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Resonance Functions for Radial Schrödinger Operators

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A characterization of resonance functions in terms of amplitude and phase is given for radial Schrödinger operators. The potential is a sum of an analytic background potential as, for example, the Coulomb potential and an exponentially decaying term. © 1987 Academic Press, Inc.

INTRODUCTION

Since the work of Gamow [5] on α-decay of nuclei, resonances have been associated with outgoing, exponentially growing solutions of the Schrödinger equation (Gamow waves). Especially in the radial case an extensive literature has developed on this subject, cf. Newton [6]. For a more detailed analysis of resonance functions it is useful to connect them with solutions of the analytically continued Lippman–Schwinger equation.

Assuming V exponentially decaying, the Lippman–Schwinger operator $VR_0(k)$ has an analytic continuation $V\tilde{R}_0(k)$ into a strip as a function taking values in the space $\mathscr{C}(h_a)$ of compact operators on an exponentially weighted space h_a. Resonances are identified as singular points of $V\tilde{R}_0(k)$ in the 4th quadrant, and a resonance function ψ at the resonance z is given by $\psi = \tilde{R}_0(k)\,\Phi$, where Φ is a solution in h_a of the equation $\Phi + V\tilde{R}_0(z)\,\Phi = 0$. This suggests a generalization to pairs $(H_1, H_1 + V)$, where $H_1 = H_0 + U$ and U is a suitable short-range potential. The key property to be established is, that the operator $VR_1(k)$ should have an analytic $\mathscr{C}(h_a)$-valued continuation $V\tilde{R}_1(k)$ into the 4th quadrant.

In Section 1 we establish this theory of resonances for radial "background" potentials U, using partial wave analysis. We give an explicit expression for $\tilde{R}_1(k)$ for each value of the angular momentum quantum number l, in terms of analytically continued generalized eigenfunctions

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(Lemma 1.4). It is also shown that if the S-matrix of (H_0, H_1) has an analytic extension into a region $\mathcal{O}$, then these analytically continued eigenfunctions exist (Lemma 1.1) and hence $VR_1(k)$ has a $\mathscr{C}(h_a)$-valued continuation into $\{k \in \mathcal{O} \mid \operatorname{Im} k > -a\}$.

An important example of potentials U, for which the S-matrix has an analytic extension, is the class of dilation-analytic, short-range potentials [1]. Here $\mathcal{O}$ is the sector $S_\alpha = \{z \mid |\operatorname{Arg} z| < \alpha\}$ of dilation-analyticity. As a consequence (Theorem 1.6) we show that the S-matrix of $(H_0, H_0 + U + V)$ has an analytic extension for U short-range, dilation-analytic and V exponentially decaying, generalizing a result of [3] in the radial case. Thus, $U + V$ can also serve as a background potential.

In Section 2 we characterize a resonance z by the existence of a regular solution ψ (the resonance function) of the Schrödinger equation $(H_0 + U + V - z^2)\psi = 0$ which is asymptotically very close to the outgoing solution of the equation $(H_0 + U - z^2)u = 0$.

Writing the Schrödinger equation as a pair of differential equations for the amplitude f and phase φ of the solution u, we derive in Section 3 some basic properties of f and φ, when z^2 is nonreal.

In Section 4 we further analyze the resonance function $\psi = fe^{i\varphi}$ in terms of certain asymptotic conditions on f and φ (Theorem 4.1). The result is precise in the following sense. Given an amplitude f satisfying these conditions, there exist a unique phase function φ and potential V, such that $\psi = fe^{i\varphi}$ is the resonance function of $H_0 + U + V$ at the prescribed resonance z. Thus, for a given background potential U and a prescribed resonance z, we have characterized the class of all functions ψ which can occur as resonance function for $H_0 + U + V$ for some $V = o(e^{-2ar})$. An analogous result is proved for antibound states (Theorem 4.4). Here the phase φ is 0, but in general the antibound state has a finite number of nodes, whereas the resonance function is node-free.

Finally, in Section 5 the theory is extended to the physically interesting case, where U is the Coulomb potential. Using the explicitly known form of the Coulomb wave functions, we obtain similar results on resonance functions and antibound states with modifications due to the logarithmic term in the Coulomb phase function.

Most of the results of Section 1–4 have been given without proof in [2].

1. Resonances for a Background Potential

Let $\mathbb{R}^+ = (0, \infty)$, $\overline{\mathbb{R}^+} = [0, \infty)$, $\mathbb{C}^+ = \{k \in \mathbb{C} \mid \operatorname{Im} k > 0\}$, $\overline{\mathbb{C}^+} = \{k \in \mathbb{C} \mid \operatorname{Im} k \geq 0\}$. For $a > 0$ we let $\mathbb{C}_a = \{k \in \mathbb{C} \mid \operatorname{Im} k > -a\}$, $\mathscr{T}_a = \{k \in \mathbb{C} \mid -a < \operatorname{Im} k < a\}$.

Let $h = L^2(\mathbb{R}^+)$, $h^2 = H^2(\mathbb{R}^+)$, the Sobolev space of order 2 on $\mathbb{R}^+$. The exponentially weighted spaces $h_{\pm a}$ and the weighted Sobolev space h^2_{-a} are defined by

$$h_{\pm a} = \{f \mid \|f\|_{\pm a} = \|e^{\pm ar} f\|_h < \infty\},$$
$$h^2_{-a} = \{f \mid \|f\|_{h^2_{-a}} = \|e^{-ar} f\|_{h^2} < \infty\}.$$

For any angular momentum quantum number $l = 0, 1, 2, \ldots$, the free Hamiltonian H_0^l is the operator in h given by

$$H_0^l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2},$$

where H_0^l is essentially self-adjoint on $C_0^\infty(\mathbb{R}^+)$ for $l \geq 1$, while the self-adjoint operator H_0^0 is the closure of its restriction to $\{u \in C_0^\infty(\overline{\mathbb{R}^+}) \mid u(0) = 0\}$.

The free resolvent $R_0^l(k) = (H_0^l - k^2)^{-1}$ is defined for $k \in \mathbb{C}^+$, and $R_0^l(k) \in B(h, h^2)$. We also define the operators

$$R_0^{l,a}(k) \in B(h_a, h^2_{-a}) \qquad \text{for} \quad k \in \mathbb{C}^+ \qquad \text{by} \quad R_0^{l,a}(k) = R_0(k) \mid h_a.$$

It is well known, that $R_0^{l,a}(k)$ has an analytic, $B(h_a, h^2_{-a})$-valued analytic continuation $\tilde{R}_0^l(k)$ from $\mathbb{C}^+$ to $\mathbb{C}_a$. If the potential V is in $\mathscr{C}(h^2_{-a}, h_a)$, we have $V\tilde{R}_0^l(k) \in \mathscr{C}(h_a)$ analytic in $\mathbb{C}_a$, and resonances of $(H_0^l, H_0^l + V)$ in $\mathbb{C}_a$ can be defined as poles of $(1 + V\tilde{R}_0^l(k))^{-1}$.

This suggests the following generalization. Let U be a symmetric, H_0^l-compact operator; $H_1^l = H_0^l + U$ is self-adjoint on $\mathscr{D}_{H_0^l}$. Let $R_1^l(k) = (H_1^l - k^2)^{-1}$ for $k \in \mathbb{C}^+$, $R_1^l(k) \in B(h, h^2)$, and let $R_1^{l,a}(k) = R_1^l(k) \mid h_a \in B(h_a, h^2_{-a})$. If $R_1^{l,a}(k)$ has an analytic continuation $\tilde{R}_1^l(k)$ from $\mathbb{C}^+$ across $\mathbb{R}^+$ to a larger domain $\mathcal{O}$, and $V \in \mathscr{C}(h^2_{-a}, h_a)$, then $V\tilde{R}_1^l(k)$ is a $\mathscr{C}(h_a)$-valued, analytic function in $\mathcal{O}$, and resonances of $(H_1^l, H_1^l + V)$ can be defined as poles of $(1 + V\tilde{R}_1(k))^{-1}$ in the 4th quadrant. In this section we shall discuss under what conditions on a multiplicative potential U such continuation exists. U will be called the background potential.

We make use of the well-known partial wave analysis, referring to [6] for general background. The potential U is assumed to be a real-valued, measurable function on $\mathbb{R}^+$ satisfying the following conditions:

(i) $\int_0^R r^2 \mid U(r)\mid^2 dr < \infty$ for all $R > 0$,

(ii) $\int_1^\infty \mid U(r)\mid dr < \infty$,

(iii) $\operatorname{ess\,sup}_{R_0 \leq r < \infty} \mid U(r)\mid < \infty$ for some $R_0 > 0$.

The class of potentials satisfying (i)–(iii) will be called $S - R$. We note that $U \in S - R$ implies U H_0^l-compact and hence $H_1^l = H_0^l + U$ self-adjoint on $\mathscr{D}_{H_0^l}$. We construct $R_1^l(k)=(H_1^l-k^2)^{-1}$ for $k\in\mathbb{C}^+$ via the Green's function for the equation

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+U(r)-k^2\right)u(r)=0. \tag{1.1}$$

We denote by $u_0^l(k,r)$ the regular solution of (1.1), defined for $k\in\mathbb{C}$ and $r>0$ by $u_0^l(k,r)\simeq r^{l+1}$ for $r\to 0$ and recall that $u_0^l(k,r)$ is entire in k^2 for every $r>0$. The outgoing and incoming solutions $u_\pm^l(k,r)$, defined for $\pm k\in\overline{\mathbb{C}^+}\setminus\{0\}$ by $u_\pm^l(k,r)\simeq e^{\pm ikr}$ for $r\to\infty$, are for every $r>0$ analytic in $\pm\mathbb{C}^+$ and continuous in $\pm\overline{\mathbb{C}^+}\setminus\{0\}$. The Jost function $\mathscr{F}_l(k)=W[u_+^l(k,\cdot),u_-^l(k,\cdot)]$ is analytic for $k\in\mathbb{C}^+$ and continuous for $k\in\overline{\mathbb{C}^+}\setminus\{0\}$. The connection between u_0^l, u_+^l, and u_-^l is given for $k\in\mathbb{R}\setminus\{0\}$, $r>0$ by

$$u_0^l(k,r)=\frac{1}{2ik}\left[\mathscr{F}_l(-k)\,u_+^l(k,r)-\mathscr{F}_l(k)\,u_-^l(k,r)\right]. \tag{1.2}$$

The S-matrix $S_l(k)$ is a continuous function of $k\in\mathbb{R}^+$ and is given by

$$S_l(k)=e^{i\pi l}\frac{\mathscr{F}_l(-k)}{\mathscr{F}_l(k)}. \tag{1.3}$$

We set $\mathscr{N}_l=\{z\in\mathbb{C}^+\mid\mathscr{F}_l(z)=0\}=\{i\lambda\mid\lambda>0,\ -\lambda^2$ is an eigenvalue of $H_1^l\}$.

