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# Degenerations for derived categories

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To Claus Michael Ringel for his sixtieth birthday

**Abstract**

We propose a theory of degenerations for derived module categories, analogous to degenerations in module varieties for module categories. In particular we define two types of degenerations, one algebraic and the other geometric. We show that these are equivalent, analogously to the Riedtmann–Zwara theorem for module varieties. Applications to tilting complexes are given, in particular that any two-term tilting complex is determined by its graded module structure.


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## 0. Introduction

Geometrical methods were introduced in representation theory of finite dimensional algebras in order to parameterize possible module structures on a given vector space by algebraic varieties. These varieties carry an action of a reductive algebraic group $G$ such that the orbits correspond to isomorphism classes of modules. One says that a module $M$ degenerates to $N$ if $N$ is in the closure $\overline{G \cdot M}$ of the orbit of $M$ under the $G$-action, and in

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this case one writes $M \leqslant N$. Riedtmann defined in [10] a relation $\leqslant_{\text{alg}}$ by setting $M \leqslant_{\text{alg}} N$ if there is a module $Z$ and a short exact sequence

$$0 \longrightarrow N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow 0$$

of $A$-modules. She showed that $M \leqslant_{\text{alg}} N$ implies $M \leqslant N$ and in [12] Zwara proved that $M \leqslant N$ implies $M \leqslant_{\text{alg}} N$.

Since the derived category became a powerful tool in representation theory, it seems desirable to study derived categories from such a geometric point of view. De Concini and Strickland [3] studied geometric properties of varieties of bounded complexes of free modules. For a finite dimensional algebra $A$, Huisgen-Zimmermann and Saorin [11] defined an affine variety which parameterizes bounded complexes of $A$-modules. For this variety no group action seems available so that the quasi-isomorphism classes correspond to orbits under the action. Bekkert and Drozd studied in [1] minimal right bounded complexes of projective modules, where quasi-isomorphism is the same as homotopy equivalence. There homotopy equivalence classes are obtained as orbits of an action of a group; however Bekkert and Drozd did not study the topology of their space and in particular they did not study degeneration.

The purpose of the present paper is to define and to study a geometric structure on a set of right bounded complexes of projective modules and to show a result analogous to the result of Zwara and Riedtmann. More precisely, we define a topological space $comproj^{\underline{d}}$ parameterizing right bounded complexes of projective modules depending on a dimension array $\underline{d}$ replacing the dimension vector for module varieties. This topological space is a projective limit of affine varieties and a projective limit $G$ of affine algebraic groups is acting on it. The $G$-orbits correspond to quasi-isomorphism classes of right bounded complexes of projective modules. For two right bounded complexes $M$ and $N$, we define $M \leqslant_{\Delta} N$ if there is a complex $Z$ and a distinguished triangle

$$N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow N[1].$$

For two right bounded complexes $M$ and $N$ in $comproj^{\underline{d}}$, we say $M \leqslant_{\text{top}} N$ if $N \in \overline{G \cdot M}$. Our main result is the following.

**Theorem.** *Let A be a finite dimensional k-algebra over an algebraically closed field k and let N and M be complexes in the bounded derived category of A-modules $D^b(A)$. Then, there is a dimension array $\underline{d}$ so that N and M belong to $comproj^{\underline{d}}$ and moreover $M \leqslant_{\Delta} N$ if and only if $M \leqslant_{\text{top}} N$.*

Using $\leqslant_{\text{alg}}$ and $\leqslant_{\Delta}$, we show that for two $A$-modules $M$ and $N$ one can choose a dimension array $\underline{d}$ so that the module $M$ degenerates to $N$ in the module variety if and only if the projective resolution of $M$ degenerates to the projective resolution of $N$ in $comproj^{\underline{d}}$. To illustrate how the topology of $comproj^{\underline{d}}$ can be used, we show that a partial two-term tilting complex is determined, up to isomorphism, by its structure as a graded module. We give an example showing that this is not true for longer tilting complexes.

The paper is organized as follows. In Section 1 we define the variety $comproj^{\underline{d}}$, define a group acting on it, and show some basic properties. In Section 2 we define $\leqslant_{\Delta}$ and show that $\leqslant_{\Delta}$ implies the topological degeneration for two complexes with bounded homology.

In Section 3 we show the converse. Section 4 finally develops consequences for complexes without self-extensions.

## 1. General definitions and elementary properties

Let $A$ be a finite dimensional algebra over an algebraically closed field $k$. Let $mod(A,d)$ denote the affine variety of $d$-dimensional $A$-modules. The general linear group $Gl_d(k)$ acts on $mod(A,d)$ by change of basis and the orbits correspond to the isomorphism classes of $d$-dimensional modules.

Let $P_1,\dots,P_l$ be a complete set of projective indecomposable $A$-modules, one in each isomorphism class. For an element $d=(d^1,\dots,d^l)\in\mathbb{N}^l$, let $\alpha(d)$ be defined by $\bigoplus_{j=1}^{l}P_j^{d^j}\in mod(A,\alpha(d))$.

For every sequence $\underline{d}:\mathbb{Z}\longrightarrow\mathbb{N}^l$ for which there is an $i_0\in\mathbb{Z}$ with $d_i=(0,\dots,0)$ for $i\leqslant i_0$ we define $comp(A,\underline{\alpha}(\underline{d}))$ to be the subset of

$$\left(\prod_{i\in\mathbb{Z}} mod(A,\alpha(d_i))\right)\times\left(\prod_{i\in\mathbb{Z}} Hom_k(k^{\alpha(d_i)},k^{\alpha(d_{i-1})})\right)$$

consisting of elements $((M_i)_{i\in\mathbb{Z}},(\partial_i)_{i\in\mathbb{Z}})$ with the properties that $\partial_i$ is an $A$-homomorphism when viewed as a map from $M_i$ to $M_{i-1}$ and $\partial_i\partial_{i-1}=0$.

The group $\prod_{i\in\mathbb{Z}}Gl_{\alpha(d_i)}$ acts on $comp(A,\underline{\alpha}(\underline{d}))$ by change of basis and the orbits correspond to isomorphism classes of complexes.

