Orientational anisotropy and plastic spin in finite elasto-plasticity

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The present paper deals with the orientational anisotropy, in the multiplicative elasto-plastic models with non-zero spin and initial orthotropic anisotropy, under the supposition of small elastic strains, while elastic rotations and plastic deformations are large. A new rate form of the model is derived in the Eulerian setting. The evolution in time for the Cauchy stress, plastic part of deformation, tensorial hardening variables and elastic rotations involves the objective derivatives associated with the same elastic spin. A common plastic spin is allowed in the model as direct consequences that follows from the adopted constitutive framework of finite elasto-plastic materials with isoclinic configurations and internal variables. In this model the orientation of the orthotropy directions are characterized in terms of the Euler angles, which replace the elastic rotations. We provided a constitutive framework valuable for the description of the evolution of the orthotropy orientation during a deformation process whose principal directions are different from the orthotropic axes. Only when the plastic spin is non-vanishing, the orientational anisotropy could develop. We proved that only when there exists an initial orthotropic axis which is orthogonal to the sheet, the rotation of the orthotropic axes remains in plane, i.e. in the plane of the sheet, during a plane deformation process. We investigate the effects of three different analytical expressions for the common plastic spin. We make comparisons with the models and the numerical results already provided in the literature.

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1. Introduction

The constitutive model is developed in the finite elasto-plasticity, based on the multiplicative decomposition of the deformation gradient \( F \) into its elastic and plastic components, denoted by \( E \) and \( P \), respectively:

\[
F = EP,
\]

(1)

under the supposition that the elastic strains are small, while the elastic rotations, \( R \), are large. We adopt the point of view developed by Cleja-Tigoiu (1990), Cleja-Tigoiu and Soós (1990) to describe elasto-plastic materials with internal variables, with respect to plastically deformed configurations (i.e. the so called local relaxed isoclinic configurations), and the material symmetry concept introduced in Cleja-Tigoiu and Soós (1989, 1990) to characterize the structural anisotropy of materials. The general constitutive framework has been applied to a transversal anisotropic elasto-plastic material in Cleja-Tigoiu (2000a), and the orthotropic finite elasto-plasticity in Cleja-Tigoiu (2000b). When we consider that the crystallographic axes are kept constant as directions in all local relaxed configurations, we practically realize the so-called isoclinic configurations in the terminology adopted by Mandel (1972) and Teodosiu (1970).

The model proposed here includes non-zero plastic spin and three types of anisotropy, namely initial structural anisotropy (the orthotropic one), induced anisotropy of kinematic hardening type and orientational anisotropy, which are emphasized as soon as plastic deformations start to develop. As a consequence of the material symmetry assumptions made in Cleja-Tigoiu and Soós (1990), the constitutive and evolution equations should be invariant with respect to the material symmetry group, which characterizes the initial material anisotropy. In our case the initial structural anisotropy is characterized by the orthotropic symmetry group which is defined in Liu (1982) by

\[
g_b = \{ Q \in \text{Ort} | Qn_i = n_i, \text{ or } Qn_i = -n_i, \quad i = 1, 2, 3 \}.
\]

(2)

Only when the elastic strains are small during elasto-plastic processes, the deformed anisotropic axis remain orthogonal if they were initially orthogonal. In our case this means \( m_i = R'n_i \).

In this paper we shortly present the model with respect to plastically deformed (i.e. isoclinic) configurations in Section 2. Although the paper deals with small elastic strain while the elastic rotations and plastic deformations are finite, we start from the large deformation formalism since the kinematic relationships that follows from the multiplicative decomposition lied to well defined
correlations between spins and rate of deformations, respectively. Subsequently, we derive in Section 3 the rate independent model in the Eulerian description, by pushing away the constitutive relationships to the deformed configuration (for the pushing away procedure) see also Cleja-Tigoiu and Maugin, 2000. The objective time derivatives associated with the elastic strain, \( \varepsilon' := R'(R)^T \), naturally follow to be involved in the differential system which describes the evolution in time for the Cauchy stress, plastic part of deformation, and internal variables. The elastic spin is expressed through the difference between the motion spin \( W \) and the plastic spin pushed away to the deformed configuration. Indeed, the common plastic spin is allowed in the model as a direct consequence of the passage from the isoclinic configurations to the deformed configurations by a pushing away procedure.

In the rate form of the model we replace the elastic rotation, which characterizes the orientation of the orthotropy directions, by Euler’s angles. This new way of numerical implementation allows us to derive the complete rate type elasto-plastic constitutive model, those projection on the current anisotropic axes \( m_i \), contains the orientational variables described through Euler angles only, in Section 3. We introduce three different analytical expressions for the common plastic spin. We proved that the special case when the orthotropic directions remain perpendicular to the sheet plane can be derived from the general system, for all three plastic spins considered.

The present paper deals with orientational anisotropy, which has been emphasized experimentally by Kim and Yin (1997). The orientational anisotropy consists of the rotation of the anisotropy axes during a deformation process, whose principal directions are different from the orthotopic axes of the metal sheet. The orthotropic sheet samples are subjected to uniaxial tensile loading having a non-zero angle with the anisotropic axis. This phenomenon is explained and simulated by Dafalias (2000) in the constitutive framework of orthotropic rigid plastic models with plastic spin. Moreover, no orientational anisotropy occurs if the plastic spin becomes zero. Motivated by the experimental data from Kim and Yin (1997), Dafalias assumed that orthotropy is preserved when the orthotropic sheets are loaded by an angle with respect to their anisotropy direction.

In Section 4 we exemplify the prediction of the model proposed in Section 3. Here we consider a sheet made up from an orthotropic material with the initial anisotropic axis arbitrarily oriented, i.e. none of the initial orthotropic directions is orthogonal to the sheet, and we assume that the body is homogeneously deformed under a plane deformation state. As a principal result we prove that for all three chosen plastic spins, if only one of the initial orthotropy directions is perpendicular to the sheet, the orientational anisotropy is developed in the sheet plane being described by the proper rotation angle. Numerical examples emphasize the peculiar behaviour of the sheet depending on the material constants (elastic constants, initial yield parameters and the hardening constants), initial orientation of the orthotropic axis, and analytical models chosen for the plastic spin.

Only if one of the symmetry directions coincides with the normal to the sheet plane, we could find the previously analyzed behaviour for Mandel type plastic spin in Cleja-Tigoiu (2007), i.e. only one of Euler’s angle, namely the proper rotation, is involved in the model. Furthermore, we also provide comparisons with the models and results of Dafalias (2000) and Kim and Yin (1997).

2. Orthotropic anisotropy

We consider an anisotropic crystalline material with the initial orthonormal symmetry directions characterized by the set \( \{n_1, n_2, n_3\} \). We denote by \( m_i = E n_i, \quad i \in \{1, 2, 3\} \), the image through the deformation of the structural anisotropy axes.

Remark. Following Mandel (1972), in the case of small elastic strains, \( \varepsilon' \), but large elastic rotations, \( R' \), the polar decomposition of the elastic part of the deformation can be represented through the following relationships:

\[
E = R'U', \quad U' = I + \varepsilon', \quad ||\varepsilon'|| \ll 1.
\]

As a consequence of (3) and (4), the change in the orientation of the orthotropy and elastic spin are characterized by \( m_i = R'n_i \), and \( m_i = R'n_i = \omega' m_i, \quad i \in \{1, 2, 3\} \).

2.1. Orthotropic elasto-plastic model

We shortly present the model with respect to the plastically deformed (isoclinic) configuration, at every fixed particle \( X \) of the body, in terms of the history of the deformation gradient \( t \sim F(X, t) \). As a straightforward consequence of the multiplicative decomposition (1), the kinematical relationships between the velocity gradient \( L \) and the rates of elastic \( L' \) and plastic \( L'' \) parts of the deformation are derived:

\[
L = FF^{-1} = E(e)^{-1} + EPP^{-1} E^{-1}, \quad L' = EE^{-1}, \quad L'' := PP^{-1} = D' + W'.
\]

Here the skew-symmetric part of the rate of plastic deformation, \( W' \) defines the plastic spin. In the case of small elastic strains and large elastic rotations, described through the relationships written in (4), when we separate the symmetric and skew-symmetric parts from (6) we obtain:

\[
D = R'\varepsilon'(R')^T + R'D'(R')^T, \quad W = R'(R')^T + R'W'(R')^T.
\]

In (7) \( D := \frac{1}{2}(FF^{-1} + (FF^{-1})^T) \), and \( W := \frac{1}{2}(FF^{-1} - (FF^{-1})^T) \) are the rate of the deformation tensor and the spin motion, respectively. Here \( A' \) denotes the transpose of the tensor \( A \).