LEMMA 1.1. *Let $\mathcal{O}$ be a domain in $\mathbb{C}\setminus(\{-\mathscr{N}\}\cup\{0\})$ having nonempty intersection with $\mathbb{R}^+$. The following statements are equivalent*:

(1) *The Jost function $\mathscr{F}_l(k)$ has an analytic continuation $\tilde{\mathscr{F}}_l(k)$ from $\overline{\mathbb{C}^+}\setminus\{0\}$ to $\mathcal{O}\cap\mathbb{C}^-$.*

(2) *The S-matrix $S_l(k)$ has a meromorphic extension $\tilde{S}_l(k)$ from $\mathcal{O}\cap\mathbb{R}^+$ to $\mathcal{O}\cap\mathbb{C}^-$ with poles at the zeros of $\tilde{\mathscr{F}}_l(k)$ and no zeros.*

(3) *For every $r>0$, $u_+^l(k,r)$ has an analytic continuation $\tilde{u}_+^l(k,r)$ from $\overline{\mathbb{C}^+}\setminus\{0\}$* to *$\mathcal{O}\cap\mathbb{C}^-$.*

Proof. (1)$\Leftrightarrow$(2) is clear from (1.3).

(1)$\Rightarrow$(3). We define $\tilde{u}_+^l(k,r)$ for $k\in\mathcal{O}\cap\overline{\mathbb{C}^-}$ by

$$\tilde{u}_+^l(k,r)=\frac{2ik}{\mathscr{F}_l(-k)}u_0^l(k,r)+\frac{\tilde{\mathscr{F}}_l(k)}{\mathscr{F}_l(-k)}u_-^l(k,r). \tag{1.4}$$

By (1.2) this agrees with $u_+^l(k,r)$ for $k\in\mathbb{R}^+\cap\mathcal{O}$ and hence for every $r>0$, $\tilde{u}_+^l(k,r)$ is an analytic continuation of $u_+^l(k,r)$ to $\mathcal{O}\cap\mathbb{C}^-$.

(3)$\Rightarrow$(1). Fix $r_0>0$ such that $u_-^l(k,r_0)\neq 0$ for $\operatorname{Im}k<0$. Then

$u^l_-(k, r_0) \neq 0$ except for k in a discrete set $\mathcal{M}(r_0)$. Define $\tilde{\mathcal{F}}_l^{r_0}(k)$ for $k \notin \mathcal{M}(r_0)$, $k \in \mathcal{O} \cap \mathbb{C}^-$ by

$$\tilde{\mathcal{F}}_l^{r_0}(k) = -2ik \frac{u_0^l(k, r_0)}{u^l_-(k, r_0)} + \mathcal{F}_l(-k) \frac{\tilde{u}^l_+(k, r_0)}{u^l_-(k, r_0)}.$$

Let $r > 0$ be fixed. We have for all $k > 0$

$$u_0^l(k, r) = \frac{1}{2ik} [\mathcal{F}_l(-k)\, u^l_+(k, r) - \mathcal{F}_l(k)\, u^l_-(k, r)].$$

By uniqueness of analytic continuation we get for all $k \in \mathcal{O} \cap \mathbb{C}^-$, $k \notin \mathcal{M}(r_0)$,

$$u_0^l(k, r) = \frac{1}{2ik} [\mathcal{F}_l(-k)\, \tilde{u}^l_+(k, r) - \tilde{\mathcal{F}}_l^{r_0}(k)\, u^l_-(k, r)].$$

Hence $\tilde{\mathcal{F}}_l^{r_0}(k) = \tilde{\mathcal{F}}_l^{r}(k)$ for $k \in \mathcal{O} \cap \mathbb{C}^-$, $k \notin \mathcal{M}(r_0) \cup \mathcal{M}(r)$.

For every $k \in \mathcal{M}(r_0)$ there exists r_1 such that $k \notin \mathcal{M}(r_1)$, since otherwise we would have $u^l_-(k, r) \equiv 0$. Then the function $\tilde{\mathcal{F}}_l^{r_0, r_1}$ defined by

$$\tilde{\mathcal{F}}_l^{r_0, r_1}(k) = \begin{cases} \tilde{\mathcal{F}}_l^{r_0}(k) & \text{for} \quad k \notin \mathcal{M}(r_0) \\ \tilde{\mathcal{F}}_l^{r_1}(k) & \text{for} \quad k \in \mathcal{M}(r_1) \end{cases}$$

is analytic also at k. This shows that all points of $\mathcal{M}(r_0)$ are removable singularities of $\tilde{\mathcal{F}}_l^{r_0}(k)$, and it follows that $\mathcal{F}_l(k)$ has an analytic continuation to $\mathcal{O} \cap \mathbb{C}^-$.

The lemma is proved.

Definition 1.2. Let $\mathcal{O}$ be a domain in $\mathbb{C} \setminus (\{-\mathcal{N}\} \cup \{0\})$, such that $\mathcal{O} \cap \mathbb{R}^+ \neq \varnothing$. The potential $U \in S - R$ is said to be $\mathcal{O}$-analytic, if the equivalent conditions (1)–(3) of Lemma 4.1 are satisfied.

To proceed further it is important to know the asymptotic behaviour of $\tilde{u}^l_+(k, r)$ for $k \in \mathcal{O} \cap \mathbb{C}^-$. We have the following result.

Lemma 1.3. *Assume that $U \in S - R$ is $\mathcal{O}$-analytic. Then*

$$\tilde{u}^l_+(k, r)\, e^{-ikr} \to 1 \qquad \text{for} \quad r \to \infty,$$

uniformly for k in compact subsets of $\mathbb{C}^+ \cup \mathcal{O}$.

Proof. We recall the following estimate (cf. [6]):

$$|u_0^l(k, r)\, e^{-ikr}| \leqslant C \tag{1.5}$$

valid for $\operatorname{Im} k \leqslant 0$, $r \geqslant 0$.

From (1.4), (1.5) and $u^l_{\pm}(k,r)\,e^{\mp ikr} \to 1$ for $r \to \infty$ it follows that for every compact subset K of $\mathcal{O}$ and $\varepsilon > 0$ there exists $C(K,\varepsilon)$ such that

$$|\tilde{u}^l_{+}(k,r)\,e^{-ikr}| \leqslant C(K,\varepsilon) \qquad \text{for} \quad k \in K,\ r \geqslant \varepsilon. \tag{1.6}$$

Now conclude from (1.6), since $u^l_{+}(k,r)\,e^{-ikr} \to 1$ for $r \to \infty$, $k \in \mathbb{C}^+ \cap \mathcal{O}$, by Vitali's convergence theorem (cf. [7]), applied to any sequence $\tilde{u}^l_{+}(k,r_n)\,e^{-ikr_n}$, $k \in \mathcal{O}$, such that $r_n \to \infty$, that

$$\tilde{u}^l_{+}(k,r_n)\,e^{-ikr_n} \xrightarrow[n\to\infty]{} 1 \qquad \text{for} \quad k \in \mathcal{O},$$

uniformly for k in compact subsets of $\mathcal{O}$. The Lemma is proved.

Based on Lemma 1.3 we obtain the following result on analytic continuation of $R_1^{l,a}(k)$:

LEMMA 1.4. *Assume that $U \in S - R$ is $\mathcal{O}$-analytic. Then the $B(h_a, h^2_{-a})$-valued function $R_1^{l,a}(k)$ has a meromorphic continuation $\tilde{R}^l_1(k)$ from $\mathbb{C}^+ \setminus \mathcal{N}$ to $\mathcal{O} \cap \mathcal{T}_a$ with poles at the zeros of $\tilde{\mathcal{F}}_l(k)$, given by*

$$\begin{aligned}(\tilde{R}^l_1(k)\,v)(r) = {}& \frac{1}{\tilde{\mathcal{F}}_1(k)}\,\tilde{u}^l_{+}(k,r)\int_0^\infty u^l_0(k,t)\,v(t)\,dt \\ &+ \frac{1}{2ik}\,\tilde{u}^l_{+}(k,r)\int_r^\infty u^l_{-}(k,t)\,v(t)\,dt \\ &- \frac{1}{2ik}\,u^l_{-}(k,r)\int_r^\infty \tilde{u}^l_{+}(k,t)\,v(t)\,dt.\end{aligned} \tag{1.7}$$

Moreover,

$$|(\tilde{R}^l_1(k)\,v)(r)| \leqslant C(k)\,r^2 \qquad \textit{for} \quad r \textit{ near } 0.$$

For every $r > 0$ the function $(\tilde{R}^l_1(k)\,v)(r)$ is meromorphic in $k \in (\mathbb{C}^+ \setminus \mathcal{N}) \cup (\mathcal{O} \cap \mathcal{T}_a)$ with poles at the zeros of $\tilde{\mathcal{F}}_l(k)$, and for $v \in h_a$, $\tilde{R}^l_1(k)\,v$ is a solution of the equation

$$(H^l_{1,-a} - k^2)\,\tilde{R}^l_1(k)\,v = v.$$

Proof. $R^l_1(k)$ is a meromorphic $B(h,h^2)$-valued and hence $R_1^{l,a}(k)$ a meromorphic $B(h_a, h^2_{-a})$-valued function on $\mathbb{C}^+ \setminus \mathcal{N}$. By the standard construction of the Green's function it is easy to show that $R_1^{l,a}(k)\,v$ is given for $k \in (\mathbb{C}^+ \setminus \mathcal{N}) \cap \mathcal{T}_a$ by

$$(R_1^{l,a}(k)\,v)(r)=\frac{1}{\mathscr{F}_l(k)}\,u_+^l(k,r)\int_0^r u_0^l(k,t)\,v(t)\,dt$$
$$+\frac{1}{\mathscr{F}_l(k)}\,u_0^l(k,r)\int_r^\infty u_+^l(k,t)\,v(t)\,dt$$
$$=\frac{1}{\mathscr{F}_l(k)}\,u_+^l(k,r)\int_0^\infty u_0^l(k,t)\,v(t)\,dt$$
$$+\frac{1}{2ik}\,u_+^l(k,r)\int_r^\infty u_-^l(k,t)\,v(t)\,dt$$
$$-\frac{1}{2ik}\,u_-^l(k,r)\int_r^\infty u_+^l(k,t)\,v(t)\,dt. \qquad (1.8)$$

By the $\mathscr{O}$-analyticity of U and (1.5), (1.6) we can define $(\tilde{R}_1^l(k,v)(r)$ for $k\in(\mathscr{O}\cap\mathscr{T}_a)\backslash\{k\,|\,\tilde{\mathscr{F}}_l(k)=0\}$ by (1.7).

