JOURNAL OF COMBINATORIAL THEORY, Series B 30, 136–143 (1981)

X^k-Digraphs

W. G. BRIDGES AND R. A. MENA

The University of Wyoming, Laramie, Wyoming 82071

Communicated by the Editors

Received November 25, 1978

Let G be a directed graph on n vertices (single loops allowed) such that there are λ directed paths of length k from P to Q for any distinct pair of vertices (P, Q). We prove that if $n > 2$ and $k > 2$, G is regular. The regular case is also discussed.

1. INTRODUCTION

A *polynomial digraph* is a directed graph (loops allowed) whose (0, 1) adjacency matrix Z satisfies

$$f(Z) = D + \lambda J \tag{1.1}$$

for some real polynomial $f(x)$, where D is diagonal, $\lambda \neq 0$, and J is the matrix all of whose entries are 1. This concept, definitively introduced in [2] based on Ryser's investigation [11] of the case $f(x) = x^2$, may be seen as a generalization of Hoffman's notion of the polynomial of a graph [4] and the subsequent extension to regular digraphs by Hoffman and MacAndrew [5]. We refer to the underlying digraph associated with (1.1) as an *$f(x)$-graph* and call Z the *carrying matrix*. We further agree to normalize by taking $f(x)$ monic and $f(0) = 0$. If H is a digraph, then by the (looped) cone over H we mean the digraph obtained by adjoining a single new (looped) vertex joined both to and from all the vertices of H. We call a digraph *regular* if it is both in and out regular ($ZJ = JZ$). Ryser [11] exhibits all the non-regular x^2-graphs. They are all cones, namely, (1) the looped cone over a graph with no edges; (2) the unlooped cone over a disjoint set of double directed edges (this is an ordinary graph, the pinwheel) and the unlooped cone over the graph H of Fig. 1.1.

In [2] the first author exhibits all non-regular $f(x)$-graphs where $f(x)$ is a quadratic polynomial. (While there are infinitely many such graphs only one is not a cone.) It is further proven in [3] that x^3-graphs on more than 2-vertices are regular and this extends to x^{2h+1}-graphs. We prove in Section 3

0095-8956/81/020136-08$02.00/0

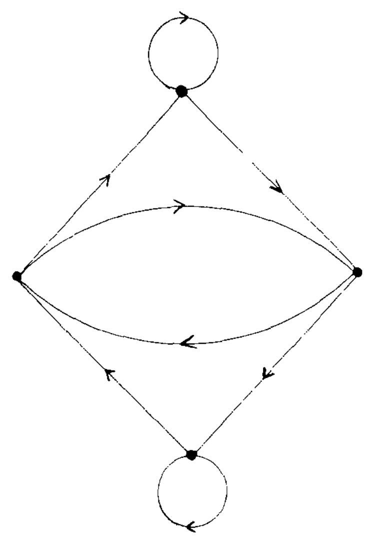

FIGURE 1

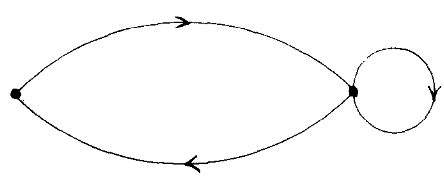

FIGURE 2

without restriction that x^k-graphs are regular for $k > 2$ excepting only the x-graph (Fig. 1.2) which is a non-regular x^k-graph for all k.

Regular x^k-graphs are themselves rather interesting and have received considerable attention [6–9, 10]. In Section 4 we discuss these and point out a brief proof of Lam's result [9] that there are no non-trivial regular x^2-graphs with circulant adjacency matrix. The next section contains background information on polynomial digraphs needed for the proof of the main theorem.

