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Global stability of two-group SIR model with random perturbation ☆

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ABSTRACT

In this paper, we discuss the two-group SIR model introduced by Guo, Li and Shuai [H.B. Guo, M.Y. Li, Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, Can. Appl. Math. Q. 14 (2006) 259–284], allowing random fluctuation around the endemic equilibrium. We prove the endemic equilibrium of the model with random perturbation is stochastic asymptotically stable in the large. In addition, the stability condition is obtained by the construction of Lyapunov function. Finally, numerical simulations are presented to illustrate our mathematical findings.

1. Introduction

Epidemiology is the study of the spread of diseases with the objective to trace factors that are responsible for or contribute to their occurrence. Significant progress has been made in the theory and application of epidemiology modeling by mathematical research. Most models for the transmission of infectious diseases descend from the classical SIR model of Kermack and McKendrick [2]. SIR model and a lot of its extensions are well investigated by many scholars [3–11]. In recent years, multigroup models have been proposed in the literature to describe the transmission dynamics of infectious diseases in heterogeneous host populations. One of the earliest works on multigroup models is the seminal paper by Lajmanovich and Yorke [12] on a class of SIS multigroup models for the transmission dynamics of Gonorrhea. They established a complete analysis of the global dynamics. The global stability of the unique endemic equilibrium is proved by using a global Lyapunov function. Subsequently, much research has been done on multigroup models, see, e.g., [1,13–16]. Similarly, the authors in [1,13,14] also study the question of global stability of the endemic equilibrium of multigroup models under certain restrictions. In light of these results, complete determination of the global dynamics of these models is essential for their application and further development.

In fact, there are real benefits to be gained in using stochastic models because real life is full of randomness and stochasticity. Recently, several authors studied stochastic biological system, see [17–26]. In addition, some stochastic epidemic models have been studied by many authors, see [27–30]. In [27–29], the situation of the parameter perturbation was considered. Dalal et al. [27,28] showed that stochastic models had nonnegative solutions and carried out analysis on the asymptotic stability of models. Tornatore et al. [29] studied the stability of disease free equilibrium of a stochastic SIR model with or without distributed time delay. On the other hand, white noise stochastic perturbations around the positive endemic equilibrium of epidemic models was considered in [30,31]. Carletti [30] investigated the stability properties of a stochastic model for phage-bacteria interaction in open marine environment both analytically and numerically. Beretta et al. [31] proved the stability of epidemic model with stochastic time delays influenced by probability under certain conditions.

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Due to the large scale and complexity of multigroup models, progresses in the mathematical analysis of deterministic and stochastic systems have been slow.

In this paper, we will consider the problem of the two-group SIR model in [1] with respect to white noise stochastic perturbations around its positive endemic equilibrium. The remain part of this paper is as follows. In the next section, we recall the deterministic multigroup SIR model and its main results given by Guo et al. [1]. Section 3 introduce the stochastic model. In Section 4, we carry out an analysis of stability of the endemic equilibrium by means of Lyapunov functions. Finally, numerical simulations are presented to illustrate our mathematical findings.

2. Deterministic multigroup SIR epidemic model and main results

Guo et al. [1] characterized that a multigroup model is, in general, formulated by dividing the population of size $N(t)$ into n distinct groups. For $1 \leqslant k \leqslant n$, the k-th group is further partitioned into three compartments: the susceptible, infectious, and recovered, whose numbers of individuals at time t are denoted by $S_k(t)$, $I_k(t)$ and $R_k(t)$, respectively. They considered the following multigroup SIR epidemic model:

$$\begin{cases} \dot{S}_k = (1-p_k)\Lambda_k - \left(d_k^S + \theta_k\right)S_k - \displaystyle\sum_{j=1}^{n} \beta_{kj} S_k I_j, \\ \dot{I}_k = \displaystyle\sum_{j=1}^{n} \beta_{kj} S_k I_j - \left(d_k^I + \epsilon_k + \gamma_k\right)I_k, \\ \dot{R}_k = p_k \Lambda_k + \theta_k S_k + \gamma_k I_k - d_k^R R_k, \end{cases} \tag{2.1}$$

where $k = 1, 2, \ldots, n$. The parameters in the model are summarized in the following list:

- β_{ij}: transmission coefficient between compartments S_i and I_j;
- $d_k^S; d_k^I; d_k^R$: natural death rates of S, I, R compartments in the k-th group, respectively;
- Λ_k: influx of individuals into the k-th group;
- p_k: fraction of new individuals into the k-th group who are immuned;
- θ_k: fraction of individuals in S_k who are vaccinated;
- γ_k: recovery rate of infectious individuals in the k-th group;
- ϵ_k: disease-caused death rate in the k-th group.

All parameter values are assumed to be nonnegative and $d_k^S, d_k^I, d_k^R, \Lambda_k > 0$ for all k.