Thus, for $v\in h_a$ the function $u(k,r)=(\tilde{R}_1^l(k)\,v)(r)$ is given by

$$u(k,r)=\int_0^\infty \mathscr{K}(k,r,t)\,v(t)\,dt, \qquad k\in\mathscr{O}\cap\mathscr{T}_a,\ r>0,$$

where $\mathscr{K}(k,r,t)$ is meromorphic in $\mathscr{O}\cap\mathscr{T}_a$ with poles at the zeros of $\tilde{\mathscr{F}}_l(k)$ for every fixed $r,t>0$. By Fubini's theorem this implies

$$\int_\Gamma u(k,r)\,dk=0 \qquad \text{for every Jordan curve}$$
$$\Gamma\subset(\mathscr{O}\cap\mathscr{T}_a)\backslash\{k\,|\,\tilde{\mathscr{F}}_l(k)=0\}$$

and by Morera's theorem $u(k,r)$ is analytic in $(\mathscr{O}\cap\mathscr{T}_a)\backslash\{k\,|\,\tilde{\mathscr{F}}_l(k)=0\}$. Clearly, the zeros of $\tilde{\mathscr{F}}_l(k)$ are poles of $u(k,r)$.

Since $u_-^l(k,r)\simeq cr^{-l}$ for $r\to 0$, by (1.4) also $\tilde{u}_+^l(k,r)\simeq cr^{-l}$ for $r\to 0$, and since $u_0^l(k,r)\simeq r^{l+1}$ for $r\to 0$ we obtain, using the expression for $u(k,r)$ given by the analytic continuation of the first formula in (1.8), that $|u(k,r)|\leqslant c(k)\,r^2$ for r near 0.

Moreover, by Fubini's and Morera's theorems, for every $v,w\in h_a$ the function

$$\langle w,\tilde{R}_1^l(k)\,v\rangle_{h_a,h_{-a}}=\int_0^\infty \bar{w}(r)\,u(k,r)\,dr$$

is meromorphic in $\mathscr{O}\cap\mathscr{T}_a$ with poles at the zeros of $\tilde{\mathscr{F}}_l(k)$, hence $\tilde{R}_1^l(k)$ is a $B(h_a,h_{-a})$-valued meromorphic function in $\mathbb{C}^+\cup(\mathscr{O}\cap\mathscr{T}_a)$ with poles at the zeros of $\tilde{\mathscr{F}}_l(k)$.

Differentiation of (1.7) yields

$$u'' = Uu - k^2 u - v = [(U - k^2)\,\tilde{R}_1^l(k) - I]\,v. \tag{1.9}$$

In view of (i) and (iii), the above implies that $U\tilde{R}_1^l(k)$ and hence by (1.8) the map $v \to (d^2/dr^2)\,\tilde{R}_1^l(k)\,v$ is a meromorphic $B(h_a, h_{-a})$-valued function in $\mathcal{O} \cap \mathcal{T}_a$. It follows that $\tilde{R}_1^l(k)$ is a $B(h_a, h^2_{-a})$-valued function in $\mathbb{C}^+ \cup (\mathcal{O} \cap \mathcal{T}_a)$ with poles at the zeros of $\mathcal{F}_l(k)$. Moreover, by (1.9)

$$(H^l_{1,-a} - k^2)\,\tilde{R}_1^l(k)\,v = v$$

and the lemma is proved.

Based on Lemma 1.4 we can now define resonances of $(H_1, H_1 + V)$ for exponentially decaying potentials V. We assume that V is a real-valued, measurable function on $\mathbb{R}^+$ satisfying (i) and

(ii′) $V(r) = W(r)\,e^{-2ar}$, where $\int_r^{r+1} |W(t)|^2\,dt \to 0$ for $r \to \infty$.

The class of potentials V satisfying (i) and (ii′) will be denoted by $\mathcal{E}_a$.

THEOREM 1.5. *Assume that $U \in S$—R is $\mathcal{O}$-analytic and $V \in \mathcal{E}_a$. Then $V\tilde{R}_1^l(k)$ is a $\mathcal{C}(h_a)$-valued analytic function and $(1 + V\tilde{R}_1^l(k))^{-1}$ a $B(h_a)$-valued meromorphic function in $\{\mathbb{C}^+ \cup (\mathcal{O} \cap \mathcal{T}_a)\} \setminus \{k \mid \mathcal{F}_l(k) = 0\}$. The operator $H_2^l = H_0^l + U + V = H_1^l + V$ is self-adjoint on $\mathcal{D}_{H_0^l} = \mathcal{D}_{H_1^l}$ with $\sigma_e(H_2^l) = [0, \infty)$. The $B(h_a, h^2_{-a})$-valued function $R_2^{l,a}(k) = (H_2^l - k^2)^{-1} | h_a$ has the meromorphic continuation $\tilde{R}_2^l(k)$ from $\mathbb{C}^+ \setminus \mathcal{N}$ to $\{\mathbb{C}^+ \setminus \mathcal{N}\} \cup \{\mathcal{O} \cap \mathcal{T}_a\}$ given by*

$$\tilde{R}_2^l(k) = \tilde{R}_1^l(k)(1 + V\tilde{R}_1^l(k))^{-1} \tag{1.10}$$

with the same poles as $(1 + V\tilde{R}_1^l(k))^{-1}$. This set of poles is symmetric with respect to the imaginary axis.

Proof. $V \in \mathcal{E}_a$ implies that $V \in \mathcal{C}(h^2_{-a}, h_a)$, and by Lemma 1.4 $V\tilde{R}_1^l(k)$ is a $\mathcal{C}(h_a)$-valued analytic function on $\{\mathbb{C}^+ \cup (\mathcal{O} \cap \mathcal{T}_a)\} \setminus \{k \mid \mathcal{F}_l(k) = 0\}$. By the analytic Fredholm theorem this implies that $(1 + V\tilde{R}_1^l(k))^{-1}$ is a meromorphic $B(h_a)$-valued function in the same region. It follows from the conditions on U and V, that H_1^l and H_2^l are self-adjoint on $\mathcal{D}_{H_0^l}$ with $\sigma_e(H_2^l) = \sigma_e(H_1^l) = [0, \infty)$. Restriction of the second resolvent identity to h_a yields $R_2^{l,a}(k) = R_1^{l,a}(k)(1 + VR_1^{l,a}(k))^{-1}$ for $k \in \{\mathbb{C}^+ \setminus \mathcal{N}\} \setminus \{i\lambda \mid \lambda > 0, -\lambda^2 \in \sigma_p(H_2)\}$. By analytic continuation we obtain (1.10) as an identity of meromorphic functions in $(\mathbb{C}^+ \setminus \mathcal{N}) \cup (\mathcal{O} \cap \mathcal{T}_a)$. Clearly $\tilde{R}_2^l(k)$ and

$(1+\tilde{R}_1^l(k))^{-1}$ have the same poles. Let $R_2^{l,a*}(k)$ be the adjoint of $R_2^{l,a}(k)$ with respects to the duality between h_a and h_{-a} defined by

$$\langle u, v\rangle_{a,-a} = \int_{\mathbb{R}^+} \bar{u}(r)\, v(r)\, dr, \qquad u\in h_a, v\in h_{-a}.$$

For $k\in\mathbb{C}^+\setminus\{i\lambda \mid \lambda>0,\ -\lambda^2\in\sigma_p(H_2)$ we have $R_2^{l*}(k)=R_2^l(-\bar{k})$; this implies $R_2^{l,a*}(k)=R_2^{l,a}(-\bar{k})$ and hence by analytic continuation

$$\tilde{R}_2^l(k)=\tilde{R}_2^l(-\bar{k}), \qquad k\in\{\mathbb{C}^+\setminus\mathcal{N}\}\cup\{\mathcal{O}\cap\mathcal{T}_a\}$$

as an identity between meromorphic functions. Hence the set of poles of $\tilde{R}_2^l(k)$ is symmetric with respect to the imaginary axis.

THEOREM 1.6. *Assume that $U\in S-R$ is $\mathcal{O}$-analytic and $V\in\mathcal{E}_a$. Then $U+V$ is $\mathcal{O}_1$-analytic, where $\mathcal{O}_1=\{\mathcal{O}\cap\mathcal{T}_a\}\setminus\{k \mid \tilde{\mathcal{F}}_l(k)=0\}$. Denoting by $\tilde{y}^l_+(k)$ and $\tilde{\mathcal{G}}_l(k)$ the analytic continuation of the outgoing solution and the Jost function of the equation*

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+U+V-k^2\right)y(k,r)=0 \tag{1.11}$$

we have

$$\frac{\tilde{y}^l_+(k)}{\tilde{\mathcal{G}}_l(k)}=(1-\tilde{R}_2(k)\,V)\,\tilde{u}^l_+(k),$$

where $\tilde{u}^l_+(k,r)$ is defined by (1.4).

Proof. Let $W=U+V$ and define $\mathcal{y}^l_+(k)$ for $k\in\mathbb{C}^+$, $k^2\notin\sigma_d(H_2)$ by

$$\mathcal{y}^l_+(k)=(1-R_2^l(k)\,W)\,w_l^{(+)}(k\cdot),$$

where $w_l^{(+)}(kr)$ is the Ricatti–Hankel function of order l (cf. [6, pp. 38, 39]).

For $k^2\notin\sigma_d(H_1)\cup\sigma_d(H_2)$ we have

$$\begin{aligned}\mathcal{y}^l_+(k)&=(1-R_2^l(k)\,V)(1-R_1^l(k)\,U)\,w_l^{(+)}(k\cdot)\\&=(1-R_2^l(k)\,V)\,u^l_+(k,\cdot)=(1-R_2^{l,a}(k)\,V)\,u^l_+(k,\cdot).\end{aligned} \tag{1.12}$$

By Theorem 1.5 $\mathcal{y}^l_+(k)$ has an h^2_{-a}-valued meromorphic continuation $\tilde{\mathcal{y}}^l_+(k)$ to $\{\mathbb{C}^+\setminus\mathcal{N}\}\cup\{\mathcal{O}\cap\mathcal{T}_a\}$ given by

$$\mathcal{y}^l_+(k)=(1-\tilde{R}_2^l(k)\,V)\,\tilde{u}^l_+(k). \tag{1.13}$$

Since $\tilde{\mathcal{y}}^l_+(k,r)\in C^1(\mathbb{R}^+)$, this is easily seen to imply that $\mathcal{y}^l_+(k,r)$ has a meromorphic continuation to $\{\mathbb{C}^+\setminus\mathcal{N}\}\cup\{\mathcal{O}\cap\mathcal{T}_a\}$ for each fixed $r>0$.

For $k>0$ we have, letting $y_0^l(k)$ denote the regular solution of (1.11),

$$\begin{aligned}\mathscr{y}_+^l(k,r) &= [(1-R_2^l(k)\,W)\,w_l^{(+)}(k\cdot)](r)\\ &= w_l^{(+)}(kr)-\frac{y_+^l(k,r)}{\mathscr{G}_l(k)}\frac{k^l e^{-i\pi l/2}}{(2l+1)!!}\\ &\quad\times\int_0^\infty y_0^l(k,t)\,W(t)\,w_l^{(+)}(k,t)\,dt\\ &\quad-\frac{y_+^l(k,r)}{2ik}\int_r^\infty y_-^l(k,t)\,W(t)\,w_l^{(+)}(k,t)\,dt\\ &\quad+\frac{y_-^l(k,r)}{2ik}\int_r^\infty y_+^l(k,t)\,W(t)\,w_l^{(+)}(k,t)\,dt\\ &= w_l^{(+)}(kr)-\frac{y_+^l(k,r)}{\mathscr{G}_l(k)}(\mathscr{G}_l(k)-1)\\ &\quad+o(e^{ikr})=\frac{y_+^l(k,r)}{\mathscr{G}_l(k)}+o(e^{ikr}),\end{aligned}$$

where we have used the identity [6, 12.144)] and the fact that $w_l^{(+)}(kr)\simeq e^{ikr}$ and $y_+^l(k,r)\simeq e^{ikr}$ for $r\to\infty$.

