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Representations by quaternary quadratic forms whose coefficients are 1, 4, 9 and 36

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ABSTRACT

Explicit formulae are determined for the number of representations of a positive integer by the quadratic forms $ax^2 + by^2 + cz^2 + dt^2$ with $a, b, c, d \in \{1, 4, 9, 36\}$, $\gcd(a, b, c, d) = 1$ and $a \leqslant b \leqslant c \leqslant d$.

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1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Set

$$\sigma(n) = \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d, \quad n \in \mathbb{N}.$$

If $n \notin \mathbb{N}$ we set $\sigma(n) = 0$.

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define

$$N(a, b, c, d; n) = \operatorname{card}\left\{(x, y, z, t) \in \mathbb{Z}^4 \;\middle|\; n = ax^2 + by^2 + cz^2 + dt^2\right\}.$$

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Clearly $N(a,b,c,d;0)=1$. For $q\in\mathbb{C}$ with $|q|<1$ we have

$$\sum_{n=0}^{\infty} N(a,b,c,d;n)q^n = \varphi\bigl(q^a\bigr)\varphi\bigl(q^b\bigr)\varphi\bigl(q^c\bigr)\varphi\bigl(q^d\bigr), \tag{1.1}$$

where $\varphi(q)$ denotes Ramanujan's theta function, namely

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

There are twenty-six quaternary quadratic forms $ax^2+by^2+cz^2+dt^2$ with $a,b,c,d\in\{1,4,9,36\}$, $\gcd(a,b,c,d)=1$ and $a\leqslant b\leqslant c\leqslant d$. Formulae for $N(a,b,c,d;n)$ $(n\in\mathbb{N})$ for the seven forms $(a,b,c,d)=(1,1,1,1)$, $(1,1,1,4)$, $(1,1,1,9)$, $(1,1,4,4)$, $(1,1,9,9)$, $(1,4,4,4)$, $(1,9,9,9)$ appear in the literature, [1,2]. In this paper we treat the remaining nineteen forms (Theorems 2.1–2.19). The form $(1,4,9,36)$ was treated by Gogišvilii [9, p. 101].

We also require the theta function $\psi(q)$ defined by

$$\psi(q) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}.$$

Definition 1.1. For $k\in\mathbb{N}$ and $q\in\mathbb{C}$ with $|q|<1$, we define

$$E_k = E_k(q) := \prod_{n=1}^{\infty}\bigl(1-q^{kn}\bigr).$$

The infinite product representations of $\varphi(q)$, $\psi(q)$, $\varphi(-q)$ are due to Jacobi

$$\varphi(q) = \frac{E_2^5}{E_1^2E_4^2}, \qquad \psi(q) = \frac{E_2^2}{E_1}, \qquad \varphi(-q) = \frac{E_1^2}{E_2}. \tag{1.2}$$

From [8, p. 357] we have

$$\bigl(3\varphi\bigl(-q^9\bigr)-\varphi(-q)\bigr)^3 = 8\frac{\psi^3(q)}{\psi(q^3)}\varphi\bigl(-q^3\bigr). \tag{1.3}$$

Then, immediately from (1.2) and (1.3), we obtain

$$3\varphi\bigl(-q^9\bigr)-\varphi(-q) = 2\frac{E_2^2E_3}{E_1E_6}. \tag{1.4}$$

It is convenient to define integers $c(n)$ $(n\in\mathbb{N})$ and $t(n)$ $(n\in\mathbb{N}_0)$ by

$$C_{1,6}(q) := qE_6^4 = \sum_{n=1}^{\infty} c(n)q^n, \tag{1.5}$$

$$T(q) := \frac{E_2^{17}E_3}{E_1^7E_4^6E_6} = \sum_{n=0}^{\infty} t(n)q^n. \tag{1.6}$$

The identity (1.4) is used to express $T(q)$ in terms of theta functions, see (6.1). It is clear from (1.5) that $c(n) = 0$ if $n \not\equiv 1 \pmod 6$. We set

$$C_{1,12}(q) := \sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 12)}}^{\infty} c(n)q^n, \qquad C_{7,12}(q) := \sum_{\substack{n=1 \\ n\equiv 7 \ (\mathrm{mod}\ 12)}}^{\infty} c(n)q^n, \tag{1.7}$$

$$T_r(q) := \sum_{\substack{n=0 \\ n\equiv r \ (\mathrm{mod}\ 12)}}^{\infty} t(n)q^n, \quad r \in \{0, 1, 2, \ldots, 11\}. \tag{1.8}$$

Klein and Fricke [10, Vol. II, p. 374] have given an arithmetic formulation of $c(n)$.

2. Statements of main results

Theorem 2.1. *For $n \in \mathbb{N}$*

$$N(1,1,1,36;n) = \begin{cases} \frac{5}{3}\sigma(n) + 2c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 2\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 3 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{3}\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 2c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{5}{3}\sigma(n) + 3\sigma(n/9) + \frac{1}{3}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{4}{3}\sigma(n) + \frac{1}{3}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.2. *For $n \in \mathbb{N}$*

$$N(1,1,4,9;n) = \begin{cases} \frac{13}{9}\sigma(n) + \frac{10}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 2\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ \frac{5}{9}\sigma(n) + \sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 3 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{28}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \sigma(n) + 3\sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{5}{9}\sigma(n) + \frac{2}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{13}{9}\sigma(n/3) + 5\sigma(n/9) - \frac{1}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{8}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.3. *For* $n \in \mathbb{N}$

$$N(1,1,4,36;n) = \begin{cases} \frac{10}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{16}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/6) + \frac{1}{9}t(n), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{10}{9}\sigma(n) + 2\sigma(n/9) + \frac{2}{9}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.4. *For* $n \in \mathbb{N}$

$$N(1,1,9,36;n) = \begin{cases} \frac{10}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) + \frac{2}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{3}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ \frac{4}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 11 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.5. *For* $n \in \mathbb{N}$

$$N(1,1,36,36;n) = \begin{cases} \frac{8}{9}\sigma(n) + \frac{4}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) + \frac{4}{3}c(n/2) + \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.6. *For $n \in \mathbb{N}$*

$$N(1,4,4,9;n)=\begin{cases}\frac{8}{9}\sigma(n)+\frac{8}{3}c(n)-\frac{2}{9}t(n), & \text{if } n\equiv 1 \pmod{12},\\ \frac{2}{3}\sigma(n)-\frac{1}{9}t(n), & \text{if } n\equiv 2 \pmod{12},\\ 0, & \text{if } n\equiv 3,7,11 \pmod{12},\\ 2\sigma(n/4)-8\sigma(n/16)+4c(n/4), & \text{if } n\equiv 4 \pmod{12},\\ \frac{20}{9}\sigma(n)-\frac{2}{9}t(n), & \text{if } n\equiv 5 \pmod{12},\\ 4\sigma(n/6)-\frac{1}{9}t(n), & \text{if } n\equiv 6 \pmod{12},\\ 4\sigma(n/4)-16\sigma(n/16), & \text{if } n\equiv 8 \pmod{12},\\ \frac{8}{9}\sigma(n)+4\sigma(n/9)-\frac{2}{9}t(n), & \text{if } n\equiv 9 \pmod{12},\\ \frac{1}{3}\sigma(n)-\frac{1}{9}t(n), & \text{if } n\equiv 10 \pmod{12},\\ 8\sigma(n/12)-32\sigma(n/48), & \text{if } n\equiv 0 \pmod{12}.\end{cases}$$

Theorem 2.7. *For $n \in \mathbb{N}$*

$$N(1,4,4,36;n)=\begin{cases}\frac{5}{9}\sigma(n)+\frac{2}{3}c(n)+\frac{1}{9}t(n), & \text{if } n\equiv 1 \pmod{12},\\ 0, & \text{if } n\equiv 2,3,6,7,10,11 \pmod{12},\\ 2\sigma(n/4)-8\sigma(n/16)+4c(n/4), & \text{if } n\equiv 4 \pmod{12},\\ \frac{8}{9}\sigma(n)+\frac{1}{9}t(n), & \text{if } n\equiv 5 \pmod{12},\\ 4\sigma(n/4)-16\sigma(n/16), & \text{if } n\equiv 8 \pmod{12},\\ \frac{5}{9}\sigma(n)+\sigma(n/9)+\frac{1}{9}t(n), & \text{if } n\equiv 9 \pmod{12},\\ 8\sigma(n/12)-32\sigma(n/48), & \text{if } n\equiv 0 \pmod{12}.\end{cases}$$

Theorem 2.8. *For $n \in \mathbb{N}$*

$$N(1,4,9,9;n)=\begin{cases}\frac{8}{9}\sigma(n)+\frac{8}{3}c(n)-\frac{2}{9}t(n), & \text{if } n\equiv 1 \pmod{12},\\ \frac{2}{3}\sigma(n)-\frac{1}{9}t(n), & \text{if } n\equiv 2 \pmod{12},\\ 2\sigma(n/9), & \text{if } n\equiv 3 \pmod{12},\\ \frac{4}{3}\sigma(n/4)-\frac{16}{3}\sigma(n/16)+\frac{8}{3}c(n/4), & \text{if } n\equiv 4 \pmod{12},\\ \frac{10}{9}\sigma(n)-\frac{1}{9}t(n), & \text{if } n\equiv 5 \pmod{12},\\ 4\sigma(n/9), & \text{if } n\equiv 6 \pmod{12},\\ \frac{4}{9}\sigma(n)-\frac{2}{9}t(n), & \text{if } n\equiv 7 \pmod{12},\\ \frac{4}{3}\sigma(n/4)-\frac{16}{3}\sigma(n/16), & \text{if } n\equiv 8 \pmod{12},\\ 6\sigma(n/9), & \text{if } n\equiv 9 \pmod{12},\\ \frac{2}{3}\sigma(n)-\frac{2}{9}t(n), & \text{if } n\equiv 10 \pmod{12},\\ \frac{2}{9}\sigma(n)-\frac{1}{9}t(n), & \text{if } n\equiv 11 \pmod{12},\\ 8\sigma(n/36)-32\sigma(n/144), & \text{if } n\equiv 0 \pmod{12}.\end{cases}$$

