CONTINUOUS BRANCHING PROCESSES AND SPECTRAL POSITIVITY *

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Certain properties of continuous-state branching processes are studied via the random time-change linking them with spectrally positive Lévy processes. The results are compared and contrasted with those for simple branching processes.

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0. Introduction

We consider the random time-change connecting continuous-time continuous-state branching processes (CB-processes) with spectrally positive Lévy processes. As consequences, we obtain a number of new results on CB-processes. One of our aims is to compare and contrast the situation with that obtaining for simple branching processes. Most of the results for CB-processes are similar to those for simple branching processes, suitably reformulated if necessary. Several of our results for CB-processes are much more explicit than their analogues for simple branching processes. In other cases, problems can be solved completely for CB-processes which remain unanswered for simple branching processes. This greater tractability of CB-processes arises because the state-space is smooth rather than discrete, and the distributions which appear are infinitely divisible.

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In Section 1 we discuss the properties of the random time-change which we shall need. In Section 2 we deal with extinction probabilities, and apply fluctuation theory to study the distribution of the total progeny of a CB-process. The distribution of the supremum over all time of a CB-process is obtained in Section 3, again using fluctuation theory. In Section 4 we discuss Grey’s martingale convergence theorem [13], and in Theorem 4.2 give our extension of the Harris–Sevastyanov theorem from Markov branching processes to general CB-processes. In Section 5 (Theorems 5.1–5.6) we describe completely the behaviour of the left tail of the random variable \( W \) of Section 4; the right tail is discussed in Section 6 (Theorems 6.1, 6.2). We close with a weak-convergence result for the critical case in Section 7 (Theorem 7.1).

1. The random time-change

The class of CB-processes \( Z = \{ Z(t): t > 0 \} \) was introduced by Jirina [19] and studied by Lamperti [24] and Siverstein [32]. To avoid trivialities, we exclude the degenerate case \( Z(t) = 0 \) for all \( t > 0 \), a.s.; then \( Z \) is stochastically continuous [26], and one may take \( Z \) to be a Hunt process [23]. Then (by [24]; see also [32]) each such \( Z \) may be obtained from a spectrally positive Lévy process (process with stationary independent increments whose Lévy measure is concentrated on \([0, \infty)\)) by a random time-change in the following way. Let \( Y = \{ Y(t): t > 0 \} \) be a spectrally positive Lévy process with exponent given by

\[
(1.1) \quad \mathcal{E} \exp \{-s[Y(t) - Y(0)]\} = \exp \{-t \Psi(s)\}, \quad s, t > 0,
\]

where

\[
(1.2) \quad \Psi(s) = as + b - \frac{1}{2} \sigma^2 s^2 + \int_{(0,1]} (1 - e^{-sx} - sx) \Pi(dx)
+ \int_{(1,\infty)} (1 - e^{-sx}) \Pi(dx)
\]

for \( \Re s > 0 \); here \( \Pi(1, \infty) \) and \( \int_{(0,1]} x^2 \Pi(dx) \) are finite, \( a \) is real, \( b, \sigma > 0 \) (see, e.g. [3]). Take \( Y(0) = x > 0 \), and write

\[
I(t) = \int_0^t \frac{du}{V(u)}
\]
(for \( t < T_x = \inf\{u > 0 : Y(u) = 0\} \), \( I(t) = \infty \) for \( t \geq T_x \)),

\[
J(t) = \inf\{u : I(u) > t\}
\]

(\( J(t) = \infty \) if \( I(u) \leq t \) for all \( u \)). Then if

(1.3) \( Z(t) = Y(J(t)) \),

\[ Z = \{Z(t) : t \geq 0\} \] is a CB-process with \( Z(0) = x \).

Conversely, if \( Z \) is such a process,

\[
J(t) = \int_0^t Z(u) \, du ,
\]

\[
I(t) = \inf\{u : J(u) > t\} ,
\]

(\( I(t) = \infty \) if \( J(u) \leq t \) for all \( t \)), then

(1.4) \( Y(t) = Z(I(t)) \)

defines a spectrally positive Lévy process with \( Y(0) = x > 0, \) stopped on first hitting zero (note that since \( Y \) is spectrally positive, first passage through levels \( y < x \) takes place continuously). We restrict our attention to conservative CB-process (with \( Z(t) < \infty \) for all \( t \), a.s.); these are the ones with \( b = 0 \) and

\[
\int_0^\infty \frac{du}{\Psi(u)} = \infty
\]

(Grey [13]).

