Uniform approximation by some Hermite interpolating splines

E. Neuman (*)

ABSTRACT

In this paper some upper bound for the error $\| s-f \|_\infty$ is given, where $f \in C^1[a, b]$, but s is a so-called Hermite spline interpolant (HSI) of degree $2q-1$ such that $f(x_i) = s(x_i)$, $f'(x_i) = s'(x_i)$, $s^{(j)}(x_i) = 0$ $(i = 0, 1, \ldots, n;\ j = 2, 3, \ldots, q-1;\ n > 0,\ q > 0)$ and the knots x_i are such that $a = x_0 < x_1 < \ldots < x_n = b$. Necessary and sufficient conditions for the existence of convex HSI are given and upper error bound for approximation of the function $f \in C^1[a, b]$ by convex HSI is also given.

1. INTRODUCTION AND NOTATION

For a given interval $[a, b]$ $(-\infty < a < b < \infty)$ and given natural number $n > 0$ let Δ denote a fixed partition of the interval $[a, b]$ such that

$$\Delta : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

By $H^{(q)}(\Delta)$ (q - integer, $q > 0$) we denote the so-called *Hermite space* of all real-valued functions $s(x)$ $(x \in [a, b])$ such that :

(i) in each subinterval (x_i, x_{i+1}) $(i = 0, 1, \ldots, n-1)$ $s(x)$ is an algebraic polynomial of degree at most $2q-1$,

(ii) $s(x) \in C^{q-1}[a, b]$.

Each function $s(x) \in H^{(q)}(\Delta)$ is called *Hermite spline.*

Let us consider the following interpolating problem : for a given real-valued function $f \in C^1[a, b]$ find the function $s(x) \in H^{(q)}(\Delta)$ $(q > 1)$ such that :

$$\text{(H)} \quad f(x_i) = s(x_i), \quad f'(x_i) = s'(x_i), \quad s^{(j)}(x_i) = 0$$
$$(i = 0, 1, \ldots, n;\ j = 2, 3, \ldots, q-1).$$

It is obvious that the problem (H) possesses exactly one solution for any arbitrary function $f \in C^1[a, b]$. This solution $s(x)$ is called *Hermite spline interpolant.*

In the section 2 of this note some fundamental functions used later are constructed. Section 3 contains some upper bound for the error $\| s-f \|_\infty$ (where function s is solution of (H)). This bound is given in terms of the modulus of continuity $\omega(f, h)$ of the functions f (Theorem 3.1), where

$$\omega(f, h) = \sup_{|x-y| \leq h} |f(x) - f(y)| \quad (x, y \in [a,b],\ 0 < h \leq b-a).$$

Additionaly the upper error bound for approximation of the function $f \in C^1[a, b]$ by a convex Hermite spline interpolant (CHSI) is given (Theorem 3.2). This bound is also given in terms of $\omega(f, h)$. Necessary and sufficient conditions for the existence of CHSI are given in Proposition 3.1. In the section 4 some remarks are given.

2. SOME FUNDAMENTAL FUNCTIONS

Now we assume that the integer $q > 1$. In the construction of the solution $s(x)$ of (H) an important role play the so-called *fundamental functions* $\phi_0(t)$, $\phi_1(t)$, $\psi_0(t)$, $\psi_1(t)$ $(t \in [0, 1])$ such that :

$$\phi_0(0) = \phi_1(1) = 1, \quad \phi_0(1) = \phi_1(0) = \phi_l^{(j)}(0) = \phi_l^{(j)}(1) = 0$$
$$(l = 0, 1;\ j = 1, 2, \ldots, q-1), \tag{2.1}$$

$$\psi_0'(0) = \psi_1' = (1),\ \psi_0'(1) = \psi_1'(0) = \psi_l^{(j)}(0) = \psi_l^{(j)}(1) = 0$$
$$(l = 0, 1;\ j = 0, 2, 3, \ldots, q-1). \tag{2.2}$$