Theorem 2.9. *For* $n \in \mathbb{N}$

$$N(1,4,9,36;n) = \begin{cases} \frac{2}{3}\sigma(n) + \frac{4}{3}c(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) - \frac{2}{3}c(n/2), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3, 7, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{2}{3}\sigma(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.10. *For* $n \in \mathbb{N}$

$$N(1,4,36,36;n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 6, 7, 10, 11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{2}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.11. *For* $n \in \mathbb{N}$

$$N(1,9,9,36;n) = \begin{cases} \frac{5}{9}\sigma(n) + \frac{2}{3}c(n) + \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 5, 8, 11 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{9}\sigma(n) + \frac{2}{3}c(n) + \frac{1}{9}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.12. *For* $n \in \mathbb{N}$

$$N(1,9,36,36;n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2, 3, 5, 7, 8, 11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{9}\sigma(n) + \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.13. *For* $n \in \mathbb{N}$

$$N(1,36,36,36;n) = \begin{cases} \frac{1}{3}\sigma(n) - \frac{2}{3}c(n) + \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,3,5,6,7,8,10,11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.14. *For* $n \in \mathbb{N}$

$$N(4,4,4,9;n) = \begin{cases} \frac{1}{3}\sigma(n) + 2c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,3,6,7,10,11 \pmod{12}, \\ 2\sigma(n/4) - 8\sigma(n/16) + 4c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n) - \frac{1}{3}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ 4\sigma(n/4) - 16\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ \frac{1}{3}\sigma(n) + 3\sigma(n/9) - \frac{1}{3}t(n), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/12) - 32\sigma(n/48), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.15. *For* $n \in \mathbb{N}$

$$N(4,4,9,9;n) = \begin{cases} \frac{4}{9}\sigma(n) + \frac{8}{3}c(n) - \frac{4}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ \frac{2}{3}\sigma(n/2) + \frac{4}{3}c(n/2) - \frac{1}{9}t(n), & \text{if } n \equiv 2 \pmod{12}, \\ 0, & \text{if } n \equiv 3,7,11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{8}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{2}{9}\sigma(n) - \frac{2}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.16. *For* $n \in \mathbb{N}$

$$N(4,4,9,36;n) = \begin{cases} \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,3,6,7,10,11 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 5 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16), & \text{if } n \equiv 8 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.17. *For $n \in \mathbb{N}$*

$$N(4,9,9,9;n) = \begin{cases} \frac{1}{3}\sigma(n) + 2c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,5,8,11 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 3 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ \frac{1}{3}\sigma(n) - \frac{2}{3}c(n) - \frac{1}{3}t(n), & \text{if } n \equiv 7 \pmod{12}, \\ 6\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{3}\sigma(n) - \frac{1}{3}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.18. *For $n \in \mathbb{N}$*

$$N(4,9,9,36;n) = \begin{cases} \frac{2}{9}\sigma(n) + \frac{4}{3}c(n) - \frac{2}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,3,5,7,8,11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/9), & \text{if } n \equiv 6 \pmod{12}, \\ 4\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ \frac{1}{9}\sigma(n) - \frac{1}{9}t(n), & \text{if } n \equiv 10 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 2.19. *For $n \in \mathbb{N}$*

$$N(4,9,36,36;n) = \begin{cases} \frac{1}{9}\sigma(n) + \frac{2}{3}c(n) - \frac{1}{9}t(n), & \text{if } n \equiv 1 \pmod{12}, \\ 0, & \text{if } n \equiv 2,3,5,6,7,8,10,11 \pmod{12}, \\ \frac{2}{3}\sigma(n/4) - \frac{8}{3}\sigma(n/16) + \frac{4}{3}c(n/4), & \text{if } n \equiv 4 \pmod{12}, \\ 2\sigma(n/9), & \text{if } n \equiv 9 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144), & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

A numerical study showed that there do not exist rational numbers u, v, w (not all zero) with $u\sigma(n) + vc(n) + wt(n) = 0$ for all $n \equiv 1 \pmod{12}$ so that the formulation of each of the above theorems cannot be simplified in the case $n \equiv 1 \pmod{12}$. Similarly for the remaining congruence classes.

3. Theta function identities

We state nineteen theta function identities (Theorems 3.1–3.19) from which the theorems of Section 2 follow.

Definition 3.1. Let $k \in \mathbb{N}$ and $i \in \mathbb{N}_0$ with $0 \leqslant i \leqslant k-1$. Then we define $L_{i,k}(q)$ and the Eisenstein series $L(q)$ by

$$L_{i,k}(q) := \sum_{\substack{n=1 \\ n \equiv i \pmod{k}}}^{\infty} \sigma(n)q^n, \qquad L(q) := 1 - 24\sum_{n=1}^{\infty} \sigma(n)q^n.$$

It is convenient to define $\alpha(q)$, $\beta(q)$ and $\gamma(q)$ by

$$\alpha(q) := L_{0,1}(q) - 4L_{0,1}(q^4), \tag{3.1}$$

$$\beta(q) := L_{1,3}(q) - 4L_{1,3}(q^4) + 2C_{1,6}(q), \tag{3.2}$$

$$\gamma(q) := L_{2,3}(q) - 4L_{2,3}(q^4). \tag{3.3}$$

Theorem 3.1.

$$\begin{aligned}\varphi^3(q)\varphi(q^{36}) = {} & 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 3L_{1,4}(q^9) + 3L_{2,4}(q^9) \\ & + 3L_{3,4}(q^9) + \frac{5}{3}L_{1,12}(q) + 2L_{2,12}(q) + \frac{1}{3}L_{3,12}(q) + \frac{8}{3}L_{5,12}(q) + L_{6,12}(q) \\ & + \frac{1}{3}L_{7,12}(q) + \frac{5}{3}L_{9,12}(q) + L_{10,12}(q) + \frac{4}{3}L_{11,12}(q) + 2C_{1,6}(q) \\ & + \frac{1}{3}\big(T_1(q) + T_2(q) + T_3(q) + T_5(q) + T_6(q) + T_7(q) + T_9(q) \\ & + T_{10}(q) + T_{11}(q)\big).\end{aligned}$$

Theorem 3.2.

$$\begin{aligned}\varphi^2(q)\varphi(q^4)\varphi(q^9) = {} & 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 5L_{1,4}(q^9) \\ & + 3L_{2,4}(q^9) + L_{3,4}(q^9) + \frac{13}{9}L_{1,12}(q) + 2L_{2,12}(q) + \frac{5}{9}L_{3,12}(q) \\ & + \frac{28}{9}L_{5,12}(q) + L_{6,12}(q) + \frac{5}{9}L_{7,12}(q) + \frac{13}{9}L_{9,12}(q) + L_{10,12}(q) \\ & + \frac{8}{9}L_{11,12}(q) + \frac{10}{3}C_{1,12}(q) + \frac{2}{3}C_{7,12}(q) - \frac{1}{9}\big(T_1(q) + T_2(q) \\ & + T_3(q) + T_5(q) + T_6(q) + T_7(q) + T_9(q) + T_{10}(q) + T_{11}(q)\big).\end{aligned}$$

Theorem 3.3.

$$\begin{aligned}\varphi^2(q)\varphi(q^4)\varphi(q^{36}) = {} & 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) \\ & + 2L_{1,4}(q^9) + L_{2,4}(q^9) + \frac{10}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{16}{9}L_{5,12}(q) \\ & + \frac{1}{3}L_{6,12}(q) + \frac{10}{9}L_{9,12}(q) + \frac{1}{3}L_{10,12}(q) + \frac{4}{3}C_{1,12}(q) \\ & + \frac{1}{9}\big(2T_1(q) + T_2(q) + 2T_5(q) + T_6(q) + 2T_9(q) + T_{10}(q)\big).\end{aligned}$$

Theorem 3.4.

$$\begin{aligned}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) = {} & 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 6L_{1,4}(q^9) \\ & + 4L_{2,4}(q^9) + 2L_{3,4}(q^9) + \frac{10}{9}L_{1,12}(q) + \frac{2}{3}L_{2,12}(q) + \frac{8}{9}L_{5,12}(q)\end{aligned}$$

$$+\frac{2}{9}L_{7,12}(q)+\frac{2}{3}L_{10,12}(q)+\frac{4}{9}L_{11,12}(q)+\frac{4}{3}C_{1,6}(q)$$
$$+\frac{1}{9}\big(2T_1(q)+T_2(q)+T_5(q)+2T_7(q)+2T_{10}(q)+T_{11}(q)\big).$$

Theorem 3.5.

$$\varphi^2(q)\varphi^2\big(q^{36}\big)=1+8\alpha\big(q^{36}\big)+\frac{4}{3}\beta\big(q^4\big)+\frac{4}{3}\gamma\big(q^4\big)+4L_{1,4}\big(q^9\big)+\frac{4}{3}L_{2,4}\big(q^9\big)$$
$$+\frac{8}{9}L_{1,12}(q)+\frac{2}{9}L_{2,12}(q)+\frac{4}{9}L_{5,12}(q)+\frac{2}{9}L_{10,12}(q)+\frac{4}{3}C_{1,6}\big(q^2\big)$$
$$+\frac{1}{9}\big(4T_1(q)+T_2(q)+2T_5(q)+2T_{10}(q)\big).$$

Theorem 3.6.

$$\varphi(q)\varphi^2\big(q^4\big)\varphi\big(q^9\big)=1+8\alpha\big(q^{12}\big)+2\beta\big(q^4\big)+4\gamma\big(q^4\big)$$
$$+4L_{1,4}\big(q^9\big)+L_{2,4}\big(q^9\big)+\frac{8}{9}L_{1,12}(q)+\frac{2}{3}L_{2,12}(q)+\frac{20}{9}L_{5,12}(q)$$
$$+\frac{1}{3}L_{6,12}(q)+\frac{8}{9}L_{9,12}(q)+\frac{1}{3}L_{10,12}(q)+\frac{8}{3}C_{1,12}(q)$$
$$-\frac{1}{9}\big(2T_1(q)+T_2(q)+2T_5(q)+T_6(q)+2T_9(q)+T_{10}(q)\big).$$