The exponent \( \Psi(s) \) of \( Y \) (which we also call the exponent of \( Z \)) describes the infinitesimal generator of the semigroup \( T \) of \( Z \):

\[
(T_t f)(x) = \mathcal{E}_x f(Z(t))
\]

(see, for example, [23]). We shall usually regard \( Y, Z \) as being specified by \( \Psi \).

We shall further restrict attention to processes with \( \int_1^\infty x \Pi(dx) \) finite. Then

(1.5) \( m = \Psi'(0+) = a + \int_1^\infty x \Pi(dx) < \infty \);

\( Z \) is called super-critical if \( m > 0 \), critical if \( m = 0 \), sub-critical if \( m < 0 \).

In contrast to the situation for simple branching processes, critical and
sub-critical CB-processes need not have extinction probability one, and may have extinction probability zero. For example, the deterministic case

\[ Y(t) = 1 - t, \quad Z(t) = e^{-t}, \quad t \geq 0, \]

gives a sub-critical CB-process with zero extinction probability. This is fairly typical behaviour: a large class of subcritical CB-processes possess non-degenerate martingale limits [13]. For convenience, we assume from now on that \( Y, Z \) are non-deterministic unless otherwise stated.

We shall also need to classify CB-processes \( Z \) according as to whether or not the process \( Y \) has a.s. non-decreasing sample-paths (i.e., is a subordinator). If \( Y \) is a subordinator, \( \int_{1}^{t} X \Pi(dx) \) converges, and \( \Psi \) may be rewritten

\[
(1.6) \quad \Psi(s) = cs + \int_{0}^{\infty} (1 - e^{-xs}) \Pi(dx)
\]

\[
= cs + s \int_{0}^{\infty} e^{-xs} \Pi(x; \infty) \, dx, \quad s \geq 0;
\]

here \( c \) is the drift of \( Y \). If \( Y \) is not a subordinator, \( Y(t) - Y(0) \) is negative for each \( t > 0 \) with positive probability. Letting \( s \to \infty \) in (1.1),

\[
\exp\{-t\Psi(s)\} \sim \mathcal{E} \exp\{-s[Y(t) - Y(0)]; Y(t) - Y(0) < 0\}
\]

\((s \to \infty)\)

and thus

\[
(1.7) \quad \Psi(s) \to -\infty, \quad s \to \infty
\]

(in contrast to the situation in (1.6)).

Since

\[
\Psi''(s) = -\int_{0}^{\infty} e^{-xs} x^2 \Pi(dx) \leq 0,
\]

\( \Psi \) is concave, and intersects each line at most twice. If \( Y \) is a subordinator, \( \Psi(s) > 0 \) for all \( s > 0 \). If \( Y \) is supercritical and not a subordinator, \( \Psi'(0) = m > 0, \Psi(\infty) = -\infty \) and there exists a unique positive \( \gamma \) with \( \Psi(\gamma) = 0 \); if \( Y \) is critical or sub-critical (in which case \( Y \) cannot be a subordinator), \( \Psi(s) < 0 \) for all \( s > 0 \). We then write \( \gamma = 0 \); thus \( \gamma \) is the largest zero of \( \Psi(s) \) in the non-subordinator case.
Write
\[ \mathcal{E}_x \exp\{-sZ(t)\} = \exp\{-x\psi(t, s)\}, \quad x > 0, \ s, t \geq 0; \]
then the branching property of \( Z \) is expressed by
\[ (1.8) \quad \psi(t, \psi(u, s)) = \psi(t + u, s). \]
Writing \( \psi_1, \psi_2 \) for the partial derivatives of \( \psi \), one has [24, 13]
\[ (1.9) \quad \Psi(s) = \psi_1(0, s), \]
\[ (1.10) \quad \Psi(\psi(t, s)) = \psi_1(t, s), \]
\[ (1.11) \quad \int_s^t \frac{du}{\Psi(u)} = t. \]

We recall [26] that the CB-processes are exactly those which can be obtained from a sequence of simple branching processes \( \{Z_r^{(n)}\} \) as limits of scaled processes \( \{Z_t^{(n)} / b_n \} : t \geq 0 \ | \ Z_0 = cb_n \} \) (in the sense of convergence of finite-dimensional distributions). We shall see in the sequel to what extent properties of branching processes are sensitive to the passage from discrete to continuous state-space thus brought about.