Thus $\mathscr{y}_+^l(k,r)\simeq e^{ikr}/\mathscr{G}_l(k)$ for $r\to\infty$, and since $\mathscr{y}_+^l(k,r)$ is a solution of (1.11) we have

$$\mathscr{y}_+^l(k,r)=\frac{y_+^l(k,r)}{\mathscr{G}_l(k)} \qquad \text{for} \quad k>0,\ r>0. \tag{1.14}$$

Moreover, for $k>0$,

$$y_0^l(k,r)=\frac{1}{2ik}\,[\mathscr{G}_l^0(-k)\,y_+^l(k,r)-\mathscr{G}_l^0(k)\,y_-^l(k,r)],$$

where $\mathscr{G}_l(k)=(k^l e^{-i\pi l/2}/(2l+1)!!)\,\mathscr{G}_l^0(k)$.

This implies

$$\mathscr{G}_l^0(k)=\frac{(-1)^l\,2iky_0(k,r)}{\mathscr{G}_l(-k)\,\mathscr{y}_+^l(k,r)-y_-^l(k,r)} \tag{1.15}$$

for all $k,r>0$ such that the denominator is not zero.

By an argument similar to the one used in proving $(3)\Rightarrow(1)$ of Lemma 1.1 we conclude that $\mathscr{G}_l^0(k)$ and hence $\mathscr{G}_l(k)$ has an analytic continuation to $\mathscr{O}_1$ with zeros at the poles of $\tilde{y}_+^l(k,r)$, i.e., the poles of $\tilde{R}_2^l(k)$. Note that the order of a zero z of $\tilde{\mathscr{G}}_l(k)$ is the same as the order of z as a pole of $\tilde{y}^l_+(k, r)$. Thus $U+V$ is $\mathcal{O}_1$-analytic. The analytic continuation $\tilde{y}^l_+(k, r)$ known to exist by Lemma 1.1 is also obtained from (1.14) as $\tilde{y}^l_+(k, r) = \tilde{\mathscr{G}}_l(k)\, \tilde{\mathscr{y}}_l^+(k, r)$.

COROLLARY 1.7. *Suppose that U is a $S-R$, dilation-analytic potential with angle of analyticity $S_\alpha = \{k(|\operatorname{Arg} k| < \alpha\}$ and $V \in \mathscr{E}_a$. Then $U+V$ is $(S_\alpha \cap \mathscr{T}_a)$-analytic.*

Proof. It is proved in [1], that the scattering matrix and hence, in the radial case, $S_l(k)$ has a meromorphic extension to S_α with no zeros, so U is S_α-analytic. Then by Theorem 1.6, $U+V$ is $(S_\alpha \cap \mathbb{C}_a)$-analytic.

COROLLARY 1.8. *Let U, V, $\mathcal{O}$, $\mathcal{O}_1$ be as in Theorem 1.6. Then for $z \in \mathcal{O}_1$ the following conditions are equivalent:*

(1) $\tilde{\mathscr{G}}_l(z) = 0$.

(2) *The S-matrix of $(H_0^l,\ H_0^l + U + V)$ has a pole at z.*

(3) *The equation*

$$\Phi + V\tilde{R}_1^l(z)\, \Phi = 0$$

has a solution $\Phi \in h_a$, $\Phi \neq 0$.

(4) *The operator-valued function $\tilde{R}_2^l(k) \in B(h_a, h^2_{-a})$ has a pole at z.*

DEFINITION 1.9. Let U, V, $\mathcal{O}$, $\mathcal{O}_1$ be as in Theorem 1.6. We denote by Σ_l the set $\{z \in \mathcal{O}_1 \mid \tilde{\mathscr{G}}_l(z) = 0\}$. If $z = \alpha - i\beta \in \Sigma_l$ and α, $\beta > 0$, z is called a resonance of $H_0^l + U + V$. If $\alpha < 0$, $\beta > 0$, z is a conjugate resonance. If $z = -i\beta$, $\beta > 0$, z is called a virtual pole. A point $z = -i\beta$, $\beta < 0$, corresponds to a bound state, $-\beta^2$ being a discrete eigenvalue of $H_0^l + U + V$.

Note that $\Sigma_l \cap (\mathbb{R} \setminus \{0\}) = \varnothing$ (cf. [6]).

2. A CHARACTERIZATION OF RESONANCE FUNCTIONS

THEOREM 2.1. *Let $U \in S-R$ be $\mathcal{O}$-analytic, let $V \in \mathscr{E}_a$, and set $\mathcal{O}_1 = \{\mathcal{O} \cap \mathscr{T}_a\} \setminus \{k \mid \tilde{\mathscr{F}}_l(k) = 0\}$. Let $z = \alpha - i\beta \in \mathcal{O}_1$, $\alpha \in \mathbb{R}$, $0 < \beta < a$. Then $z \in \Sigma_l$ if and only if there exists a function $\psi \in C^1(\overline{\mathbb{R}^+})$ with ψ'* loc. a.c. *on $\mathbb{R}^+$, satisfying the following conditions:*

(1) $(-(d^2/dr^2) + (l(l+1)/r^2) + U(r) + V(r) - z^2)\, \psi(r) = 0$ *for* $r \in \mathbb{R}^+$.

(2) $\psi(r) \simeq c r^{l+1}$ *for* $r \to 0$, *where* $c \neq 0$.

If $\alpha \neq 0$, $\psi(r) \neq 0$ *for* $r > 0$.

(3) $\psi(r) = \tilde{u}^l_+(z, r) + o(e^{(\beta - 2a)r})$ *for* $r \to \infty$.

ψ is uniquely determined by (1)-(3) *and is given by*

$$\psi = \tilde{R}_1^l(z)\,\Phi, \qquad \Phi = -V\psi, \tag{2.1}$$

where

$$\frac{1}{\tilde{\mathscr{F}}_l(z)} \int_0^\infty u_0(z, t)\,\Phi(t)\,dt = 1. \tag{2.2}$$

Proof. (A) Assume that $z \in \mathcal{O}_1$, $-a < \operatorname{Im} z < 0$, and let $\Phi \in h_a$ satisfy

$$\Phi + V\tilde{R}_1^l(z)\,\Phi = 0.$$

Define ψ by (2.1). Clearly $\psi \in C^1(\mathbb{R}^+)$, $\psi' \in L^l_{\text{loc}}(\mathbb{R}^+)$ and by Lemma 1.4, (1) holds. Also by Lemma 1.4, $|\psi(r)| \leqslant Cr^2$ for small r, hence ψ is a multiple of the regular solution of (1), so $\psi(r) \simeq cr^{l+1}$ for $r \to 0$ with $c \neq 0$.

For a proof of the fact that ψ has no positive nodes if $\alpha \neq 0$, see Theorem 3.2 and Remark 3.3. It remains to prove (3). We estimate the last two terms on the r.h.s. of (1.7) as follows: We have $|\tilde{u}^l_-(z, r)| \leqslant Ce^{-\beta r}$ and by (1.6) $|\tilde{u}^l_+(z, r)| \leqslant Ce^{\beta r}$ for large r. Since $\Phi \in h_a$, we have $\Phi = e^{-ar}\chi$, $\chi \in L^2(\mathbb{R}^+)$. Then we get by Schwarz' inequality for large r,

$$\begin{aligned} \left| \int_r^\infty u^l_-(z, t)\,\Phi(t)\,dt \right| &\leqslant C \int_r^\infty e^{-(a+\beta)t}\,|\chi(t)|\,dt \\ &\leqslant C(a+\beta)^{-1/2}\,e^{-(a+\beta)r} \left\{ \int_r^\infty |\chi(t)|^2\,dt \right\}^{1/2} \end{aligned}$$

and hence

$$\frac{1}{2iz}\,\tilde{u}^l_+(z, r) \int_r^\infty u^l_-(z, t)\,\Phi(t)\,dt = o(e^{-ar}) \qquad \text{for} \quad r \to \infty. \tag{2.3}$$

Similarly we get

$$-\frac{1}{2iz}\,u^l_-(z, r) \int_r^\infty \tilde{u}^l_+(z, t)\,\Phi(t)\,dt = o(e^{-ar}) \qquad \text{for} \quad r \to \infty. \tag{2.4}$$

If $\int_0^\infty u^l_0(k, t)\,\Phi(t)\,dt = 0$, by (1.7), (2.3), and (2.4) ψ would be a square-integrable solution of (4), i.e., an eigenfunction of H^l with eigenvalue z^2. For $\alpha \neq 0$ this is obviously impossible. For $\alpha = 0$ it would imply that both $i\beta$ and $-i\beta$ lie in Σ_l, which is impossible (cf. [6, p. 360]). Hence $\int_0^\infty u^l_0(k, t)\,\Phi(t)\,dt \neq 0$, and we can normalize ψ by (2.2). Using (2.3), (2.4), $|u_+(z, r)| \leqslant Ce^{\beta r}$ and the condition (ii') ($V = We^{-2ar}$, $\int_r^{r+1} |W(t)|^2\,dt \to 0$ for $r \to \infty$) we obtain from (1.7):

$$\Phi(r) = -V(r)\,\psi(r) = W(r)\,O(e^{(\beta - 2a)r}). \tag{2.5}$$

Using (2.5), we get the improved estimates

$$\left|\int_r^\infty u_-^l(z,t)\,\Phi(t)\,dt\right| \leqslant C\int_r^\infty e^{-2at}\,|W(t)|\,dt$$

$$= C\sum_{n=1}^\infty \int_{r+n-1}^{r+n} e^{-2ar}\,|W(t)|\,dt$$

$$\leqslant C\sum_{n=1}^\infty e^{-2a(r+n-1)}\left\{\int_{r+n-1}^{r+n} |W(t)|^2\,dt\right\}^{1/2}$$

$$\leqslant C\sup_{r\leqslant s<\infty}\left\{\int_s^{s+1} |W(t)|^2\,dt\right\}^{1/2}\int_{r-1}^\infty e^{-2at}\,dt$$

$$= o(e^{-2ar}).$$

Hence

$$\frac{1}{2iz}\tilde{u}_+^l(z,t)\int_r^\infty u_-^l(z,t)\,\Phi(t)\,dt = o(e^{(\beta-2a)r}). \tag{2.6}$$

Similarly we get

$$-\frac{1}{2iz}u_-^l(z,t)\int_r^\infty \tilde{u}_+^l(z,t)\,\Phi(t)\,dt = o(e^{(\beta-2a)r}). \tag{2.7}$$

From (2.2), (2.6), and (2.7), (3) follows.