2. Background on Polynomial Digraphs

Let $A_1, \ldots, A_t$ be $(0, 1)$ matrices of orders $n_i \times n_i$, $i = 1, \ldots, t$. By the complementary direct sum [2] of the A_i we mean the $(0, 1)$ $\sum n_i \times \sum n_i$ matrix

$$c \cdot d \cdot s \cdot (A_i)_1^t = J - \sum_{i=1}^{t} \oplus (J - A_i), \tag{2.1}$$

where $\sum \oplus$ is the usual direct sum. The following theorem appears in [2]:

Theorem 2.1. *Let Z carry an $f(x)$-graph. Then to within permutation similarity Z is a complementary direct sum of t regular $f(x)$-graph adjacency matrices where* $t \leqslant \operatorname{degree}(f(x))$.

In this theorem if $Z = c \cdot d \cdot s \cdot (Z_i)$, we have $f(Z_i) = d_i I + \lambda_i J$, say, but possibly $\lambda_i = 0$. We refer to the Z_i, or the associated subgraphs, as the regular constituents of Z. For $f(x) = x^k$ the above theorem would say at most k constituents, but this is refined in [2] to:

Theorem 2.2. *A non-regular x^k-graph has two valences.*

(Notice for $f(x)$-graphs the in degree of every vertex is the same as the out degree.) We then have the following structure to consider:

Lemma 2.3. *Let Z carry a non-regular x^k-graph. Then essentially*

$$Z = \begin{pmatrix} Z_1 & J \\ J & Z_2 \end{pmatrix}, \tag{2.2}$$

where Z_i is $n_i \times n_i$ $(i = 1, 2)$ and

$$Z_i J = J Z_i = k_i J, \qquad Z_i^k = d_i I + \mu_i J. \tag{2.3}$$

If $Z^k = D + \lambda J$, then

$$D = \operatorname{diag}(\underbrace{d_1, \ldots, d_1}_{n_1}, \underbrace{d_2, \ldots, d_2}_{n_2})$$

and with

$$R = \begin{pmatrix} k_1 & n_2 \\ n_1 & k_2 \end{pmatrix} \tag{2.4}$$

we have

$$R^k = \begin{pmatrix} d_1 + \lambda n_1 & \lambda n_2 \\ \lambda n_1 & d_2 + \lambda n_2 \end{pmatrix}. \tag{2.5}$$

The final remark may be obtained by restricting $Z^k = D + \lambda J$ to the subspace $\{(x_1,..., x_n) | x_1 = \cdots = x_{n_1}; x_{n_1+1} = \cdots = x_n\}$.

3. x^k-Graphs Are Regular

In this section we prove

Theorem 3.1. *Let n and k be positive integers both at least* 3. *An x^k-graph on n-vertices is regular.*

Proof. We continue in the notation of Lemma 2.3. We define λ_i and β_i by

$$R^i = \lambda_i R + \beta_i I, \qquad i = 0, 1, 2,..., \tag{3.1}$$

and set

$$\gamma_i = n_i - k_i \geqslant 0, \qquad i = 1, 2. \tag{3.2}$$

We then have

$$(\lambda_0, \beta_0) = (0, 1), (\lambda_1, \beta_1) = (1, 0), (\lambda_2, \beta_2) = (k_1 + k_2, k_1\gamma_2 + k_2\gamma_1 + \gamma_1\gamma_2), \tag{3.3}$$

$$\lambda_{i+1} = \lambda_i \lambda_2 + \beta_i, \qquad \beta_{i+1} = \lambda_i \beta_2, \qquad i \geqslant 1. \tag{3.4}$$

Now since $Z_i^k = d_i I + \mu_i J$ and Z_i has line sums k_i easily by induction,

$$0 \leqslant d_i + \mu_i \leqslant k_i^{k-1}, \tag{3.5}$$

as no entry of Z_i^k exceeds this bound. Moreover if $n_i \geqslant 2$ we must have

$$k_i^{k-1} \geqslant \mu_i \geqslant 0. \tag{3.6}$$

Now we have also from Lemma 2.3

$$k_i^k = d_i + \mu_i n_i, \qquad i = 1, 2. \tag{3.7}$$

We define for $i \geqslant 0$

$$e_i = \beta_i - \gamma_1 \lambda_i,$$
$$f_i = \beta_i - \gamma_2 \lambda_i,$$
$$\tau_i = \frac{1}{n_1}(k_1^i - e_i), \tag{3.8}$$
$$\phi_i = \frac{1}{n_2}(k_2^i - f_i).$$