We have a projection $\pi_M: comp(A,\underline{\alpha}(\underline{d}))\longrightarrow\prod_{i\in\mathbb{Z}}mod(A,\alpha(d_i))$ and we define

$$comproj^{\underline{d}}:=\pi_M^{-1}\left(\prod_{i\in\mathbb{Z}}\bigoplus_{j=1}^{l}P_j^{d_i^j}\right).$$

We say that $\underline{d}$ is bounded if there is an $i_1\in\mathbb{Z}$ with $d_i=(0,\dots,0)$ for $i\geqslant i_1$. In this case we identify $comp(A,\underline{\alpha}(\underline{d}))$ with the affine variety of bounded complexes defined by Huisgen-Zimmermann and Saorin in [11]; in particular it has the Zariski topology. Also $comproj^{\underline{d}}$ is then an affine variety, being a closed subset of $comp(A,\underline{\alpha}(\underline{d}))$.

Naive truncation on the left induces surjective morphisms of varieties

$$\varphi_n: comproj^{\underline{d}_n}\longrightarrow comproj^{\underline{d}_{n-1}}$$
$$\left(\prod_{i\leqslant n}M_i,\prod_{i\leqslant n}\partial_i\right)\mapsto\left(\prod_{i\leqslant n-1}M_i,\prod_{i\leqslant n-1}\partial_i\right)$$

and similarly surjective maps

$$\pi_n: comproj^{\underline{d}}\longrightarrow comproj^{\underline{d}_n}$$
$$\left(\prod_{i\in\mathbb{Z}}M_i,\prod_{i\in\mathbb{Z}}\partial_i\right)\mapsto\left(\prod_{i\leqslant n}M_i,\prod_{i\leqslant n}\partial_i\right).$$

We give $comproj^{\underline{d}}$ the weak topology with respect to the maps $\{\pi_n\}$. So, the open sets in $comproj^{\underline{d}}$ are of the form $U = \bigcup_{n \geqslant n_0} \pi_n^{-1}(U_n)$ for open sets $U_n$ in $comproj^{\underline{d}_n}$ and $n_0 \in \mathbb{Z}$. Similarly, the closed sets in $comproj^{\underline{d}}$ are of the form $C = \bigcap_{n \geqslant n_0} \pi_n^{-1}(C_n)$ for closed sets $C_n$ in $comproj^{\underline{d}_n}$ and an $n_0 \in \mathbb{Z}$. Note that $comproj^{\underline{d}}$ is the projective limit of the varieties $comproj^{\underline{d}_n}$ in the category of topological spaces.

The group

$$G := \prod_{i \in \mathbb{Z}} Stab_{Gl_{\alpha(d_i)}} \left( \bigoplus_{j=1}^{l} P_j^{d_i^j} \right) \cong \prod_{i \in \mathbb{Z}} Aut_A \left( \bigoplus_{j=1}^{l} P_j^{d_i^j} \right)$$

acts on the space $comproj^{\underline{d}}$ by conjugation and the orbits correspond to isomorphism classes of complexes of projective $A$-modules. The action of $G$ on $comproj^{\underline{d}}$ induces naturally an action of $G$ on $comproj^{\underline{d}_n}$ for all $n$ such that $\pi_n$ and $\varphi_n$ are $G$-equivariant maps.

We see that $G$ is a connected algebraic group if $\underline{d}$ is bounded since the endomorphism ring is a linear space, hence irreducible, and the automorphism group is an open dense subvariety. Moreover, the action of $G$ is the action of a connected algebraic group on an affine variety if $\underline{d}$ is bounded.

The following lemma is well known to the experts, but we could not find a reference and include a proof below. We do not require the field to be algebraically closed for the remainder of this section.

**Lemma 1.** *Let* $X = (\bigoplus_{i \in \mathbb{Z}} Q_i, \bigoplus_{i \in \mathbb{Z}} \partial_i^X)$ *and* $Y = (\bigoplus_{i \in \mathbb{Z}} Q_i, \bigoplus_{i \in \mathbb{Z}} \partial_i^Y)$ *be two right bounded complexes of projective $A$-modules with the same homogeneous components $Q_i$ in each degree $i \in \mathbb{Z}$. Then, $X$ is isomorphic to $Y$ if and only if $X$ is homotopy equivalent to $Y$.*

**Proof.** If $X$ is isomorphic to $Y$ in the category of complexes, then clearly $X$ is homotopy equivalent to $Y$. So, suppose that $X$ is homotopy equivalent to $Y$, that is there is a mapping of complexes $\varphi : X \longrightarrow Y$ and a mapping $\psi : Y \longrightarrow X$ of complexes so that there is a map $h$ of degree 1 so that $\varphi\psi - id_X = h\partial^X + \partial^X h$ and likewise there is an $h'$ with $\psi\varphi - id_Y = h'\partial^Y + \partial^Y h'$. We shall show that $X \simeq X' \oplus N_X$ where $im(\partial^X|_{X'}) \subseteq rad(X')$ and $N_X$ is contractible, and likewise for $Y$.

Suppose for the moment that this is shown. Then, since $N_X$ and $N_Y$ are contractible, we get that $X'$ and $Y'$ are quasi-isomorphic and therefore, since both are right bounded complexes of projective modules, homotopy equivalent. Once we can show that then $X'$ and $Y'$ are isomorphic as complexes, then also $N_X$ and $N_Y$ are isomorphic. Indeed, $N_X$ and $N_Y$ are isomorphic as graded modules. Now, since $N_X$ and $N_Y$ are contractible, they are both isomorphic to a direct sum of copies of shifted copies of complexes of the form

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\simeq} M \longrightarrow 0 \longrightarrow \cdots$$

Comparing the direct factors, and using the fact that $N_X$ and $N_Y$ are isomorphic as graded modules, one sees that $N_X \simeq N_Y$ as complexes. So, we suppose for the moment in the statement of the lemma that $im(\partial^X) \subseteq rad(X)$. But then, $\varphi\psi - id_X = h\partial^X + \partial^X h$, and therefore $(\varphi\psi - id_X)(X_i) \subseteq rad(X_i)$ for any degree $i$. Nakayama's Lemma implies that $\varphi\psi$ is invertible. Likewise $\psi\varphi$ is invertible. Hence, $\varphi$ is an isomorphism.