Theorem 3.7.

$$\varphi(q)\varphi^2\big(q^4\big)\varphi\big(q^{36}\big)=1+8\alpha\big(q^{12}\big)+2\beta\big(q^4\big)+4\gamma\big(q^4\big)+L_{1,4}\big(q^9\big)+\frac{5}{9}L_{1,12}(q)$$
$$+\frac{8}{9}L_{5,12}(q)+\frac{5}{9}L_{9,12}(q)+\frac{2}{3}C_{1,12}(q)+\frac{1}{9}\big(T_1(q)+T_5(q)+T_9(q)\big).$$

Theorem 3.8.

$$\varphi(q)\varphi\big(q^4\big)\varphi^2\big(q^9\big)=1+8\alpha\big(q^{36}\big)+\frac{4}{3}\beta\big(q^4\big)+\frac{4}{3}\gamma\big(q^4\big)+6L_{1,4}\big(q^9\big)$$
$$+4L_{2,4}\big(q^9\big)+2L_{3,4}\big(q^9\big)+\frac{8}{9}L_{1,12}(q)+\frac{2}{3}L_{2,12}(q)+\frac{10}{9}L_{5,12}(q)$$
$$+\frac{4}{9}L_{7,12}(q)+\frac{2}{3}L_{10,12}(q)+\frac{2}{9}L_{11,12}(q)+\frac{8}{3}C_{1,12}(q)-\frac{1}{9}\big(2T_1(q)+T_2(q)$$
$$+T_5(q)+2T_7(q)+2T_{10}(q)+T_{11}(q)\big).$$

Theorem 3.9.

$$\varphi(q)\varphi\big(q^4\big)\varphi\big(q^9\big)\varphi\big(q^{36}\big)=1+8\alpha\big(q^{36}\big)+\frac{4}{3}\beta\big(q^4\big)+\frac{4}{3}\gamma\big(q^4\big)+4L_{1,4}\big(q^9\big)$$
$$+\frac{4}{3}L_{2,4}\big(q^9\big)+\frac{2}{3}L_{1,12}(q)+\frac{2}{9}L_{2,12}(q)+\frac{2}{3}L_{5,12}(q)+\frac{2}{9}L_{10,12}(q)$$
$$+\frac{4}{3}C_{1,12}(q)-\frac{2}{3}C_{1,6}\big(q^2\big).$$

Theorem 3.10.

$$\varphi(q)\varphi(q^4)\varphi^2(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 2L_{1,4}(q^9) + \frac{4}{9}L_{1,12}(q) + \frac{2}{9}L_{5,12}(q) + \frac{2}{9}T_1(q) + \frac{1}{9}T_5(q).$$

Theorem 3.11.

$$\varphi(q)\varphi^2(q^9)\varphi(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 6L_{1,4}(q^9) + 4L_{2,4}(q^9) + 2L_{3,4}(q^9) + \frac{5}{9}L_{1,12}(q) + \frac{1}{9}L_{7,12}(q) + \frac{1}{3}L_{10,12}(q) + \frac{2}{3}C_{1,6}(q) + \frac{1}{9}\big(T_1(q) + T_7(q) + T_{10}(q)\big).$$

Theorem 3.12.

$$\varphi(q)\varphi(q^9)\varphi^2(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9) + \frac{4}{9}L_{1,12}(q) + \frac{1}{9}L_{10,12}(q) + \frac{2}{9}T_1(q) + \frac{1}{9}T_{10}(q).$$

Theorem 3.13.

$$\varphi(q)\varphi^3(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 2L_{1,4}(q^9) + \frac{1}{3}L_{1,12}(q) - \frac{2}{3}C_{1,12}(q) + \frac{1}{3}T_1(q).$$

Theorem 3.14.

$$\varphi^3(q^4)\varphi(q^9) = 1 + 8\alpha(q^{12}) + 2\beta(q^4) + 4\gamma(q^4) + 3L_{1,4}(q^9) + \frac{1}{3}L_{1,12}(q) + \frac{4}{3}L_{5,12}(q) + \frac{1}{3}L_{9,12}(q) + 2C_{1,12}(q) - \frac{1}{3}\big(T_1(q) + T_5(q) + T_9(q)\big).$$

Theorem 3.15.

$$\varphi^2(q^4)\varphi^2(q^9) = 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9) + \frac{4}{9}L_{1,12}(q) + \frac{2}{9}L_{2,12}(q) + \frac{8}{9}L_{5,12}(q) + \frac{2}{9}L_{10,12}(q) + \frac{8}{3}C_{1,12}(q) + \frac{4}{3}C_{1,6}(q^2) - \frac{1}{9}\big(4T_1(q) + T_2(q) + 2T_5(q) + 2T_{10}(q)\big).$$

Theorem 3.16.

$$\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{4}{3}\beta(q^4) + \frac{4}{3}\gamma(q^4) + 2L_{1,4}(q^9) + \frac{2}{9}L_{1,12}(q) + \frac{4}{9}L_{5,12}(q) + \frac{4}{3}C_{1,12}(q) - \frac{2}{9}T_1(q) - \frac{1}{9}T_5(q).$$

Theorem 3.17.

$$\begin{aligned}\varphi(q^4)\varphi^3(q^9) = 1 &+ 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 6L_{1,4}(q^9) + 4L_{2,4}(q^9)\\ &+ 2L_{3,4}(q^9) + \frac{1}{3}\big(L_{1,12}(q) + L_{7,12}(q) + L_{10,12}(q)\big)\\ &+ 2C_{1,12}(q) - \frac{2}{3}C_{7,12}(q) - \frac{1}{3}\big(T_1(q) + T_7(q) + T_{10}(q)\big).\end{aligned}$$

Theorem 3.18.

$$\begin{aligned}\varphi(q^4)\varphi^2(q^9)\varphi(q^{36}) = 1 &+ 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 4L_{1,4}(q^9) + \frac{4}{3}L_{2,4}(q^9)\\ &+ \frac{2}{9}L_{1,12}(q) + \frac{1}{9}L_{10,12}(q) + \frac{4}{3}C_{1,12}(q) - \frac{2}{9}T_1(q) - \frac{1}{9}T_{10}(q).\end{aligned}$$

Theorem 3.19.

$$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) = 1 + 8\alpha(q^{36}) + \frac{2}{3}\beta(q^4) + 2L_{1,4}(q^9) + \frac{1}{9}L_{1,12}(q) + \frac{2}{3}C_{1,12}(q) - \frac{1}{9}T_1(q).$$

4. The (p, k)-parametrization

Following [6, p. 178] we set

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \qquad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}. \tag{4.1}$$

Duplication and change of sign principles. (See [6, Theorems 9 and 11].)

$$p(q^2) = \frac{1 + p - p^2 - ((1-p)(1+p)(1+2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1-p)(1+p)(1+2p))^{1/2})k}{2},$$

$$p(-q) = \frac{-p}{1+p}, \qquad k(-q) = (1+p)^2 k.$$

Proposition 4.1. *Define p and k by* (4.1). *Then*

(a) $\varphi(q) = (1+2p)^{3/4}k^{1/2}$,

(b) $\varphi(q^3) = (1+2p)^{1/4}k^{1/2}$,

(c) $\varphi(q^9) = \frac{1}{3}(1+2p)^{1/12}\big((1+2p)^{2/3} + 2^{2/3}(1-p)^{1/3}(2+p)^{1/3}\big)k^{1/2}$.

Proof. (a)(b) See [4, p. 222]. (c) See [1, p. 156]. □

By using Proposition 4.1, we obtain the following result.

Proposition 4.2.

$$\text{(a)}\quad \varphi(q)-\varphi\big(q^9\big)=\frac{2}{3}(1+2p)^{3/4}k^{1/2}-\frac{2^{2/3}}{3}(1+2p)^{1/12}(1-p)^{1/3}(2+p)^{1/3}k^{1/2},$$

$$\text{(b)}\quad \varphi(q)-3\varphi\big(q^9\big)=-2^{2/3}(1+2p)^{1/12}(1-p)^{1/3}(2+p)^{1/3}k^{1/2} + \frac{k}{3}2^{5/3}(1+2p)^{5/6}(1-p)^{1/3}(2+p)^{1/3},$$

$$\text{(c)}\quad \varphi^2\big(q^9\big)=\frac{k}{9}\big[(1+2p)^{3/2}+2^{4/3}(1+2p)^{1/6}(1-p)^{2/3}(2+p)^{2/3} + 2^{5/3}(1+2p)^{5/6}(1-p)^{1/3}(2+p)^{1/3}\big].$$

We use the following basic properties of $\varphi(q)$, see for example [5]:

$$\varphi(q)+\varphi(-q)=2\varphi\big(q^4\big), \tag{4.2}$$

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2\big(q^2\big), \tag{4.3}$$

$$\varphi(q)-\varphi(-q)=2\big(\varphi(q)-\varphi\big(q^4\big)\big), \tag{4.4}$$

$$\varphi^2\big(q^2\big)=\varphi^2(q)-2\varphi(q)\varphi\big(q^4\big)+2\varphi^2\big(q^4\big), \tag{4.5}$$

$$\varphi(q)\varphi(-q)=\varphi^2\big(-q^2\big). \tag{4.6}$$

The change of sign principle and Proposition 4.1 yield Proposition 4.3.