Model. For any \( X \), at every time \( t \), the unknowns \( \varepsilon', R', P, \varepsilon \) are to be determined from the following set of relationships:

1. The elastic type constitutive equation which renders the Piola–Kirchhoff stress tensor, denoted by \( \pi \), and the evolution equations for both the plastic part of the deformation and the internal variables:

\[
\pi = E(e'), \quad \varepsilon : Sym \rightarrow Sym \quad \text{linear map}
\]

\[
P = \mu B(\pi, x), \quad D = \mu N'(\pi, x), \quad W' = \mu W'(\pi, x), \quad x = \mu x(\pi, x), \quad R'(R')^T = D + R'D'(R')^T, \quad R'(R')^T = W - R'W'(R')^T.
\]

2. The variation in time of the plastic part of the deformation and the internal variables are associated with the yield function \( F(\pi, x) \), by rate independent evolution equations, in terms of the plastic factor (or multiplier) – \( \mu \), which satisfies:

\[
\mu > 0, \quad F \leq 0, \quad \mu F = 0 \quad \text{Khun–Tucker conditions}, \quad \mu \varepsilon = 0 \quad \text{consistency condition}.
\]
3. Material symmetry assumptions: All of the constitutive and evolution functions, $\mathcal{E}, \mathcal{F}, \mathbf{N}^0, \Omega^0$ and $I$, are invariant with respect to $g_0$.

4. Corresponding initial conditions:
\[ \mathbf{e}^f(t_0) = 0, \quad \mathbf{R}^f(t_0) = I, \quad \mathbf{P}(t_0) = I, \quad \mathbf{x}(t_0) = 0. \] (10)

Remark. If we eliminate the Piola–Kirchhoff stress, $\pi$, via the linear elastic constitutive Eq. (8), a rate-type constitutive model is provided. Consequently, $\mathbf{e}^f, \mathbf{R}^f, \mathbf{P}$ and $\mathbf{x}$ can be viewed as solutions of a differential system resulting from (8)-(10) to be solved for a given history of the deformation gradient $t \rightarrow \mathbf{F}(X,t)$ for any fixed material point.

Remark. According to the general representation theorem for an anisotropic function proved by S. Liu (1982) and applied to the invariant functions relative to the orthogonal group $g_0$, there exist the functions $\mathcal{E}, \mathcal{F}, \mathbf{N}^0, \Omega^0$ and $I$ depending on $(\pi, x)$ and the orientational variables $(\mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2)$, which are isotropic with respect to all of their arguments, i.e., for any function $f - g_0$ invariant there exists an isotropic function $f$ such that:
\[ f(\pi, x) = \tilde{f}(\pi, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2). \] (11)

In the case of small elastic deformations, the following relationships between the fields in the plastically deformed configurations and their expressions derived by pushing away to the deformed configuration hold:
\[ \mathbf{T} = \mathbf{R}^f \pi(\mathbf{R}^f)^T, \quad \mathbf{e}^f = \mathbf{R}^f \mathbf{e} (\mathbf{R}^f)^T, \quad \mathbf{a} = \mathbf{R}^f \mathbf{a} (\mathbf{R}^f)^T, \quad \mathbf{m}_i = \mathbf{R}^f \mathbf{m}_i, \quad i = 1, 2, 3, \] (12)

for the elastic strain $\mathbf{e}^f$, the Cauchy stress $\mathbf{T}$ and hardening variables $\mathbf{a}$.

When we take the time derivatives of the above fields, the objective derivatives associated with the elastic spin, $\omega^f$, for $\mathbf{e}^f, \mathbf{T}$ and $\mathbf{a}$, are defined via the formulae:
\[ \frac{D}{Dt}(\mathbf{e}^f) := \dot{\mathbf{e}}^f - \omega^f \mathbf{e} + \mathbf{e} \dot{\omega}^f = \mathbf{R}^f \dot{\mathbf{e}} (\mathbf{R}^f)^T, \]
\[ \frac{D}{Dt}(\mathbf{e}^f) := \dot{\mathbf{T}} - \omega^f \mathbf{T} + \mathbf{T} \dot{\omega}^f = \mathbf{R}^f \dot{\mathbf{T}} (\mathbf{R}^f)^T, \]
\[ \frac{D}{Dt}(\mathbf{a}) := \dot{\mathbf{a}} - \omega^f \mathbf{a} + \mathbf{a} \dot{\omega}^f = \mathbf{R}^f \dot{\mathbf{a}} (\mathbf{R}^f)^T, \quad \frac{D}{Dt}(\mathbf{m}_i) := \dot{\mathbf{m}}_i - \omega^f \mathbf{m}_i. \] (13)

These derivatives are involved in the differential system which describes the evolution in time for the Cauchy stress, plastic part of the deformation, and internal variables.

2.2. Orientational anisotropy in terms of Euler’s angles

In the problem investigated herein, three types of orthogonal axes have been considered, namely:

- $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$, which correspond to fixed directions,
- $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, the initial orthotropic axis, and
- $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$, the rotated orthotropic axis, which satisfy the initial condition $\mathbf{m}_i(t_0) = \mathbf{n}_i, i = 1, 2, 3$.

Let us denote by $\mathbf{R} \in \text{O}rth$ the rotation of the deformed orthotropic axis $\mathbf{m}_i$ with respect to the fixed axes $\mathbf{j}_i$:
\[ \mathbf{R}_k = \mathbf{m}_k, \quad \text{for } k = 1, 2, 3, \quad \text{where } \mathbf{m}_k = \mathbf{R}_k \mathbf{j}_k, \quad \text{and } \mathbf{R}_k = \mathbf{j}_k, \mathbf{R}_k. \] (14)

$\mathbf{R}_k$ characterizes the rotation of the orthotropic axis $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, at the initial moment, from the fixed axis, and $\mathbf{R}(t_0) = \mathbf{R}_0$.

The elastic rotation $\mathbf{R}^e \in \text{O}rth$ is related to $\mathbf{R}$ by
\[ \mathbf{R}^e_0 = \mathbf{m}_k, \quad \text{for } k = 1, 2, 3, \] and $\mathbf{R}^e(t_0) = I$.
\[ \mathbf{R}^e(t) = \mathbf{R}(t) \mathbf{R}(t_0)^{-1}. \] (15)

The rotation tensor $\mathbf{R}$ is expressed in terms of Euler’s angles (see e.g. Beju et al., 1983), which are denoted by $\psi$ – the precession, $\theta$ – the nutation, and $\phi$ – the proper rotation. Euler’s angles are defined as follows: $\theta$ is the angle between the axis $\mathbf{m}_i$ and $\mathbf{j}_3$, $\psi$ measures the angle between $\mathbf{j}_1$ and the nodal axis ON, which coincides with the intersection line of the planes ($\mathbf{j}_1, \mathbf{j}_3$) and ($\mathbf{m}_1, \mathbf{m}_2$), while $\phi$ is the angle between ON and $\mathbf{m}_1$. The matrix associated with the rotation tensor $\mathbf{R}$ is expressed with respect to the tensorial basis $\mathbf{j}_1 \otimes \mathbf{j}_2$ by
\[ \mathbf{R} = \begin{pmatrix}
\cos \psi \cos \theta - \sin \psi \sin \phi & - \cos \psi \sin \theta & \sin \psi \sin \phi \\
\sin \psi \cos \phi & - \cos \psi \sin \phi & \sin \psi \cos \phi \\
- \sin \cos \theta & 0 & \cos \theta
\end{pmatrix}. \] (16)

**Proposition 1**

(a) The rotation tensor $\mathbf{R}$ is composed by three successive plan rotations of angles: $\psi$ – round about $\mathbf{j}_3$, $\theta$ – round about nodal axis ON, and $\phi$ – round about $\mathbf{m}_3$, respectively. Thus:
\[ \mathbf{R} = \mathbf{R}_1 \mathbf{R}_2 \mathbf{R}_3, \quad \text{where } \mathbf{R}_1 = \begin{pmatrix}
\cos \psi & - \sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}, \]
\[ \mathbf{R}_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & - \sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix}
\cos \phi & - \sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}. \] (17)

where $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ are written with respect to their appropriate bases.

(b) The spin $\omega^e = \mathbf{R}^e \omega^f = \mathbf{R} \omega^f$ has the component representation in the tensorial basis $\mathbf{m}_i \otimes \mathbf{m}_j$:
\[ \omega^e = \begin{pmatrix}
0 & - \phi - \psi \cos \theta & \psi \sin \phi - \psi \sin \cos \phi \\
\phi + \psi \cos \theta & 0 & - \psi \cos \phi - \psi \sin \phi \sin \theta \\
- \psi \sin \phi \sin \cos \phi & \psi \cos \phi + \psi \sin \phi \sin \theta & 0
\end{pmatrix}. \] (18)

In the rate form of the model, we replace the elastic rotation which characterizes the orientation of the orthotropy directions by Euler’s angles.