We need to show that $X \simeq X' \oplus N_X$ for a contractible $N_X$ and a complex $X'$ with $im(\partial^X|_{X'}) \subseteq rad(X')$. Since $\partial^X(rad(X_i)) \subseteq rad(X_{i-1})$, the complex $X$ induces a complex $(X/rad(X), \overline{\partial}^X)$ of semisimple modules. Let $m$ be the smallest degree such that $X_{m-1}$ is non-zero. Denote $\overline{X} := X/rad(X)$. If

$$0 \neq \overline{\partial}^X_m : \overline{X_m} \longrightarrow \overline{X}_{m-1}, \text{ then } \overline{X}_{m-1} \simeq \overline{X}'_{m-1} \oplus \overline{X}''_{m-1}$$

so that $\overline{\partial}^X_m : \overline{X}_m \longrightarrow \overline{X}''_{m-1}$ is surjective. But then, $\partial^X_m$ is also surjective onto the projective cover $X''_{m-1}$ of $\overline{X}''_{m-1}$. Since $X''_{m-1}$ is projective, there is a splitting $\sigma_{m-1} : X''_{m-1} \longrightarrow X_m$ of $\partial^X_m$ and therefore, $X_m \simeq X'_m \oplus X''_{m-1}$ and $X_{m-1} \simeq X'_{m-1} \oplus X''_{m-1}$ so that $\partial^X_m$ is transformed by these isomorphisms into $\begin{pmatrix} \partial^{X'}_m & 0 \\ 0 & id_{X''_{m-1}} \end{pmatrix}$. Now, by induction define $N_X := X''$ and one gets $X \simeq X' \oplus N_X$ and $\partial^X|_{X'}$ induces the 0-mapping modulo the radical. This is tantamount to saying that $im(\partial^X|_{X'}) \subseteq rad(X')$. $\square$

As a consequence of the lemma, we see that the orbits in $comproj^{\underline{d}}$ under the action of $G$ correspond to homotopy equivalence classes, or equivalently quasi-isomorphism classes, of right bounded complexes of projective modules with fixed dimension array $\underline{d}$. Note however that $\underline{d}$ is not preserved under quasi-isomorphism.

**Lemma 2.** *Let $M$ and $N$ be right bounded complexes of finitely generated projective modules. Then, there is a dimension array $\underline{d}$ and homotopy equivalent complexes $M \simeq M'$ and $N \simeq N'$ so that $M', N' \in comproj^{\underline{d}}$.*

**Proof.** Let $n$ be the smallest degree such that the homogeneous component of $M$ or $N$ is non-zero.

$$M'_m := \bigoplus_{j=n}^{m} M_j \oplus \bigoplus_{j=n}^{m} N_j \text{ and } N'_m := \bigoplus_{j=n}^{m} N_j \oplus \bigoplus_{j=n}^{m} M_j,$$

where the differential $d_{M'}$ is chosen to be $d_M$ on $M_m$, and the differential $d_{N'}$ is chosen to be $d_N$ on $N_m$. Moreover,

$$d_{M'}|_{M_k} = \begin{cases} id & \text{if } m-k>0 \text{ is even} \\ 0 & \text{else} \end{cases} \quad \text{whereas } d_{M'}|_{N_k} = \begin{cases} id & \text{if } m-k \text{ is odd} \\ 0 & \text{else.} \end{cases}$$

Define the differential on $N'$ likewise, and get this way that $M'_m \simeq N'_m$ for all $m$, and also $M \simeq M'$ as well as $N \simeq N'$. $\square$

We define for a complex $X$ the complex $X[1]$ shifted by one degree to the left by $(X[1])_m := X_{m-1}$ and $\partial^{X[1]}_m := -\partial^X_{m-1}$.

## 2. Algebraic relation implies topological relation

Let $A$ be an algebra over an algebraically closed field $k$, let $D^-(A)$ be the derived category of right bounded complexes of finitely generated $A$-modules, and let $D^b(A)$ be its full subcategory formed by bounded complexes of $A$-modules. Let $K^-(A)$ be the homotopy category of right bounded complexes of finitely generated projective $A$-modules and $K^{-,b}(A)$ the image of $D^b(A)$ in $K^-(A)$ under the equivalence $K^-(A) \simeq D^-(A)$. Concerning conventions for derived categories we shall follow [7].

For any $X$ and $Y$ in $D^-(A)$, let $X \leqslant_\Delta Y$ if there is a distinguished triangle

$$Y \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow Y[1]$$

for an object $Z$ in $D^-(A)$.

On the topological side we define a relation $\leqslant_{\text{top}}$ by

$$X \leqslant_{\text{top}} Y :\Leftrightarrow Y \in \overline{G \cdot X}$$

for $X, Y \in comproj^{\underline{d}}$.

We denote by $\underline{dim}(X)$ the dimension array of a complex $X \in K^-(A)$.

Observe that if $X$ and $Y$ are in $D^b(A)$, then $X \leqslant_\Delta Y$ implies $[X] = [Y]$ in $K_0(D^b(A))$.

**Theorem 1.** *Let $M$ and $N$ be right bounded complexes of finitely generated projective modules with the same dimension array $\underline{d}$. Then, $M \leqslant_\Delta N$ implies $M \leqslant_{\text{top}} N$ in $comproj^{\underline{d}}$.*

**Proof.** Let $U$ be a subset of $comproj^{\underline{d}}$. We show that $\overline{U} = \bigcap_n \pi_n^{-1}(\overline{\pi_n(U)})$. The inclusion $\subseteq$ is obvious. Let $C$ be a closed subset of $comproj^{\underline{d}}$ with $U \subseteq C$. Then $C = \bigcap_n \pi_n^{-1}(C_n)$ for closed subsets $C_n \subseteq comproj^{\underline{d}_n}$. Hence $U \subseteq \pi_n^{-1}(C_n)$ and so $\overline{\pi_n(U)} \subseteq C_n$ for every $n$, which proves the other inclusion.

Now, if one can prove that whenever $M \leqslant_\Delta N$, then $\pi_n(M) \leqslant_\Delta \pi_n(N)$, and moreover, if this implies that $\pi_n(N) \in \overline{G \cdot \pi_n(M)}$, then by the above, $N \in \overline{G \cdot M}$. This means that $M$ degenerates to $N$ in the topological sense.