For each k, adding the three equations in (2.1), gives

$$\begin{aligned} (S_k + I_k + R_k)' &= \Lambda_k - d_k^S - \left(d_k^I + \epsilon_k\right)I_k - d_k^R R_k \\ &\leqslant \Lambda_k - d_k^*(S_k + I_k + R_k), \end{aligned}$$

where $d_k^* = \min\{d_k^S, d_k^I + \epsilon_k, d_k^R\}$. Hence $\limsup_{t\to\infty}(S_k + I_k + R_k) \leqslant \Lambda_k / d_k^*$. Similarly, it follows from the first equation in (2.1) that

$$\limsup_{t\to\infty} S_k \leqslant \frac{(1-p_k)\Lambda_k}{d_k^S + \theta_k}.$$

Observe that the variable R_k does not appear in the first two equations of (2.1). This allows us to consider first the following reduced system for S_k and I_k

$$\begin{cases} \dot{S}_k = (1-p_k)\Lambda_k - \left(d_k^S + \theta_k\right)S_k - \displaystyle\sum_{j=1}^{n} \beta_{kj} S_k I_j, \\ \dot{I}_k = \displaystyle\sum_{j=1}^{n} \beta_{kj} S_k I_j - \left(d_k^I + \epsilon_k + \gamma_k\right)I_k, \end{cases} \tag{2.2}$$

where $k = 1, 2, \ldots, n$, in the feasible region

$$\Gamma = \left\{ (S_1, I_1, \ldots, S_n, I_n) \in \mathbb{R}_+^{2n} \colon S_k \leqslant \frac{(1-p_k)\Lambda_k}{d_k^S + \theta_k},\ S_k + I_k \leqslant \frac{\Lambda_k}{d_k^*},\ k = 1, 2, \ldots, n \right\}. \tag{2.3}$$

Behaviors of R_k can then be determined from the last equation in (2.1). It can be verified that Γ in (2.3) is positively invariant with respect to (2.2). Let $\operatorname{int}\Gamma$ denote the interior of Γ. The following results will be stated for system (2.2) in Γ, and can be translated straightforwardly to system (2.1).

System (2.2) always has the disease-free equilibrium

$$P_0 = \left(S_1^0, 0, S_2^0, 0, \ldots, S_n^0, 0\right),$$

where

$$S_k^0 = \frac{(1-p_k)\Lambda_k}{d_k^S + \theta_k}, \quad k = 1, 2, \ldots, n, \tag{2.4}$$

is the equilibrium of the S_k population in the absence of disease ($I_1 = I_2 = \cdots = I_n = 0$). An endemic equilibrium $P^* = (S_1^*, I_1^*, S_2^*, I_2^*, \ldots, S_n^*, I_n^*)$ of (2.2) belongs to int Γ, namely, $S_k^* > 0$, $I_k^* > 0$, $k = 1, 2, \ldots, n$. Set

$$R_0 = \rho(M_0), \tag{2.5}$$

where

$$M_0 = M\left(S_1^0, S_2^0, \ldots, S_n^0\right) = \left(\frac{\beta_{ij} S_i^0}{d_i^I + \epsilon_i + \gamma_i}\right)_{n\times n}, \tag{2.6}$$

and ρ denotes the spectral radius. The basic reproduction number R_0 is the key threshold parameter whose values completely characterize the global dynamics of (2.2). Note that the matrix $B = (\beta_{ij})_{n\times n}$ denotes the contact matrix, where $\beta_{ij} \geqslant 0$. The following results were proved in [1] and just recalled:

Proposition 2.1. *Assume $B = (\beta_{ij})$ is irreducible. Then the following hold.*

(1) *If $R_0 \leqslant 1$, then P_0 is the unique equilibrium of (2.2) and it is globally stable in Γ.*
(2) *If $R_0 > 1$, then P_0 is unstable and system (2.2) is uniformly persistent in* int Γ.

Lemma 2.2. *Assume $B = (\beta_{ij})$ is irreducible. If $R_0 > 1$, then (2.2) has at least one endemic equilibrium.*

Lemma 2.3. *Assume $B = (\beta_{ij})$ is irreducible. If $R_0 > 1$, then there exists a unique endemic equilibrium P^*, and P^* is globally asymptotically stable in* int Γ.

The authors proved that, when $R_0 > 1$, the endemic equilibrium of the model is unique and globally asymptotically stable. Their proof of global stability of the endemic equilibrium relies on the use of a class of global Lyapunov functions and graph theory. For this class of multigroup models, the results completely resolve the open problem on the uniqueness and global stability of endemic equilibrium.

3. Stochastic model derivation

In this paper, we only consider the case of $k = 2$ in system (2.1):

$$\begin{cases} \dot{S}_k = (1-p_k)\Lambda_k - \left(d_k^S + \theta_k\right)S_k - \beta_{k1} S_k I_1 - \beta_{k2} S_k I_2, \\ \dot{I}_k = \beta_{k1} S_k I_1 + \beta_{k2} S_k I_2 - \left(d_k^I + \epsilon_k + \gamma_k\right) I_k, \\ \dot{R}_k = p_k \Lambda_k + \theta_k S_k + \gamma_k I_k - d_k^R R_k, \end{cases} \tag{3.1}$$

where $k = 1, 2$. The global stability of the endemic equilibrium of system (3.1) has been discussed in the proof of Theorem 2.3 in [1].

Here we also assume $B = (\beta_{ij})_{2\times 2}$ is irreducible. From Lemma 2.3 in Section 2, if $R_0 > 1$, then there exists a unique endemic equilibrium P^*.