(B) Assume now that ψ satisfies (1)–(3) and set $\Phi = -V\psi$. By (3) and (ii′), $\Phi = O(e^{(\beta-2a)r})\,W(r)$, hence $\Phi\in h_a$. Set $\psi_1 = \tilde{R}_1^l(z)\,\Phi$. We shall prove that $\psi=\psi_1\cdot\psi_1$ satisfies the conditions

(1′) $(-(d^2/dr^2)+(l(l+1)/r^2)+U(r)-z^2)\,\psi_1=\Phi$,

(2′) $|\psi_1(r)|\leqslant Cr^2$ for r small.

(3′) $\psi_1(r)=c\tilde{u}_+^l(z,r)+\eta(r)$, $\eta(r)=o(e^{(\beta-2a)r})$,

where

$$c=\frac{1}{\tilde{\mathscr{F}}_l(z)}\int_0^\infty u_0(z,t)\,\Phi(t)\,dt.$$

Here (1′) and (2′) follow from Lemma 1.4, and (3′) is proved as above using $\Phi(r)=O(e^{(\beta-2a)r})\,W(r)$.

By (1) and (1′) the function $\psi_0=\psi-\psi_1$ satisfies

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+U(r)-z^2\right)\psi_0=0. \tag{2.8}$$

By (2) and (2'), $|\psi_0(r)| \leqslant Cr^2$ for r small, hence for some α

$$\psi_0(r) = \alpha u_0(z, r) = \frac{\alpha}{2ik}\,[\mathscr{F}_l(-z)\,\tilde{u}^l_+(z, r) - \tilde{\mathscr{F}}_l(z)\,u^l_-(z, r)]. \tag{2.9}$$

Finally, by (3) and (3')

$$\psi_0(r) = (1 - c)\,\tilde{u}^l_+(z, r) + o(e^{(\beta - 2a)r}). \tag{2.10}$$

From (2.9) and (2.10), $\alpha = 0$, $c = 1$, $\psi_0 \equiv 0$ follows. Hence $\psi = \tilde{R}^l_1(z)\,\Phi$, $\Phi = -V\psi$ and (2.2) holds.

If ψ_1 and ψ_2 satisfy (1)–(3), then $\psi_1 - \psi_2$ satisfies (1), (2) and $(\psi_1 - \psi_2)(r) = o(e^{(\beta - 2a)r})$ for $r \to \infty$; hence $\psi_1 - \psi_2 \equiv 0$, so ψ is uniquely determined by (1)–(3).

DEFINITION 2.2. Let $z = \alpha - i\beta \in \Sigma_l$, $0 < \beta < \alpha$. If $\alpha > 0$, the function ψ of Theorem 2.1 is called the resonance function of H^l at z. If $\alpha < 0$, ψ is called the conjugate resonance function of H^l at z (in fact, ψ is the complex conjugate of the resonance function at $-\bar{z}$). If $\alpha = 0$, then ψ is called the antibound state of H^l at z.

Remark 2.3. The resonances and resonance functions of H^l are in fact independent of the decomposition of the total potential $Y = U + V$. In a standard way, resonances are zeros of the Jost function $\tilde{\mathscr{G}}_l(k)$ (cf. Theorem 1.6), and the resonance function ψ at the resonance z is the regular solution of (1), which is equal to the outgoing solution normalized by $\psi(r) \simeq e^{ikr}$ for $r \to \infty$.

If the total potential $Y = U + V$ has another decomposition $Y = U_1 + V_1$ as in Theorem 2.1, then one obtains a new fucntion $\Phi_1 = -V_1\psi$ with $\psi = \tilde{R}^l_1(z)\,\Phi_1$, where $\tilde{R}^l_1(k) = (H^l_0 + U_1 - k^2)^{-1\sim}$. Then ψ is characterized by being asymptotically very close to the outgoing solution of (2.8) with U replaced by U_1. The interest of Theorem 2.1 lies in the fact, that certain splittings of Y are natural, as for example, if $Y = r^{-\gamma} + V$, $V \in \mathscr{E}_a$, $1 \leqslant \gamma < \frac{3}{2}$ (for the case $\alpha = 1$ see Sect. 5). Also, keeping the background potential U fixed and letting V vary over $\mathscr{E}_a$, we obtain from Theorem 2.1 a complete characterization of the class of all resonance functions, as we shall see in Section 4.

3. DIFFERENTIAL EQUATIONS FOR AMPLITUDE AND PHASE

We consider a differential equation of the form

$$-u'' + Wu - (E - i\Gamma)\,u = 0, \tag{3.1}$$

where $W \in L^1_{\text{loc}}(I)$, I is a (finite or infinite) open interval (a, b), and E and Γ are given real numbers with $\Gamma \neq 0$. A solution u of (3.1) is a complex-valued function on I, such that u and u' are loc. a.c. on I, i.e., absolutely continuous on every closed interval contained in I, and (3.1) holds a.e. on I. Let f be the amplitude of u, i.e.,

$$u = f\theta, \qquad f = (\bar{u}u)^{1/2}. \tag{3.2}$$

THEOREM 3.1. *Let u be a solution of* (3.1) *on I. Then u has at most one node in I, i.e., there is at most one point $r_0 \in I$ such that $u(r_0) = 0$.*

If u has no node in I, $f \in C^1(I)$ and f' is loc. a.c. on I. There exists a real-valued phase function $\varphi \in C^2(I)$ with φ'' loc. a.c. on I, such that $u = fe^{i\varphi}$. The pair (f, φ) satisfies the differential equations

$$-f'' + Vf + \varphi'^2 f - Ef = 0, \tag{3.3}$$

$$f\varphi'' + 2f'\varphi' - \Gamma f = 0. \tag{3.4}$$

The function φ' is given in terms of f for any $c > 0$ by

$$\varphi'(r) = f^{-2}(r)\left\{\Gamma \int_c^r f^2(t)\, dt + \varphi'(c) f^2(c)\right\}. \tag{3.5}$$

If u has a node $r_0 \in I$, then f' is loc. a.c. on $I \setminus \{r_0\}$, and f' has the limits $f'_\pm(r_0) = \pm\, |u'(r_0)|$. There exists a phase function $\varphi \in C^2(I \setminus \{r_0\})$ with φ'' loc. a.c. on $I \setminus \{r_0\}$ and with the limits $\varphi_+(r_0) = \varphi_-(r_0) + \pi$, $\varphi'_\pm(r_0) = 0$, $\varphi''_\pm(r_0) = \Gamma/3$, such that $u = fe^{i\varphi}$ on I.

The pair (f, φ) satisfies (3.3) *and* (3.4) *on $(a, r_0) \cup (r_0, b)$, and* (3.5) *holds for every $c \in I$ and $r \in I$, where $\varphi'(r_0)$ is replaced by* 0 *and the r.h.s. for $r = r_0$ means the limit for $r \to r_{0\pm}$.*

Proof. First, assume that u has no node in I. Then by (3.2) $f \in C^1(I)$ with f' loc. a.c. on I. Clearly, $\theta = u/f \in C^1(I)$ with θ' loc. a.c. on I, and there exists a continuous phase fuction φ such that $\theta = e^{i\varphi}$. Since $\varphi' = -i\theta'/\theta$, φ' is loc. a.c. on I. Inserting $u = fe^{i\varphi}$ in (3.1), we obtain (3.3) and (3.4) a.e. on I. By (3.4), φ'' can be taken to be loc. a.c. on I with (2.4) holding for all $r \in I$, and solving (3.4) for φ' we get (3.5).

Assume now that u has at least one node r_0. If r_0 were an accumulation point of the set of nodes of u, we would have $u(r_0) = u'(r_0) = 0$, implying $u \equiv 0$. Hence r_0 is isolated in this set. Let I_1 and I_2 be the maximal open intervals to the left and right of r_0 such that $u(r) \neq 0$ for $r \in I_1 \cup I_1$. Then $f \in C^1(I_1 \cup I_2)$, $\theta \in C^1(I_1 \cup I_2)$. Moreover, for $r \in I_i$, $i = 1, 2$,

$$\frac{f(r)}{r - r_0} = (-1)^i \left[\frac{\bar{u}(r)}{r - r_0}\frac{u(r)}{r - r_0}\right]^{1/2} \xrightarrow[r \to r_{0\pm}]{} \pm\, |u'(r_0)| = f'_\pm(r_0). \tag{3.6}$$

Furthermore,

$$\theta(r)=\frac{u(r)/(r-r_0)}{f(r)/(r-r_0)} \xrightarrow[r\to r_{0\pm}]{} \pm\frac{u'(r_0)}{|u'(r_0)|}:=\theta_\pm(r_0). \tag{3.7}$$

By (3.2), for $r\in I_1\cup I_2$

$$f'(r)=\tfrac{1}{2}[\bar{u}'(r)\,\theta(r)+u'(r)\,\bar{\theta}(r)]. \tag{3.8}$$

From (3.6)–(3.8) follows

$$f'(r)\xrightarrow[r\to r_0]{} \pm|u'(r_0)|=f'_\pm(r_0). \tag{3.9}$$

By (3.7) there exists a continuous phase function φ on $I_1\cup I_2$ such that φ has limits $\varphi_\pm(r_0)$ at r_0 and

$$\theta(r)=e^{i\varphi(r)} \quad \text{for} \quad r\in I_1\cup I_2, \qquad \theta_\pm(r_0)=e^{i\varphi_\pm(r_0)}. \tag{3.10}$$

Clearly, $\varphi'=-i\theta'/\theta$ is loc a.c. on $I_1\cup I_2$, and (f,φ) satisfies (3.3) and (3.4) a.e. on $I_1\cup I_2$. By (3.4), φ'' can be taken loc. a.c. on $I_1\cup I_2$.

The solutions of (3.4) for φ' in terms of f are given on I_i, $i=1, 2$, by

$$\varphi'(r)=c_i f^{-2}(r)+\Gamma f^{-2}(r)\int_{r_0}^{r} f^2(t)\,dt. \tag{3.11}$$

By (3.11) and l'Hospital's rule

$$\lim_{r\to r_{0\pm}}\frac{\int_{r_0}^{r} f^2(y)\,dt}{f^2(r)}=\lim_{r\to r_{0\pm}}\frac{f(r)}{2f'(r)}=0.$$

By (3.9), (3.11), and (3.12),

$$\varphi'(r)\simeq C_i\,|u'(r_0)|^{-2}\,(r-r_0)^{-2} \qquad \text{for} \quad r\to r_{0\pm}.$$

This contradicts the existence of $\varphi_\pm(r_0)$ unless $C_1=C_2=0$; we obtain for $r\in I_1\cup I_2$

$$\varphi'(r)=\Gamma f^{-2}(r)\int_{r_0}^{r} f^2(t)\,dt \tag{3.13}$$

and it follows from (3.12) and (3.13) that $\varphi'_\pm(r_0)=0$. This together with (3.13) implies that $\varphi'\in C^1(I_1\cup I_2)$ and that φ' extends by continuity across r_0. Moreover, by (3.13)

$$\varphi''(r)=-2\Gamma f'(r)\frac{\int_0^r f^2(r)\,dt}{f^3(r)}+\Gamma \tag{3.14}$$

and it follows from (3.14) by l'Hospital's rule that $\varphi''_\pm(r_0)=\Gamma/3$.