And note:

$$d_1 = e_k, \quad d_2 = f_k, \quad \mu_1 = \tau_k, \quad \mu_2 = \phi_k. \tag{3.9}$$
$$e_{i+1} = k_1 e_i - n_1 f_i; \quad f_{i+1} = k_2 f_i - n_2 e_i.$$

Direct calculation gives

$$e_4 = (k_1^3 + 2k_1^2 k_2 + k_1 k_2^2 + k_1^2 \gamma_2 + 2k_1 \gamma_1 \gamma_2 + k_2^2 \gamma_1 + 2k_1 k_2 \gamma_1)(\gamma_2 - \gamma_1) + \gamma_1^2 \gamma_2^2,$$
$$f_4 = (k_2^3 + 2k_1^2 k_2 + k_1 k_2^2 + k_2^2 \gamma_1 + 2k_2 \gamma_1 \gamma_2 + k_1^2 \gamma_2 + 2k_1 k_2 \gamma_2)(\gamma_1 - \gamma_2) + \gamma_1^2 \gamma_2^2. \tag{3.10}$$

The theorem for $k = 3$ is established in [3] but we note it is available here by the observation that $e_3 + \tau_3 < 0$ and if $k = 3$, $e_3 + \tau_3 = d_1 + \mu_1 \geqslant 0$ using (3.9) and (3.5). We assume now $k \neq 3$.

LEMMA 3.2. *For* $i \geqslant 4$

$$e_i = p_i(\gamma_2 - \gamma_1) + (-1)^i s_i \tag{3.11}$$

and

$$f_i = q_i(\gamma_1 - \gamma_2) + (-1)^i t_i \tag{3.12}$$

for suitable $p_i > k_1^{i-1}, q_i > k_2^{i-1}$, *and* $s_i = t_i = (\gamma_1 \gamma_2)^{i/2}$ *if* i *is even and* $\gamma_2 s_i = \gamma_1 t_i = (\gamma_1 \gamma_2)^{(i+1)/2}$ *if* i *is odd.*

Proof. We proceed by induction on i citing (3.10) for $i = 4$ and establish the claim for e_i as the case for f_i follows by the symmetric role of Z_1 and Z_2. Now the recursion (3.9) gives

$$\begin{aligned} e_{i+1} &= k_1 e_i - n_1 f_i = k_1 p_i(\gamma_2 - \gamma_1) + (-1)^i k_1 s_i - n_1 q_i(\gamma_1 - \gamma_2) \\ &\quad - (-1)^i n_1 t_i \\ &= (\gamma_2 - \gamma_1)(k_1 p_i + n_1 q_i) + (-1)^i (k_1 s_i - n_1 t_i) \end{aligned} \tag{3.13}$$

and

$$k_1 p_i + n_1 q_i > k_1^i \qquad \text{by induction.}$$

As to s_{i+1} suppose i is even so $s_i = (\gamma_1 \gamma_2)^{i/2} = t_i$. Then

$$(-1)^i (k_1 s_i - n_1 t_i) = (\gamma_1 \gamma_2)^{i/2} (k_1 - n_1)$$
$$= (-1)^{i+1} (\gamma_1 \gamma_2)^{i/2} \gamma_1$$

so $s_{i+1} = (\gamma_1 \gamma_2)^{i/2} \gamma_1$ as required. If i is odd, $s_i = (\gamma_1 \gamma_2)^{(i-1)/2} \gamma_1$ and $t_i = (\gamma_1 \gamma_2)^{(i-1)/2} \gamma_2$ and then

$$(-1)^i (k_1 s_i - n_1 t_i) = (-1)^i (\gamma_1 \gamma_2)^{(i-1)/2} (k_1 \gamma_1 - n_1 \gamma_2)$$
$$= (-1)^i (\gamma_1 \gamma_2)^{(i-1)/2} (k_1 \gamma_1 - k_1 \gamma_2 - \gamma_1 \gamma_2)$$
$$= (\gamma_2 - \gamma_1)(k_1(\gamma_1 \gamma_2)^{(i-1)/2}) + (-1)^{i+1} (\gamma_1 \gamma_2)^{(i+1)/2}.$$