Easy calculation shows that functions ϕ_0 and ϕ_1 are such

$$\phi_1(t) = \frac{1}{a_q} \int_0^t x^{q-1}(1-x)^{q-1} dx,$$

$$\phi_0(t) + \phi_1(t) = 1 \quad (t \in [0, 1]), \tag{2.3}$$

$$a_q = \int_0^1 x^{q-1}(1-x)^{q-1} dx.$$

From this also follows that $\phi_0(t) \geq 0$, $\phi_1(t) \geq 0$ for

(*) E. Neuman, Institute of Computer Science, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

$t \in [0, 1]$. For functions ψ_0 and ψ_1 we have explicit formulas

$$\psi_0(t) = t\,[1-\beta_q(t)], \qquad \psi_1(t) = (1-t)\,\gamma_q(t),$$

$$\beta_q(t) = \frac{1}{b_q}\int_0^t x^{q-2}(1-x)^{q-1}\,dx, \tag{2.4}$$

$$\gamma_q(t) = -\frac{1}{b_q}\int_0^t x^{q-1}(1-x)^{q-2}\,dx \qquad (t \in [0, 1]),$$

$$b_q = \int_0^1 x^{q-2}(1-x)^{q-1}\,dx.$$

Hence also follows that

$$\psi_0(t) \geqslant 0, \qquad \psi_1(t) \leqslant 0 \quad \text{for } t \in [0, 1]. \tag{2.5}$$

Elementary calculations show that

$$\frac{a_q}{b_q} = \frac{q-1}{2q-1} \tag{2.6}$$

3. MAIN RESULTS

For simplicity of further notations let $f_i = f(x_i)$, $f_i' = f'(x_i)$ $(i = 0, 1, \dots, n)$, $J_i = (x_i, x_{i+1})$, $h_i = x_{i+1} - x_i$ $(i = 0, 1, \dots, n-1)$.

In virtue of (2.1) - (2.2) the function $s(x) \in H^{(q)}(\Delta)$ which is a solution of (H) is given (for $x \in J_i$) in the following way

$$s(x) = f_i\phi_0(t) + f_{i+1}\phi_1(t) + h_i[f_i'\psi_0(t) + f_{i+1}'\psi_1(t)] \qquad \left(t = \frac{x - x_i}{h_i}\right). \tag{3.1}$$

Now we are able to formulate the first of our theorems.

THEOREM 3.1. *For a given function $f \in C^1[a, b]$ let $s(x) \in H^{(q)}(\Delta)$ be a solution of the Hermite interpolating problem* (H). *Then we have the following estimation*

$$\|s - f\|_\infty \leqslant \omega(f, h) + F h\, a_q,$$

where $F = \max\limits_{0 \leqslant i \leqslant n} |f_i'|$, $h = \max\limits_{0 \leqslant i \leqslant n-1} h_i$ *and*

$$a_q = 2^{-q}\, \frac{{}_2F_1\left(2-q, q; 1+q; \frac{1}{2}\right)}{{}_2F_1(2-q, q; 1+q; 1)} \tag{3.2}$$

(here ${}_2F_1(a, b; c; x)$ denotes an hypergeometric function with parameters a, b, c).

Proof

For $x \in J_i$ $(i = 0, 1, \dots, n-1)$ in virtue of (3.1) and we have

$$s(x) - f(x) = [f_i - f(x)]\phi_0(t) + [f_{i+1} - f(x)]\phi_1(t) + h_i[f_i'\psi_0(t) + f_{i+1}'\psi_1(t)]. \tag{2.3}$$

From (2.3) and (2.5) we have

$$|s(x) - f(x)| \leqslant \omega(f, h) + F h \max_{0 \leqslant t \leqslant 1}[\psi_0(t) - \psi_1(t)] \tag{3.3}$$