Proposition 4.3.

$$\text{(a)}\quad \varphi(-q)=(1-p)^{3/4}(1+p)^{1/4}k^{1/2},$$

$$\text{(b)}\quad \varphi\big(-q^3\big)=(1-p)^{1/4}(1+p)^{3/4}k^{1/2},$$

$$\text{(c)}\quad \varphi\big(-q^9\big)=\frac{1}{3}\big(2^{2/3}(1+2p)^{1/3}(1-p)^{1/12}(1+p)^{1/4}(2+p)^{1/3} + (1-p)^{3/4}(1+p)^{1/4}\big)k^{1/2},$$

$$\text{(d)}\quad \varphi(-q)-\varphi\big(-q^9\big)=\frac{2}{3}(1-p)^{3/4}(1+p)^{1/4}k^{1/2} - \frac{1}{3}2^{2/3}(1+2p)^{1/3}(1-p)^{1/12}(1+p)^{1/4}(2+p)^{1/3}k^{1/2},$$

$$\text{(e)}\quad \varphi(-q)-3\varphi\big(-q^9\big)=-2^{2/3}(1+2p)^{1/3}(1-p)^{1/12}(1+p)^{1/4}(2+p)^{1/3}k^{1/2}.$$

It was shown in [1, p. 156] that

$$\varphi^4\big(q^3\big)=3\varphi(q)\varphi^3\big(q^9\big)-3\varphi^2(q)\varphi^2\big(q^9\big)+\varphi^3(q)\varphi\big(q^9\big), \tag{4.7}$$

$$\varphi^2(q)\varphi^2\big(q^9\big)=\frac{1}{6}\varphi^4(q)-\frac{2}{3}\varphi^4\big(q^3\big)+\frac{3}{2}\varphi^4\big(q^9\big)+\frac{8}{3}C_{1,6}(q). \tag{4.8}$$

By appealing to (4.7) and (4.8), we obtain

$$C_{1,6}(q) = -\frac{1}{16}\left[\varphi(q) - \varphi\left(q^9\right)\right]\left[\varphi(q) - 3\varphi\left(q^9\right)\right]\left[\varphi^2(q) + 3\varphi^2\left(q^9\right)\right]. \tag{4.9}$$

Theorem 4.1.

$$\begin{aligned}&\left[\varphi(q) - \varphi\left(q^9\right)\right]\left[\varphi(q) - 3\varphi\left(q^9\right)\right]\left[\varphi^2(q) + 3\varphi^2\left(q^9\right)\right]\\ &\quad = -\left[\varphi(-q) - \varphi\left(-q^9\right)\right]\left[\varphi(-q) - 3\varphi\left(-q^9\right)\right]\left[\varphi^2(-q) + 3\varphi^2\left(-q^9\right)\right].\end{aligned}$$

Proof. The assertion follows immediately from (1.5) and (4.9). □

Theorem 4.2.

$$\begin{aligned}&2\varphi(q)\varphi\left(q^9\right)\left[\varphi^2(-q) - 2\varphi(-q)\varphi\left(-q^9\right) + 3\varphi^2\left(-q^9\right)\right]\\ &\quad = \varphi^2(q)\left[\varphi(-q) - \varphi\left(-q^9\right)\right]^2 + \varphi^2\left(q^9\right)\left[\varphi(-q) - 3\varphi\left(-q^9\right)\right]^2.\end{aligned}$$

Proof. Appealing to Propositions 4.1(a)(c), 4.2(c) and 4.3(a)(c), we see that each side of asserted equation is

$$\begin{aligned}&\frac{4k^2}{9}\left[(1+2p)^{3/2}(1-p)^{3/2}(1+p)^{1/2}\right.\\ &\quad + 2^{2/3}(1+2p)^{5/6}(1-p)^{11/6}(1+p)^{1/2}(2+p)^{1/3}\\ &\quad + 2^{1/3}(1+2p)^{13/6}(1-p)^{1/6}(1+p)^{1/2}(2+p)^{2/3}\\ &\quad \left. + 2(1+2p)^{3/2}(1-p)^{1/2}(1+p)^{1/2}(2+p)\right]. \qquad \square\end{aligned}$$

Theorem 4.3.

$$\text{(a)}\quad L\left(q^6\right) - 2L\left(q^{12}\right) = -\left(1 + 2p - p^3 - \frac{1}{2}p^4\right)k^2,$$

$$\text{(b)}\quad L_{1,2}\left(q^3\right) = \left(\frac{1}{8}p^3 + \frac{1}{16}p^4\right)k^2,$$

$$\text{(c)}\quad 8L_{1,2}\left(q^3\right) - L\left(q^6\right) + 2L\left(q^{12}\right) = \varphi^4\left(q^3\right).$$

Proof. (a) See [7, Eq. (3.11), p. 33]. (b) See [6, Eq. (14.6), p. 189].
(c) Follows from (a), (b) and Proposition 4.1(b). □

Theorem 4.4.

$$\begin{aligned}\text{(a)}\quad L_{2,3}(q) &= \left(\frac{1}{27} + \frac{8}{27}p + \frac{2}{3}p^2 + \frac{8}{27}p^3 + \frac{1}{27}p^4\right)k^2\\ &\quad + \left(\frac{-1}{27} - \frac{1}{9}p + \frac{1}{9}p^2 + \frac{1}{27}p^3\right)2^{2/3}\left((1-p)(1+2p)(2+p)\right)^{1/3}k^2\\ &\quad + \left(\frac{1}{54} - \frac{1}{27}p + \frac{1}{54}p^2\right)2^{1/3}\left((1-p)(1+2p)(2+p)\right)^{2/3}k^2,\end{aligned}$$

$$\text{(b)}\quad L_{2,3}(q^4) = \left(\frac{1}{27} + \frac{2}{27}p - \frac{1}{27}p^3 + \frac{1}{108}p^4\right)k^2 + \left(-\frac{1}{27} - \frac{1}{18}p + \frac{1}{108}p^3\right)2^{2/3}\big((1-p)(1+2p)(2+p)\big)^{1/3}k^2 + \left(\frac{1}{54} + \frac{1}{54}p + \frac{1}{216}p^2\right)2^{1/3}\big((1-p)(1+2p)(2+p)\big)^{2/3}k^2.$$

Proof. (a) See [6, Eq. (15.9), p. 190].

(b) Apply the duplication principle to part (a) twice. □

Theorem 4.5.

$$\gamma(q) = \frac{3}{4}\varphi^2(q^9)\big[\varphi(q) - \varphi(q^9)\big]^2, \qquad \gamma(q^4) = \frac{3}{4}\varphi^2(q^{36})\big[\varphi(q^4) - \varphi(q^{36})\big]^2.$$

Proof. Appealing to (3.3) and Theorem 4.4, we obtain

$$\gamma(q) = \frac{1}{9}(-1 + 6p^2 + 4p^3)k^2 - \frac{2^{1/3}}{18}(1+2p)\big((1+2p)(1-p)(2+p)\big)^{2/3}k^2 + \frac{2^{2/3}}{9}(1+p+p^2)\big((1+2p)(1-p)(2+p)\big)^{1/3}k^2.$$

Set $X = (1+2p)^{3/2}$, $Y = (1+2p)^{1/6}(1-p)^{2/3}(2+p)^{2/3}$ and $Z = (1+2p)^{5/6}(1-p)^{1/3}(2+p)^{1/3}$. By Proposition 4.2(a)(c), we obtain

$$\begin{aligned}
\frac{3}{4}\varphi^2(q^9)\big[\varphi(q) - \varphi(q^9)\big]^2 &= \frac{k^2}{108}\big(X + 2^{4/3}Y + 2\cdot 2^{2/3}Z\big)\big(4X + 2^{4/3}Y - 4\cdot 2^{2/3}Z\big)\\
&= \frac{k^2}{108}\big(4X^2 + 5\cdot 2^{4/3}XY + 4\cdot 2^{2/3}XZ - 8YZ + 4\cdot 2^{2/3}Y^2 - 16\cdot 2^{1/3}Z^2\big)\\
&= \frac{k^2}{9}(-1 + 6p^2 + 4p^3) - \frac{k^2}{18}\cdot 2^{1/3}(1+2p)\big((1+2p)(1-p)(2+p)\big)^{2/3}\\
&\quad + \frac{k^2}{9}\cdot 2^{2/3}(1+p+p^2)\big((1+2p)(1-p)(2+p)\big)^{1/3}.
\end{aligned}$$

The second assertion follows on replacing q by q^4 in the first one. □

Theorem 4.6.

$$\beta(q) = \frac{3}{2}\varphi^3(q^9)\big[\varphi(q) - \varphi(q^9)\big], \qquad \beta(q^4) = \frac{3}{2}\varphi^3(q^{36})\big[\varphi(q^4) - \varphi(q^{36})\big].$$

Proof. It was proved in [1, Theorem 2.5, p. 158] that

$$\begin{aligned}
\varphi^3(q)\varphi(q^9) &= 2L_{1,3}(q) + 4L_{2,3}(q) - 8L_{1,3}(q^4) - 16L_{2,3}(q^4)\\
&\quad - L(q^6) + 2L(q^{12}) + 8L_{1,2}(q^3) + 4C_{1,6}(q).
\end{aligned} \tag{4.10}$$

Substituting (3.2), (3.3) and Theorem 4.3(c) into (4.10), we obtain

$$\varphi^3(q)\varphi(q^9) = 2\beta(q) + 4\gamma(q) + \varphi^4(q^3). \tag{4.11}$$

Appealing to (4.11), Theorem 4.5 and (4.7), we obtain

$$\beta(q) = \frac{1}{2}\left[\varphi^3(q)\varphi(q^9) - 4\gamma(q) - \varphi^4(q^3)\right] = \frac{3}{2}\varphi^3(q^9)\left[\varphi(q) - \varphi(q^9)\right].$$

The second assertion follows on replacing q by q^4 in the first one. □

5. Identities involving $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

We give four identities, (5.1)–(5.4), of degree 4 of the form

$$\sum_{\substack{i,j,k,l=0 \\ i+j+k+l=4}}^{4} a(i,j,k,l)\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36}) = 0.$$

We list the coefficients $a(i,j,k,l)$ of identities (5.1)–(5.4) in Table 5.1.