3. Eulerian description of the model with orientational variables

We describe the behaviour of the elasto-plastic material in the Eulerian description, starting from the model previously presented in (8) and (9). The following steps have to be pursued (for a detailed presentation of the procedure see Cleja-Tigoiu, 2000a):

- The hat functions have been introduced, by the procedure listed in (11).
- By pushing away to the deformed configuration, the new variables listed in (12) will be introduced in the hat functions, based on their isotropy, as for instance for $\mathbf{N}^0$:
\[ \mathbf{R}^e \mathbf{N}^0(\pi, \mathbf{n}_1 \otimes \mathbf{n}_1, \mathbf{n}_2 \otimes \mathbf{n}_2) (\mathbf{R}^e)^T = \mathbf{N}^0(\mathbf{T}, \mathbf{a}, \mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2); \] (19)
- Finally, we introduce the definitions and expressions for the objective derivatives listed in (13).
3.1. Eulerian model

**M.1** The linear elastic type constitutive equation gives rise to a Cauchy stress in terms of the small elastic strain measure, pushed away to the actual configuration:

$$T = \tilde{\varepsilon} (m_1 \otimes m_1, m_2 \otimes m_3)|\varepsilon^e|, \quad \varepsilon^e = R^e \varepsilon R^e T. \quad (20)$$

The expression for $\tilde{\varepsilon}$ is written in (60) and (62).

**M.2** The yield condition for kinematic hardening material, which is quadratic with respect to $\pi - \alpha$ and the invariant $\bar{g}_{66}$, and is pushed away to the actual configuration, allows the representation:

$$\mathcal{F} = C_1 \bar{S} \cdot \bar{S} + C_2 \bar{S} \cdot (n_1 \otimes n_1) + C_3 \bar{S} \cdot (n_2 \otimes n_2)$$

$$+ C_4 (\bar{S} n_1 - n_1)^2 + C_5 (\bar{S} n_2 - n_2)^2$$

$$+ C_6 (\bar{S} n_1 - n_1) (\bar{S} n_2 - n_2) - k. \quad (21)$$

where $\bar{S} = T - a$ represents the effective stress. Here a classical form of Hill’s criterion (Hill, 1950) with kinematic hardening in terms of six (anisotropic) yield parameters $C_j, j = 1, 2, 3$. (21) appears to be identical to expression $A_1$ from Dafalias and Rashid (1989), where the stress is used instead of the effective stress.

**Remark.** The plastic criterion considered in (21) is not explicitly dependent on $\text{tr} \mathcal{F}$, which means that $\mathcal{F}$ is not dependent on the hydrostatic pressure. Note that for anisotropic materials, see the expression for $\text{tr} \mathcal{N}^p$ that holds as a consequence of (24) and Cleja-Tigou (2000), even if $\bar{S}$ is a deviatoric tensor, $\text{tr} \mathcal{F}^{-1}$ is generally non-zero. Consequently, we do not introduced here an additional hypothesis that $T$ and $a$ are deviatoric tensors, but we could choose the material constants in such a way that $\text{tr} \mathcal{F}^{-1} = 0$.

**M.3** The evolution equations are associated with the yield condition written in (21) with the new plastic factor $\mu = \mu$ which satisfies the appropriate conditions that follows from (9):

$$\dot{\mu} \geq 0, \quad \dot{\mathcal{F}} \leq 0, \quad \dot{\mu} \dot{\mathcal{F}} = 0 \quad \text{and} \quad \mu_0 \frac{d}{dt} (\mathcal{F}) = 0. \quad (22)$$

**M.4** The symmetric part of the rate of the plastic deformation in the actual configuration is described as a flow rule associated with the yield condition (21):

$$R'^p (R')^T = \dot{\mu} \mathcal{N}^p (T, a, m_1 \otimes m_1, m_2 \otimes m_2), \quad \mathcal{N}^p = \dot{\varepsilon}_t \mathcal{F}. \quad (23)$$

if the flow rule has been previously postulated by $\mathcal{N}^p (\pi, x) = \dot{\varepsilon}_t \mathcal{F}$. Then the flow rule (23) is defined through the constitutive function

$$\mathcal{N}^p = 2 C_1 S + C_2 (S (m_1 \otimes m_1) + (m_2 \otimes m_1) + (m_2 \otimes m_1) + (m_2 \otimes m_2) \mathcal{S})$$

$$+ C_4 (S (m_2 \otimes m_2) + (m_2 \otimes m_2) \mathcal{S})$$

$$+ 2 C_5 \left( S (m_1 \otimes m_1) \right) (m_1 \otimes m_1)$$

$$+ 2 C_6 \left( S (m_2 \otimes m_2) \right) (m_2 \otimes m_2)$$

$$+ C_7 \left( S (m_1 \otimes m_2) + m_1 \otimes m_2 \right) (m_2 \otimes m_2) + \mathcal{S} (m_2 \otimes m_2) (m_1 \otimes m_1) \right],$$

with $S = T - a$. \quad (24)

**M.5** The expression of the plastic spin pushed forward to the actual configuration leads to a constitutive representation with the isotropic function $\Omega^p$:

$$R'^p (R')^T = \mu \Omega^p (T, a, m_1 \otimes m_1, m_2 \otimes m_2). \quad (25)$$

Three types of equations defining the plastic spin should be used in this paper, namely:

- **The Mandel type plastic spin** has been proposed in Cleja-Tigou (2007) in the form:

$$\Omega^p = A_1 \left( (m_1 \otimes m_1) \mathcal{S} \right)^a + A_2 \left( (m_2 \otimes m_2) \mathcal{S} \right)^a, \quad (26)$$

that is derived from a Mandel type quadratic yield condition in terms of $\Sigma = E^T \mathcal{E}$ in the case of large part of deformation, see Cleja-Tigou (2000b).

- The expression of a plastic spin, called the Liu–Wang type plastic spin, has been introduced by

$$\Omega^p = \eta_1 \left( \mathcal{N}^p (m_1 \otimes m_1) \right)^a + \eta_2 \left( \mathcal{N}^p (m_2 \otimes m_2) \right)^a$$

$$+ \eta_3 \left( (m_1 \otimes m_1) \mathcal{N}^p (m_2 \otimes m_2) \right)^a, \quad (27)$$

as a consequence of Wang’s representation theorem in Wang (1970), and which is applied for a skew-symmetric tensor field dependent on the orientational tensors and symmetric tensor fields.

**Remark.** A three material parameter representation has been postulated earlier by Dafalias (1985), through a formula which is similar to (27), but with $T$ instead of $\mathcal{N}^p$.

- The Dafalias type plastic spin is viewed as a non-coaxility of the rate of plastic deformation, but with respect to the effective stress:

$$\Omega^p = \eta \left( \mathcal{N}^p - \mathcal{N}^p S \right), \quad (28)$$

apart from that proposed in Dafalias (1985, 2000), where $\mathcal{S}$ is replaced by $T$.

**M.6** The evolution equation for the tensorial hardening variable in the actual configuration is described by a Prager–Ziegler hardening rule adapted to the orthotropic material:

$$R'^p (R')^T = \mu \left( T, a, m_1 \otimes m_1, m_2 \otimes m_2 \right),$$

$$I = C_0 \mathcal{N}^p + C_1 \left[ \mathcal{N}^p (m_1 \otimes m_1) + (m_1 \otimes m_1) \mathcal{N}^p \right]$$

$$+ C_2 \left[ \mathcal{N}^p (m_2 \otimes m_2) + (m_2 \otimes m_2) \mathcal{N}^p \right], \quad (29)$$

generally with the material functions $c_m$ depending on the orthotropic invariants of $N^p$, but here constant.

**Conclusions.** The spin $w^p$ which occurs in the expression for the objective derivatives can be expressed by

$$w^p = W - \mu \dot{\Omega}^p, \quad (30)$$

as a direct consequence of (7), together with the appropriate constitutive representation of the plastic spin (25) via $\dot{\Omega}^p$.

**Remark.** Under the hypothesis that the evolution equation for hardening variable and the skew symmetric part of $\mathcal{P}^{-1}$ are generated by the symmetric part of the rate of plastic deformation only, but being $\bar{g}_{66} -$ invariant, the appropriate form for the spin representation is that written in (27). If the flow rule is adopted to be associated with a representation for the yield surface, which is quadratic with respect to the stress, then Eq. (27) expanded in reference to the orthotropic axes in the deformed configuration acquires the form of Dafalias (1985) or Dafalias and Rashid (2000).

Liu–Wang and Mandel type spins are linear with respect to the effective stress, apart from the Dafalias spin which is quadratic with respect to the effective stress. Note that the Liu–Wang type plastic spin is more general than the Mandel type spin.
3.2. Rate type model with orientational variables

If we take the time differential for the elastic type constitutive Eqs. (20), we can reformulate the constitutive model.

