alpha \geqslant 0$, if

$$\frac{|B_K(H)|}{|H|} < 1 + \alpha. \tag{3}$$

Now let B be our S-edge colored finite graph and suppose that

$$\frac{|Q_L^B|}{|V(B)|} > 1 - \beta. \tag{4}$$

The exact values of β and L shall be given later. Assume that H is of type (K, α), where

$$H \subset B_{L/100}(1) \quad \text{and} \quad K < \frac{L}{10}. \tag{5}$$

Now consider all tiles in B in the form $T_x(H)$, where $x \in Q_L^B$. Note that no vertex of B is covered by more than $|H|$ tiles. Indeed, if z is covered, then the $L/2$-neighborhood of z in B is colored isomorphic to the $L/2$-neighborhood of 1 in G. Hence if $z \in T_x(H)$, then $z \in Q_{L/2}^B$ and $x \in T_z(H^{-1})$. Summarizing these:

- for any $y \in V(B)$, $\sum_{x: y \in T_x(H)} 1 \leqslant |H|$;
- $\sum_{x \in Q_L^B} |T_x(H)| = |Q_L^B||H| \geqslant (1-\beta)|V(B)||H|$.

Consequently, the tiles $\{T_x(H)\}_{x \in Q_L^B}$ form a δ-even covering of $V(B)$, where $\delta = 1-\beta$. Then by [6, Lemma 4], there exists an ϵ_1-disjoint subcollection of tiles, $\bigcup_{x \in I} T_x(H)$ such that they form a $\epsilon_1(1-\beta)$-covering of $V(B)$.

Now suppose that the number of vertices in $V(B)$ not covered by this subcollection above is greater than $(\epsilon/2)|V(B)|$. Let B_1 be the graph induced by the uncovered vertices. We would like to estimate the quotient: $|Q_K^{B_1}|/|V(B)|$. Note that if $y \in V(B_1)$ and

$$y \notin \bigcup_{x \in I} \big(T_x\big(B_K(H)\big) \backslash T_x(H)\big)$$

then $y \in Q_K^{B_1}$. Hence by ϵ_1-disjointness,

$$\left|\bigcup_{x \in I} \big(T_x\big(B_K(H)\big) \backslash T_x(H)\big)\right| \leqslant \alpha \sum_{x \in I} |H| \leqslant \alpha(1-\epsilon_1)^{-1}\big|V(B)\big|.$$

Hence

$$|Q_K^{B_1}| \geqslant \big|V(B_1)\big| - \alpha(1-\epsilon_1)^{-1}\big|V(B)\big|,$$

that is

$$\frac{|Q_K^{B_1}|}{|V(B)|} \geqslant 1 - \beta_1, \tag{6}$$

where $\beta_1 = \alpha(1-\epsilon_1)^{-1}(2/\epsilon)$. Also note that

$$\frac{\epsilon}{2}\big|V(B)\big| \leqslant \big|V(B_1)\big| \leqslant \big(1 - \epsilon_1(1-\beta)\big)\big|V(B)\big|.$$

6. The proof of Theorem 2

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of real numbers tending to zero and let $\{s_k\}_{k=1}^{\infty}$ be a sequence of real numbers tending to infinity, satisfying the following inequalities:

$$s_k \geqslant 1, \qquad s_{k+1} \geqslant 10 s_k.$$

We call a subsequence of the Følner exhaustion $\{F_n\}_{n=1}^{\infty}$ an (α, s)-good subsequence if it satisfies the following conditions:

- $1 \in F_{n_1} \subset B_{s_1}(1) \subset F_{n_2} \subset B_{s_2}(1) \subset F_{n_3} \subset \cdots$;
- $F_{n_{i+1}}$ is of type $(100s_i, \alpha_i)$.

Obviously one can choose $\{s_k\}_{k=1}^{\infty}$ for any fixed $\{\alpha_k\}_{k=1}^{\infty}$ to have such (α, s)-good subsequences. Now let M be an integer such that

$$\left(1 - \frac{\epsilon_1}{2}\right)^M < \frac{\epsilon}{100}. \tag{7}$$

Also, pick $\beta > 0$ so that

$$\beta M < \frac{\epsilon}{100}. \tag{8}$$

And finally fix a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that

$$\alpha_i (1 - \epsilon_1)^{-1} \frac{2}{\epsilon} < \beta. \tag{9}$$

Now let B a finite S-colored graph such that

$$\frac{Q_{100s_M}^B}{|V(B)|} > 1 - \beta,$$

where β, M are as above. Then by the argument of the previous section we can $\epsilon_1(1 - \beta)$-cover the vertices of B by ϵ_1-disjoint tiles of type F_{n_M}. If B_1 is the graph induced by the uncovered vertices of $V(B)$, by (6):

$$\frac{|Q_{s_M}^{B_1}|}{|V(B_1)|} > 1 - \beta_1,$$

where $\beta_1 = \alpha_M (1 - \epsilon_1)^{-1}(2/\epsilon)$. Now we can $\epsilon_1(1 - \beta_1)$-cover $V(B_1)$ by tiles of type $F_{n_{M-1}}$. If B_2 denotes the graph induced by the uncovered part of $V(B_1)$ then

$$\frac{|Q_{s_{M-1}}^{B_2}|}{|V(B_2)|} > 1 - \beta_2,$$

where $\beta_2 = \alpha_{M-1}(1 - \epsilon_1)^{-1}(2/\epsilon)$.