Table 5.1

$\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36})$	$a(i,j,k,l)$			
	(5.1)	(5.2)	(5.3)	(5.4)
$\varphi^4(q)$	1	1	1	0
$\varphi^3(q)\varphi(q^4)$	−4	−4	−4	0
$\varphi^3(q)\varphi(q^9)$	−4	−4	−4	0
$\varphi^3(q)\varphi(q^{36})$	4	4	4	0
$\varphi^2(q)\varphi^2(q^4)$	12	4	0	1
$\varphi^2(q)\varphi(q^4)\varphi(q^9)$	12	12	12	0
$\varphi^2(q)\varphi(q^4)\varphi(q^{36})$	−24	−8	0	−2
$\varphi^2(q)\varphi^2(q^9)$	6	6	6	0
$\varphi^2(q)\varphi(q^9)\varphi(q^{36})$	−12	−12	−12	0
$\varphi^2(q)\varphi^2(q^{36})$	12	4	0	1
$\varphi(q)\varphi^3(q^4)$	−16	0	8	−2
$\varphi(q)\varphi^2(q^4)\varphi(q^9)$	−24	−8	0	−2
$\varphi(q)\varphi^2(q^4)\varphi(q^{36})$	48	0	−24	6
$\varphi(q)\varphi(q^4)\varphi^2(q^9)$	−12	−12	−12	0
$\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36})$	48	16	0	4
$\varphi(q)\varphi(q^4)\varphi^2(q^{36})$	−48	0	24	−6
$\varphi(q)\varphi^3(q^9)$	−12	−12	−12	0
$\varphi(q)\varphi^2(q^9)\varphi(q^{36})$	36	36	36	0
$\varphi(q)\varphi(q^9)\varphi^2(q^{36})$	−72	−24	0	−6
$\varphi(q)\varphi^3(q^{36})$	48	0	−24	6
$\varphi^4(q^4)$	8	0	−4	1
$\varphi^3(q^4)\varphi(q^9)$	16	0	−8	2
$\varphi^3(q^4)\varphi(q^{36})$	−32	0	16	−4
$\varphi^2(q^4)\varphi^2(q^9)$	12	4	0	1
$\varphi^2(q^4)\varphi(q^9)\varphi(q^{36})$	−48	0	24	−6
$\varphi^2(q^4)\varphi^2(q^{36})$	48	0	−24	6
$\varphi(q^4)\varphi^3(q^9)$	12	12	12	0
$\varphi(q^4)\varphi^2(q^9)\varphi(q^{36})$	−72	−24	0	−6
$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36})$	144	0	−72	18
$\varphi(q^4)\varphi^3(q^{36})$	−96	0	48	−12
$\varphi^4(q^9)$	9	9	9	0
$\varphi^3(q^9)\varphi(q^{36})$	−36	−36	−36	0
$\varphi^2(q^9)\varphi^2(q^{36})$	108	36	0	9
$\varphi(q^9)\varphi^3(q^{36})$	−144	0	72	−18
$\varphi^4(q^{36})$	72	0	−36	9

Identities (5.1) and (5.2) are consequences of Theorems 4.1 and 4.2 respectively using (4.2). Identities (5.3) and (5.4) are linear combinations of (5.1) and (5.2).

6. $T_r(q)$ in terms of $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

By (1.2), (1.4), (1.6) and (4.2), we obtain

$$T(q) = \frac{1}{2}\varphi^3(q)\big(\varphi(q) - 2\varphi\big(q^4\big) - 3\varphi\big(q^9\big) + 6\varphi\big(q^{36}\big)\big). \tag{6.1}$$

Let ω denote any 12th root of unity. Then

$$\varphi(\omega q) = \sum_{\substack{n=-\infty \\ n\equiv 0 \ (\mathrm{mod}\ 6)}}^{\infty} q^{n^2} + \omega \sum_{\substack{n=-\infty \\ n\not\equiv 0 \ (\mathrm{mod}\ 2) \\ n\not\equiv 0 \ (\mathrm{mod}\ 3)}}^{\infty} q^{n^2} + \omega^4 \sum_{\substack{n=-\infty \\ n\equiv 0 \ (\mathrm{mod}\ 2) \\ n\not\equiv 0 \ (\mathrm{mod}\ 3)}}^{\infty} q^{n^2} + \omega^9 \sum_{\substack{n=-\infty \\ n\not\equiv 0 \ (\mathrm{mod}\ 2) \\ n\equiv 0 \ (\mathrm{mod}\ 3)}}^{\infty} q^{n^2}.$$

Thus we obtain

$$\varphi(\omega q) = \omega\varphi(q) + \big(\omega^4 - \omega\big)\varphi\big(q^4\big) + \big(\omega^9 - \omega\big)\varphi\big(q^9\big) + \big(1 + \omega - \omega^4 - \omega^9\big)\varphi\big(q^{36}\big). \tag{6.2}$$

In particular we have

$$\varphi\big(\omega^4 q^4\big) = \varphi\big(q^{36}\big) + \omega^4\big(\varphi\big(q^4\big) - \varphi\big(q^{36}\big)\big), \tag{6.3}$$

$$\varphi\big(\omega^9 q^9\big) = \varphi\big(q^{36}\big) + \omega^9\big(\varphi\big(q^9\big) - \varphi\big(q^{36}\big)\big). \tag{6.4}$$

Replacing q by ωq in (6.1), and appealing to (6.3) and (6.4), we obtain

$$T(\omega q) = \frac{1}{2}\varphi^3(\omega q)\big(\varphi(\omega q) - 2\varphi\big(\omega^4 q^4\big) - 3\varphi\big(\omega^9 q^9\big) + 6\varphi\big(q^{36}\big)\big). \tag{6.5}$$

Next for $r \in \{0, 1, \dots, 11\}$ we have with ω chosen to be $e^{2\pi i/12}$

$$T_r(q) = \frac{1}{12}\sum_{n=0}^{\infty} t(n)q^n \sum_{k=0}^{11} \omega^{(n-r)k} = \frac{1}{12}\sum_{k=0}^{11} \omega^{-rk} \sum_{n=0}^{\infty} t(n)\big(\omega^k q\big)^n,$$

that is

$$T_r(q) = \frac{1}{12}\sum_{k=0}^{11} \omega^{-rk} T\big(\omega^k q\big). \tag{6.6}$$

Appealing to (6.2)–(6.5), using MAPLE, we calculate $T(\omega^k q)$, and then by substituting the values of $T(\omega^k q)$ in (6.6), we obtain $T_r(q)$ ($r \in \{0, 1, \dots, 11\}$) in the form

$$T_r(q) = \sum_{\substack{i,j,k,l=0 \\ i+j+k+l=4}}^{4} b_r(i, j, k, l)\varphi^i(q)\varphi^j\big(q^4\big)\varphi^k\big(q^9\big)\varphi^l\big(q^{36}\big).$$

The coefficients $b_r(i, j, k, l)$ are given in Table 6.1. (We note that we used the identities (5.2) and (5.4) to simplify $T_4(q)$ to the form given.)

Table 6.1

$\varphi^i(q)\varphi^j(q^4)\varphi^k(q^9)\varphi^l(q^{36})$	$b_r(i,j,k,l)$ $(r=0,1,\ldots,11)$											
	$T_0(q)$	$T_1(q)$	$T_2(q)$	$T_3(q)$	$T_4(q)$	$T_5(q)$	$T_6(q)$	$T_7(q)$	$T_8(q)$	$T_9(q)$	$T_{10}(q)$	$T_{11}(q)$
$\varphi^4(q)$	0	0	0	0	0	0	0	0	0	0	0	0
$\varphi^3(q)\varphi(q^4)$	0	0	0	0	0	0	0	1	0	0	0	0
$\varphi^3(q)\varphi(q^9)$	1/2	0	0	0	0	0	0	0	0	0	0	0
$\varphi^3(q)\varphi(q^{36})$	−1/2	0	0	5/2	0	0	0	−1	0	0	0	0
$\varphi^2(q)\varphi^2(q^4)$	0	0	0	0	0	0	0	−3	0	0	0	0
$\varphi^2(q)\varphi(q^4)\varphi(q^9)$	−3/2	0	0	−3/2	0	0	0	−3	0	0	0	0
$\varphi^2(q)\varphi(q^4)\varphi(q^{36})$	3/2	0	0	−6	0	0	9/2	6	0	0	0	0
$\varphi^2(q)\varphi^2(q^9)$	−3/2	0	0	0	0	0	0	0	−3/2	0	0	0
$\varphi^2(q)\varphi(q^9)\varphi(q^{36})$	3	0	0	−6	0	0	0	3	3	0	0	3
$\varphi^2(q)\varphi^2(q^{36})$	−3/2	0	9/2	6	0	0	−9/2	−3	−3/2	0	0	−3
$\varphi(q)\varphi^3(q^4)$	0	−1	0	0	0	0	0	3	0	0	0	0
$\varphi(q)\varphi^2(q^4)\varphi(q^9)$	3/2	0	0	3	0	0	−9/2	6	0	0	0	0
$\varphi(q)\varphi^2(q^4)\varphi(q^{36})$	−3/2	3	0	9/2	0	0	−9/2	−9	0	3/2	0	0
$\varphi(q)\varphi(q^4)\varphi^2(q^9)$	3	0	0	3	0	0	0	3	3	0	0	−6
$\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36})$	−6	0	0	6	0	0	0	−12	−6	0	0	6
$\varphi(q)\varphi(q^4)\varphi^2(q^{36})$	3	−3	−9	−9	0	6	9	9	3	−3	0	0
$\varphi(q)\varphi^3(q^9)$	3/2	0	0	0	3/2	0	0	0	3	0	0	0
$\varphi(q)\varphi^2(q^9)\varphi(q^{36})$	−9/2	0	0	9/2	−9/2	0	0	−9/2	−9	0	0	0
$\varphi(q)\varphi(q^9)\varphi^2(q^{36})$	9/2	0	−9	−9	9/2	0	9/2	9	9	0	9/2	0
$\varphi(q)\varphi^3(q^{36})$	−3/2	9/2	9	9/2	−3/2	−6	−9/2	−9/2	−3	3/2	−9/2	0
$\varphi^4(q^4)$	0	1	0	0	−1	0	0	−1	0	0	0	0
$\varphi^3(q^4)\varphi(q^9)$	−1/2	1	0	−3/2	0	0	9/2	−3	0	−5/2	0	0
$\varphi^3(q^4)\varphi(q^{36})$	0	−4	0	−1	4	0	0	4	0	1	0	0
$\varphi^2(q^4)\varphi^2(q^9)$	−3/2	0	−9/2	−3	0	0	9/2	−3	−3/2	0	0	6
$\varphi^2(q^4)\varphi(q^9)\varphi(q^{36})$	3	−3	9	0	0	−3	−9	9	3	6	0	−9
$\varphi^2(q^4)\varphi^2(q^{36})$	0	6	0	3	−6	−3	0	−6	0	−3	0	3
$\varphi(q^4)\varphi^3(q^9)$	−3/2	0	0	−3/2	−3/2	0	0	−9/2	−3	0	0	6
$\varphi(q^4)\varphi^2(q^9)\varphi(q^{36})$	9/2	0	9	0	9/2	0	−9/2	18	9	0	−9/2	−18
$\varphi(q^4)\varphi(q^9)\varphi^2(q^{36})$	−9/2	9/2	−9	9/2	−9/2	0	9/2	−45/2	−9	−9/2	9/2	18
$\varphi(q^4)\varphi^3(q^{36})$	0	−9	0	−3	12	6	0	9	0	3	0	−6
$\varphi^4(q^9)$	−3/2	0	0	0	−3/2	0	0	0	−3/2	0	0	0
$\varphi^3(q^9)\varphi(q^{36})$	6	0	0	−3	6	0	0	6	6	0	0	−3
$\varphi^2(q^9)\varphi^2(q^{36})$	−9	0	0	9	−9	0	0	−18	−9	0	0	9
$\varphi(q^9)\varphi^3(q^{36})$	6	−6	0	−9	6	3	0	18	6	3	0	−9
$\varphi^4(q^{36})$	0	6	0	3	−9	−3	0	−6	0	−3	0	3