**Theorem 1.** Let a history of the deformation process be given, i.e. \( t \rightarrow \mathbf{F}(t) \) for a fixed material point with \( \mathbf{D} = \{(\mathbf{F}(t))^{-1}\)^t\} and \( \mathbf{W} = \{(\mathbf{F}(t))^{-1}\)^t\}. The time evolution of the set of variables \( \{\mathbf{T}, \mathbf{a}, \mathbf{m}_k\} \) is described, in terms of the objective derivatives defined in (13), by the differential system:

\[
\frac{D}{Dt} \mathbf{T} = \mathbf{T} - \mathbf{a} = \mathbf{\hat{F}}(\mathbf{m}_1 \otimes \mathbf{m}_2 \otimes \mathbf{m}_3)[\mathbf{Np}], \quad \frac{D}{Dt} \mathbf{a} = \mathbf{\dot{a}}, \quad \frac{D}{Dt} \mathbf{m}_k = 0.
\]

The initial conditions are given by

\[
\mathbf{\tilde{S}}(t_0) = 0, \quad \mathbf{\dot{S}}(t_0) = \Psi_0, \quad \mathbf{\theta}(t_0) = \Phi_0.
\]

In order to derive the explicit form of the system of differential equations for the components of the effective stress, we introduce the following notations for the components of the rate of deformation and the motion spin, respectively, with respect to the rotated orthotopic directions:

\[
\mathbf{\tilde{D}}_j = \mathbf{m}_j \cdot \mathbf{D} \mathbf{m}_j, \quad \mathbf{\tilde{W}}_j = \mathbf{m}_j \cdot \mathbf{W} \mathbf{m}_j,
\]

and the new six yield parameters:

\[
K_{11} = C_1 + C_2 + C_4, \quad K_{12} = C_6, \quad K_{22} = C_1 + C_3 + C_5, \quad K_{33} = C_1, \quad K_{m1} = 2C_1 + C_2 + C_3, \quad K_{m2} = 2C_1 + C_2.
\]

which allow us to represent the current yield surface (21) in terms of the projection on the directions \( \mathbf{m}_k \otimes \mathbf{m}_k \).

\[
\mathbf{\tilde{F}} \left( \mathbf{\tilde{S}}, \mathbf{a} \right) = K_{11} \mathbf{\tilde{S}}_{11} + K_{22} \mathbf{\tilde{S}}_{22} + K_{33} \mathbf{\tilde{S}}_{33} + K_{m1} \mathbf{\tilde{S}}_{12} + K_{m2} \mathbf{\tilde{S}}_{13} + (2K_{12} + K_{m1} - K_{m2}) \mathbf{\tilde{S}}_{23} - k.
\]

In (36) the term \( \mathbf{\tilde{S}} \cdot \mathbf{\tilde{D}} \mathbf{m}, \mathbf{\tilde{S}} \mathbf{\tilde{W}} \mathbf{m} \) can be calculated directly using the formula (62) in which \( D_1 \) are replaced by \( \mathbf{\tilde{D}}_j \). The components \( \mathbf{\tilde{S}} \cdot \mathbf{\tilde{N}} \mathbf{m} \) are listed in (63) together with (64) were calculated using again formula (62) in which \( D_1 \) are replaced by the components listed in (41). Finally \( \mathbf{\tilde{S}}, \mathbf{\tilde{a}} \) are calculated from (29) bearing in mind that the components for \( \mathbf{\tilde{N}} \) derived in (41), see formulae (65) and (66). The following theorems are directly obtained:

**Theorem 4.** We suppose that the plastic spin is described by Mandel type expression (26). Then the differential system which describes the effective stress components in the basis \( \mathbf{m}_k \otimes \mathbf{m}_k \) and Euler’s angles as time dependent functions is expressed in the form:

\[
\mathbf{\tilde{S}}_{11} = -\mu \left[ S_{11} (B_{1111} + P_{1111}) + S_{22} (B_{1122} + P_{1122}) + 5 S_{33} B_{1133} \right] + D_{11} A_{11} + D_{22} A_{12} + D_{33} A_{13},
\]

\[
\mathbf{\tilde{S}}_{22} = -\mu \left[ S_{11} (B_{2211} + P_{2211}) + S_{22} (B_{2222} + P_{2222}) + 5 S_{33} B_{2233} \right] + D_{11} A_{12} + D_{22} A_{22} + D_{33} A_{23},
\]

\[
\mathbf{\tilde{S}}_{33} = -\mu \left[ S_{11} (B_{3311} + P_{3311}) + S_{22} (B_{3322} + P_{3322}) + 5 S_{33} B_{3333} \right] + D_{11} A_{13} + D_{22} A_{23} + D_{33} A_{33},
\]

\[
\mathbf{\tilde{S}}_{12} = -\mu S_{12} (B_{1212} + P_{1212}) + D_{12} A_{14},
\]

\[
\mathbf{\tilde{S}}_{13} = -\mu S_{13} (B_{1313} + P_{1313}) + D_{13} A_{66},
\]

\[
\mathbf{\tilde{S}}_{23} = -\mu S_{23} (B_{2323} + P_{2323}) + D_{23} A_{55},
\]

\[
\text{if and only if } \sin \theta \neq 0, \text{ i.e.}
\]

\[
\phi = \frac{\mu}{2} \left[ (A_{13} - A_{23}) S_{12} - \cot \theta (A_{13} S_{13} \cos \phi + A_{23} S_{23} \sin \phi) \right] - \cot \theta \left( \tilde{W}_{11} \cos \phi + \tilde{W}_{12} \sin \phi \right),
\]

\[
\dot{\theta} = \frac{\mu}{2} \sin \theta \left[ (A_{23} S_{23} \cos \phi + A_{13} S_{13} \sin \phi) + \tilde{W}_{11} (-\sin \phi + \tilde{W}_{12} \cos \phi) \right],
\]

\[
\dot{\phi} = \frac{\mu}{2} \sin \theta \left[ (A_{23} S_{23} \cos \phi + A_{13} S_{13} \sin \phi) + \tilde{W}_{11} (-\sin \phi + \tilde{W}_{12} \cos \phi) \right].
\]
The constant parameters are enumerated in the Appendix, (64) and (66).

The initial conditions are given by 
\[ S_i(t_0) = 0, \psi(t_0) = \Psi_0, \theta(t_0) = \theta_0 \neq 0, \varphi(t_0) = \Phi_0. \]

In the above differential system, the yield function is defined by (21), the plastic factor (32) is expressed through the components of \( \mathbf{S} \) and \( \mathbf{D} \) in terms of \( \beta \):
\[
\mu = \frac{1}{\dot{S}}[\beta(\mathbf{j})],
\]

\[
\beta = \left( B_{1111}S_{11} + B_{1122}S_{22} + B_{1133}S_{33} \right) \dot{D}_{11}
+ \left( B_{2222}S_{11} + B_{2233}S_{22} + B_{2244}S_{33} \right) \dot{D}_{22}
+ \left( B_{3333}S_{11} + B_{3333}S_{22} + B_{3344}S_{33} \right) \dot{D}_{33}
+ 2B_{3322}S_{12} \dot{D}_{12} + 2B_{3333}S_{13} \dot{D}_{13} + 2B_{3333}S_{23} \dot{D}_{23},
\]

with the hardening parameter \( h_c \) expressed by
\[
h_c = H_{1111}S_{11} + H_{1122}S_{22} + H_{1133}S_{33} + H_{2222}S_{22}
+ H_{3333}S_{33} + 2H_{1233}S_{12} + 2H_{1313}S_{13} + 2H_{2333}S_{23},
\]

The constant parameters in the above expressions are given in (67).

**Theorem 5.** If the plastic spin is described by the Liu–Wang type expression (27), then the differential system which describes the effective stress components in the basis \( \mathbf{m} \circ \mathbf{m} \) is given by (42) and Euler's angles as time dependent functions, again for \( \theta(t_0) \neq 0 \), are expressed in the form:
\[
\phi = \frac{1}{2} \left( \left[ -\eta_1 + \eta_2 + \eta_3 \right] \kappa \right) \sin \theta,
\]

\[
\theta = \frac{1}{2} \left( \left[ -\eta_1 \kappa \right] \sin \theta \right),
\]

\[
\psi = \frac{1}{2} \left( \left[ -\eta_2 \kappa \right] \sin \theta \right).
\]

(46)

In (46) \( \mathbf{W} = 0 \). For arbitrarily given \( \mathbf{W} \) the terms containing its components, which appear in (43), have to be added in the right-hand side of (46).