It is now easy to prove that u has at most one node. Assume that r_1, $r_2 \in I$ and

$$f(r_1) = f(r_2) = 0, \qquad f(r) > 0 \quad \text{for} \quad r_1 < r < r_2.$$

By (3.13) with r_0 replaced by r_1 we have for $r_1 < r < r_2$

$$\varphi'(r) = \Gamma f^{-2}(r) \int_{r_1}^{r} f^2(t)\, dt \xrightarrow[r \to r_{2-}]{} \infty,$$

contradicting $\varphi'_-(r_2) = 0$.

Thus, if u has a node r_0, we have $I_1 = (a, r_0)$, $I_2(r_0, b)$, and the theorem is proved. It only remains to note, that (3.5) for $c = r_0$ follows from (3.4) and (3.13), and by (3.7) φ can be chosen such that $\varphi_+(r_0) = \varphi_-(r_0) + \pi$.

THEOREM 3.2. *Let W be a real-valued, measurable function on $\mathbb{R}^+$ satisfying the condition*

$$\int_0^R r\,|W(r)|\, dr < \infty \qquad \textit{for every} \quad R > 0. \tag{$*$}$$

Let $l = 0, 1, 2, \ldots$, be fixed, and let u_0^l be the regular solution on $\mathbb{R}^+$ of the differential equation

$$-u'' + \frac{l(l+1)}{r^2} u + Wu - k^2 u = 0 \tag{3.15}$$

defined by

$$u_0^l(r) \sim r^{l+1} \qquad \textit{for} \quad r \to 0, \tag{3.16}$$

where k is fixed with $k^2 = E - i\Gamma$, $E \in \mathbb{R}$, $\Gamma > 0$.

Then u_0^l has no nodes in $\mathbb{R}^+$. There exists a continuous phase function on $\mathbb{R}^+$ such that $u_0^l = fe^{i\varphi}$, and the pair (f, φ) satisfies the following conditions:

(1) $f \in C^1(\overline{\mathbb{R}^+})$, f' loc a.c. *on* $\overline{\mathbb{R}^+}$, $f(r) > 0$ *for* $r > 0$,

(2) $f(r) \simeq r^{l+1}$, $f'(r) \simeq (l+1)\, r^l$ *for* $r \to 0$,

(3) $\varphi \in C^2(\overline{\mathbb{R}^+})$, φ'' loc. a.c. *on* $\overline{\mathbb{R}^+}$,

(4) $\varphi'_+(0) = 0$, $\varphi''_+(0) = \Gamma/(2l+3)$,

(5) $\varphi'(r) = \Gamma(\int_0^r f^2(t)\, dt)/f^2(r)$,

(6) $-f'' + (l(l+1)/r^2) f + Wf + \varphi'^2 f - Ef = 0$,

(7) $f\varphi'' + 2f'\varphi' - \Gamma f = 0$.

Proof. By Theorem 3.1, u_0^l has at most one positive node.

Assume that $u_0^l(r_0) = 0$, $r_0 > 0$. By Theorem 3.1, $f \in C^1(0, r_0)$ with

f' loc. a.c. on $(0, r_0)$, and there exists $\varphi \in C^2(0, r_0)$ with φ'' loc. a.c. on $(0, r_0)$ such that $u_0^l = fe^{i\varphi}$. Moreover, (6) and (7) are satisfied on $(0, r_0)$.

For $l = 0$, (3.16) amounts to

$$u_0^0(0) = 0, \qquad u_0^{0'}(0) = 1. \tag{3.17}$$

For $l \geqslant 1$, by (3.16)

$$\frac{u_0^l(r)}{r} \xrightarrow[r \to 0]{} 0 = u_{0+}^{l'}(0)$$

Also, by (3.15), (3.16), and (*),

$$u_0^{l''} = Vu_0^l + \frac{l(l+1)}{r^2} u_0^l - k^2 u_0^l \in L^1(0, 1), \tag{3.18}$$

and hence

$$u_0^{l'}(r) = u_0^{l'}(1) - \int_r^1 u_0^{l''}(t)\, dt \xrightarrow[r \to 0]{} u_0^{l'}(1) - \int_0^1 u_0^{l''}(t)\, dt = u_0^{l'}(0) = 0. \tag{3.19}$$

From (3.16), (3.18), and (3.19) we get

$$u_0^{l'}(r) = \int_0^r u_0^{l''}(t)\, dt \simeq (l+1)\, r^l \qquad \text{for} \quad r \to 0. \tag{3.20}$$

By (3.16) and (3.20), for $0 < r < r_0$,

$$\frac{u_0^{l'}(r)}{u_0(r)} = \frac{f'(r)}{f(r)} + i\varphi'(r) \simeq (l+1)\, r^{-1} \qquad \text{for} \quad r \to 0. \tag{3.21}$$

From (3.17) it follows that (3.19) holds also for $l = 0$, which implies that for $l = 0, 1, 2, \ldots,$

$$\frac{f'(r)}{f(r)} r \xrightarrow[r \to 0]{} l + 1 \tag{3.22}$$

and

$$\varphi'(r) \xrightarrow[r \to 0]{} 0. \tag{3.23}$$

By (3.16),

$$f(r) \simeq r^{l+1} \qquad \text{for} \quad r \to 0 \tag{3.24}$$

and hence by (3.22),

$$f'(r) \simeq (l+1)\, r^l \qquad \text{for} \quad r \to 0 \tag{3.25}$$

proving (2). By Theorem 3.1, for $0 < r < r_0$,

$$\varphi'(r) = \Gamma f^{-2}(r) \int_{r_0}^{r} f^2(t)\, dt. \tag{3.26}$$

From (3.24) and (3.26),

$$\varphi'(r) \simeq -\Gamma \int_0^{r_0} f^2(t)\, dt \cdot r^{-2l-2} \qquad \text{for} \quad r \to 0,$$

follows, contradicting (3.23). We conclude that u_0^l has no positive nodes and that (6) and (7) hold on $\mathbb{R}^+$. Also,

$$\varphi'(r) = Cf^{-2}(r) + \Gamma f^{-2}(r) \int_0^r f^2(t)\, dt$$

is the general solution of (7) on $\mathbb{R}^+$. By (3.23) and (3.24) we must have $C = 0$, and (5) is proved. By (5),

$$\varphi''(r) = -2\frac{f'(r)\,\varphi'(r)}{f(r)} + \Gamma. \tag{3.27}$$

By (5), $\varphi'(r) \simeq (\Gamma/(2l+3))\, r$ for $r \to 0$ and hence, by (3.22) and (3.27),

$$\varphi''(r) \to -2\Gamma\frac{l+1}{l+3} + \Gamma = \frac{\Gamma}{2l+3} \qquad \text{for} \quad r \to 0,$$

which concludes the proof of (4), and the theorem is proved.

Remark 3.3. The fact that any nonzero solution u of (3.1) has at most one node can be proved by noting that if $r_1 < r_2$ were two such zeros, the boundary value problem defined by (3.1) on (r_1, r_2) and $u(r_1) = u(r_2) = 0$ would have a nonreal eigenvalue. Similarly, the regular solution $u_0^l(k, r)$ has no nodes, because if $u_0^l(k, r_0) = 0$ then the Dirichlet problem in the ball $\{\bar{r} \mid |\bar{r}| < r_0\}$ would have a nonreal eigenvalue.

4. Resonance Functions Characterized by Amplitude and Phase

Let $U \in S - R$ be an $\mathcal{O}$-analytic potential and let $z = \alpha - i\beta \in \mathcal{O}$, $\alpha > 0$, $0 < \beta < a$, $\tilde{\mathscr{F}}_l(z) \neq 0$. Let $\tilde{u}_+^l(z, r)$ be the outgoing solution, defined by (1.4), of the equation

$$(H_1^l - z^2)\, u(z, r) = 0, \tag{4.1}$$

where

$$H_1^l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + U(r).$$

By Theorem 3.1, $\tilde{u}^l_+(z, r)$ has at most one node $r_0 > 0$, and $\tilde{u}_+(z, r)$ can be written as

$$\tilde{u}^l_+(z, r) = f_l(r)\, e^{i\varphi_l(r)}, \tag{4.2}$$

where φ_l is continuous on $\mathbb{R}^+$, if $\tilde{u}^l_+(z, r)$ has no node, and continuous on $\mathbb{R}^+ \backslash \{r_0\}$ with $\varphi_{l_+}(r_0) = \varphi_{l_-}(r_0) + \pi$ if $\tilde{u}^l_+(z, r_0) = 0$. Moreover, since by Lemma 1.3 $\tilde{u}^l_+(z, r) \simeq e^{izr}$ for $r \to \infty$, we can choose φ_l such that $\varphi_l(r) - \alpha r \to 0$ for $r \to \infty$. Also, $f_l(r) \simeq e^{\beta r}$ for $r \to \infty$.

If $V \in \mathscr{E}_a$ and z is a resonance of $H_1^l + V$ with resonance function ψ, then by Theorems 2.1 and 3.2 ψ can be written as

$$\psi(r) = f(r)\, e^{i\varphi(r)}, \tag{4.3}$$

where φ is continuous on $[0, \infty)$ and $\varphi(r) - \varphi_l(r) \to 0$ for $r \to \infty$.

In what follows it is understood, that the phase functions φ_l and φ are chosen as indicated above. With the notations (4.2) and (4.3) and the above normalizations of the phase functions we can now formulate the main result of this section.