7. $C_{1,12}(q)$ and $C_{7,12}(q)$ in terms of $\varphi(q^i)$, $i \in \{1, 4, 9, 36\}$

It was shown in [5, Lemma 2.3, p. 16] that

$$\begin{aligned}\varphi(iq) &= \varphi(q^4) + i\big(\varphi(q) - \varphi(q^4)\big),\\ \varphi(-iq) &= \varphi(q^4) - i\big(\varphi(q) - \varphi(q^4)\big).\end{aligned} \tag{7.1}$$

Theorem 7.1.

$$\begin{aligned}\text{(a)}\quad C_{1,12}(q) = {} & \frac{1}{16}\varphi^4(q) - \frac{1}{4}\varphi^3(q)\varphi(q^4) - \frac{1}{4}\varphi^3(q)\varphi(q^9) + \frac{1}{4}\varphi^3(q)\varphi(q^{36})\\ & + \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) + \frac{3}{8}\varphi^2(q)\varphi^2(q^9) - \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36})\\ & + \frac{1}{4}\varphi(q)\varphi^3(q^4) - \frac{3}{4}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9)\\ & + \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) - \frac{3}{4}\varphi(q)\varphi^3(q^9) + \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36})\\ & - \frac{3}{4}\varphi(q)\varphi^3(q^{36}) - \frac{1}{4}\varphi^3(q^4)\varphi(q^9) + \frac{3}{4}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) + \frac{3}{4}\varphi(q^4)\varphi^3(q^9)\\ & - \frac{9}{4}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + \frac{9}{16}\varphi^4(q^9) - \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) + \frac{9}{4}\varphi(q^9)\varphi^3(q^{36}),\\ \text{(b)}\quad C_{7,12}(q) = {} & -\frac{1}{8}\varphi^4(q) + \frac{1}{4}\varphi^3(q)\varphi(q^4) + \frac{1}{2}\varphi^3(q)\varphi(q^9) - \frac{1}{4}\varphi^3(q)\varphi(q^{36})\\ & - \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) - \frac{3}{4}\varphi^2(q)\varphi^2(q^9) + \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) - \frac{1}{4}\varphi(q)\varphi^3(q^4)\\ & + \frac{3}{4}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9) - \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^{36})\\ & + \frac{3}{2}\varphi(q)\varphi^3(q^9) - \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi^3(q^{36}) + \frac{1}{4}\varphi^3(q^4)\varphi(q^9)\\ & - \frac{3}{4}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) - \frac{3}{4}\varphi(q^4)\varphi^3(q^9) + \frac{9}{4}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36})\\ & - \frac{9}{8}\varphi^4(q^9) + \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) - \frac{9}{4}\varphi(q^9)\varphi^3(q^{36}).\end{aligned}$$

Proof. Recall that $c(n) = 0$ for $n \not\equiv 1 \pmod 6$. We have

$$\begin{aligned}C_{1,12}(q) - C_{7,12}(q) &= \sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ 6)\\ n\equiv 1 \ (\mathrm{mod}\ 4)}}^{\infty} c(n)q^n - \sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ 6)\\ n\equiv 3 \ (\mathrm{mod}\ 4)}}^{\infty} c(n)q^n\\ &= \sum_{\substack{n=1\\ n\equiv 1 \ (\mathrm{mod}\ 6)}}^{\infty} c(n)q^n\left(\frac{i^n - (-i)^n}{2i}\right) = \frac{1}{2i}C_{1,6}(iq) - \frac{1}{2i}C_{1,6}(-iq).\end{aligned}$$

Appealing to (4.9) and (7.1), we obtain

$$\begin{aligned}
C_{1,12}(q) - C_{7,12}(q) = {}& \frac{1}{4}\varphi^3(q)\varphi(q^4) - \frac{1}{4}\varphi^3(q)\varphi(q^{36}) - \frac{3}{4}\varphi^2(q)\varphi^2(q^4) \\
& - \frac{3}{4}\varphi^2(q)\varphi(q^4)\varphi(q^9) + \frac{3}{2}\varphi^2(q)\varphi(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi^2(q)\varphi(q^9)\varphi(q^{36}) \\
& - \frac{3}{4}\varphi^2(q)\varphi^2(q^{36}) + \frac{1}{2}\varphi(q)\varphi^3(q^4) + \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^9) \\
& - \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{4}\varphi(q)\varphi(q^4)\varphi^2(q^9) - 3\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36}) \\
& + \frac{3}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) - \frac{9}{4}\varphi(q)\varphi^2(q^9)\varphi(q^{36}) + \frac{9}{2}\varphi(q)\varphi(q^9)\varphi^2(q^{36}) \\
& - \frac{3}{2}\varphi(q)\varphi^3(q^{36}) - \frac{1}{2}\varphi^3(q^4)\varphi(q^9) - \frac{3}{4}\varphi^2(q^4)\varphi^2(q^9) + \frac{3}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) \\
& - \frac{3}{4}\varphi(q^4)\varphi^3(q^9) + \frac{9}{2}\varphi(q^4)\varphi^2(q^9)\varphi(q^{36}) - \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) \\
& + \frac{9}{4}\varphi^3(q^9)\varphi(q^{36}) - \frac{27}{4}\varphi^2(q^9)\varphi^2(q^{36}) + \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}), \\
C_{1,12}(q) + C_{7,12}(q) = {}& \frac{3}{4}\varphi(q)\varphi^3(q^9) - \frac{3}{8}\varphi^2(q)\varphi^2(q^9) + \frac{1}{4}\varphi^3(q)\varphi(q^9) \\
& - \frac{1}{16}\varphi^4(q) - \frac{9}{16}\varphi^4(q^9).
\end{aligned}$$

The assertions follow by adding and subtracting the above two equations and using identity (5.2). □

Theorem 7.2.

$$C_{1,6}(q^2) = \frac{1}{4}\varphi^2(q)\varphi^2(q^{36}) + \frac{1}{4}\varphi^2(q^4)\varphi^2(q^9) - \frac{1}{2}\varphi(q)\varphi(q^4)\varphi(q^9)\varphi(q^{36}).$$

Proof. Sun [11] has shown that for $a, b \in \mathbb{N}$ and $q \in \mathbb{R}$ with $|q| < 1$

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-q^{an})(1-q^{bn}) = 1 + \sum_{n=1}^{\infty}\frac{1}{2}\big[&R\big(a+b, 12(a-b), 36(a+b); 24n+a+b\big) \\
&- R\big(4(a+b), 12(a-b), 9(a+b); 24n+a+b\big)\big]q^n,
\end{aligned}$$

where $R(a,b,c;n) = \mathrm{card}\{(x,y) \in \mathbb{Z}^2 \mid n = ax^2 + bxy + cy^2\}$. Take $a = b = 12$. Then

$$\begin{aligned}
\prod_{n=1}^{\infty}(1-q^{12n})^2 &= 1 + \sum_{n=1}^{\infty}\frac{1}{2}\big[R(24,0,864;24n+24) - R(96,0,216;24n+24)\big]q^n \\
&= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\big[R(1,0,36;n+1) - R(4,0,9;n+1)\big]q^n.
\end{aligned}$$

Multiplying both sides by $2q$, we obtain

$$2q\prod_{n=1}^{\infty}\left(1-q^{12n}\right)^2 = 2q + \sum_{n=2}^{\infty} R(1,0,36;n)q^n - \sum_{n=2}^{\infty} R(4,0,9;n)q^n$$
$$= \varphi(q)\varphi\left(q^{36}\right) - \varphi\left(q^4\right)\varphi\left(q^9\right),$$

as $R(1,0,36;0)=1$, $R(1,0,36;1)=2$, $R(4,0,9;0)=1$, $R(4,0,9;1)=0$. Squaring both sides, we obtain the asserted identity by (1.5). □

8. Evaluation of $L_{i,j}(q)$

Theorem 8.1.

$$L_{1,2}(q) = \frac{1}{2}\varphi\left(q^4\right)\left[\varphi^3(q) - 3\varphi^2(q)\varphi\left(q^4\right) + 4\varphi(q)\varphi^2\left(q^4\right) - 2\varphi^3\left(q^4\right)\right].$$

Proof. From [3, Theorem 2.4(i)] we have

$$L_{1,2}(q) = \frac{1}{4}\left(\varphi(q) - \varphi(-q)\right)\varphi^2\left(q^2\right)\varphi\left(q^4\right).$$

The asserted result now follows using (4.4) and (4.5). □

Lemma 8.1. $L_{2,4}(q) = 3L_{1,2}(q^2)$.