We assume stochastic perturbations are of white noise type, which are directly proportional to distances $S(t), I(t), R(t)$ from values of S^*, I^*, R^*, influence on the $\dot{S}(t), \dot{I}(t), \dot{R}(t)$ respectively. So system (3.1) results in

$$\begin{cases} \dot{S}_k = (1-p_k)\Lambda_k - \left(d_k^S + \theta_k\right)S_k - \beta_{k1} S_k I_1 - \beta_{k2} S_k I_2 + \sigma_{1k}\left(S_k - S_k^*\right)\dot{B}_{1k}(t), \\ \dot{I}_k = \beta_{k1} S_k I_1 + \beta_{k2} S_k I_2 - \left(d_k^I + \epsilon_k + \gamma_k\right) I_k + \sigma_{2k}\left(I_k - I_k^*\right)\dot{B}_{2k}(t), \\ \dot{R}_k = p_k \Lambda_k + \theta_k S_k + \gamma_k I_k - d_k^R R_k + \sigma_{3k}\left(R_k - R_k^*\right)\dot{B}_{3k}(t), \end{cases} \tag{3.2}$$

where $B_{1k}(t), B_{2k}(t), B_{3k}(t)$ $(k = 1, 2)$ are independent standard Brownian motions and $\sigma_{ik}^2 > 0$ represent the intensities of $B_{ik}(t)$ $(i = 1, 2, 3)$, respectively.

Obviously, stochastic system (3.2) has the same equilibrium points as system (3.1). In the next section, we will investigate the stability of the equilibrium P^* of system (3.2). Below we will construct a class of different Lyapunov functions from those used in [1] to achieve our proof under certain conditions.

4. Stochastic stability of the endemic equilibrium

In this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B_{ik}(t)$ be the Brownian motions defined on this probability space.

If $R_0 > 1$, then the stochastic system (3.2) can be centered at its endemic equilibrium $P^* = (S_1^*, I_1^*, R_1^*, S_2^*, I_2^*, R_2^*)$, by the change of variables

$$u_k = S_k - S_k^*, \qquad v_k = I_k - I_k^*, \qquad w_k = R_k - R_k^*.$$

By the way, we obtain the following system

$$\begin{cases} \dot{u}_k = -\left(d_k^S + \theta_k - \beta_{k1} I_1^* - \beta_{k2} I_2^*\right) u_k - \beta_{k1} S_k^* v_1 - \beta_{k2} S_k^* v_2 - \beta_{k1} u_k v_1 - \beta_{k2} u_k v_2 + \sigma_{1k} u_k \dot{B}_{1k}(t), \\ \dot{v}_k = \left(\beta_{k1} I_1^* + \beta_{k2} I_2^*\right) u_k - \left(d_k^I + \epsilon_k + \gamma_k\right) v_k + \beta_{k1} S_k^* v_1 + \beta_{k2} S_k^* v_2 + \beta_{k1} u_k v_1 + \beta_{k2} u_k v_2 + \sigma_{2k} v_k \dot{B}_{2k}(t), \\ \dot{w}_k = \theta_k u_k + \gamma_k v_k - d_k^R w_k + \sigma_{3k} w_k \dot{B}_{3k}(t). \end{cases} \tag{4.1}$$

It is easy to see that the stability of the system (3.2) equilibrium is equivalent to the stability of zero solution of system (4.1).

Before proving the main theorem we put forward a lemma in [19].

Consider the d-dimensional stochastic differential equation [19]

$$dx(t) = f\big(x(t), t\big)\,dt + g\big(x(t), t\big)\,dB(t), \quad \text{on } t \geqslant t_0. \tag{4.2}$$

Assume that the assumptions of the existence-and-uniqueness theorem are fulfilled. Hence, for any given initial value $x(t_0) = x_0 \in \mathbb{R}^d$, Eq. (4.2) has a unique global solution that is denoted by $x(t; t_0, x_0)$. Assume furthermore that

$$f(0,t) = 0 \quad \text{and} \quad g(0,t) = 0 \quad \text{for all } t \geqslant t_0.$$

So Eq. (4.2) has the solution $x(t) \equiv 0$ corresponding to the initial value $x(t_0) = 0$. This solution is called the trivial solution or equilibrium position.

Denote by $C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty]$ such that they are continuously twice differentiable in x and once in t. Define the differential operator L associated with Eq. (4.2) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \left[g^T(x,t) g(x,t)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_i}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(x,t) = V_t(x,t) + V_x(x,t) f(x,t) + \frac{1}{2} trace\left[g^T(x,t) V_{xx}(x,t) g(x,t)\right].$$

Definition 4.1. (i) The trivial solution of Eq. (4.2) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$P\left\{\left|x(t; t_0, x_0)\right| < r \text{ for all } t \geqslant t_0\right\} \geqslant 1 - \varepsilon$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

(ii) The trivial solution is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$P\left\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} \geqslant 1 - \varepsilon$$

whenever $|x_0| < \delta_0$.

(iii) The trivial solution is said to be stochastically asymptotically stable in the large if it is stochastically asymptotically stable and, moreover, for all $x_0 \in \mathbb{R}^d$

$$P\left\{\lim_{t\to\infty} x(t; t_0, x_0) = 0\right\} = 1.$$

Lemma 4.2. *If there exists a positive-definite decrescent radially unbounded function $V(x,t) \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$ such that $LV(x,t)$ is negative-definite, then the trivial solution of Eq.* (4.2) *is stochastically asymptotically stable in the large.*

The proof of this theorem can be found in [19].