So here $p_{i+1} = k_1 p_i + n_1 q_i + k_1(\gamma_1 \gamma_2)^{(i-1)/2}$ and $s_{i+1} = (\gamma_1 \gamma_2)^{(i+1)/2}$ so the lemma is established.

LEMMA 3.3. *If* $n_1 > 1$ *and* $n_2 > 1$, $k \leqslant 2$.

Proof. We assume $\gamma_2 > \gamma_1$ (note $\gamma_1 = \gamma_2$ is regularity). Suppose $k \geqslant 4$. If k is odd by Lemma 3.2 and (3.9) $d_2 = f_k < -k_2^{k-1}$ and from (3.6) and (3.9) $\phi_k = \mu_2 \leqslant k_2^{k-1}$ whence $d_2 + \mu_2 < 0$ in conflict with (3.5). If k is even, we have $e_k > k_1^{k-1}$ and since $\mu_1 \geqslant 0$, $d_1 + \mu_1 = e_k + \mu_1 > k_1^{k-1}$ in conflict with (3.5) completing the proof.

This leaves us with the case that $n_1 = 1$ say, the graph is a cone. Suppose $\gamma_2 > \gamma_1 \geqslant 0$. If $\gamma_1 = 1$, then $k_1 = 0$ and we have from (3.9) that $e_{i+1} = -f_i$ and $f_{i+1} = k_2 f_i + n_2 f_{i-1}$ for $i \geqslant 1$. Now from (3.8)

$$\phi_{i+1} = \frac{1}{n_2} (k_2^{i+1} - f_{i+1}) = k_2 \phi_i - f_{i-1}$$

so that

$$f_{i+1} + \phi_{i+1} = k_2(f_i + \phi_i) + (n_2 - 1) f_{i-1}. \tag{3.14}$$

Now one can check that $f_2 \leqslant 0$ and $f_3 < 0$ directly so that $f_i < 0$ for $i \geqslant 3$ and also one may compute

$$f_3 + \phi_3 = k_2^2(2 - \gamma_2) - 2k_2 - \gamma_2 - \gamma_2^2 - k_2 \gamma_2(k_2 - 1) < 0$$

in this case. Then (3.14) gives $f_i + \phi_i < 0$ for $i \geqslant 3$ contrary to $d_2 + \mu_2 = f_k + \phi_k \geqslant 0$.

Now if $\gamma_1 = 0$, $k_1 = 1$ and, from Lemma 3.2, $d_2 = f_k < -k_2^{k-1}$ and with

(3.6), $d_2 + \mu_2 < 0$ and again (3.5) is violated. This leaves only the case $\gamma_2 = 0$, i.e., $Z_2 = J$ so that $\gamma_1 = 1$ and $k_1 = 0$. Direct calculation reveals that the (1, 2) entry of Z^i is less than the (2, 3) entry for every i in this case so that unless $n = 2$ (the graph of Fig. 2) $Z^i \neq D + \lambda J$. This completes the proof of Theorem 3.1.

4. The Regular Case

We begin with the following easy observation.

THEOREM 4.1. *Let $\mathscr{A}$ be an algebra of $n \times n$ real symmetric matrices containing I. If A is normal and $A^m \in \mathscr{A}$, then $AA^t \in \mathscr{A}$.*

Proof. Since $\mathscr{A}$ contains only symmetric matrices we have $(A^m)^t = (A^t)^m = A^m \in \mathscr{A}$. Since A is normal, $A^m(A^t)^m = (AA^t)^m \in \mathscr{A}$. Now $(AA^t)^m = X \in \mathscr{A}$ say and X is positive semi-definite evidently. Thus X has a unique positive semi-definite mth root which, being a polynomial in X, lies in $\mathscr{A}$ [12]. But this root is AA^t.