Proof. Recall that for all $n \in \mathbb{N}$, we have

$$\sigma(2n) = 3\sigma(n) - 2\sigma(n/2). \tag{8.1}$$

Hence, appealing to (8.1), we deduce

$$L_{2,4}(q) = \sum_{\substack{n=1 \\ n\equiv 2 \ (\mathrm{mod}\ 4)}}^{\infty} \sigma(n)q^n = \sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} \sigma(2n)q^{2n}$$
$$= 3\sum_{\substack{n=1 \\ n\equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} \sigma(n)q^{2n} = 3L_{1,2}\left(q^2\right). \qquad \square$$

Theorem 8.2.

$$\text{(a)} \quad L_{1,4}(q) = \frac{1}{2}\varphi^3\left(q^4\right)\left(\varphi(q) - \varphi\left(q^4\right)\right),$$
$$\text{(b)} \quad L_{2,4}(q) = \frac{3}{4}\varphi^2\left(q^4\right)\left(\varphi(q) - \varphi\left(q^4\right)\right)^2,$$
$$\text{(c)} \quad L_{3,4}(q) = \frac{1}{2}\varphi\left(q^4\right)\left(\varphi(q) - \varphi\left(q^4\right)\right)^3.$$

Proof. (a) From [3, Theorem 2.4(ii)] we have by (4.4),

$$L_{1,4}(q)=\frac{1}{4}\left[\varphi(q)-\varphi(-q)\right]\varphi^3\left(q^4\right)=\frac{1}{2}\left(\varphi(q)-\varphi\left(q^4\right)\right)\varphi^3\left(q^4\right).$$

(b) Replacing q by q^2 in (4.5), we have

$$\varphi^2\left(q^2\right)-\varphi^2\left(q^4\right)=2\left(\varphi\left(q^2\right)-\varphi\left(q^8\right)\right)\varphi\left(q^8\right). \tag{8.2}$$

Substituting (4.5) into (8.2), we obtain

$$\left(\varphi(q)-\varphi\left(q^4\right)\right)^2=2\varphi\left(q^8\right)\left(\varphi\left(q^2\right)-\varphi\left(q^8\right)\right). \tag{8.3}$$

Appealing to Lemma 8.1, Theorem 8.1, (4.5) and (8.3) we obtain

$$\begin{aligned}L_{2,4}(q)&=\frac{3}{2}\varphi\left(q^8\right)\left[\varphi^3\left(q^2\right)-3\varphi^2\left(q^2\right)\varphi\left(q^8\right)+4\varphi\left(q^2\right)\varphi^2\left(q^8\right)-2\varphi^3\left(q^8\right)\right]\\&=\frac{3}{2}\varphi\left(q^8\right)\left[\varphi\left(q^2\right)-\varphi\left(q^8\right)\right]\left[\varphi^2\left(q^2\right)-2\varphi\left(q^2\right)\varphi\left(q^8\right)+2\varphi^2\left(q^8\right)\right]\\&=\frac{3}{4}\varphi^2\left(q^4\right)\left[\varphi(q)-\varphi\left(q^4\right)\right]^2.\end{aligned}$$

(c) From [3, Theorem 2.4(iii)] we have

$$L_{3,4}(q)=\frac{1}{4}\left[\varphi(q)-\varphi(-q)\right]\left[\varphi\left(q^2\right)-\varphi\left(-q^2\right)\right]\varphi\left(q^4\right)\varphi\left(q^8\right).$$

Then, by (4.4) and (8.3), we obtain

$$\begin{aligned}L_{3,4}(q)&=\varphi\left(q^4\right)\left[\varphi(q)-\varphi\left(q^4\right)\right]\varphi\left(q^8\right)\left[\varphi\left(q^2\right)-\varphi\left(q^8\right)\right]\\&=\varphi\left(q^4\right)\left[\varphi(q)-\varphi\left(q^4\right)\right]\frac{1}{2}\left[\varphi(q)-\varphi\left(q^4\right)\right]^2=\frac{1}{2}\varphi\left(q^4\right)\left[\varphi(q)-\varphi\left(q^4\right)\right]^3.\qquad\square\end{aligned}$$

Theorem 8.3.

$$\begin{aligned}&\text{(a)}\quad L_{1,4}\left(q^9\right)=\frac{1}{2}\varphi^3\left(q^{36}\right)\left(\varphi\left(q^9\right)-\varphi\left(q^{36}\right)\right),\\&\text{(b)}\quad L_{2,4}\left(q^9\right)=\frac{3}{4}\varphi^2\left(q^{36}\right)\left(\varphi\left(q^9\right)-\varphi\left(q^{36}\right)\right)^2,\\&\text{(c)}\quad L_{3,4}\left(q^9\right)=\frac{1}{2}\varphi\left(q^{36}\right)\left(\varphi\left(q^9\right)-\varphi\left(q^{36}\right)\right)^3.\end{aligned}$$

Proof. Replace q by q^9 in Theorem 8.2. □

Theorem 8.4.

$$\text{(a)}\quad \alpha(q)=\frac{1}{8}\big(-1+\varphi^4(q)\big),$$

$$\text{(b)}\quad \alpha\big(q^{12}\big)=\frac{1}{8}\big(-1+3\varphi\big(q^4\big)\varphi^3\big(q^{36}\big)-3\varphi^2\big(q^4\big)\varphi^2\big(q^{36}\big)+\varphi^3\big(q^4\big)\varphi\big(q^{36}\big)\big),$$

$$\text{(c)}\quad \alpha\big(q^{36}\big)=\frac{1}{8}\big(-1+\varphi^4\big(q^{36}\big)\big).$$

Proof. (a) The number of representations of $n\in\mathbb{N}$ as the sum of four squares is $8\sigma(n)-32\sigma(n/4)$ so

$$\varphi^4(q)-1=\sum_{n=1}^{\infty}\big(8\sigma(n)-32\sigma(n/4)\big)q^n=8L_{0,1}(q)-32L_{0,1}\big(q^4\big)=8\alpha(q).$$

(b) Replacing q by q^4 in (4.7), we obtain

$$\varphi^4\big(q^{12}\big)=3\varphi\big(q^4\big)\varphi^3\big(q^{36}\big)-3\varphi^2\big(q^4\big)\varphi^2\big(q^{36}\big)+\varphi^3\big(q^4\big)\varphi\big(q^{36}\big). \tag{8.4}$$

The assertion follows from (a) on replacing q by q^{12} and then using (8.4).

(c) Replace q by q^{36} in part (a). □

Theorem 8.5. *Let $\omega=e^{2\pi i/3}$. Then, for $s=1,2,\ldots,11$,*

$$L_{s,12}(q)=\frac{1}{3}L_{s,4}(q)+\frac{1}{3}\omega^{2s}L_{s,4}(\omega q)+\frac{1}{3}\omega^{s}L_{s,4}\big(\omega^2 q\big).$$

Proof. We have

$$\begin{aligned}L_{s,12}(q)&=\sum_{\substack{n=1\\ n\equiv s \ (\mathrm{mod}\ 3)\\ n\equiv s\ (\mathrm{mod}\ 4)}}^{\infty}\sigma(n)q^n\\ &=\frac{1}{3}\sum_{\substack{n=1\\ n\equiv s\ (\mathrm{mod}\ 4)}}^{\infty}\sigma(n)q^n\big(1+\omega^{n-s}+\omega^{2(n-s)}\big)\\ &=\frac{1}{3}L_{s,4}(q)+\frac{1}{3}\omega^{2s}L_{s,4}(\omega q)+\frac{1}{3}\omega^{s}L_{s,4}\big(\omega^2 q\big).\qquad\square\end{aligned}$$

Theorem 8.6. *For $s\in\{1,2,3,5,6,7,9,10,11\}$, we have*

$$L_{s,12}(q)=\sum_{\substack{i,j,k,l=0\\ i+j+k+l=4}}^{4}c_s(i,j,k,l)\varphi^i(q)\varphi^j\big(q^4\big)\varphi^k\big(q^9\big)\varphi^l\big(q^{36}\big).$$

The coefficients $c_s(i,j,k,l)$ are listed in Table 8.1. *We note that $L_{s,12}(q)$ is shortened as $L_{s,12}$.*

Table 8.1

$c_s(i,j,k,l)$	$s=1$ $L_{1,12}$	$s=2$ $L_{2,12}$	$s=3$ $L_{3,12}$	$s=5$ $L_{5,12}$	$s=6$ $L_{6,12}$	$s=7$ $L_{7,12}$	$s=9$ $L_{9,12}$	$s=10$ $L_{10,12}$	$s=11$ $L_{11,12}$
$c_s(4,0,0,0)$	0	0	0	0	0	0	0	0	0
$c_s(3,1,0,0)$	0	0	0	0	0	1/2	0	0	0
$c_s(3,0,1,0)$	0	0	0	0	0	0	0	0	0
$c_s(3,0,0,1)$	0	0	1/2	0	0	−1/2	0	0	0
$c_s(2,2,0,0)$	0	0	0	0	0	−3/2	0	3/4	0
$c_s(2,1,1,0)$	0	0	3/2	0	0	−3/2	0	0	0
$c_s(2,1,0,1)$	0	0	−3	0	3/2	3	0	−3/2	0
$c_s(2,0,2,0)$	0	0	0	0	0	0	0	0	0
$c_s(2,0,1,1)$	0	0	−3	0	0	3/2	0	0	3/2
$c_s(2,0,0,2)$	0	3/4	3	0	−3/2	−3/2	0	3/4	−3/2
$c_s(1,3,0,0)$	1/2	0	0	0	0	3/2	0	−3/2	0
$c_s(1,2,1,0)$	0	0	−3	0	3/2	3	0	−3/2	0
$c_s(1,2,0,1)$	−3/2	0	9/2	0	−9/2	−9/2	3/2	9/2	0
$c_s(1,1,2,0)$	0	0	−3	0	0	3/2	0	0	3/2
$c_s(1,1,1,1)$	0	3	12	0	−6	−6	0	3	−6
$c_s(1,1,0,2)$	3/2	−9/2	−9	3/2	9	9/2	−3	−9/2	9/2
$c_s(1,0,3,0)$	0	0	0	0	0	0	0	0	0
$c_s(1,0,2,1)$	0	0	9/2	0	0	0	0	0	−9/2
$c_s(1,0,1,2)$	0	−9/2	−9	0	9/2	0	0	0	9
$c_s(1,0,0,3)$	0	9/2	9/2	−3/2	−9/2	0	3/2	0	−9/2
$c_s(0,4,0,0)$	−1/2	0	0	0	0	−1/2	0	3/4	0
$c_s(0,3,1,0)$	−1/2	0	3/2	0	−3/2	−3/2	1/2	3/2	0
$c_s(0,3,0,1)$	2	0	−2	0	3	2	−2	−3	0
$c_s(0,2,2,0)$	0	3/4	3	0	−3/2	−3/2	0	3/4	−3/2
$c_s(0,2,1,1)$	3/2	−9/2	−9	3/2	9	9/2	−3	−9/2	9/2
$c_s(0,2,0,2)$	−3	9/2	6	−3	−9	−3	6	9/2	−3
$c_s(0,1,3,0)$	0	0	3/2	0	0	0	0	0	−3/2
$c_s(0,1,2,1)$	0	−9/2	−9	0	9/2	0	0	0	9
$c_s(0,1,1,2)$	0	27/2	27/2	−9/2	−27/2	0	9/2	0	−27/2
$c_s(0,1,0,3)$	0	−9	−6	6	9	0	−6	0	6
$c_s(0,0,4,0)$	0	0	0	0	0	0	0	0	0
$c_s(0,0,3,1)$	0	0	−3/2	0	0	−3/2	0	0	3
$c_s(0,0,2,2)$	0	9/2	9/2	0	−9/4	9/2	0	−9/4	−9
$c_s(0,0,1,3)$	−3/2	−9	−9/2	3	9/2	−9/2	−3/2	9/2	9
$c_s(0,0,0,4)$	3/2	9/2	3/2	−3	−9/4	3/2	3/2	−9/4	−3