From the above lemma, we can obtain the stochastically asymptotically stability of equilibrium as following:

Theorem 4.3. *Assume that $B = (\beta_{ij})_{2\times 2}$ is irreducible and $R_0 > 1$, then if the following condition is satisfied*

$$\sigma_{1k}^2 < 2\big(d_k^S + \theta_k\big), \qquad \sigma_{2k}^2 < \frac{2(d_k^I + \epsilon_k + \gamma_k)(\beta_{k1} I_1^* + \beta_{k2} I_2^*)}{\beta_{k1} I_1^* + \beta_{k2} I_2^* + d_k^S + \theta_k + d_k^I + \epsilon_k + \gamma_k}, \qquad \sigma_{3k}^2 < 2d_k^R, \tag{4.3}$$

the endemic equilibrium P^* is stochastically asymptotically stable in the large.

Proof. It is easy to see that we only need to prove the zero solution of (4.1) is stochastically asymptotically stable in the large. Let $x_k(t) = (u_k(t), v_k(t), w_k(t))^T$, $k = 1, 2$ and $x(t) = (x_1(t), x_2(t))^T$.

We define the Lyapunov function $V(x(t))$ as follows

$$V(x) = \frac{1}{2}\sum_{k=1}^{2}\big[a_k v_k^2 + b_k(u_k + v_k)^2 + c_k w_k^2\big], \quad k = 1, 2, \tag{4.4}$$

where $a_k > 0$, $b_k > 0$, $c_k > 0$ are real positive constants to be chosen later. Then it can be described as the quadratic form

$$V(x) = \frac{1}{2}\sum_{k=1}^{2} x_k^T Q_k x_k,$$

where

$$Q_k = \begin{pmatrix} b_k & b_k & 0 \\ b_k & a_k + b_k & 0 \\ 0 & 0 & c_k \end{pmatrix}$$

is a symmetric positive-definite matrix. So it is obviously that $V(x)$ is positive-definite and decrescent.

For sake of simplicity, (4.4) may be divided into three functions: $V(x) = V_1(x) + V_2(x) + V_3(x)$, where

$$V_1(x) = \frac{1}{2}\sum_{k=1}^{2} a_k v_k^2, \qquad V_2(x) = \frac{1}{2}\sum_{k=1}^{2} b_k(u_k + v_k)^2, \qquad V_3(x) = \frac{1}{2}\sum_{k=1}^{2} c_k w_k^2, \quad k = 1, 2.$$

Using Itô's formula, we compute

$$\begin{aligned}
LV_1 &= \sum_{k=1}^{2} a_k v_k\big[\big(\beta_{k1} I_1^* + \beta_{k2} I_2^*\big)u_k - \big(d_k^I + \epsilon_k + \gamma_k\big)v_k + \beta_{k1} S_k^* v_1 + \beta_{k2} S_k^* v_2 + \beta_{k1} u_k v_1 + \beta_{k2} u_k v_2\big] \\
&\quad + \frac{1}{2}\sum_{k=1}^{2} a_k \sigma_{2k}^2 v_k^2 \\
&= \sum_{k=1}^{2} a_k\big[-\big(d_k^I + \epsilon_k + \gamma_k\big)v_k^2 + \big(\beta_{k1} S_k^* v_1 + \beta_{k2} S_k^* v_2\big)v_k\big] \\
&\quad + \sum_{k=1}^{2} a_k\left[\frac{1}{2}\sigma_{2k}^2 v_k^2 + \big(\beta_{k1} I_1^* + \beta_{k2} I_2^*\big)u_k v_k + (\beta_{k1} v_1 + \beta_{k2} v_2)u_k v_k\right] \\
&= \sum_{k=1}^{2} a_k I_k^*\left[-\big(d_k^I + \epsilon_k + \gamma_k\big)I_k^* \frac{v_k^2}{I_k^{*2}} + \left(\beta_{k1} S_k^* I_1^* \frac{v_1}{I_1^*} + \beta_{k2} S_k^* I_2^* \frac{v_2}{I_2^*}\right)\frac{v_k}{I_k^*}\right] \\
&\quad + \sum_{k=1}^{2} a_k\left[\frac{1}{2}\sigma_{2k}^2 v_k^2 + \big(\beta_{k1} I_1^* + \beta_{k2} I_2^*\big)u_k v_k + (\beta_{k1} v_1 + \beta_{k2} v_2)u_k v_k\right] \\
&= \sum_{k=1}^{2} a_k I_k^*\left[-(\bar\beta_{k1} + \bar\beta_{k2})\left(\frac{v_k}{I_k^*}\right)^2 + \left(\bar\beta_{k1}\frac{v_1}{I_1^*} + \bar\beta_{k2}\frac{v_2}{I_2^*}\right)\frac{v_k}{I_k^*}\right] \\
&\quad + \sum_{k=1}^{2} a_k\left[\frac{1}{2}\sigma_{2k}^2 v_k^2 + \big(\beta_{k1} I_1^* + \beta_{k2} I_2^*\big)u_k v_k + (\beta_{k1} v_1 + \beta_{k2} v_2)u_k v_k\right].
\end{aligned} \tag{4.5}$$