We proceed inductively. In each step the new tiles are disjoint from all previous ones. Also,

$$V(B_i) \leqslant V(B_{i-1})\left(1 - \frac{\epsilon_1}{2}\right).$$

Hence by our conditions, in at most M steps we obtain an ϵ-disjoint $(1 - \epsilon)$-covering of $V(B)$.

7. The proof of Theorem 1

Let G be a finitely generated amenable group, $A \in \mathrm{Mat}_{d\times d}(\mathbb{C}G)$ and $\{B_n\}_{n=1}^{\infty}$ be a sequence converging to C_G. Let $\{T_n\}_{n=1}^{\infty}$ be the sequence of approximating operators as in Section 3.

Proposition 7.1. *For any pair $\delta, \epsilon > 0$ there exists $k_{\delta,\epsilon} > 0$ such that if $k \geqslant k_{\delta,\epsilon}$ then*

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \geqslant \big(\dim_G(\ker A) - \delta\big)(1-\epsilon).$$

Proof. Let $1 \in F_1 \subseteq F_2 \subseteq \cdots$ be a Følner exhaustion of G, such that all the F_n's have property $A(\epsilon, \delta, -)$ (see Lemma 2.1). Let $\{H_1, H_2, \ldots, H_s\}$ be an ϵ-quasi-tiling of B_k by tiles from this Følner sequence. Such ϵ-quasi-tiling exists by Theorem 2 if k is large enough. For $1 \leqslant i \leqslant s$ let $K_i \subset H_i$ be a subset such that:

- $|K_i|/|H_i| > 1 - \epsilon$;
- $K_i \cap K_j = \emptyset$ if $i \neq j$.

Since the F_n's have property $A(\epsilon, \delta, -)$ there exist subspaces $V_i \subset (l^2(B_n))^d$ such that:

- if $f \in V_i$ then $\operatorname{supp} f \subseteq K_i$;
- $f \in \ker T_k$;
- $\dim_{\mathbb{C}} V_i/|H_i| \geqslant \dim_G(\ker A) - \delta$.

Now consider the subspace $\bigoplus_{i=1}^{s} V_i \subseteq \ker T_k$. Then

$$\dim_{\mathbb{C}}\left(\bigoplus_{i=1}^{s} V_i\right) \geqslant \left(\sum_{i=1}^{s} |H_i|\right)\big(\dim_G(\ker A) - \delta\big) \geqslant (1-\epsilon)\big|V(B_k)\big|\big(\dim_G(\ker A) - \delta\big).$$

That is

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \geqslant \big(\dim_G(\ker A) - \delta\big)(1-\epsilon). \qquad \square$$

Proposition 7.2. *For any pair $\delta, \epsilon > 0$ there exists $m_{\delta,\epsilon} > 0$ such that if $k \geqslant m_{\delta,\epsilon}$ then*

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \leqslant (1-\epsilon)^{-1}\big(\dim_G(\ker A) + \delta\big) + \epsilon.$$

Proof. Again let $1 \in F_1 \subseteq F_2 \subseteq \cdots$ be a Følner exhaustion of G, such that all the F_n's have property $A(\delta, +)$ (see Lemma 2.1). Consider the ϵ-quasi-tilings of the previous proposition. Now let $W_i \subset l^2(H_i)$ be the restriction of $\ker T_k$ onto H_i. By our assumption,

$$\dim_G(\ker T_k) \leqslant |H_i|\big(\dim_G(\ker A) + \delta\big).$$

Since $\{H_i\}_{i=1}^s$ form an ϵ-covering

$$\dim_{\mathbb{C}}(\ker T_k) \leqslant \epsilon \left|V(B_k)\right| + \sum_{i=1}^{s} |H_i| \bigl(\dim_G(\ker A) + \delta\bigr).$$

Note that by ϵ-disjointness

$$\sum_{i=1}^{s} |H_i| \leqslant (1-\epsilon)^{-1} \left|V(B_k)\right|.$$

Thus

$$\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \leqslant (1-\epsilon)^{-1} \bigl(\dim_G(\ker A) + \delta\bigr) + \epsilon. \qquad \square$$

Clearly, Propositions 7.1 and 7.2 imply Theorem 1.

References

[1] I. Benjamini, O. Schramm, Recurrence of distributional limits of finite planar graphs, Electron. J. Probab. 6 (23) (2001), 13 pp. (electronic).
[2] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, Approximating L^2-invariants and the Atiyah conjecture, Comm. Pure Appl. Math. 56 (7) (2003) 839–873.
[3] G. Elek, On the analytic zero divisor conjecture of Linnell, Bull. London Math. Soc. 35 (2) (2003) 236–238.
[4] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues, Geom. Funct. Anal. 4 (1994) 455–481.
[5] W. Lück, L^2-Invariants: Theory and Applications to Geometry and K-theory, Ergeb. Math. Grenzgeb. (3), vol. 44, Springer-Verlag, Berlin, 2002.
[6] O.S. Ornstein, B. Weiss, Entropy and isomorphism theorems for actions of amenable groups, J. Anal. Math. 48 (1987) 1–141.