Proof. We do the proof only for $L_{1,12}(q)$ as the other ones can be done similarly. By Theorem 8.5, we have

$$L_{1,12}(q) = \frac{1}{3}L_{1,4}(q) + \frac{1}{3}\omega^2 L_{1,4}(\omega q) + \frac{1}{3}\omega L_{1,4}(\omega^2 q). \tag{8.5}$$

Appealing to Theorem 8.2(a), (6.2) and (6.3), we obtain

$$\begin{aligned} L_{1,4}(\omega q) = {} & \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) + 3\varphi(q)\varphi^3(q^{36}) \\ & + \frac{1}{2}\varphi^3(q^4)\varphi(q^9) - 2\varphi^3(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) \\ & + 9\varphi^2(q^4)\varphi^2(q^{36}) + \frac{9}{2}\varphi^4(q^{36}) - 12\varphi(q^4)\varphi^3(q^{36}) - \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}) \\ & + 9\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + \omega A(q), \end{aligned}$$

$$\begin{aligned}L_{1,4}(\omega^2 q) = {} & -\frac{1}{2}\varphi(q)\varphi^3(q^4) + 3\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - \frac{9}{2}\varphi(q)\varphi(q^4)\varphi^2(q^{36}) \\ & + \frac{3}{2}\varphi(q)\varphi^3(q^{36}) + \frac{1}{2}\varphi^4(q^4) + \varphi^3(q^4)\varphi(q^9) - 4\varphi^3(q^4)\varphi(q^{36}) - 6\varphi(q^4)\varphi^3(q^{36}) \\ & - \frac{9}{2}\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) + 9\varphi^2(q^4)\varphi^2(q^{36}) + \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) - \omega A(q),\end{aligned}$$

where

$$\begin{aligned}A(q) = {} & \frac{1}{2}\varphi(q)\varphi^3(q^4) - \frac{3}{2}\varphi(q)\varphi^2(q^4)\varphi(q^{36}) + \frac{3}{2}\varphi(q)\varphi^3(q^{36}) \\ & - \frac{1}{2}\varphi^4(q^4) - \frac{1}{2}\varphi^3(q^4)\varphi(q^9) + 2\varphi^3(q^4)\varphi(q^{36}) + \frac{9}{2}\varphi^4(q^{36}) \\ & - 6\varphi(q^4)\varphi^3(q^{36}) - \frac{9}{2}\varphi(q^9)\varphi^3(q^{36}) + \frac{9}{2}\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}).\end{aligned}$$

Substituting $L_{1,4}(q)$, $L_{1,4}(\omega q)$, $L_{1,4}(\omega^2 q)$ into (8.5) completes the proof. □

We note that $L_{4,12}(q)$ and $L_{8,12}(q)$ can be obtained in a similar manner. They are excluded as they are not needed in this paper.

9. Proofs of theorems of Section 3

Theorem 3.10 follows from Theorems 4.5, 4.6, 8.3(a), 8.4(c), Tables 6.1 and 8.1 immediately. We just prove Theorem 3.7. The rest can be proved similarly.

By Theorems 4.5, 4.6, 7.1(a), 8.3(a), 8.4(b), Table 6.1, Table 8.1 and the identity (5.3), the right-hand side of Theorem 3.7 becomes

$$\begin{aligned}& \frac{1}{24}\big[\varphi^4(q) - 4\varphi^3(q)\varphi(q^4) - 4\varphi^3(q)\varphi(q^9) + 4\varphi^3(q)\varphi(q^{36}) + 12\varphi^2(q)\varphi(q^4)\varphi(q^9) \\ & \quad + 6\varphi^2(q)\varphi^2(q^9) - 12\varphi^2(q)\varphi(q^9)\varphi(q^{36}) + 8\varphi(q)\varphi^3(q^4) - 12\varphi(q)\varphi^3(q^9) \\ & \quad - 24\varphi(q)\varphi^2(q^4)\varphi(q^{36}) - 12\varphi(q)\varphi(q^4)\varphi^2(q^9) + 24\varphi(q)\varphi(q^4)\varphi^2(q^{36}) \\ & \quad + 36\varphi(q)\varphi^2(q^9)\varphi(q^{36}) - 24\varphi(q)\varphi^3(q^{36}) - 4\varphi^4(q^4) - 8\varphi^3(q^4)\varphi(q^9) \\ & \quad + 16\varphi^3(q^4)\varphi(q^{36}) + 24\varphi^2(q^4)\varphi(q^9)\varphi(q^{36}) - 24\varphi^2(q^4)\varphi^2(q^{36}) + 9\varphi^4(q^9) \\ & \quad - 72\varphi(q^4)\varphi(q^9)\varphi^2(q^{36}) + 48\varphi(q^4)\varphi^3(q^{36}) + 12\varphi(q^4)\varphi^3(q^9) - 36\varphi^4(q^{36}) \\ & \quad + 72\varphi(q^9)\varphi^3(q^{36}) - 36\varphi^3(q^9)\varphi(q^{36})\big] + \varphi(q)\varphi^2(q^4)\varphi(q^{36}) \\ & = \varphi(q)\varphi^2(q^4)\varphi(q^{36}).\end{aligned}$$

10. Proofs of theorems of Section 2

We deduce Theorem 2.7 from Theorem 3.7. The rest can be proved similarly. By Theorem 3.7 and (1.1), (1.5), (1.7), (1.8), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,4,4,36;n)q^n \\
&= 1 + 8\sum_{n=1}^{\infty}\sigma(n)q^{12n} - 32\sum_{n=1}^{\infty}\sigma(n)q^{48n} + 2\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty}\sigma(n)q^{4n} \\
&\quad - 8\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty}\sigma(n)q^{16n} + 4\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n)q^{4n} - 16\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n)q^{16n} \\
&\quad + \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty}\sigma(n)q^{9n} + \frac{5}{9}\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}\sigma(n)q^{n} + \frac{8}{9}\sum_{\substack{n=1\\ n\equiv 5 \pmod{12}}}^{\infty}\sigma(n)q^{n} \\
&\quad + \frac{5}{9}\sum_{\substack{n=1\\ n\equiv 9 \pmod{12}}}^{\infty}\sigma(n)q^{n} + \frac{2}{3}\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}c(n)q^{n} + 4\sum_{\substack{n=1\\ n\equiv 1 \pmod 6}}^{\infty}c(n)q^{4n} \\
&\quad + \frac{1}{9}\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}t(n)q^{n} + \sum_{\substack{n=1\\ n\equiv 5 \pmod{12}}}^{\infty}t(n)q^{n} + \sum_{\substack{n=1\\ n\equiv 9 \pmod{12}}}^{\infty}t(n)q^{n}\right) \\
&= 1 + 8\sum_{n=1}^{\infty}\sigma(n/12)q^{n} - 32\sum_{n=1}^{\infty}\sigma(n/48)q^{n} + 2\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty}\sigma(n/4)q^{n} \\
&\quad - 8\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty}\sigma(n/16)q^{n} + 4\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n/4)q^{n} - 16\sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty}\sigma(n/16)q^{n} \\
&\quad + \sum_{\substack{n=1\\ n\equiv 1 \pmod 4}}^{\infty}\sigma(n/9)q^{n} + \frac{5}{9}\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}\sigma(n)q^{n} + \frac{8}{9}\sum_{\substack{n=1\\ n\equiv 5 \pmod{12}}}^{\infty}\sigma(n)q^{n} \\
&\quad + \frac{5}{9}\sum_{\substack{n=1\\ n\equiv 9 \pmod{12}}}^{\infty}\sigma(n)q^{n} + \frac{2}{3}\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}c(n)q^{n} + 4\sum_{\substack{n=1\\ n\equiv 4 \pmod 6}}^{\infty}c(n/4)q^{n} \\
&\quad + \frac{1}{9}\left(\sum_{\substack{n=1\\ n\equiv 1 \pmod{12}}}^{\infty}t(n)q^{n} + \sum_{\substack{n=1\\ n\equiv 5 \pmod{12}}}^{\infty}t(n)q^{n} + \sum_{\substack{n=1\\ n\equiv 9 \pmod{12}}}^{\infty}t(n)q^{n}\right).
\end{aligned}$$

Equating the coefficients of q^n completes the proof.

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