We still have to show that if $M \leqslant_\Delta N$ then $\pi_n(N) \in \overline{G \cdot \pi_n(M)}$. We shall use the very same proof as in the module case by Riedtmann [10]. Let $M \leqslant_\Delta N$. Then, there is a complex $Z$ of projective modules so that

$$N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow N[1]$$

is a distinguished triangle. This implies that

$$Z[-1] \longrightarrow N \longrightarrow M \oplus Z \longrightarrow (Z[-1])[1]$$

is a distinguished triangle. Hence, $M \oplus Z \simeq cone(Z[-1] \longrightarrow N)$ in the homotopy category. Now, we use that the dimension array of $N$ and of $M$ coincide. Indeed,

$$\begin{aligned}\underline{dim}(cone(Z[-1] \longrightarrow N)) &= \underline{dim}(Z) + \underline{dim}(N) = \underline{dim}(Z) + \underline{dim}(M)\\ &= \underline{dim}(M \oplus Z).\end{aligned}$$

Hence, $cone(Z[-1] \longrightarrow N) \simeq M \oplus Z$ in the category of complexes and so there is a sequence

$$0 \longrightarrow N \xrightarrow{(\phi,\alpha)} Z \oplus M \xrightarrow{\binom{\beta}{\psi}} Z \longrightarrow 0$$

which is exact in the category of complexes. This shows at once that $M \leqslant_{\Delta} N$ implies $\pi_n(M) \leqslant_{\Delta} \pi_n(N)$ for any $n$.

The first assertion is that $\beta$ is invertible if and only if $\alpha$ is invertible and in this case,

$$0 \longrightarrow N \xrightarrow{(\phi,\alpha)} Z \oplus M \xrightarrow{\binom{\beta}{\psi}} Z \longrightarrow 0$$

is isomorphic to

$$0 \longrightarrow N \xrightarrow{(0,\alpha)} Z \oplus M \xrightarrow{\binom{\beta}{0}} Z \longrightarrow 0$$

and therefore $N \simeq M$. Indeed, we get an isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "(\phi,\alpha)"] \arrow[d, equal] & Z \oplus M \arrow[r, "\binom{\beta}{\psi}"] \arrow[d, "\begin{pmatrix} id_Z & 0 \\ \psi\beta^{-1} & id_M \end{pmatrix}"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "(0,\alpha)"] & Z \oplus M \arrow[r, "\binom{\beta}{0}"] & Z \arrow[r] & 0
\end{tikzcd}$$

and likewise for $\alpha$ invertible.

For any $t \in k$ we have a homomorphism of complexes $\binom{\beta+t\cdot id_Z}{\psi}$. Let

$$N_t := ker\left(\binom{\beta + t \cdot id_Z}{\psi}\right)$$

in the category of complexes. For any $t$ with $f_t := \binom{\beta+t\cdot id_Z}{\psi}$ being surjective we have that $f_t$ is locally split. Here we call a homomorphism of complexes $g$ locally split if $g$ is split in each degree, but not necessarily split as a homomorphism of complexes. For all such $t$ we see that $N_t$ is a complex of projective modules with the same dimension array as $N$. We now consider $\pi_n(N_t)$, $\pi_n(N)$, $\pi_n(M)$, $\pi_n(Z)$ and the induced mappings on the truncated complexes. Of course, we still have $ker(\pi_n(f_t)) = \pi_n(N_t)$.

We shall prove that

$$t \mapsto \pi_n(N_t) \in comproj^{\underline{dim}\pi_n(N)}$$

is a rational morphism of varieties, imitating Christine Riedtmann's proof in [10].

There is an open neighborhood $U$ of 0 in $k$ so that $\pi_n(f_t)$ is surjective for all $t \in U$, using the fact that being surjective is an open condition and that $\pi_n(f_0)$ is surjective.

Let

$$(B, \partial^B) \xrightarrow{f} (A, \partial^A) \longrightarrow 0$$

be a surjective map of complexes of projective modules. We want to compute the kernel $(C, \partial^C)$ of this map. Since the structure of $C$ as graded module is clear, we may choose bases in $B$ so that we can identify $B$ with $C \oplus A$ as graded modules. Let $g = (g_C, g_A) : C \longrightarrow B$ be the inclusion of the kernel. We have $f = \binom{f_C}{f_A}$ where $f_A$ is an isomorphism. Then $g_C$ is an isomorphism as well and we may assume that $g_C = id_C$. But then,

$$g_A = -f_C f_A^{-1}$$

since $f_A$ is invertible. The differential on $ker(f)$ depending on $f$ is

$$\partial_C = (id_C, -f_C f_A^{-1}) \cdot \partial_B \cdot \begin{pmatrix} id_C \\ 0 \end{pmatrix}.$$

Thus we get a rational morphism of varieties $Hom(B, A) \longrightarrow comproj^{\underline{dim}(C)}$ defined on the open neighborhood of $f \in Hom(B, A)$ for which $f_A$ is an isomorphism.

We may now apply this construction to the map $f_0$ and by composing with the map

$$t \mapsto \begin{pmatrix} \beta + t \cdot id_Z \\ \psi \end{pmatrix}$$

we get the promised rational morphism of varieties.

Finally, for those $t$ for which $\pi_n(\beta + t \cdot id_Z)$ is an isomorphism, that is for all but the finite number of eigenvalues of $-\beta$, we get $\pi_n(N_t) \simeq \pi_n(M)$ and for $t = 0$ we get $\pi_n(N_0) \simeq \pi_n(N)$. Therefore

$$\pi_n(N) \in \overline{G \cdot \pi_n(M)}. \qquad \square$$

## 3. Geometric relation implies algebraic relation

We shall prove in this section that under some conditions the inverse implication of Theorem 1 is true as well.

Let $\underline{d} = (d_n, \ldots, d_m)$ be a bounded dimension array. We associate to the affine variety $comproj^{\underline{d}}(k)$ an affine $k$-scheme $comproj^{\underline{d}}(-)$. This $k$-scheme has the following functorial description. Let $R$ be a commutative $k$-algebra. Let $comproj^{\underline{d}}(R)$ denote the subset of

$$\prod_{i \in \mathbb{Z}} Hom_R(R^{\alpha(d_i)}, R^{\alpha(d_{i-1})})$$

consisting of elements $(\partial_i)_{i \in \mathbb{Z}}$ with the properties that $\partial_i$ is an $R \bigotimes_k A$-homomorphism when viewed as a map from $\bigoplus_{j=1}^{l} (R \bigotimes_k P_j^{d_i^j})$ to $\bigoplus_{j=1}^{l} (R \bigotimes_k P_j^{d_{i-1}^j})$ and $\partial_i \partial_{i-1} = 0$. For a $k$-algebra homomorphism $f : S \longrightarrow R$, there is naturally a corresponding map $f^* : comproj^{\underline{d}}(S) \longrightarrow comproj^{\underline{d}}(R)$ sending a tuple of matrices $(\partial_i)$ to the tuple $(f(\partial_i))$. Similarly we may associate to the affine algebraic group $G$ a smooth affine group scheme $G(-)$ over $k$. The action of $G$ on $comproj^{\underline{d}}$ extends to an action of $G(-)$ on $comproj^{\underline{d}}(-)$.