**Theorem 6.** We suppose that the plastic spin is described this time by the Dafalias type expression (28). Then the differential system which describes the effective stress components in the basis \( \mathbf{m} \circ \mathbf{m} \) is given by (42), while the time derivatives of Euler's angles are expressed in the form (again written for \( \mathbf{W} = 0 \), and \( \theta(t_0) \neq 0 \))
\[
\phi = \frac{1}{2} \mu \left( k_0 S_{11} + k_0 S_{22} S_{12} + k_0 S_{33} S_{13} \right)
\]

\[
- \cos \theta \left( k_0 S_{11} S_{12} + k_0 S_{22} S_{23} + k_0 S_{33} S_{32} \right) \sin \phi,
\]

\[
\theta = \frac{1}{2} \left( \left[ -k_0 \kappa \right] \theta \right)
\]

\[
\psi = \frac{1}{2} \left( \left[ -k_0 \kappa \right] \psi \right),
\]

(47)

Here the new plastic constants \( k_i \), \( i = 1, \ldots, 7 \), are given by
\[
k_1 := (K_{m1} + K_{m2} - 2K_{11}),
\]

\[
k_2 := 2K_{22} - K_{12} - K_{m1},
\]

\[
k_3 := 2K_{33} + K_{m1} - K_{m2},
\]

\[
k_4 := -K_{m2} - K_{m1},
\]

\[
k_5 := 2K_{33} - K_{m2},
\]

\[
k_6 := K_{m2} - 2K_{11},
\]

\[
k_7 := 2K_{33} + K_{m1} - K_{m2} - 2K_{22},
\]

and they are expressed in terms of the yield constants.

**Theorem 7.** For all three plastic spins, when \( \sin \theta \) is vanishing at every time \( t \), the special case when \( \sin \theta = 0 \) at every time \( t \) can be derived from the general system (42), together with either (43) for the Mandel type plastic spin, or (46) for the Liu–Wang plastic spin, or (47) for the Dafalias plastic spin. The effective shear stresses \( S_{13} \) and \( S_{23} \) necessarily become zero. Only one of Euler's angles is involved in the problem, namely the proper rotation \( \phi \). This happens if and only if the shear component of the effective stress \( S_{12} \) is not zero. The rotational anisotropy is described by
\[
\phi = \eta \left( k_0 S_{11} + k_0 S_{22} S_{12} \right),
\]

(48)

for the Dafalias plastic spin.

where \( \eta = \left( x_1 - x_2 \right) / \left( x_1 + x_2 \right) \) for the Mandel spin, or \( \eta = \left( x_1 + x_2 \right) / \left( x_1 - x_2 \right) \) for the Liu–Wang spin, and \( k_1, k_2 \) are given by (48).

In order to prove the aforementioned result, let us remark that from the last equation written in (43), multiplied by \( \sin \theta \), and from the evolution for \( \theta \) we get:
\[
A_1 S_{11} \cos \phi + A_2 S_{23} \sin \phi = 0,
\]

\[
A_3 S_{13} \cos \phi + A_4 S_{23} \cos \phi = 0,
\]

(50)

when \( \theta = 0 \). Thus as a consequence of (50) \( S_{13} = S_{23} = 0 \), and when we come back to the system (43), the only equation which describes the rotation of the orthotropic axes is (49).

A similar result can be proven for the Liu–Wang plastic spin, via (27), and for the Dafalias plastic spin via (28), respectively.

The qualitative difference is emphasized in Theorem 7, namely \( \phi \) is quadratic in the effective stress components for the Dafalias spin, while \( \phi \) is linearly dependent on the effective shear stress component for the other two.

4. Homogeneous deformation of the sheet

We simulate a local homogeneous process in order to emphasize the orientational anisotropy.

Consider an orthotropic sheet with the geometrical axes \( \{ j_k \} \), and whose initial orthotropic axes \( \{ m_i \} \) are different from the geometrical axes of the sheet, see Fig. 1. In this application we consider a homogeneous deformation process:
\[
F = \lambda_1(t) j_1 \otimes j_1 + \lambda_2(t) j_2 \otimes j_2 + \lambda_3(t) j_1 \otimes j_3,
\]

with \( \lambda_3(t) = 1 \).

(51)

The rate of deformation and the spin motion are given by
\[
\mathbf{L} = \mathbf{F}^{-1} = \frac{\lambda_1}{\lambda_2} j_1 \otimes j_1 + \frac{\lambda_2}{\lambda_2} j_2 \otimes j_2,
\]

\[
\mathbf{D} = \mathbf{L}, \quad \mathbf{W} = 0.
\]

(52)

**Proposition 2.** The components of the rate of deformation tensor \( \mathbf{D} \) with respect to the actual orthotropic directions \( \mathbf{m}_i \circ \mathbf{m}_j \) are calculated using the rotation tensor \( \mathbf{R} \), in terms of the component of \( \mathbf{D} \) in the fixed axes \( \mathbf{j}_k \circ \mathbf{j}_k \) via formula (38). In matrix form, these components are given by
\[
(\mathbf{D})_{ij} = \begin{pmatrix}
\frac{1}{2} R_{11}^{11} & \frac{1}{2} R_{12}^{12} & \frac{1}{2} R_{13}^{13} \\
\frac{1}{2} R_{12}^{12} & \frac{1}{2} R_{22}^{22} & \frac{1}{2} R_{23}^{23} \\
\frac{1}{2} R_{13}^{13} & \frac{1}{2} R_{23}^{23} & \frac{1}{2} R_{33}^{33}
\end{pmatrix}
\]

(53)
Fig. 1. Orthotropic sheet with current orthotropic axes \( \{ \mathbf{m}_j \} \) different from its geometrical directions \( \{ \mathbf{j}_j \} \).

**Remark.** The differential systems which describe the evolution in time of the effective stress \( \tilde{S}_0 \) and Euler angles, in the projections on the actual orthotropic axes for the applied homogeneous deformation process, are described by **Theorems 4–6** related to various choices for the plastic spin.

**Theorem 8.** The rotation of the orthotropic axes remains in plane, i.e. in the plane of the sheet, only if one of the initial orthotropic axes is orthogonal to the sheet, for all three plastic spins considered.

**Proof.** We suppose that none of the orthotropic axes is orthogonal to the sheet surface, and the rotation of the orthotropic axes remains in the plane. The initial conditions at time \( t = 0 \), \( P = \mathbf{1} \), \( \mathbf{a} = 0 \), \( \varepsilon' = 0 \), \( \theta = \theta_0 \neq 0 \), \( \psi = \Psi_0 \), \( \phi = \Phi_0 \), are associated with the differential system (42) and (43), written for the deformation process given in (51). No variation in time for Euler angles, for the plastic deformation, as well as for hardening variable \( \mathbf{a} \), occurs in the elastic behaviour. We proved in **Theorem 7** that \( S_{13} = S_{23} = 0 \) necessarily hold, during the considered deformation process with the motion spin vanishing. The elastic solution can be calculated directly from (60)-(62). First we calculate the small strains that corresponds to the homogeneous deformation (51), with respect to the axes \( \mathbf{j}_j \), \( \varepsilon_{11} = 1/2 (\varepsilon^2 - 1) \), \( \varepsilon_{22} = 1/2 (\varepsilon^2 - 1) \), and \( \varepsilon_{13} = 0 \), \( \varepsilon_{ij} = 0 \) for \( i \neq j \). By the hypothesis of the proof, we found that \( \mathbf{n}_1 \cdot \mathbf{Tn}_n \) and \( \mathbf{n}_2 \cdot \mathbf{Tn}_n \) are vanishing, and \( \theta_0 \neq 0 \). From the elastic response, see (62), we found that the appropriate components of the small strains \( \varepsilon_1 \) and \( \varepsilon_{23} \) have to be zero. Since the components of the elastic strain in the initial orthotropic axes can be similarly calculated using (53), we arrive at the following algebraic system:
\[ \varepsilon_{13} := \varepsilon_{13}^0 R_{13}(0) + \varepsilon_{23}^0 R_{23}(0) = 0, \]
\[ \varepsilon_{23} := \varepsilon_{13}^0 R_{13}(0) R_{13}(0) + \varepsilon_{23}^0 R_{23}(0) R_{23}(0) = 0. \]  

(54)

In order to have non-zero elastic strain, the determinant of (54) should be zero, i.e. \( \cos \varphi \sin \psi \sin \frac{3}{2} \theta = 0 \). Under the hypothesis \( \varphi = 0 \), it is sufficient to consider \( \sin \theta = 0 \). Then from (16) it follows that \( R_{13}(0) = 0 \), and \( \varepsilon_{23} = 0 \) at every time \( t \), along the elastic solution. Consequently, at the moment \( t_0 \), when the stress state is on the yield surface, we have the initial conditions \( \varphi = 0, \phi = 0, \psi = 0, \) and \( S_{13} = S_{23} = 0, \forall t \). From the differential system (42) and (43) we obtain the following necessary conditions:

\[ D_{13} = 0, \quad D_{23} = 0, \quad \varphi = \frac{1}{2} \mu [A_1 - A_2] S_{12}, \quad \theta = 0, \quad \psi = 0. \]  

(55)

We arrive at the algebraic system \( D_{13} = 0, D_{23} = 0 \) with respect to \( \frac{\varphi}{2} \) and \( \frac{\psi}{2} \), which is similar to (54). Then \( \sin(2\varphi) = 0 \), i.e. \( \varphi = 0 \), and consequently \( S_{12} = 0, \forall t \geq t_0 \). At the moment \( t_0 \), \( S_{12} = T_{12} = 0 \), and from the elastic constitutive law (62) written in projection to the orthotropic axes we obtain \( \varepsilon_{12}(t_0) = 0 \). Bearing in mind the appropriate formula (53) together with (16), it follows \( \varepsilon_{12}(t_0) = -\sin(2\varphi_0)\varepsilon_{11}(t_0) = 0 \). Thus we arrived at a contradiction.