Set $\bar\beta_{ij} = \beta_{ij} S_i^* I_j^*$. Here we choose $a_1 I_1^* = \bar\beta_{21}$, $a_2 I_2^* = \bar\beta_{12}$, i.e., $a_1 = \beta_{21} S_2^*$, $a_2 = \beta_{12} S_1^*$. Substituting this into (4.5), yields

$$\begin{aligned}
LV_1 &= \bar\beta_{21}\left[-(\bar\beta_{11}+\bar\beta_{12})\left(\frac{v_1}{I_1^*}\right)^2+\left(\bar\beta_{11}\frac{v_1}{I_1^*}+\bar\beta_{12}\frac{v_2}{I_2^*}\right)\frac{v_1}{I_1^*}\right]\\
&\quad+\bar\beta_{12}\left[-(\bar\beta_{21}+\bar\beta_{22})\left(\frac{v_2}{I_2^*}\right)^2+\left(\bar\beta_{21}\frac{v_1}{I_1^*}+\bar\beta_{22}\frac{v_2}{I_2^*}\right)\frac{v_2}{I_2^*}\right]\\
&\quad+\sum_{k=1}^2 a_k\left[\frac12\sigma_{2k}^2v_k^2+\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)u_kv_k+(\beta_{k1}v_1+\beta_{k2}v_2)u_kv_k\right]\\
&= -\bar\beta_{12}\bar\beta_{21}\left(\frac{v_1}{I_1^*}-\frac{v_2}{I_2^*}\right)^2+\sum_{k=1}^2 a_k\left[\frac12\sigma_{2k}^2v_k^2+\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)u_kv_k+(\beta_{k1}v_1+\beta_{k2}v_2)u_kv_k\right]\\
&\leqslant \sum_{k=1}^2 a_k\left[\frac12\sigma_{2k}^2v_k^2+\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)u_kv_k+(\beta_{k1}v_1+\beta_{k2}v_2)u_kv_k\right]. \qquad (4.6)
\end{aligned}$$

Similarly, from Itô's formula, we obtain

$$\begin{aligned}
LV_2 &= \sum_{k=1}^2 b_k(u_k+v_k)\left[-\left(d_k^S+\theta_k\right)u_k-\left(d_k^I+\epsilon_k+\gamma_k\right)v_k\right]+\frac12\sum_{k=1}^2\left(a_k\sigma_{1k}^2u_k^2+b_k\sigma_{2k}^2v_k^2\right)\\
&= \sum_{k=1}^2 b_k\left[-\left(d_k^S+\theta_k-\frac12\sigma_{1k}^2\right)u_k^2-\left(d_k^I+\epsilon_k+\gamma_k-\frac12\sigma_{2k}^2\right)v_k^2-\left(d_k^S+\theta_k+d_k^I+\epsilon_k+\gamma_k\right)u_kv_k\right],\\
LV_3 &= \sum_{k=1}^2 c_kw_k\left(\theta_ku_k+\gamma_kv_k-d_k^Rw_k\right)+\frac12\sum_{k=1}^2 c_k\sigma_{3k}^2w_k^2\\
&= \sum_{k=1}^2 c_k\left[-\left(d_k^R-\frac12\sigma_{3k}^2\right)w_k^2+\theta_ku_kw_k+\gamma_kv_kw_k\right]. \qquad (4.7)
\end{aligned}$$

Then we compute

$$\begin{aligned}
LV &= LV_1+LV_2+LV_3\\
&\leqslant \sum_{k=1}^2 a_k\left[\frac12\sigma_{2k}^2v_k^2+\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)u_kv_k+\left(\beta_{k1}v_1+\beta_{k2}v_2\right)u_kv_k\right]\\
&\quad+\sum_{k=1}^2 b_k\left[-\left(d_k^S+\theta_k-\frac12\sigma_{1k}^2\right)u_k^2-\left(d_k^I+\epsilon_k+\gamma_k-\frac12\sigma_{2k}^2\right)v_k^2-\left(d_k^S+\theta_k+d_k^I+\epsilon_k+\gamma_k\right)u_kv_k\right]\\
&\quad+\sum_{k=1}^2 c_k\left[-\left(d_k^R-\frac12\sigma_{3k}^2\right)w_k^2+\theta_ku_kw_k+\gamma_kv_kw_k\right]\\
&= \sum_{k=1}^2\left\{-b_k\left(d_k^S+\theta_k-\frac12\sigma_{1k}^2\right)u_k^2-\left[b_k\left(d_k^I+\epsilon_k+\gamma_k-\frac12\sigma_{2k}^2\right)-\frac12a_k\sigma_{2k}^2\right]v_k^2\right.\\
&\quad+\left[a_k\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)-b_k\left(d_k^S+\theta_k+d_k^I+\epsilon_k+\gamma_k\right)\right]u_kv_k\\
&\quad\left.-c_k\left(d_k^R-\frac12\sigma_{3k}^2\right)w_k^2+c_k\theta_ku_kw_k+c_k\gamma_kv_kw_k\right\}+\sum_{k=1}^2 a_k(\beta_{k1}v_1+\beta_{k2}v_2)u_kv_k\\
&= L_0V+\sum_{k=1}^2 a_k(\beta_{k1}v_1+\beta_{k2}v_2)u_kv_k, \qquad (4.8)
\end{aligned}$$

where

$$\begin{aligned}
L_0V =: \sum_{k=1}^2&\left\{-b_k\left(d_k^S+\theta_k-\frac12\sigma_{1k}^2\right)u_k^2-\left[b_k\left(d_k^I+\epsilon_k+\gamma_k-\frac12\sigma_{2k}^2\right)-\frac12a_k\sigma_{2k}^2\right]v_k^2\right.\\
&+\left[a_k\left(\beta_{k1}I_1^*+\beta_{k2}I_2^*\right)-b_k\left(d_k^S+\theta_k+d_k^I+\epsilon_k+\gamma_k\right)\right]u_kv_k
\end{aligned}$$

$$- c_k\left(d_k^R - \frac{1}{2}\sigma_{3k}^2\right)w_k^2 + c_k\theta_k u_k w_k + c_k\gamma_k v_k w_k\Bigg\} \tag{4.9}$$

is the linear part of the right-hand side of inequality.