THEOREM 4.1. *Assume that $\mathscr{U} \in S - R$ is $\mathcal{O}$-analytic.*

(A) *Let $V \in \mathscr{E}_a$, and assume that $z = \alpha - i\beta \in \mathcal{O}$ is a resonance of $(H_0^l, H_0^l + V)$ with resonance function ψ. The pair (f, φ) satisfies the following conditions, where $z^2 = E - i\Gamma$:*

(1) $f \in C^1(\overline{\mathbb{R}^+})$, f' loc. a.c. *on* $\overline{\mathbb{R}^+}$, $f(r) > 0$ *for* $r > 0$,

(2) $f(r) \simeq cr^{l+1}$, $f'(r) \simeq c(l+1)\, r$ *for* $r \to 0$,

(3) $f(r) = f_l(r) + o(e^{(\beta - 2a)r})$ *for* $r \to \infty$,

(4) $f'' \in L^2_{\text{loc}}(\mathbb{R}^+)$, $\int_0^1 |r(f''(r)/f(r)) - (l(l+1)/r)|^2\, dr < \infty$,

(5) $f''(r) = f''(r) + e^{(\beta - 2a)r} g_l(r)$, *where*

$$\int_r^{r+1} |g_l(t)|^2\, dr \to 0 \quad \textit{for} \quad r \to \infty,$$

(6) $\int_0^r f^2(t)\, dt + \int_r^\infty p_l(t)\, dt = \Gamma^{-1} \varphi_l'(r) f_l^2(r)$ *for every* $r > 0$, *where*

$$p_l(r) = f^2(r) - f_l^2(r),$$

(7) $\varphi \in C^2(\overline{\mathbb{R}^+})$, φ'' loc. a.c. *on* $\overline{\mathbb{R}^+}$,

(8) $\varphi'(0)=0$, $\varphi''(0)=\Gamma/(2l+3)$,

(9) $\varphi'(r)=\Gamma f^{-2}(r)\int_0^r f^2(t)\,dt$,

(10) $\varphi'(r)=\varphi_l'(r)+o(e^{-2ar})$ *for* $r\to\infty$,

(11) $\varphi(r)=\varphi_l(r)+\Gamma\int_r^\infty \{(p_l(t)\int_0^t f^2(s)\,ds/f_l^2(t)\,f^2(t))+(\int_t^\infty p_l(s)\,ds/f_l^2(t))\}\,dt$,

(12) $\varphi(r)=\varphi_l(r)+o(e^{-2ar})$ *for* $r\to\infty$,

(13) $-f''+(U+V+(l(l+1)/r^2))f+\varphi'^2 f-Ef=0$,

(14) $2f'\varphi'+f\varphi''-\Gamma f=0$.

(B) *Let* $z=\alpha-i\beta\in\mathcal{O}$, $\alpha>0$, $0<\beta<a$, $\tilde{\mathscr{F}}_l(z)\neq 0$, $z^2=E-i\Gamma$. *Assume that* f *satisfies conditions* (1)–(6) *and define* φ *by* (11) *and* V *by* (13). *Then* $V\in\mathscr{E}_a$, *and* $\Psi=fe^{i\varphi}$ *is the resonance function of* (H_1^l, H_1^l+V) *at the resonance* z.

Proof. (A) Assume that $z=\alpha-i\beta$ is a resonance of (H_1^l, H_1^l+V), where $V\in\mathscr{E}_a$, and let $\psi=fe^{i\varphi}$ be the resonance function at z. Then (1), (2), (7), (8), (9), (13), and (14) follow from $\psi\in C^1(\overline{\mathbb{R}^+})$, $\psi'\in L^1_{\text{loc}}(\mathbb{R}^+)$ and (1), (2) of Theorem 2.1, and Theorem 3.2. Now (4) follows from condition (1) on $V\in\mathscr{E}_a$, (1), (2), (7), and (13). From (3) of Theorem 2.1, (3) and (12) follow. We now turn to the proof of (6) and (10). By (9), for any $c>0$,

$$\begin{aligned}\varphi'(r)&=\Gamma f^{-2}(r)\int_0^c f^2(t)\,dt+\Gamma f^{-2}(r)\int_c^r f^2(t)\,dt\\&=\Gamma f_l^{-2}(r)\int_0^c f^2(t)\,dt+\Gamma f_l^{-2}(r)\int_c^r f_l^2(t)\,dt\\&\quad-\Gamma\frac{p_l(r)}{f^2(r)f_l^2(r)}\int_0^r f^2(t)\,dt+\Gamma f_l^{-2}(r)\int_c^\infty p_l(t)\,dt\\&\quad-\Gamma f_l^{-2}(r)\int_r^\infty p_l(t)\,dt.\end{aligned}\tag{4.4}$$

By (3.11) and (3.13) of Theorem 3.1, $\tilde{u}_+^l(z,r)$ has at most one node, and for any $c>0$,

$$\varphi_l'(r)=\varphi_l'(c)f_l^2(c)f_l^{-2}(r)+\Gamma f_l^{-2}(r)\int_c^r f_l^2(t)\,dt.\tag{4.5}$$

Introducing (4.5) into (4.4) and using (3) and $p_l(r)=f^2(r)-f_l^2(r)=o(e^{(2\beta-2a)r})$ for $r\to\infty$, we get

$$\varphi'(r)=\varphi_l'(r)+Cf_l^{-2}(r)+\Gamma\frac{p_l(r)}{f^2(r)f_l^2(r)}$$

$$\times\int_0^r f^2(t)\,dt-\Gamma f_l^{-2}(r)\int_r^\infty p_l(t)\,dt$$

$$=\varphi_l'(r)+Cf_l^{-2}(r)+o(e^{-2ar}), \qquad (4.6)$$

where $C=\Gamma\int_0^c f^2(t)\,dt+\Gamma\int_c^\infty p_l(t)\,dt-\varphi_l'(c)f_l^2(c)$.

Clearly (4.6) contradicts (12) unless $C=0$. This yields (6). Inserting $C=0$ in (4.6) we get (10), and taking account again of (12), we obtain (11). Finally (5) follows from (3), (10), (13) and condition (ii′) on V.

(B) Assume now that f satisfies (1)–(6) with $z=\alpha-i\beta$, $\alpha>0$, $0<\beta<\alpha$, $z^2=E-i\Gamma$. Define φ by (11) and V by (13). From (11) and (6) we conclude that φ' is given by (3.21) with $C=0$. By (4.5) this implies that φ' satisfies (9), and hence (f, φ) is a solution of (14). Since (13) is satisfied by definition of V, it follows that $\psi=fe^{i\varphi}$ satisfies (1) of Theorem 2.1. Properties (2) and (3) as well as (i) of $V\in\mathscr{E}_a$ are easy consequences of the properties of f and φ. Finally, we verify (ii′) of $V\in\mathscr{E}_a$ as follows, using (3), (5), and (10),

$$V(r)=\frac{f''(r)}{f(r)}-\frac{l(l+1)}{r^2}-\varphi_l'^2(r)+E$$

$$=\frac{f_l''(r)+e^{(\beta-2a)}g_l(r)}{f_l(r)+o(e^{(\beta-2a)r})}-\frac{l(l+1)}{r^2}$$

$$-\varphi_l'^2(r)+o(e^{-2ar})+E$$

$$=\frac{f_l''(r)}{f_l(r)}-\frac{l(l+1)}{r^2}$$

$$-\varphi_l'^2+E+e^{-2a}W(r)=e^{-2ar}W(r),$$

where

$$\int_r^{r+1}|W(t)|^2\,dt\to 0 \qquad \text{for} \quad r\to\infty,$$

since (f_l, φ_l) satisfies (13) with $V=0$.

It now follows from Theorem 2.1, that ψ is the resonance function of H_1^l+V at z.

COROLLARY 4.2. *Assuming $U\in C^n(\mathbb{R}^+)$ we have $V\in C^n(\mathbb{R}^+)$ if and only if $f\in C^{n+2}(\mathbb{R}^+)$, $\varphi\in C^{n+3}(\mathbb{R}^+)$. Assuming U analytic in a sector S_α, we have V analytic in S_α if and only if f and φ are analytic in S_α. $V=o(e^{-br})$ for all*

$b>0$ *if and only if* $f(r)-f_l(r)=o(e^{-br})$, $\varphi(r)-\varphi_l(r)=o(e^{-br})$ *for all* $b>0$. $V(r)=0$ *for* $r>R$ *if and only if* $f(r)=f_l(r)$, $\varphi(r)=\varphi_l(r)$, $p_l(r)=0$ for $r>R$.

In this case, (3), (4), (10), (12) are automatic, and (6) becomes for $r\geqslant R$

$$\text{(6c)} \qquad \int_0^r f^2(t)\,dt=\Gamma^{-1}\varphi_l'(r)f_l^2(r).$$

Note that (6_c) cannot be satisfied with $r=r_0$ if $f_l(r_0)=0$ for some $r_0\geqslant R$. Thus, a necessary condition for the existence of a resonance at z for any such V that $\tilde{u}_+^l(z,r)$ has no node $r_0\geqslant R$.

Remark 4.3. Theorem 2.1 also covers the case $z=-\alpha-i\beta$, $\alpha>0$, $0<\beta<a$, i.e., conjugate resonances. Accordingly, Theorem 4.1 extends to conjugate resonance states. The conjugate resonance state at $-\alpha-i\beta$ is simply $\bar{\psi}=fe^{-i\varphi}$, where Ψ is the resonance state at $\alpha-i\beta$.

It remains to discuss the case when $\alpha=0$. With the usual normalization of phases, the function ψ is real, $\varphi(r)\equiv 0$. This simplifies considerably the results. On the other hand the absence of nodes of the resonance function ψ is linked to the existence of a nontrivial phase function. When $\alpha=0$, ψ has in general nodes.

We obtain the following results on antibound states.

THEOREM 4.4. *Assume that* $U\in S-R$ *is* $\mathcal{O}$*-analytic and* $z=-i\beta\in\mathcal{O}$, $0<\beta<a$.

(A) *Let* $V\in\mathcal{E}_a$ *and let* z *be a virtual pole of* H_1^l+V *with antibound state* ψ. *Then* ψ *is a real-valued function satisfying the following conditions*:

(1) $\psi\in C^1(\overline{\mathbb{R}^+})$, ψ' loc. a.c. on $\overline{\mathbb{R}^+}$,

(2) ψ has at most a finite number of nodes,

(3) $\psi(r)\simeq cr^{l+1}$ *for* $r\to 0$, $c\neq 0$,

(4) $\psi(r)=\tilde{u}_+^l(-i\beta r)+o(e^{(\beta-2a)r})$ *for* $r\to\infty$,

(5) $\psi''\in L^2_{\mathrm{loc}}(\mathbb{R}^+)$, $\int_0^1 |r(\psi''(r)/\psi(r))-(l(l+1)/r^2)|^2\,dr<\infty$,

(6) $\psi''(r)=\tilde{u}_+^{l''}(-i\beta r)+e^{(\beta-2a)r}g_l(r)$, *where* $\int_r^{r+1}|g_l(t)|^2\,dt\to 0$ *for* $r\to 0$,

(7) $(-(d^2/dr^2)+(l(l+1)/r^2)+U(r)+V(r)+\beta^2)\psi(r)=0$ *for* $r\in\mathbb{R}^+$.

(B) *Assume that* ψ *satisfies* (1)–(6) *with* $0<\beta<a$ *and define* V *by* (7). *Then* $V\in\mathcal{E}_a$, *and* ψ *is the antibound state of* H_1^l+V *at* $-i\beta$.

Proof. (A) Assume that $V \in \mathscr{E}_a$ and ψ is the antibound state of $H_1^l + V$ at $-i\beta$. Then (1), (3), (4), (7) are satisfied by Theorem 2.1. It follows from (4) that ψ has no nodes for large r, since $\tilde{u}_+^l(-ir) \simeq e^{\beta r}$ for $r \to \infty$. If $r_0 > 0$ were an accumulation point of positive nodes, we would have $\psi(r_0) = \psi'(r_0) = 0$, hence $\psi \equiv 0$, so the positive nodes are isolated. Finally $r_n \to_n 0$ $(r_n \neq r_m$ for $n \neq m)$ and $\psi(r_n) = 0$ for all n would contradict (3), and (2) is proved. From (4), (7) and $\mathscr{E}_a$(ii'), we obtain (6), and (5) follows from (1), (7), and $\mathscr{E}_a$(i).