**Numerical solutions:** The appropriate differential systems, see Theorems 4–7, were numerically integrated for \( \lambda(t) = \lambda(t) = 1 \) and various initial conditions using Matlab codes. The differential equations are rate independent, and the time variable could be replaced by \( \lambda(t) \) if its variation in time is not vanishing, i.e. \( \dot{\lambda}(t) \neq 0 \).

The solution of the differential systems \( S^i_{ij} \) (in the orthotropic basis) and Euler angles, together with the components of the tensorial hardening variables \( a_i \) in the same basis, denoted by \( a_{ij} \), allow us to find numerically the components \( T^i_{ij} \) in the geometrical basis \( \mathbf{j}_i \), \( i, j = 1, 2, 3 \), and \( \psi, \varphi, \theta \), \( \psi, \varphi, \varphi_0, \theta \), which characterize the elastic rotation tensor, as well as the hardening parameter \( h \), which has to remain non-negative, and \( \beta \) to check the switch from the elastic to the plastic solutions. Let us introduce the equivalent plastic deformation, which corresponds to the nominal strain \( \varepsilon = \ln \left( \lambda(t) \right) \) by the following definition:

\[ e^p(x) = \int_{x_0}^x \sqrt{\frac{2}{3} D^p(u) \cdot D^p(u) du}. \]  

(56)

where \( x = \ln \left( \lambda(t) \right) \) and \( x_0 = \ln \left( \lambda(t_1) \right) \) are calculated at the moment \( t_0 \), when the yield condition is reached in the uniaxial tensile test. Thus the functions depending on \( \ln \left( \lambda(t) \right) \) could be represented in terms of \( e^p \).

The set of numerical values has been considered for the dimensionless material parameters (i.e. the original ones divided by \( 10^8 \, \text{Pa} \)).

**Fig. 4.** Dependence of the stress components \( T_{11} \) and \( T_{22} \) on the material constant \( K_{22} \) and initial Euler angle \( \Phi_0 \). For the initial condition \( \psi_0 = \pi/3, \theta_0 = \pi/3 \). The failure is produced for the material constant \( K_{22} \) and the initial condition \( \Phi_0 = \pi/4 \) and \( \Phi_0 = \pi/3 \), respectively. Linear graphics correspond to the behaviour that remains elastic.

**Fig. 5.** Dependence of the Euler angles \( \Theta - \Theta_0 (\text{with } \varphi = \psi), \theta - \theta_0 \) on the material constant \( K_{22} \) and on the initial Euler angle \( \Phi_0 \). For the initial condition \( \psi_0 = \pi/3, \theta_0 = \pi/3 \). The failure is produced for the material constant \( K_{22} \) and the initial condition \( \Phi_0 = \pi/4 \) and \( \Phi_0 = \pi/3 \), respectively.
Remark. When we make the comparison with the experimental data in the numerical simulations, the same material constants have to be considered, but only three ratio for plastic material constants are given in Kim and Yin (1997) (and used in Dafalias, 2000 too). \( f = F/N = 0.3613, \ g = G/N = 0.3535, \ h = H/N = 0.4957. \)

Thus, as a consequence of the identification of Hill material constants with our plastic constitutive parameters which are involved in (21):

\[
H + G = K_{11}/k, \quad H + F = K_{22}/k, \quad 2H = K_{12}/k, \quad 2N = K_{m1}/k,
\]

we calibrated the model to be compatible with these three numerical values for appropriate ratios.

We exemplify the numerical calculation. The stress components \( T_{11} \) and \( T_{12} \) are plotted in Fig. 2 and two of the elastic rotation angles, \( \psi - \theta_0 \) and \( \theta - \theta_0 \), are shown in Fig. 3. We recall that \( \sigma_3 = 1/\sqrt{K_{11}} \) and \( \sigma_n = 1/\sqrt{K_{22}} \) are the initial yield stress in the orthotropic directions \( n_1 \) and \( n_2 \), respectively. Qualitative changes have been observed in the graphics for \( T_{11} \) and \( T_{12} \) versus \( \varepsilon_{11} \) (or by \( \varepsilon_{11} = 1/((\varepsilon_{11})^2 - 1)) \), when \( K_{11} \) is replaced by \( K_{11}/2 \), in this case \( \sigma_n \) becomes smaller than \( \sigma_3 \). The influence of the initial condition \( \Phi_0 \) and the material constant \( K_{11} \) on Euler angles \( \psi - \theta_0 \) and \( \theta - \theta_0 \) can be observed in Fig. 3.

Note that for certain values of the material constants listed in (57) and corresponding to certain initial values, the hardening parameter \( h \) could become zero and, consequently, \( e^h \to \infty \). Thus the failure has been produced. In Figs. 4 and 5 one can see the dependence of the stress components and Euler angles on the
material constant $K_{22}$ and the angle $\phi_0$. The failure points have been found for $K_{22}/2$ (in this case $\sigma_0$ becomes smaller than $\sigma_0^c$) and corresponding to the initial values $\phi_0 = \pi/3$ and $\phi_0 = \pi/4$, respectively. The failure points are marked in Figs. 4 and 5. From the same figures it can be seen that for the same $K_{22}/2$ and the initial condition $\phi_0 = \pi/6$ the behaviour of the material remains elastic, with no variation in time for Euler angles, due to the fact that $\beta$, i.e. $\mu$, remains negative. No plastic loading occurs in this case.

Remarks about the plastic spin: Large differences can be observed in the graphics plotted for the stress components and Euler angles, when Dafalias plastic spin or Mandel plastic spin has been considered, in contrast with Mandel type and Liu–Wang type plastic spins which lead to similar representations, see Figs. 6 and 7. The oscillating behaviour can be seen in the graphs associated with the Dafalias type plastic spin. We also formulate the supposition that for elasto-plastic materials with higher values of the plastic yield stresses, i.e. for smaller $K_0$, the effect of the Dafalias plastic spin is significant due to the fact that the plastic spin depends on the square of the stresses, and supported also by Figs. 6 and 7.

Interpretation of numerical results for the plane rotation of the orthotropic axes that can be assured if the orthotropic axes are initially described by Euler angles ($\theta_0 = 0, \psi_0 = 0, \phi_0$).

Uniaxial yield stress: $\sigma_u$, defined by $\sigma_u := \mathbf{j}_1 \cdot \mathbf{T}_1 = T_{11}$ is plotted in Fig. 8(a) as a function of the angle $\phi_0$ for different values of the equivalent plastic strain.

The ratio $r_s$ is defined by Kuroda and Tvergaard (2000) as the ratio of the width to thickness plastic strain rates. The ratio $r_s$ is assumed to be independent of the equivalent plastic strain for any fixed value of the angle of the orthotropic axis, a supposition which has been related to the representation of the yield function, similarly to that proposed by Barlat et al. (2005).

In our constitutive framework this is expressed in terms of the effective stress by

$$1/r_s = R = \frac{m_1 \cdot \mathbf{D}^\mathbf{m}_1}{m_2 \cdot \mathbf{D}^\mathbf{m}_2} = \frac{2K_{12}S_{23}}{K_{22}S_{22} + K_{12}S_{11}}. \quad (58)$$

Thus $R$ becomes a function of the nominal strain $\ln (\lambda_1)$, or $\epsilon$, along the solutions of the differential equations. In Fig. 8(b), we remark that $R$-ratio can be considered nearly constant for large values of $\phi_0$, while for its smaller values a very strong dependence of the nominal strain $\ln (\lambda_1)$ is observed for small values of $\epsilon$.