In (4.9) we choose

$$a_k\big(\beta_{k1}I_1^* + \beta_{k2}I_2^*\big) - b_k\big(d_k^S + \theta_k + d_k^I + \epsilon_k + \gamma_k\big) = 0,$$

then

$$b_k = \frac{a_k(\beta_{k1}I_1^* + \beta_{k2}I_2^*)}{d_k^S + \theta_k + d_k^I + \epsilon_k + \gamma_k}, \tag{4.10}$$

i.e.,

$$b_1 = \frac{\beta_{21}S_2^*(\beta_{11}I_1^* + \beta_{12}I_2^*)}{d_1^S + \theta_1 + d_1^I + \epsilon_1 + \gamma_1}, \qquad b_2 = \frac{\beta_{12}S_1^*(\beta_{21}I_1^* + \beta_{22}I_2^*)}{d_2^S + \theta_2 + d_2^I + \epsilon_2 + \gamma_2}.$$

Moreover, using Cauchy inequality to $\theta_k u_k w_k$ and $\gamma_k v_k w_k$, we can obtain

$$\begin{aligned}
\theta_k u_k w_k &\leqslant \frac{1}{4}\left(d_k^R - \frac{1}{2}\sigma_{3k}^2\right)w_k^2 + \frac{\theta_k^2 u_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2},\\
\gamma_k v_k w_k &\leqslant \frac{1}{4}\left(d_k^R - \frac{1}{2}\sigma_{3k}^2\right)w_k^2 + \frac{\gamma_k^2 v_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2}.
\end{aligned} \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9), yields

$$\begin{aligned}
L_0 V &\leqslant \sum_{k=1}^{2}\Bigg\{-\left[b_k\left(d_k^S + \theta_k - \frac{1}{2}\sigma_{1k}^2\right) - c_k\frac{\theta_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2}\right]u_k^2\\
&\quad - \left[b_k\left(d_k^I + \epsilon_k + \gamma_k - \frac{1}{2}\sigma_{2k}^2\right) - \frac{1}{2}a_k\sigma_{2k}^2 - c_k\frac{\gamma_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2}\right]v_k^2\\
&\quad - \frac{1}{2}c_k\left(d_k^R - \frac{1}{2}\sigma_{3k}^2\right)w_k^2\Bigg\}\\
&= -\sum_{k=1}^{2}\big(A_k u_k^2 + B_k v_k^2 + D_k w_k^2\big),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
A_k &= b_k\left(d_k^S + \theta_k - \frac{1}{2}\sigma_{1k}^2\right) - c_k\frac{\theta_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2},\\
B_k &= b_k\left(d_k^I + \epsilon_k + \gamma_k - \frac{1}{2}\sigma_{2k}^2\right) - \frac{1}{2}a_k\sigma_{2k}^2 - c_k\frac{\gamma_k^2}{d_k^R - \frac{1}{2}\sigma_{3k}^2},\\
D_k &= \frac{1}{2}c_k\left(d_k^R - \frac{1}{2}\sigma_{3k}^2\right).
\end{aligned}$$

Let us choose c_k such that

$$0 < c_k < \min\left\{\frac{d_k^R - \frac{1}{2}\sigma_{3k}^2}{\theta_k^2}b_k\left(d_k^S + \theta_k - \frac{1}{2}\sigma_{1k}^2\right), \frac{d_k^R - \frac{1}{2}\sigma_{3k}^2}{\gamma_k^2}\left[b_k\big(d_k^I + \epsilon_k + \gamma_k\big) - \frac{1}{2}(a_k + b_k)\sigma_{2k}^2\right]\right\}.$$

On the other hand, the condition in (4.3) is satisfied, so A_k, B_k, D_k are positive constants. Let $\lambda = \min_{k=1,2}\{A_k, B_k, D_k\}$, then $\lambda > 0$. From (4.12), one sees that

$$LV \leqslant -\lambda\big|x(t)\big|^2 + \circ\big(\big|x(t)\big|^2\big).$$

Hence $LV(x,t)$ is negative-definite in a sufficiently small neighborhood of $x = 0$ for $t \geqslant 0$. According to Lemma 4.1, we therefore conclude that the zero solution of (4.1) is stochastically asymptotically stable in the large.