We may verify Grunewald–O'Halloran's conditions which are necessary to apply [5, Theorem 1.2].

**Theorem 2.** *Let* $\underline{d}$ *be a dimension array. Let* $M, N \in comproj^{\underline{d}}$ *be two complexes with bounded homology. If* $M \leqslant_{\mathrm{top}} N$ *then* $M \leqslant_{\Delta} N$.

**Proof.** Let $n$ be an integer such that the homology of $M$ and $N$ vanishes in all degrees larger than or equal to $n$. We will construct a short exact sequence of complexes

$$0 \longrightarrow \pi_n(N) \longrightarrow \pi_n(M) \oplus Z_{(n)} \longrightarrow Z_{(n)} \longrightarrow 0,$$

where $Z_{(n)}$ is a complex of projective $A$-modules. We are going to follow the steps of Zwara's proof for the module case.

By Grunewald–O'Halloran's result [5, Theorem 1.2], there is a discrete valuation $k$-algebra $R$ with maximal ideal $\mathfrak{m}$ and residue field $k$ and with over $k$ finitely generated quotient field $K$ of transcendence degree 1 and a complex $Y$ in $comproj^{\underline{d}_n}(R)$ so that $k \bigotimes_R Y = \pi_n(N)$ and as complexes of $K \bigotimes_R A$-modules, $K \bigotimes_R Y = g \cdot (K \bigotimes_k \pi_n(M))$ for a $g \in G(K)$. Since the valuation on $R$ is discrete, $\mathfrak{m}$ is principal, generated by an element $f$.

Since $\underline{d}_n$ is bounded, there is a non-zero element $z \in R$ so that $zg$ is a tuple of matrices with entries in $R$. Using the explicit definition of the action, we get

$$K \bigotimes_R Y = g \cdot (K \bigotimes_k \pi_n(M)) = zg \cdot (K \bigotimes_k \pi_n(M)).$$

So, we may assume that $g$ is a tuple of matrices with entries in $R$. Restricting the multiplication with $g$ to $R \bigotimes_k \pi_n(M)$ gives a morphism of complexes of $R \bigotimes_k A$-modules $\varphi : R \bigotimes_k \pi_n(M) \longrightarrow Y$. Let $X$ denote the image of this morphism. Both $X$ and $Y$ are complexes of free $R$-modules, with equal rank in all degrees; therefore there exists some $s$ such that $\mathfrak{m}^s Y \subseteq X$.

Now we take the point of view that the complexes $X$ and $Y$ are graded $R \bigotimes_k A$-modules with differentials. Fix a $k$-basis $\mathscr{B}$ of $R$. As complexes of $A$-modules we have

$$X = \bigoplus_{b \in \mathscr{B}} X_b,$$

where $X_b = \varphi(\langle b \rangle \bigotimes_k \pi_n(M)) \cong \pi_n(M)$ and where $\langle b \rangle$ denotes the $k$-subspace of $R$ generated by $b$.

For each $h$ we have a short exact sequence of complexes

$$0 \longrightarrow Y/\mathfrak{m}Y \longrightarrow Y/\mathfrak{m}^{h+1}Y \longrightarrow Y/\mathfrak{m}^h Y \longrightarrow 0.$$

We will show that there exists an $h$ such that $Y/\mathfrak{m}^{h+1}Y \simeq \pi_n(M) \oplus (Y/\mathfrak{m}^h Y)$ as complexes of $A$-modules where the mapping $(Y/\mathfrak{m}Y) \simeq (\mathfrak{m}^h Y/\mathfrak{m}^{h+1}Y) \longrightarrow (Y/\mathfrak{m}^{h+1}Y)$ is induced by multiplication by $f^h$ and canonical inclusion. Let $V = \bigoplus_i V_i$ be a graded vector space formed by taking vector space complements of $X_i$ in $Y_i$ in each degree $i$. Note that $V$ is a finite dimensional vector space since $Y$ is bounded and $\mathfrak{m}^s Y \subseteq X$. Let $Z_0$ be the smallest $A$-subcomplex of $Y$ containing $V$. Then $Z_0$ is a finite dimensional complex of $A$-modules, since $Y$ is bounded. Now $Y = X + Z_0$ as complexes of $A$-modules. Let $\mathscr{V}$ be a finite subset of $\mathscr{B}$ such that $Z_0 \cap \bigoplus_{b \in \mathscr{V}} X_b = Z_0 \cap X$. Such a subset exists since $Z_0$ is finite dimensional over $k$. Let $Z_1 = Z_0 + \bigoplus_{b \in \mathscr{V}} X_b$. Then $Y = Z_1 \oplus \bigoplus_{b \notin \mathscr{V}} X_b$. Since $\mathscr{V}$ is finite there exists an integer $t$ such that $\mathfrak{m}^{t+1}X \cap \bigoplus_{b\in\mathscr{V}} X_b = 0$. Thus there is a finite subset $\mathscr{W}$ of $\mathscr{B}$ such that

$$\mathfrak{m}^{t+1}X \oplus \bigoplus_{b\in\mathscr{W}} X_b \oplus \bigoplus_{b\in\mathscr{V}} X_b = X.$$

Let $Z_2 = Z_1 + \bigoplus_{b\in\mathscr{W}} X_b$. Then $Y = \mathfrak{m}^{t+1}X \oplus Z_2$.

It follows that we have a chain of inclusions

$$\mathfrak{m}^{s+t+2}Y \subseteq \mathfrak{m}^{t+2}X \subseteq \mathfrak{m}^{t+1}X \subseteq Y,$$

where the last two inclusions have direct complements as complexes of $A$-modules. Thus

$$\begin{aligned} Y/\mathfrak{m}^{s+t+2}Y &\cong (\mathfrak{m}^{t+2}X/\mathfrak{m}^{s+t+2}Y) \oplus \pi_n(M) \oplus (Y/\mathfrak{m}^{t+1}X) \\ &\simeq \pi_n(M) \oplus (Y/\mathfrak{m}^{s+t+1}Y), \end{aligned}$$

where the last isomorphism follows since

$$\mathfrak{m}^{t+2}X/\mathfrak{m}^{s+t+2}Y \simeq \mathfrak{m}^{t+1}X/\mathfrak{m}^{s+t+1}Y \text{ and}$$
$$\mathfrak{m}^{t+1}X/\mathfrak{m}^{s+t+1}Y \oplus Y/\mathfrak{m}^{t+1}X \simeq Y/\mathfrak{m}^{s+t+1}Y.$$

Now since $Y/\mathfrak{m}Y \cong \pi_n(N)$ we get the promised short exact sequence of complexes $0 \longrightarrow \pi_n(N) \longrightarrow \pi_n(M) \oplus Z_{(n)} \longrightarrow Z_{(n)} \longrightarrow 0$ by choosing $Z_{(n)} = Y/\mathfrak{m}^{s+t+1}Y$.