Principal stresses: In the case of a plane rotation, the stress state is described with respect to the fixed axes $\mathbf{j}_1$, by $T_{11}$ and stresses $T_{12}, T_{22}$. The angle made by the first eigenvector with the axis $\mathbf{j}_1$, say $\alpha$, is expressed by

$$\tan(2\alpha) = \frac{2T_{12}}{T_{11} - T_{22}}. \quad (59)$$

In Fig. 9, it can be seen that in an anisotropic elasto-plastic material the principal axes of the stress do not coincide with the orthotropy

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**Fig. 8.** (a) Uniaxial yield stress $\sigma_u := \mathbf{j}_1 \cdot \mathbf{T}_1$ distribution versus the angle $\phi_0$, for various equivalent plastic strains, with $\theta_0 = 0, \psi_0 = 0$. (b) The $R$ – ratio $= \frac{m_1}{m_2}$ as a function of the nominal strain $\ln \lambda_1$, for various values of $\phi_0 (\theta_0 = 0, \psi_0 = 0)$.

**Fig. 9.** The angle $\alpha$, which measures the rotation of the principal stresses in plane, versus the equivalent plastic strain $\epsilon$, (a) for Dafalias spin and (b) for Mandel spin.
during the plastic deformation, and a very large changes occur for the small values of the plastic deformations. The graphs have been plotted for the Dafalias and Mandel spins.

**Hardening:** The kinematic hardening is not directly influenced by the plastic spin as it can be seen from the Eqs. (8) and (11), as a consequence of the Prager and Ziegler hardening rule adapted to the orthotropic material, see formula (29). This rule is dependent only on the symmetric part of the rate of the plastic deformation \( \dot{\mathbf{P}}_\mathbf{P} \)

\[
\frac{\mathbf{C}_0}{1 + g_s}.
\]

In the evolution equation one could introduce for \( \alpha \)

\[
\mathbf{A}_1^2 = \frac{\mathbf{C}_0}{1 + g_s}.
\]

which is essentially dependent on the plastic spin. However, it should be clearly emphasized that the plastic spin influence is inferred in the model through the solution of the differential equations formulated in Theorems 4–7, (i.e. the Euler angles which replaced the elastic rotation in the Eulerian setting are also dependent on the choice of the plastic spins). The same kinematic hardening parameters \( c_0, c_1, c_2 \) have been used in Figs. 10 and 12 and they modeled a weakening hardening, as results from the order of their magnitude in comparison with the elastic constants, see Eqs. (57) and (67).

To show the kinematic hardening an illustration of the evolution of the back stress \( \alpha \) through its component \( A_{12} \) is provided in Fig. 11(a), for the three initial values of initial anisotropy \( \phi_0 = 30^\circ, 45^\circ, 60^\circ \). The influence of the plastic spins on the hardening, via the solution of the differential equations systems, are similar for Mandel and Liu–Wang spins, while the Dafalias spin reveals a different character at the beginning of the plastic deformation. The graphs have an appropriate horizontal asymptotic dependent on the initial orientation of the anisotropy direction, which means a strong influence of the orientational anisotropy. Similar conclusions are noticed in the resulting shear stress \( T_{12} \) versus deformation \( \varepsilon_{11} \) in Fig. 11(b).

Comparisons with the experimental data: In figures Figs. 10 and 12 the kinematic hardening parameters \( c_0, c_1, c_2 \) are the same. Only the yield (plastic) material constants are taken to be different for
the curves plotted in Figs. 10 and 12. The curves are plotted in Fig. 10 for $K_y$ listed in (57), while the curves in Fig. 12 are plotted for $K_y \times 100$, this means that the appropriate yield stresses, say $\sigma_Y = 1/\sqrt{K_y}$ are ten times smaller than the corresponding values employed to obtain Fig. 10. This is rationale for which the material was called a "softer material" for the data given in Fig. 12. The experimental data provided by Kim and Yin (1997) on the orientational change of the orthotropic axes with the uniaxial strain have been plotted in Figs. 10 and 12. The discrepancies, which appear in the graphs plotted in Figs. 12 and 10 show that the material response for the solutions of the appropriate differential systems is dependent on the plastic material constants and not only through the peculiar ratios of these parameters. Only when we choose in our model the softer material, we obtain that the Dafalias plastic spin expression, with a fixed material spin parameter, for the orientational anisotropy produces a satisfactory fit of the experimental data as shown in Fig. 12(a), see also Fig. 12(b). We presented Fig. 12(b), showing the variation of $\varphi - \Phi_0$ for the Mandel type plastic spin, where the same kinematic hardening parameters and yield constants have been used, exactly as proceeded with the Dafalias spin in Fig. 12(a). Although not illustrated, it should be mentioned that a similar conclusion can be drawn for the Liu–Wang plastic spin, like for the Mandel one. Concerning the attribute of opposite rotation when $\varphi - \Phi_0 = -30$ than when $\varphi - \Phi_0 = -45$ or $-60$, we notice that Kim and Yin experimental data are best simulated only with the Dafalias spin. For the other plastic spins considered in this work, i.e. Mandel and Liu–Wang plastic spins, the orthotropic axis $\mathbf{n}_1$ is rotated towards $\mathbf{m}_1$ in the same direction, for the aforementioned values of the angle $\Phi_0$. It should be mentioned that a monotonous behaviour of the angle $\varphi - \Phi_0$ with respect to the initial direction of the anisotropic axes has been emphasized for the Liu–Wang plastic spin, as well as the Mandel plastic spin, see Fig. 12(b). The attribute of opposite rotation in Kim and Yin experimental data, for one initial orthotropy direction in contrast with the other two, is best simulated only with the Dafalias spin, because of the quadratic dependence of the plastic spin on the stress, as already mentioned in the remark following Theorem 7. We have also performed the numerical simulations reproduced in Fig. 12 for different plastic spins without hardening (i.e. $\epsilon_2 = \epsilon_3 = \epsilon_0 = 0$) and with hardening parameters one order larger than the previous ones given in (57) (i.e. which corresponded to weak hardening), see Fig. 13.
and Kim and Yin (1997), we found a drastically reorientation of the orthotropic axes, see Fig. 10.

5. Comments and conclusions

In this paper, we provided a constitutive framework with only one plastic spin (unlike the multiple spins model by Dafalias, 1993) for a realistic description of the evolution of the orthotropy orientation during the deformation process. A new formulation in the Eulerian setting of the model, involving the Euler angles among the variables. The Euler angles measure the change in the orientation of the orthotropic axes, the orientational anisotropy is essentially due to the presence of the plastic spin in the constitutive framework. We formulated the theorems describing, in an unified manner, the evolution of the elasto-plastic material, regardless of the existence of a plane rotation, but for different types of plastic spins considered here. The sufficient condition, formulated in Theorem 8 together with Theorem 7, ensures the plane rotation of the anisotropic spins and leads to practical conclusions: for all plastic spins considered in this paper, in an anisotropic three-dimensional body, a plane rotation of the orthotropic directions is realized in a plane deformation process if one of the anisotropic axis, say $j$, is normal to the plane at the initial moment.

The change in the anisotropy was experimentally emphasized by Kim and Yin (1997) in orthotropic sheets. In their performed experiments the samples have been pre-strained, and peculiar hardening effects are emphasized for second applied prestrains, see e.g. Figs. 3, 4(b) and (d). In the model proposed herein, we supposed that the current yield surfaces are derived by a kinematic hardening rule, which could be in agreement with hardening effects from Kim and Yin and with the Baushinger effect experimentally observed under pre-stressing by Phillips et al. (1972). A weak hardening is considered only. The kinematic hardening is not directly influenced by the plastic spin, as a consequence of the Prager and Ziegler hardening rules adapted to the orthotropic material. This rule is dependent only on the symmetric part of the rate of the plastic deformation.

The corotational formulation has been derived with respect to the same elastic spin, whose constitutive formulation naturally follows to be expressed in terms of the plastic spin. The change in the anisotropy can be realized by the proposed model if and only if the plastic spin is non-zero. For an isotropic elasto-plastic material, all shear components become zero and no variation in time has been observed for Euler’s angles, which means that orientational anisotropy could not appear.

We noticed that for an appropriate choice of the plastic constants, only through the Dafalias spin with a peculiar constant value for the material plastic spin we could find a satisfactory fit of the experimental data from Kim and Yin (1997). Moreover, in these circumstances the rotations supported by the orthotropy axes during the numerical simulations are in agreement with experiments.

No change in the values of the appropriate spin parameters is necessary in order to fit separately the experimental data given for the three reference angles. We mentioned the adapted values for the constant $\eta := \eta_s$, were $\eta$ is the spin material constant and $\eta_s$ is the principal stress, made by Dafalias (2000), whereas Kim and Yin (1997) employed some arbitrarily chosen values for a certain constant to show the prediction of the empirical formula in Fig. 5.

We conclude that the variations between the considered models can be seen as the influence of the plastic spin on the orientational anisotropy only. It should be clearly emphasized that the plastic spin influence is inferred in the model through the solution of the differential equations formulated in Theorems 4–7.

Moreover, for an anisotropic elasto-plastic material the principal axes of the stress tensor do not coincide with the orthotropy directions, as shown in the model proposed herein.