The proof is complete. □

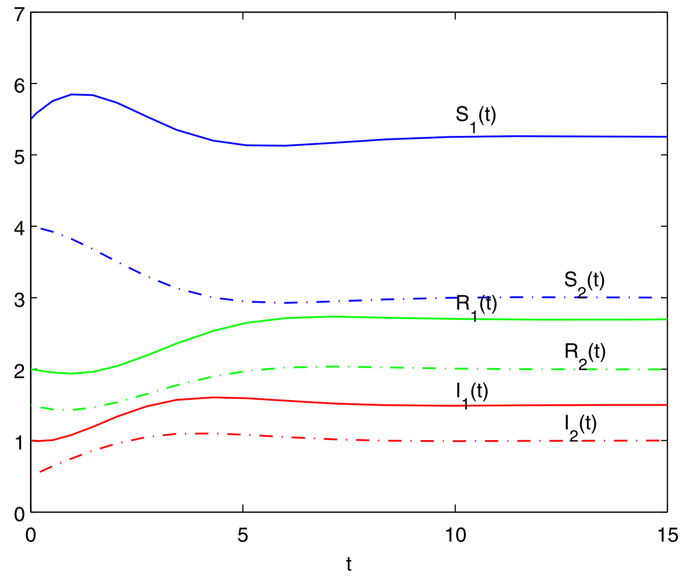

Fig. 1. Deterministic trajectories of SIR model (3.1) for initial condition $S_1(0)=5.5$, $I_1(0)=1$, $R_1(0)=2$, $S_2(0)=4$, $I_2(0)=0.5$, $R_2(0)=1.5$.

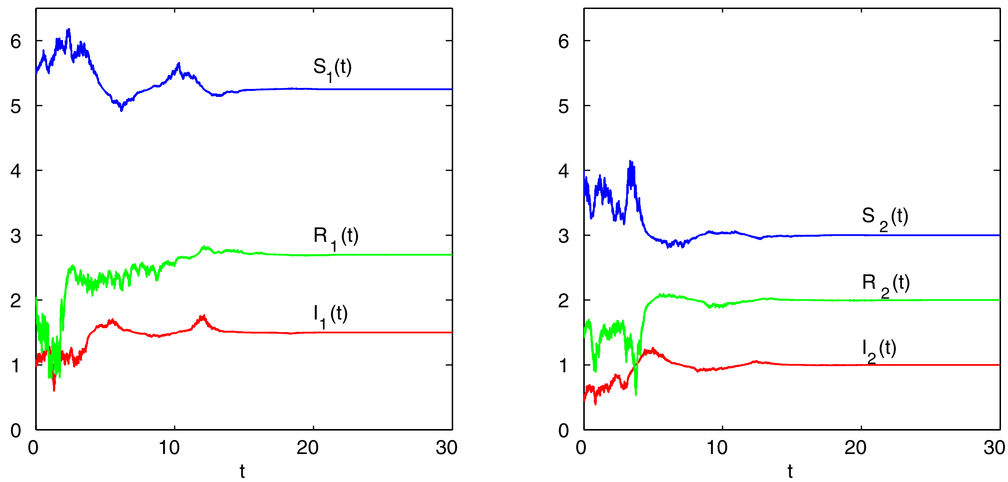

Fig. 2. Stochastic trajectories of SIR model (3.2) for $\sigma_{11}=0.5$, $\sigma_{21}=0.7$, $\sigma_{31}=0.8$, $\sigma_{12}=0.8$, $\sigma_{22}=0.6$, $\sigma_{32}=0.75$ and $\Delta t=10^{-3}$.

5. Numerical simulation

Computer simulations of this mathematical model agree well with mathematical theory. In order to confirm the stability results of Section 4 we numerically simulated the solution of the stochastic system (3.2). For simplicity, we assume $p_k=\theta_k=\epsilon_k=0$ and $d_k=d_k^S=d_k^I=d_k^R$ in systems (3.1) and (3.2). Furthermore, let $\Lambda_1=4.725$, $d_1=0.5$, $\gamma_1=0.9$, $\beta_{11}=0.15$, $\beta_{12}=0.175$; $\Lambda_2=2.4$, $d_2=0.4$, $\gamma_2=0.8$, $\beta_{21}=0.1$, $\beta_{22}=0.25$. Hence we obtain $S_1^*=5.25$, $I_1^*=1.5$, $R_1^*=2.7$; $S_2^*=3$, $I_2^*=1$, $R_2^*=2$. Moreover, we always choose initial value $(S_1(0),I_1(0),R_1(0),S_2(0),I_2(0),R_2(0))^T=(5.5,1,2,4,0.5,1.5)^T$.

In the absence of noise, we simulate the global stability of the endemic equilibrium of deterministic system (3.1) in Fig. 1.

On the other hand, we show the numerical simulation of the stochastic system (3.1). Given the discretization of system (3.2) for $t=0,\Delta t,2\Delta t,\dots,n\Delta t$,

$$\begin{cases} S_{k,i+1}=S_{k,i}+(\Lambda_k-d_kS_{k,i}-\beta_{k1}S_{k,i}I_{1,i}-\beta_{k2}S_{k,i}I_{2,i})\Delta t+\sigma_{1k}\big(S_{k,i}-S_k^*\big)\sqrt{\Delta t}\varepsilon_{1k,i},\\ I_{k,i+1}=I_{k,i}+\big(\beta_{k1}S_{k,i}I_{1,i}+\beta_{k2}S_{k,i}I_{2,i}-(d_k+\gamma_k)I_{k,i}\big)\Delta t+\sigma_{2k}\big(I_{k,i}-I_k^*\big)\sqrt{\Delta t}\varepsilon_{2k,i},\\ R_{k,i+1}=R_{k,i}+(\gamma_kI_{k,i}-d_kR_{k,i})\Delta t+\sigma_{3k}\big(R_{k,i}-R_k^*\big)\sqrt{\Delta t}\varepsilon_{3k,i}, \end{cases}$$