2. Extinction probabilities and total progeny

Let \( q_t = P\{Z(t) = 0\} \) be the probability that extinction occurs by time \( t \) (this can be positive only in the non-subordinator case when \( \Psi(s) < 0 \) for large \( s \)), \( q = \lim_{t \to \infty} q_t = P\{Z(t) = 0 \text{ for some } t \geq 0\} \) be the probability that \( Z \) becomes extinct. We say that extinction is possible iff \( q > 0 \) (equivalently, iff \( q_t > 0 \) for some \( t \)). The following result is proved in [13] (cf. also [18, p. 107].

Proposition 2.1. Extinction is possible iff
\[ (2.1) \quad \int_0^\infty \frac{du}{[-\Psi(u)]} < \infty. \]
When (2.1) holds,
\[ (2.2) \quad q = e^{-\gamma}. \]
Proof. We have \( q_t = \exp\{-\psi(t, \infty)\} \) (where \( \psi(t, \infty) = \lim_{s \to \infty} \psi(t, s) \)); thus extinction is possible iff \( \psi(t, \infty) \) is finite for some \( t \). Let \( s \to \infty \) in (1.11). If \( \psi(t, \infty) < \infty \), (2.1) holds (or the limit of the left-hand side of (1.11) would be infinite); if (2.1) holds \( \psi(t, \infty) < \infty \) (or the limit of the left-hand side of (1.11) would be zero). This proves the first part. Assume now that (2.1) holds; then we can write

\[
\int_\infty^\infty \frac{du}{\psi(t, s)} = t .
\]

Let \( t \to \infty \); \( q_t \) increases to \( q \), and \( \psi(t, \infty) \) decreases to a limit (\( c \), say) with \( q = e^{-c} \). Then

\[
\int_c^\infty \frac{du}{\psi(u)} = \infty, \quad \int_x^\infty \frac{du}{\psi(u)} < \infty, \quad x > c .
\]

Since \( \gamma > 0 \) is the largest zero of \( \Psi(s) \) and \( \Psi'(\gamma) < 0 \) (\( \Psi \) being properly concave when \( Y \) is non-deterministic), one has \( c = \gamma \), which completes the proof. \( \square \)

We note [13].

**Proposition 2.2.** \( \epsilon_x Z(t) = x e^{mt} \).

**Proof.** Differentiating (1.9) w.r.t. \( s \) and putting \( s = 0 \),

\[
\epsilon_x Z(t) = x \psi_2(t, 0) .
\]

Differentiating (1.8) w.r.t. \( s \) and putting \( s = 0 \),

\[
\psi_2(t, 0) \psi_2(u, 0) = \psi_2(t + u, 0) .
\]

So by Abel's equation, \( \psi_2(t, 0) = e^{ct} \) for some \( c \). Differentiating (1.10) w.r.t. \( s \) and putting \( s = 0 \),

\[
\psi_{21}(t, 0) = c e^{ct} = \Psi'\left(\psi_2(t, 0)\right) \psi_2(t, 0) = \psi'(0) e^{ct} = m e^{ct} ;
\]

thus \( c = m \) and the result follows. \( \square \)