The graphs for the stress components show that the proposed model is anisotropic in strength, ductility, as well as in the plastic flow. Note that for certain materials and corresponding to the initial orientation of the orthotropic axes, the failure has been produced at different strengths and plastic deformations.

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Appendix A

(A1) Linear orthotropic elastic type constitutive equation.

The linear elastic and orthotropic type constitutive equation, which corresponds to $g_e$ invariant elastic type constitutive equation with respect to the local relaxed (isoclinic) configuration in terms of the Piola–Kirchhoff tensor, is expressed with nine elastic coefficients in Cleja-Tigoiu (2000) by

$$
T = \hat{\varepsilon}[\varepsilon'] = \hat{\varepsilon}[(\mathbf{m}_1 \otimes \mathbf{m}_1, \mathbf{m}_2 \otimes \mathbf{m}_2)[\varepsilon']]
= \{[\varepsilon' \cdot \mathbf{I} + b \varepsilon \mathbf{m}_1 \otimes \mathbf{m}_1 + c \varepsilon \mathbf{m}_2 \otimes \mathbf{m}_2] \mathbf{I} + \mathbf{d} \varepsilon' + e \varepsilon' (\mathbf{m}_1 \otimes \mathbf{m}_1) + (\mathbf{m}_2 \otimes \mathbf{m}_2) \mathbf{e} \} [\varepsilon']
+ f[\varepsilon' (\mathbf{m}_2 \otimes \mathbf{m}_2) + (\mathbf{m}_2 \otimes \mathbf{m}_2) \mathbf{e}']
+ [\varepsilon' \cdot \mathbf{I} + h \varepsilon \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + \varepsilon \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)](\mathbf{m}_1 \otimes \mathbf{m}_1)
+ [\varepsilon' \cdot \mathbf{I} + n \varepsilon \cdot (\mathbf{m}_1 \otimes \mathbf{m}_1) + p \varepsilon \cdot (\mathbf{m}_2 \otimes \mathbf{m}_2)](\mathbf{m}_2 \otimes \mathbf{m}_2).
$$

Here $T$ denotes the Cauchy stress tensor and $\varepsilon$ is the small deformation tensor. An equivalent representation for the linear orthotropic elastic type constitutive equation can be formulated in terms of the elastic material constants $a_{ij} := g_{ij}$:

\[
\begin{align*}
\varepsilon_{11} & = a_1 + 2b + d + 2e + h, \\
\varepsilon_{12} & = a_1 + a_2 + b + c + n, \\
\varepsilon_{13} & = a_1 + b, \\
\varepsilon_{22} & = a_1 + 2c + d + 2f + p, \\
\varepsilon_{23} & = a_1 + c, \\
\varepsilon_{33} & = a_1 + d, \\
\varepsilon_{44} & = d + e + f, \\
\varepsilon_{55} & = d + e + a_6, \\
\varepsilon_{66} & = d + e,
\end{align*}
\]

in the form written below in the component presentation written for $\hat{\varepsilon}[\mathbf{D}]$, in the orthotropic basis:

\[
\begin{align*}
\hat{\varepsilon}[\mathbf{D}]_{11} & = a_{11} \varepsilon_{11} + a_{12} \varepsilon_{22} + a_{13} \varepsilon_{33}, \\
\hat{\varepsilon}[\mathbf{D}]_{22} & = a_{22} \varepsilon_{11} + a_{22} \varepsilon_{22} + a_{23} \varepsilon_{33}, \\
\hat{\varepsilon}[\mathbf{D}]_{33} & = a_{33} \varepsilon_{11} + a_{33} \varepsilon_{22} + a_{33} \varepsilon_{33}, \\
\hat{\varepsilon}[\mathbf{D}]_{12} & = a_{44} \varepsilon_{12}, \\
\hat{\varepsilon}[\mathbf{D}]_{13} & = a_{55} \varepsilon_{13}, \\
\hat{\varepsilon}[\mathbf{D}]_{23} & = a_{66} \varepsilon_{23}.
\end{align*}
\]

We give here the components of the symmetric tensor $\hat{\varepsilon}[\mathbf{N}]$ on the orthotropic basis $(\mathbf{m}_1 \otimes \mathbf{m}_1)$:

\[
\begin{align*}
\hat{\varepsilon}[\mathbf{N}]_{11} & = b_{1111} \varepsilon_{11} + b_{1122} \varepsilon_{22} + b_{1133} \varepsilon_{33}, \\
\hat{\varepsilon}[\mathbf{N}]_{12} & = b_{1212} \varepsilon_{12}, \\
\hat{\varepsilon}[\mathbf{N}]_{13} & = b_{1313} \varepsilon_{13}, \\
\hat{\varepsilon}[\mathbf{N}]_{22} & = b_{2222} \varepsilon_{22} + b_{2233} \varepsilon_{33}, \\
\hat{\varepsilon}[\mathbf{N}]_{23} & = b_{2323} \varepsilon_{23}, \\
\hat{\varepsilon}[\mathbf{N}]_{33} & = b_{3333} \varepsilon_{33}.
\end{align*}
\]
The coefficients written in (63) are dependent on the material constants through the relationships:

\[
\begin{align*}
B_{1111} &= 2a_11K_{11} + a_22K_{12}, & B_{1122} &= a_11K_{12} + 2a_12K_{22}, \\
B_{1212} &= a_21K_{11}, & B_{1133} &= 2a_13K_{33}, & B_{2211} &= 2a_22K_{11} + a_22K_{12}, \\
B_{2222} &= 2a_12K_{12} + 2a_22K_{22}, & B_{2233} &= a_23K_{33}, & B_{1311} &= 2a_13K_{11} + a_23K_{12}, \\
B_{3322} &= a_13K_{12} + 2a_23K_{22}, & B_{3333} &= 2a_33K_{33}, & B_{1133} &= a_06K_{n2}, \\
B_{2323} &= a_05(2K_{33} + K_{m1} - K_{m2}).
\end{align*}
\]

(A2) The components of the symmetric tensorial function \( l \) which describes the evolution of the tensorial hardening variable on the basis of the material constants \( \mathbf{m}_i \) are calculated via the following formulæ:

\[
\begin{align*}
\dot{l}_{11} &= p_{1111} \mathbf{s}_{11} + p_{1122} \mathbf{s}_{22}, & \dot{l}_{12} &= p_{2211} \mathbf{s}_{11} + p_{2222} \mathbf{s}_{22}, \\
\dot{l}_{13} &= p_{3333} \mathbf{s}_{33}, & \dot{l}_{12} &= p_{1212} \mathbf{s}_{12}, & \dot{l}_{13} &= p_{1313} \mathbf{s}_{13}, & \dot{l}_{12} &= p_{2323} \mathbf{s}_{23},
\end{align*}
\]

with the coefficients expressed in terms of the six yield constants as

\[
\begin{align*}
p_{1111} &= 2(c_0 + 2c_1)K_{11}, & p_{1122} &= (c_0 + 2c_1)K_{12}, \\
p_{2211} &= (c_0 + 2c_1)K_{11}, & p_{2222} &= (c_0 + 2c_1)K_{12}, & p_{3333} &= 2c_6K_{33}, \\
p_{1212} &= (c_0 + c_1 + c_2)K_{n1}, & p_{1313} &= (c_0 + c_1)K_{n2}, & p_{2323} &= (c_0 + c_1)(2K_{33} + K_{m1} - K_{m2}).
\end{align*}
\]

(A3) The coefficient in the expression of the hardening \( h_k \) are calculated in terms of the material constants \( \mathbf{C}_i \) as

\[
\begin{align*}
H_{1111} &= 2B_{1111}K_{11} + B_{2211}K_{12} + 2p_{1111}K_{11} + 2p_{2221}K_{12}, \\
H_{1122} &= B_{1111}K_{12} + 2B_{1122}K_{11} + 2B_{2211}K_{11} + 2B_{2221}K_{12} + 2p_{1112}K_{11} + 2p_{2211}K_{12}, \\
H_{1133} &= 2B_{1133}K_{11} + B_{2233}K_{12} + 2B_{2211}K_{12}, \\
H_{1222} &= B_{1122}K_{11} + 2B_{2222}K_{12} + 2p_{1122}K_{12} + 2p_{2222}K_{22}, \\
H_{2222} &= B_{1122}K_{12} + 2B_{2222}K_{12}, \\
H_{3333} &= 2B_{3333}K_{33} + 2p_{3333}K_{33}, \\
H_{1212} &= 2B_{1212}K_{12} + p_{1212}K_{12}, \\
H_{1313} &= 2B_{1313}K_{n2} + 2p_{1313}K_{n2}, \\
H_{2323} &= 2B_{2323}(2K_{33} + K_{m1} - K_{m2}) + 2p_{2323}(2K_{33} + K_{m1} - K_{m2}).
\end{align*}
\]