Now construct a complex $N'$ by splicing $\pi_n(N)$ with a projective resolution $P_N$ of $H_n(\pi_n(N))$. Similarly we form a complex $Z$ by splicing a projective resolution $P_Z$ of $H_n(Z_{(n)})$ with $Z_{(n)}$. By the horseshoe lemma, there exists a short exact sequence $0 \longrightarrow P_N \longrightarrow P_{M\oplus Z} \longrightarrow P_Z \longrightarrow 0$ of projective resolutions where $P_{M\oplus Z} \simeq P_N \oplus P_Z$ as graded modules and where $P_{M\oplus Z}$ is a projective resolution of $H_n(\pi_n(M) \oplus Z_{(n)}) \cong H_n(\pi_n(M)) \oplus H_n(Z_{(n)})$. Moreover we have a short exact sequence of complexes

$$0 \longrightarrow N' \longrightarrow M' \longrightarrow Z \longrightarrow 0,$$

where the complex $M'$ is formed by splicing $P_{M\oplus Z}$ with the complex $\pi_n(M) \oplus Z_{(n)}$. Now $N'$, $M'$ are homotopy equivalent to $N$, $M \oplus Z$, respectively. Thus we get a triangle $N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow N[1]$, which completes the proof of the theorem. $\square$

## 4. Consequences for the geometry of complexes

We continue with some consequences and observations on $comproj^{\underline{d}}$ and the orders $\leqslant_\Delta$ and $\leqslant_{\text{top}}$.

**Example 4.1.** We consider the quiver $Q$ defined by $\bullet_1 \longrightarrow \bullet_2$. Then, up to isomorphism, there are 3 indecomposable $kQ$-modules: the indecomposable projective module $P_1$ corresponding to the vertex 1 and the two simple modules $S_1$ and $P_2$. Moreover, in the representation variety $mod(kQ, (1, 1))$ of 2-dimensional $kQ$-modules with two different composition factors, one has that the projective module with top 1 degenerates to the direct sum of the two simple modules. The projective indecomposable module with top 1 can be considered as being in $comproj^{\left(\binom{0}{0},\binom{1}{0}\right)}$, where $\binom{a}{b}$ indicates that in a certain degree the module is $P_1^a \oplus P_2^b$. The semi-simple module $S_1 \oplus S_2$ is in $comproj^{\left(\binom{0}{1},\binom{1}{1}\right)}$. So, the modules are represented in different varieties $comproj^{\underline{d}}$ and here it is not possible to consider degenerations between them if one declares that a complex $X$ degenerates to a complex $Y$ if $Y$ is in the closure of the orbit of $X$. Nevertheless, one may consider another non-minimal projective resolution of $P_1$ as

$$P_2 \xrightarrow{(id_{P_2},0)} P_2 \oplus P_1.$$

This complex can be seen as being in $comproj^{\left(\binom{0}{1},\binom{1}{1}\right)}$, and the minimal projective resolution of $S_1 \oplus P_2$ is

$$P_2 \xrightarrow{(0,\imath)} P_2 \oplus P_1$$

for $\imath$ being the embedding $P_2 \longrightarrow P_1$. Therefore, $P_1$ and $S_1 \oplus P_2$ can be both visualized in $comproj^{\left(\binom{0}{1},\binom{1}{1}\right)}$. Moreover it is easy to see that $P_1 \leqslant_{\text{top}} S_1 \oplus P_2$. This observation is one of the motivations not to ask for the complexes to be minimal as is done in [1] and to allow zero homotopic direct summands.

Let $M$ and $N$ be $d$-dimensional $A$-modules. We write $M \leqslant N$ if $M$ degenerates to $N$ in $mod(A,d)$.

**Proposition 3.** *Let $M, N \in mod(A,d)$ for some dimension d and let $P_M, P_N \in comproj^{\underline{d}}$ for some dimension array $\underline{d}$ be a projective resolution of M and N, respectively. Then, $M \leqslant N$ in $mod(A,d)$ if and only if $P_M \leqslant_{\text{top}} P_N$ in $comproj^{\underline{d}}$.*

**Proof.** If $M \leqslant N$, then by Zwara's theorem [12] there is an exact sequence

$$0 \longrightarrow N \longrightarrow Z \oplus M \longrightarrow Z \longrightarrow 0$$

for an $A$-module $Z$. This implies a distinguished triangle

$$P_N \longrightarrow P_Z \oplus P_M \longrightarrow P_Z \longrightarrow P_N[1]$$

in $K^-(A)$ where $P_Z$ is a projective resolution of $Z$. Hence, $P_M \leqslant_\Delta P_N$ and so by Theorem 1 we have $P_M \leqslant_{\text{top}} P_N$.

Conversely, suppose $P_M \leqslant_{\text{top}} P_N$ and so by Theorem 2 we have $P_M \leqslant_\Delta P_N$. Then, there is a complex $Z$ and a distinguished triangle

$$P_N \longrightarrow Z \oplus P_M \longrightarrow Z \longrightarrow P_N[1].$$

Taking homology of this triangle gives a long exact sequence

$$\longrightarrow H_{i+1}(Z) \longrightarrow H_i(P_N) \longrightarrow H_i(Z) \oplus H_i(P_M) \longrightarrow H_i(Z) \longrightarrow H_{i-1}(P_N) \longrightarrow$$

where $H_i(P_N) = H_i(P_M) = 0$ for $i > 0$. For $i = 0$ one gets an exact sequence

$$0 \longrightarrow H_1(Z) \longrightarrow H_1(Z) \longrightarrow N \longrightarrow H_0(Z) \oplus M \longrightarrow H_0(Z) \longrightarrow 0.$$

This implies that

$$0 \longrightarrow N \longrightarrow H_0(Z) \oplus M \longrightarrow H_0(Z) \longrightarrow 0$$

is a short exact sequence and hence $M \leqslant N$ in $mod(A, d)$, again by Zwara's theorem [12]. This proves the statement. □