(B) Assume that ψ satisfies (1)–(6) with $0 < \beta < a$ and define V by (7). It is easy to show that $V \in \mathscr{E}_a$, and it follows from Theorem 2.1 that ψ is an antibound state at $-i\beta$.

Remark 4.5. Theorem 2.1 and hence Theorem 4.4 are valid also for $-a < \beta < 0$, giving a characterization of bound states of $H_1^l + V$ at eigenvalues $-i\beta \in \mathscr{T}_a$.

The proof is identical with that for $0 < \beta < a$.

5. Resonances with Coulomb Potential as Background

In this section we extend the results of Section 4 to the case, where the short-range potential U is replaced by the Coulomb potential. The regular, outgoing and incoming solutions are known explicitly in terms of confluent, hypergeometric functions, and the analytically continued resolvent $\tilde{R}_c^l(k) \in \mathscr{B}(\mathscr{h}_a, \mathscr{h}_{-a}^2)$ is constructed by means of these solutions. The result differs from the short-range case because of the asymptotically logarithmic term in the Coulomb phase function and the resulting difference in the asymptotic behaviour of the amplitude and phase of the resonance function.

The unperturbed operator H_c^l is given by

$$H_c^l = -\frac{d^2}{dr^2} + \frac{2\gamma}{r} + \frac{l(l+1)}{r^2},$$

where $\gamma = z_1 z_2 e^2 \mu$, and $z_1 e$, $z_2 e$ are the charges of the two particles and μ their reduced mass.

The regular, outgoing and incoming solutions u_c^l, u_{c+}^l, and u_{c-}^l of the equation

$$\left(-\frac{d^2}{dr^2} + \frac{2\gamma}{r} + \frac{l(l+1)}{r^2} - k^2\right) u = 0 \tag{5.1}$$

are given by

$$u_c^l(k, r) = r^{l+1} e^{ikr} \Phi\left(l+1+i\frac{\gamma}{k}, 2l+2, -2ikr\right),$$

$$u_{c^+}^l(k, r) = (-2kr)^{l+1}\, ie^{i(kr-(\pi l/2))}\, e^{\pi\gamma/2k} \times \Psi\left(l+1+i\frac{\gamma}{k}, 2l+2, -2ikr\right),$$

$$u_{c^-}^l(k, r) = u_{c^+}^l(-k, r),$$

and

$$W_l(k) = W(u_c^l(k, r), u_{c^+}^l(k, r)) = -(2k)^{-l}\, e^{(\pi\gamma/2k)+(i(\pi l/2))} \frac{\Gamma(2l+2)}{\Gamma(l+1+i(\gamma/k))}, \tag{5.2}$$

where ϕ and ψ are the regular and irregular confluent hypergeometric functions (cf. [4, 6]).

The function $u_c^l(k, r)$ is analytic in k for $k \neq 0$ with a simple pole at 0 and entire in r, whereas $\psi(a, c; x)$ is a multi-valued function of x with a logarithmic singularity at 0. Hence for fixed $r > 0$ $u_{c^+}^l$ continues analytically from $\mathbb{C}^+$ into the 4th quadrant. The asymptotic behaviour in r for fixed $k \neq 0$ is given (cf. [4, 6]) by

$$u_c^l(k, r) \simeq 2(2k)^{-l-1}\, e^{\gamma\pi/2k} \frac{\Gamma(2l+2)}{|\Gamma(l+1+i(\gamma/k))|} \times \sin\left(kr - \frac{\gamma}{k}\log 2kr - \frac{\pi l}{2} + \eta_l\right) \qquad \text{for} \quad r \to \infty,$$

where

$$\eta_l = \operatorname{Arg} \Gamma(l+1+i(\gamma/k)), \tag{5.3}$$

$$u_{c\pm}^l(k, r) \simeq e^{\pm i(kr-(\gamma/k)\log(\pm 2kr))} \qquad \text{for} \quad r \to \infty, \tag{5.4}$$

$$u_c^l(k, r) \simeq r^{l+1} \qquad \text{for} \quad r \to 0, \tag{5.5}$$

$$u_{c^+}^l(k, r) \simeq Cr^{-l} \qquad \text{for} \quad r \to 0. \tag{5.6}$$

We have the following estimates for fixed $k \neq 0$, $k = \alpha - i\beta$,

$$|u_{c^+}^l(k, r)| \leqslant Ce^{\beta r} r^{\beta\gamma/(\alpha^2+\beta^2)}, \tag{5.7}$$

$$|u_{c^-}^l(k, r)| \leqslant Ce^{-\beta r} r^{\beta\gamma/(\alpha^2+\beta^2)}. \tag{5.8}$$

Due to the estimate (5.7) we can continue analytically $R_c^{l,a}(k) \in \mathscr{B}(\mathscr{h}_a, \mathscr{h}_{-a}^2)$ from $\mathbb{C}^+$ to $\mathbb{R}^+ \cup \{k \mid -a < \operatorname{Im} k < 0\}$. Explicitly, we have

LEMMA 5.1. *$R_c^{l,a}(k)$ has an analytic continuation $\tilde{R}_c^l(k)$ from $\mathbb{C}^+$ to $\mathbb{R}^+ \cup \{k \mid -a < Im\, k < 0\}$, given by*

$$
\begin{aligned}
(\tilde{R}_c^l(k)\, v)(r) = &-\frac{1}{W_l(k)} u_{c+}^l(k, r) \int_0^\infty u_c^l(k, t)\, v(t)\, dt \\
&+\frac{1}{2ik} u_{c+}^l(k, r) \int_r^\infty u_{c-}^l(k, t)\, v(t)\, dt \\
&-\frac{1}{2ik} u_{c-}^l(k, r) \int_r^\infty u_{c+}^l(k, t)\, v(t)\, dt. \qquad (5.9)
\end{aligned}
$$

Here the function $W_l(k)$, given by (5.2), has zeros at $k = -i\gamma(p+l)^{-1}$, $p \in \mathbb{N}$, corresponding to the bound states of the hydrogen atom if $\gamma < 0$ and the antibound states if $\gamma > 0$. Thus, $\tilde{R}_c^l(k)$ has a logarithmic branch point at 0 and poles at the points $\{-i\gamma(p+l)^{-1} \mid p \in \mathbb{N}\} \cap \mathbb{C}_a$.

Based on Lemma 5.1 and the asymptotic expressions (5.3)–(5.6) we can now extend the analysis of resonance functions to the pair of operators $(H_c^l, H_c^l + V)$, where $V \in \mathscr{E}_a$.

We notice that Theorem 2.2, which is proved for short range potentials, also holds for $(\gamma/r) + V$ due to the asymptotic estimates (5.5), (5.6) for the Coulomb wave functions.

By Lemma 5.1, for $V \in \mathscr{E}_a$ the $\mathscr{C}(\mathscr{h}_a)$-valued function $VR_c^{l,a}(k)$ has the analytic continuation $V\tilde{R}_c^l(k)$ to $\mathbb{R}^+ \cup \{k \mid -a < \operatorname{Im} k < 0\}$ with poles at the points $\{k = -i\gamma(p+l)^{-1} \mid p \in \mathbb{N}\}$. It follows that $(1 + V\tilde{R}_c^l(k))^{-1}$ has a meromorphic continuation to this region. The set of poles of $(1 + V\tilde{R}_c^l(k))^{-1}$ is denoted by Σ_{cl}. It is divided as usual into resonances, antibound states and bound states of $H_c^l + V$. From Lemma 5.1, (5.7), and (5.8) we obtain the following characterization of resonances and virtual poles of $H_c^l + V$. We omit the proof, which is similar to the previous cases.

THEOREM 5.2. *Let $z = \alpha - i\beta$, $\alpha \in \mathbb{R}$, $0 < \beta < a$, $z \neq -i\gamma(p+l)^{-1}$, $p \in \mathbb{N}$. Then $z \in \Sigma_{cl}$ if and only if there exists a function ψ on $\mathbb{R}^+$ satisfying conditions* (1) *and* (2) *of Theorem* 2.1 *with V replaced by $(2\gamma/r) + V$ as well as*

$$
(3') \qquad \psi(r) = u_{c+}^l(r) + o(e^{(\beta - 2a)r} r^{\beta\gamma/(\alpha^2+\beta^2)}), \textit{for} \quad r \to \infty.
$$

ψ is uniquely determined by (1), (2), *and* (3′) *and is given by*

$$
\psi = \tilde{R}_c^l(z)\, \Phi, \qquad \Phi = -V\psi,
$$

where

$$-\frac{1}{W_l(z)}\int_0^\infty u_c^l(z,t)\,\Phi(t)\,dt=1.$$

From Lemma 5.2 *we obtain the following characterization of resonance functions of* (H_c^l, H_c^l+V).

THEOREM 5.3. (A) *Assume that* $V\in\mathscr{E}_a$, *and let* $z=\alpha-i\beta$, $\alpha>0$, $0<\beta<\alpha$, *be a resonance of* (H_c^l, H_c^l+V) *with resonance function* $\Psi=fe^{i\varphi}$, *and set* $u_{c+}^l(zr)=f_c^l e^{i\varphi_c^l}$ *with the usual normalization of phases. Then the pair* (f,φ) *satisfies conditions* (1)–(14) *of Theorem* 4.1 *with* (f_l,φ_l) *replaced by* (f_c^l,φ_c^l), *u replaced by* $2\gamma/r$ *in* (13) *and* $e^{(\beta-2a)r}$ *replaced by* $e^{(\beta-2a)r}r^{\beta\gamma/(\alpha^2+\beta^2)}$ *in* (3) *and* (5).

(B) *Assume that f satisfies* (1)–(6) *of Theorem* 3.3 *with this modification, where* $\alpha>0$, $0<\beta<a$. *Define* φ *by* (11) *and V by*

$$V=\frac{f''}{f}-\frac{l(l+1)}{r^2}-\frac{2\gamma}{r}-\varphi'^2+E.$$

Then $V\in\mathscr{E}_a$, *and* $z=\alpha-i\beta$ *is a resonance of* (H_c^l, H_c^l+V) *with resonance function* $\psi=fe^{i\varphi}$.

The characterization of antibound states and bound states is obtained as before. Thus, Theorem 4.4 holds with H_1^l replaced by H_c^l for $\beta\neq\gamma(p+l)^{-1}$ if $\gamma>0$, $\tilde{u}_+^l$ replaced by u_{c+}^l, u by $(2\gamma/r)$ in (7) and $e^{(\beta-2a)r}$ by $e^{(\beta-2a)r}r^{\beta\gamma/(\alpha^2+\beta^2)}$ in (4) and (6).

Remark 4.5 extends in the same way for $-a<\beta<0$, $\beta\neq\gamma(p+l)^{-1}$ if $\gamma<0$.

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