Now let $\psi(t) = \psi_0(t) - \psi_1(t)$ $(t \in [0,1])$. Below we prove that $\max\limits_{0 \leqslant t \leqslant 1} \psi(t) = \psi(\frac{1}{2}) = a_q$, where a_q is given by (3.2). From (2.4) and (2.5) follows that $\psi(0) = \psi(1) = 0$, $\psi(t) \geqslant 0$ and $\psi''(t) \leqslant 0$ for $t \in [0,1]$. Hence there exists exactly one point t_0 in $(0, 1)$ for which $\max\limits_{0 \leqslant t \leqslant 1} \psi(t) = \psi(t_0)$. This point is a solution of the equation

$$\int_0^t x^{q-2}(1-x)^{q-2}\,dx = b_q.$$

Easy calculation shows that $t_0 = \frac{1}{2}$. Further from (2.4) we obtain $\psi(\frac{1}{2}) = -\gamma_q(\frac{1}{2})$. Using the identity [1]

$$\int_0^u \frac{x^{k-1}}{(1+\beta x)^l}\,dx = \frac{u^k}{k}\, {}_2F_1(l, k; 1+k; -\beta u)$$

we obtain $\psi(\frac{1}{2}) = a_q$. Hence and from (3.3) follows the thesis of our theorem.

Now necessary and sufficient conditions for the convexity (in each subinterval J_i) for the Hermite spline interpolant $s(x) \in H^{(q)}(\Delta)$ are given.

PROPOSITION 3.1. *With the assumptions of* Theorem 3.1 *the Hermite spline interpolant $s(x)$ $(x \in J_i)$ is convex if and only if the following inequalities are satisfied*

$$\frac{q f_i' + (q-1) f_{i+1}'}{2q-1} \leqslant \frac{f_{i+1} - f_i}{h_i} \leqslant \frac{(q-1) f_i' + q f_{i+1}'}{2q-1} \qquad (i = 0, 1, \dots, n-1). \tag{3.4}$$

Proof

As it follows from (3.1), (2.3) and (2.4) the inequality $s''(x) \geqslant 0$ $(x \in J_i)$ is equivalent to the following one

$$\frac{q-1}{a_q}\,\frac{f_{i+1} - f_i}{h_i}(1-2t) + \frac{(2q-1)t - q}{b_q} f_i' + \frac{(2q-1)t + 1 - q}{b_q} f_{i+1}' \geqslant 0$$

$(t \in [0, 1])$. Hence in virtue of (2.6) we obtain (3.4).

Now we can prove the following

THEOREM 3.2. *With the assumptions of* Theorem 3.1 *and with the additional assumption that the Hermite spline interpolant is convex in each subinterval*

J_i *(i = 0, 1,..., n - 1) we have the following estimation*

$$\|s - f\|_\infty \leqslant \omega(f, h) + Fh(1 + a_q),$$

where F, h, a_q *are the same as in* Theorem 3.1.

Proof

For $x \in J_i$ $(i = 0, 1, \ldots, n-1)$ in virtue of (3.1), (2.3) and (2.5) we have

$$|s(x) - f(x)| \leqslant |f_i - f(x)| + \frac{|f_{i+1} - f_i|}{h_i} h\phi_1(t) + Fh[\psi_0(t) - \psi_1(t)].$$

From (3.4) we have $\frac{|f_{i+1} - f_i|}{h_i} \leqslant F$. Hence and from the above inequality we obtain the thesis.

4. REMARKS

1. For the Hermite interpolating problem (H) the estimation of the error $\|s - f\|_\infty$ (where s and f are the same as in Theorem 3.1) in terms of $\omega(f', h)$ was given in [3].

2. Assuming additionaly that for all $i = 0, 1, \ldots, n$ is $f_i' = 0$ Passow [2] proved : if $f_i > f_{i+1}$ $(f_i < f_{i+1})$ $(i = 0, 1, \ldots, n-1)$ then the spline function s (x) is nondecreasing (nonincreasing) in each subinterval J_i $(i = 0, 1, \ldots, 1 - n)$.

REFERENCES

1. ERDÈLYI, A.; MAGNUS, W.; OBERHETTINGER, F. and TRICOMI, F. G. : "Tables of integral transforms", v. I, McCraw-Hill Book Company, Inc., New York 1954.

2. PASSOW, E. : "Piecewise monotone spline interpolation", J. Approximation Theory, 12 (1974), pp. 240-241.

3. SCHWARTZ, B. K. and VARGA, R. S. : Error bounds for spline and L-spline interpolation", J. Approximation Theory, 6 (1972), pp. 6-49.