**Lemma 4.** *Let $\underline{d}$ be any dimension array and let $T$ be an element in $comproj^{\underline{d}}$ so that $G \cdot T$ is open. Then, $T$ is a minimal element for $\leqslant_{\Delta}$ and for $\leqslant_{\mathrm{top}}$.*

**Proof.** If $G \cdot T$ is open, then $comproj^{\underline{d}} \backslash \{G \cdot T\}$ is closed and for any $X \not\simeq T$ one has that

$$\overline{G \cdot X} \subseteq comproj^{\underline{d}} \backslash \{G \cdot T\}.$$

Hence, $T$ is minimal with respect to $\leqslant_{\mathrm{top}}$, and since $\leqslant_{\Delta}$ implies $\leqslant_{\mathrm{top}}$, the complex $T$ is minimal also with respect to $\leqslant_{\Delta}$. □

Observe that we only used the topology of the space in the previous argument. We shall see that for bounded $\underline{d}$ the orbits of $T$ with $Hom_{D^b(A)}(T, T[1]) = 0$ are open.

**Lemma 5.** *Let $\underline{d}$ be a bounded dimension array and let $X$ be a complex in $comproj^{\underline{d}}$. If $Hom_{D^b(A)}(X, X[1]) = 0$, then $G \cdot X$ is open in $comproj^{\underline{d}}$.*

**Proof.** First assume that $\underline{d}$ is a bounded dimension array. From Theorem 7 in [11] we see that the orbit of $X$ in $comp(A, \underline{\alpha}(\underline{d}))$ is open if $Hom_{D^b(A)}(X, X[1]) = 0$. The result now follows since $comproj^{\underline{d}}$ is a subvariety of $comp(A, \underline{\alpha}(\underline{d}))$ and since $G \cdot X = comproj^{\underline{d}} \cap (Gl_{\underline{\alpha}(\underline{d})} \cdot X)$. □

**Lemma 6.** *The relation $\leqslant_{\mathrm{top}}$ is a partial order on the set of isomorphism classes of complexes with bounded homology with fixed dimension array $\underline{d}$.*

**Proof.** If $N \in \overline{G \cdot M}$ and $M \in \overline{G \cdot L}$, then clearly $N \in \overline{G \cdot L}$. If $N \in \overline{G \cdot M}$ and $M \in \overline{G \cdot N}$, then by the proof of Theorem 1 we get $\pi_n(N) \in \overline{G \cdot \pi_n(M)}$ and $\pi_n(M) \in \overline{G \cdot \pi_n(N)}$ for all $n \in \mathbb{Z}$. This implies $\pi_n(N) \simeq \pi_n(M)$ for all $n \in \mathbb{Z}$.

We show that whenever $X$ is a complex with bounded homology in $comproj^{\underline{d}}$, then denoting by $m$ an integer so that the homology of $X$ is 0 in all degrees higher than $m$, then $Y \in G \cdot X$ if and only if $Y \in \pi_{\ell}^{-1}(G \cdot \pi_{\ell}(X))$ for all $\ell \geqslant m + 1$. Indeed, assume that $Y \in \pi_{\ell}^{-1}(G \cdot \pi_{\ell}(X))$ for all $\ell \geqslant m + 1$. Then we have an isomorphism of homology groups $H_n(Y) \simeq H_n(X)$ for all $n$, which shows that $H_n(Y) = 0$ for all $n \geqslant m + 1$. Then there is an isomorphism $\pi_{m+1}(Y) \simeq \pi_{m+1}(X)$, which lifts to a homotopy equivalence $Y \simeq X$ and so $Y \in G \cdot X$. The reverse implication is trivial.

Hence, one has $N \simeq M$. □

**Remark 4.2.** Saorin and Huisgen-Zimmermann [11, Theorem 7] cited in the proof of Lemma 5 shows that the tangent space of the variety of complexes $comp(A, \underline{\alpha})$ at some point $X$ modulo the tangent space of the orbit of $X$ under the group which is acting at $X$

is isomorphic to $Hom_{D^b(A)}(X, X[1])$. A similar result can be proven for $comproj^{\underline{d}}$ and the action of our smaller group. We also mention that Lemma 5 has a converse in the case where $\underline{d}$ is bounded. Namely, if $G \cdot X$ is open in $comproj^{\underline{d}}$ then $Hom_{D^b(A)}(X, X[1]) = 0$. This can again be seen from [11, Theorem 7].

**Corollary 7.** *Complexes with* $Hom_{D^b(A)}(X, X[1]) = 0$ *are minimal with respect to both* $\leqslant_\Delta$ *and* $\leqslant_{\text{top}}$. *In particular, partial tilting complexes are minimal with respect to both orders.*

We also give a consequence which does not require an algebraically closed field.

**Corollary 8.** *Let A be an algebra over a field K. Then, up to homotopy equivalence there is at most one two-term partial tilting complex*

$$T = \ldots \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0 \longrightarrow \ldots$$

*with fixed homogeneous components* $P_0$ *and* $P_1$.

**Proof.** Since two-term complexes of projective modules are entirely determined by their homology, and since for any field extension $L$ of $K$ one has $H(L \bigotimes_K X) \simeq L \bigotimes_K H(X)$ for any complex $X$, we may assume that $K$ is algebraically closed. Let $\alpha_i = \underline{dim}(P_i)$ for $i \in \{0, 1\}$ and $\underline{\alpha} := (\alpha_1, \alpha_0)$. The variety $comproj^{\underline{d}(\underline{\alpha})}$ is an affine space, and therefore irreducible as algebraic variety. Moreover, since $T$ is a partial tilting complex, the orbit $G \cdot T$ is open in $comproj^{\underline{d}(\underline{\alpha})}$. Therefore, $G \cdot T$ is dense. Let $S$ be another partial tilting complex in $comproj^{\underline{d}(\underline{\alpha})}$. Also $G \cdot S$ is open and dense, and therefore $S \leqslant_{\text{top}} T$ as well as $T \leqslant_{\text{top}} S$. Hence, $S \simeq T$ by Lemma 6. $\square$

**Example 4.3.** Corollary 8 does not hold for general dimension arrays. Let $A$ be given by the quiver

$$\bullet_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet_2$$

with relations $\alpha\beta\alpha = \beta\alpha\beta = 0$. For this algebra, take the indecomposable complex (unique up to isomorphism so that $P_2$ is in degree 0)

$$T_1 := \cdots \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 0 \longrightarrow \cdots$$

and $T_2 := P_1 \longrightarrow P_2$. Then, $T := T_1 \oplus T_2$ is a tilting complex. Let $\underline{d}$ be the dimension array of $T$. The complex $S$

$$P_1 \xrightarrow{(0,id)} P_1 \oplus P_1 \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}} P_2 \oplus P_2$$

is homotopy equivalent to the tilting complex $S'$

$$P_1 \xrightarrow{(\alpha,0)} P_2 \oplus P_2.$$

Here both tilting complexes $T$ and $S$ have the same dimension array, but are not isomorphic, and therefore belonging to different irreducible components of $comproj^{\underline{d}}$. Using [8] and a slightly more detailed examination of $comproj^{\underline{d}}$, one observes that $comproj^{\underline{d}}$ has exactly two irreducible components.