The integral \( J(t) = \int_0^t Z(u) \, du \) is the CB-analogue of the total number of particles (individuals) in the first \( n \) generations of a simple branching process, and \( J(\infty) \) is the CB-analogue of the total number of particles occurring (the total progeny). In the subordinator case, \( Z(t) \geq x > 0 \) for
all \( t \), so \( J(\infty) = \infty \). In the non-subordinator case, one has \( J(\infty) \leq y \) iff \( I(y) = \infty \) iff \( y \geq T_x \), and so \( J(\infty) = T_x \), the first-passage time of the spectrally positive process \( Y \) from \( x > 0 \) to zero (or the spectrally negative process \( Y(0) - Y(t) \) from 0 to \( x > 0 \)). From results on the drift and oscillation of Lévy processes (Rogozin [28]), \( T_x = J(\infty) < \infty \) a.s. if \( m < 0 \) (\( Y \) oscillates or drifts to \(-\infty\)), while \( \mathbb{P}(T_x = J(\infty) = \infty) > 0 \) if \( m > 0 \) (\( Y \) drifts to \(+\infty\)). In either case, the (possibly defective) distribution of \( J(\infty) \) may be written down, since this is the first-passage time distribution for a spectrally negative process across a positive level, and as such is well-known [37, 3]. Since \(-\Psi(s)\) is positive, continuous and strictly increasing for \( s > \gamma > 0 \) (with \(-\Psi(s) \to \infty \) as \( s \to \infty \)), it possesses a continuous strictly increasing inverse function \( \eta(s) \) (with \( \eta(s) \to \infty \) as \( s \to \infty \)), and \( \eta \) is the Lévy exponent of the first-passage process of \( Y(0) - Y(t) \). This gives:

**Proposition 2.3.** The distribution of the total progeny

\[
J(\infty) = \int_0^\infty Z(u) \, du
\]

of a CB-process with \( Z(0) = x > 0 \) is given by

\[
\mathcal{E}_x \exp\{-sJ(\infty)\} = \exp\{-x\eta(s)\}.
\]

Then \( J(\infty) < \infty \) a.s. if \( m < 0 \), while if \( m > 0 \) (and so \( \gamma > 0 \)),

\[
\mathbb{P}_x \{J(\infty) < \infty\} = \exp\{-x\gamma\}.
\]

The analogue of Proposition 2.2 for simple branching processes is, of course, well-known (Feller [10, XII.5]).

**Remark 2.4.** For a simple branching process, the total progeny is finite iff extinction occurs. In view of this, one may regard Proposition 2.3 as the correct CB-analogue of the familiar result that for simple branching processes extinction is certain if the process is critical or sub-critical and not if it is super-critical.

3. The supremum

We turn now to the distribution of \( \sup\{Z(t) : t \geq 0\} \). This is infinite a.s. in the subordinator case. In the non-subordinator case, the function
$s/[-\Psi(s)]$ is completely monotone to the right of the largest zero $\gamma$ of $\Psi(s)$, and so

$$\int_{0}^{\infty} e^{-sx} W(dx) = s/[-\Psi(s)], \quad s > \gamma,$$

for some measure $W$ [34: 87, 3].

**Proposition 3.1.** If $Y$ is not a subordinator, then

$$P_x \{Z(t) < x + y \text{ for all } t \geq 0\} = W(y)/W(x + y), \quad y \geq 0.$$

**Proof.** By (1.3) and (1.4), the range of $Z$ coincides with the range of $Y$ before it is stopped on hitting zero. So we require the probability that the spectrally positive process $Y$ with $Y(0) = x > 0$ stays below the level $x + y$ before it hits the level 0; that this is $W(y)/W(x + y)$ follows from a result of Takács ([34: 87]; cf. [3, §6]). □

**Corollary 3.2.** $W(y)/W(x + y) \rightarrow P_x \{\sup_{[0, \infty)} Z(\cdot) < \infty\} = e^{-x\gamma}, \quad y \rightarrow \infty$. So $W(\log x)$ varies regularly at infinity with exponent $\gamma$.

**Proof.** One has

$$\sup Z(\cdot) = \sup_{[0, \infty)} Y(\cdot) = \sup_{[0, T_x)} Y(\cdot)$$

as $J(\infty) = T_x$. But the sample-paths of a Lévy process are a.s. locally bounded. So a.s., $\sup_{[0, \infty)} Z(\cdot) < \infty$ iff $T_x < \infty$, and since this happens with probability $e^{-x\gamma}$ ([34: 87, 3]), the result follows. □

The measure $W$ is thus exponentially large if $\gamma > 0$ ($m > 0$). Since $s/[-\Psi(s)] \rightarrow -1/\Psi'(0) = -1/m$ as $s \rightarrow 0$, $W$ is bounded iff $m < 0$, in which case $-mW$ is a probability measure (and is familiar from, for example, the Pollaczek–Hinčin formula).