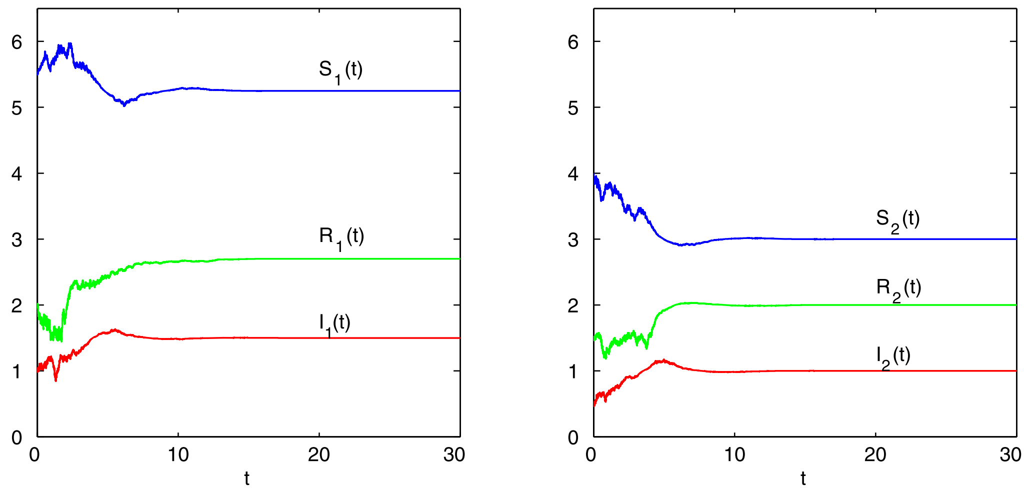

Fig. 3. Stochastic trajectories of SIR model (3.2) for $\sigma_{11}=0.4$, $\sigma_{21}=0.45$, $\sigma_{31}=0.39$, $\sigma_{12}=0.3$, $\sigma_{22}=0.42$, $\sigma_{32}=0.35$ and $\Delta t=10^{-3}$.

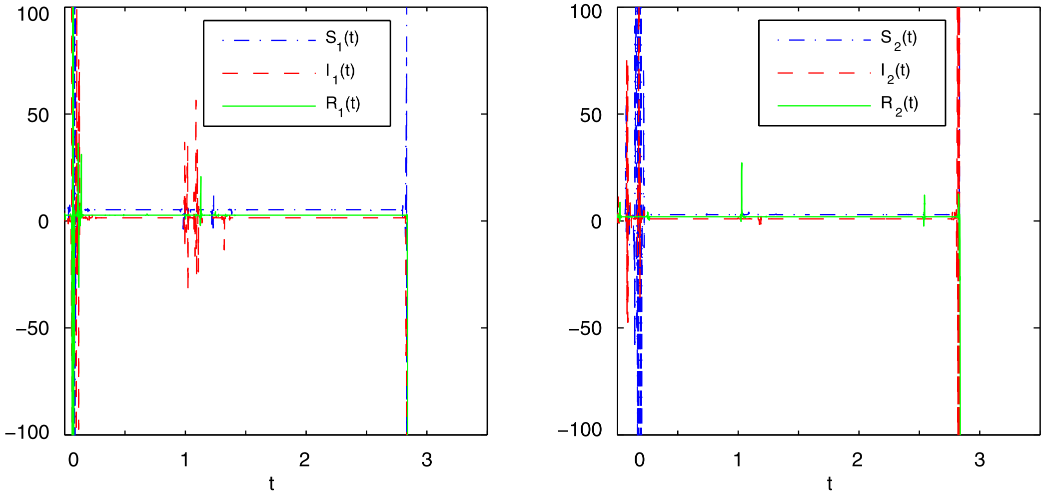

Fig. 4. Stochastic trajectories of SIR model (3.2) for $\sigma_{11}=39.23$, $\sigma_{21}=39.30$, $\sigma_{31}=39.24$, $\sigma_{12}=39.25$, $\sigma_{22}=39.27$, $\sigma_{32}=39.28$ and $\Delta t=10^{-3}$.

where time increment $\Delta t>0$, and $\varepsilon_{1k,i}, \varepsilon_{2k,i}, \varepsilon_{3k,i}$, $k=1,2$ are $N(0,1)$-distributed independent random variables which can be generated numerically by pseudo-random number generators.

In Fig. 2, the numerical simulation shows that the endemic equilibrium of stochastic system (3.2) is global asymptotically stable under the condition (4.3). Fig. 2 shows a realization of the dynamics of this system for $\sigma_{11}=0.5$, $\sigma_{21}=0.7$, $\sigma_{31}=0.8$, $\sigma_{12}=0.8$, $\sigma_{22}=0.6$, $\sigma_{32}=0.75$, whilst Fig. 2 corresponds to $\sigma_{11}=0.4$, $\sigma_{21}=0.45$, $\sigma_{31}=0.39$, $\sigma_{12}=0.3$, $\sigma_{22}=0.42$, $\sigma_{32}=0.35$. Moreover, comparison of Figs. 2 and 3 suggests that fluctuations reduce as the noise level decreases. Note that the condition (4.3) is just a sufficient condition. When this condition is not satisfied, the stochastic system (3.2) may be unstable. If we choose $\sigma_{11}=39.23$, $\sigma_{21}=39.30$, $\sigma_{31}=39.24$, $\sigma_{12}=39.25$, $\sigma_{22}=39.27$, $\sigma_{32}=39.28$, then the solution of the stochastic system (3.2) is not asymptotically stable but explode to infinity at the finite time (see Fig. 4).

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