The complex $T_1 \oplus P_1[2]$ is a tilting complex as well and denote by $\underline{e}$ the dimension array of $T_1 \oplus P_1[2]$. A short examination yields that $comproj^{\underline{e}}$ has two irreducible components, one $C_3$ of dimension 3 and another component $C_4$ of dimension 4. The orbit of $T_1 \oplus P_1[2]$ is open in $C_3$, whereas the complexes corresponding to the points in $C_4$ are not partial tilting complexes. Observe, however, in $C_4$ that there is an open orbit of a complex $U \simeq P_1[2] \oplus P_2$ with $Hom_{D^b(A)}(U, U[1]) = 0 \neq Hom_{D^b(A)}(U, U[2])$.

**Remark 4.4.** Observe that a tilting complex $T$ over $A$ is the image $F(B)$ of an equivalence $F : D^b(B) \longrightarrow D^b(A)$ of triangulated categories. By Rickard's and Keller's main theorem [9,6] there is a so-called two-sided tilting complex $X$ of $A \bigotimes_K B^{op}$-modules which are projective on the left and on the right, so that $X \bigotimes^{\mathbb{L}}_B -$ is an equivalence. For any dimension array $\underline{d}$, let $X \otimes \underline{d}$ be the dimension array which is obtained by tensoring a complex with dimension array $\underline{d}$ by $X$, and taking the total complex of the resulting bi-complex. Then, by definition $X \bigotimes_B -$ induces a morphism of varieties

$$comproj(X) : comproj^{\underline{d}}_B \longrightarrow comproj^{X \otimes \underline{d}}_A .$$

It should be an interesting question to study the image of this morphism inside $comproj^{X \otimes \underline{d}}_A$. Note that studying varieties using functors is already far from trivial in the module case (see [2,13]).

There is another consequence of these statements. Indeed, define for any two complexes $X$ and $Y$ in $K^{-,b}(A)$

$$X \leqslant_{Hom} Y :\Leftrightarrow \forall U \in D^b(A) : \; dim_k(Hom_{D^b(A)}(U, X)) \leqslant dim_k(Hom_{D^b(A)}(U, Y)).$$

**Lemma 9.** *Let $X$ and $Y$ be two complexes in $comproj^{\underline{d}}$ for bounded dimension array $\underline{d}$. Then, $X \leqslant_{\mathrm{top}} Y \Rightarrow X \leqslant_{Hom} Y$.*

**Proof.** Define for any two complexes $X$ and $Y$ with appropriate bounded dimension array $\underline{d}$ and $\underline{e}$ the mapping

$$\varphi_{X,Y} : \prod_{i \in \mathbb{Z}} Hom_A(X_i, Y_{i+1}) \longrightarrow Hom_{C^b(A)}(X, Y)$$

by $\varphi_{X,Y}(f) := \partial_X f + f \partial_Y$. It is clear that this image is exactly the set of 0-homotopic homomorphisms. Hence, we have that

$$dim_k(Hom_{D^b(A)}(X, Y)) = dim_k(Hom_{C^b(A)}(X, Y)) - dim_k(im(\varphi_{X,Y})).$$

We use the argument from [4, Section 3, Theorem 2, special case] to show that

$$\{U\} \times comproj^{\underline{d}} \longrightarrow \mathbb{N}$$

given by $(U, X) \mapsto dim_k(Hom_{D^b(A)}(U, X))$ is upper semi-continuous. Then, setting $n = dim_k(Hom_{D^b(A)}(U, X))$, one gets $\{Z|dim_k(Hom_{D^b(A)}(U, Z)) \geqslant n\}$ is closed, and if $Y \in \overline{G \cdot X}$, then $Y \in \{Z|dim_k(Hom_{D^b(A)}(U, Z)) \geqslant n\}$. Hence,

$$dim_k(Hom_{D^b(A)}(U, Y)) \geqslant dim_k(Hom_{D^b(A)}(U, X)).$$

This proves the statement. □

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## References

[1] V. Bekkert, Y. Drozd, Tame–wild dichotomy for derived categories, math.RT/0310352v1.
[2] K. Bongartz, A geometric version of the Morita equivalence, J. Algebra 139 (1991) 159–171.
[3] C. de Concini, E. Strickland, On the variety of complexes, Adv. Math. 41 (1981) 57–77.
[4] W. Crawley-Boevey, Geometry of representations of algebras, lecture notes available on http://www.amsta.leeds.ac.uk/~pmtwc/geomreps.pdf.
[5] F. Grunewald, J. O'Halloran, A characterisation of orbit closure and applications, J. Algebra 116 (1988) 163–175.
[6] B. Keller, A remark on tilting theory and DG algebras, Manuscripta Math. 79 (1993) 247–253.
[7] S. König, A. Zimmermann, Derived equivalences for group rings (with contributions by B. Keller, M. Linckelmann, J. Rickard, R. Rouquier), Lecture Notes in Mathematics, vol. 1685, Springer, Berlin, 1998.
[8] V. Lakshmibai, P. Magyar, Degeneracy schemes, Quiver schemes and Schubert varieties, Internat. Math. Res. Notices 12 (1998) 627–640.
[9] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43 (1991) 436–456.
[10] C. Riedtmann, Degenerations for representations of quivers with relations, Ann. Sci. de École Norm. Supérieure 4ème série 19 (1986) 275–301.
[11] M. Saorin, B Huisgen-Zimmermann, Geometry of chain complexes and outer automorphisms under derived equivalence, Trans. Amer. Math. Soc. 353 (2001) 4757–4777.
[12] G. Zwara, Degenerations of finite-dimensional modules is given by extensions, Compositio Math. 121 (2000) 205–218.
[13] G. Zwara, Smooth morphisms of module schemes, Proc. London Math. Soc. 84 (2002) 539–558.