COROLLARY 4.2. *Let A carry a regular x^m-graph. Suppose A is normal. Then A is the incidence matrix of a (v, k, λ)-configuration.*

Proof. Apply the theorem to $\mathscr{A} = \{aI + bJ \mid a, b \in \mathbb{R}\}$.

COROLLARY 4.3. *The carrying matrix of a regular x^2-graph is normal if and only if it is symmetric.*

Proof. If A is nonsingular, since A sends A, A^t, J into the two-dimensional algebra $\mathscr{A}$ above, these matrices must be linearly dependent. If A is singular, we have $A^2 = \mu J$, $AA^t = kJ$ easily, then $\mu = k$ and in any event $A = A^t$.

COROLLARY 4.4 (Lam [9]). *There are no non-trivial circulant* (0, 1) *matrices A satisfying $A^2 = dI + \lambda J$.*

Proof. By Corollary 4.3, A would be symmetric and thus correspond to a cyclic difference set with -1 as a multiplier which is known to be impossible [1].

COROLLARY 4.5. *A symmetric matrix A carrying an x^{2m}-graph actually carries an x^2-graph.*

We conclude by noting that a circulant $n \times n$ matrix carrying an x^2-graph would correspond to an "addition set," Y, in the cyclic group Z_n, i.e., every non-zero residue appears λ times among the sums $x + y$ $x, y \in Y$. While these provide only trivial x^2-graphs we note that the set $\{\alpha, \beta, \alpha\beta\}$ is an "addition set" in the dihedral group $D_4 = \langle \alpha, \beta | \alpha^4 = \beta^2 = e; \beta\alpha\beta = \alpha^3 \rangle$ giving rise to

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

satisfying $A^2 = I + J$. This solution, of trace zero, is non-isomorphic to Ryser's solution [11] with these parameters.

REFERENCES

1. L. D. BAUMERT , "Cyclic Difference Sets," Lecture Notes in Mathematics No. 182, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
2. W. G. BRIDGES, The polynomial of a non-regular digraph, *Pacific J. Math.* **38** (1971), 325–342.
3. W. G. BRIDGES, The regularity of x^3-graphs, *J. Combinatorial Theory* **12** (1972), 174–176.
4. A. J. HOFFMAN, On the polynomial of a graph, *Amer. Math. Monthly* **70** (1963), 30–36.
5. A. J. HOFFMAN AND M. H. MCANDREW, On the polynomial of a directed graph, *Proc. Amer. Math. Soc.* **10** (1965), 303–309.
6. W. H. LAM, A generalization of cyclic difference sets, I, *J. Combinatorial Theory Ser. A* **19** (1975), 51–65.
7. C. W. H. LAM, A generalization of cyclic difference sets, II, *J. Combinatorial Theory Ser. A* **19** (1975), 177–191.
8. C. W. H. LAM, On some solutions of $A^k = dI + \lambda J$, *J. Combinatorial Theory Ser. A* **23** (1977), 140–147.
9. C. W. H. LAM, "Rational g-Circulants Satisfying the Matrix Equation $A^2 = dI + \lambda J$," Ph. D thesis, California Institute of Technology, 1974.
10. C. W. H. LAM AND J. H. VANLINT, Directed graphs with unique paths of fixed length, *J. Combinatorial Theory Ser. B* **24** (1978), 331–337.
11. H. J. RYSER, A generalization of the matrix equation $A^2 = J$, *Linear Algebra and Appl.* **3** (1970), 451–460.
12. H. WIELANDT, "Topics in the Analytic Theory of Matrices," Lecture Notes, Department of Mathematics, University of Wisconsin, Madison, 1967.