The random time-change thus reduces the problem of finding the distribution of $\sup\{Z(t) : t \geq 0\}$ to a standard result in fluctuation theory. By contrast, it is interesting to note that the problem of finding the distribution of $\sup\{Z_n : n \geq 0\}$ for a simple branching process remains open.
4. The martingale convergence theorem

We turn now to the super-critical case. The Martingale Convergence Theorem for simple branching processes (see, e.g. [1]) possesses a complete analogue for CB-processes, which was recently proved by Grey [13]; we summarise Grey's results in Proposition 4.1 below. First note that for fixed \( t > 0 \), \( \psi(t, s) \) increases from 0 to \( \psi(t, \infty) \), and possesses an inverse function \( \eta(t, s) \) defined on \([0, \psi(t, \infty))\) and increasing from 0 to \( \infty \).

**Proposition 4.1.** Choose and fix \( p \in (0, \gamma) \). Then \( W_t = \eta(t, \rho) Z(t) \), \( W_t = \exp(-W_t) \) is a martingale, and converges a.s. as \( t \to \infty \) to a non-degenerate limiting random variable \( w = \exp(-W) \) with \( P\{W = 0\} = e^{-\gamma} \). For fixed \( u > 0 \),

\[
\eta(t + u, \rho) / \eta(t, \rho) \to e^{-mu}, \quad t \to \infty.
\]

One has \( \eta(t, \rho) \sim c e^{-mt} \) as \( t \to \infty \) for some constant \( c \) iff

\[
\int_1^\infty x \log x \Pi(dx) < \infty
\]

or equivalently,

\[
\int_0^1 \left[ \frac{m}{\psi(s)} - \frac{1}{s} \right] ds < \infty.
\]

We introduce some notation. Since each \( Z(t), W_t \) is infinitely divisible, so is \( W \). Write the Laplace–Stieltjes transform of \( W \) as

\[
\mathcal{E} \exp\{-sW\} = \int_0^\infty e^{-sx} dH(x) = \phi(s) = \exp\{-\Phi(s)\}.
\]

Then \( \Phi \) is the exponent of some subordinator. If this has drift \( c \geq 0 \) and Lévy measure \( \mu \),

\[
\Phi(s) = cs + \int_0^\infty (1 - e^{-xs}) \mu(dx).
\]

Then \( \Phi \) is strictly increasing and concave; let \( \Theta(s) \) denote its inverse function.

We next show that one can find \( \Theta \), and hence \( \Phi \), explicitly in terms of \( \psi \).
Theorem 4.2. $\Theta'(s)/\Theta(s) = m/\Psi(s)$, $0 \leq s < \gamma$. If (4.2) (or (4.2a)) holds,

$$\Theta(s) = s \exp \left( \int_0^s \left[ \frac{m}{\Psi(u)} - 1 \right] du \right), \quad 0 \leq s < \gamma.$$ 

Corollary 4.3. $\Psi(\Phi(s)) = ms\Phi'(s)$.

Proof. We first prove the corollary and then deduce the theorem. Differentiating (1.8) w.r.t. $s$ and putting $t = 0$, we obtain

(4.4) $\psi_2(0, s) = 1$.

Write $e^{s\exp{-sW_t}} = \exp{-\Phi_t(s)}$. Then

$$\Phi_t(s) = -\log e^{\exp{-s\eta(t, \rho) Z(t)}}$$

$$= \psi(t, s\eta(t, \rho)) \to \Phi(s) \quad (t \to \infty).$$

Thus

$$\Phi_{t+u}(s) = \psi(t+u, s\eta(t+u, \rho))$$

$$= \psi(u, \psi(t, s\eta(t, \rho) \cdot \eta(t+u, \rho)/\eta(t, \rho)))$$

$$\to \psi(u, \Phi(s e^{-mu})) \quad (t \to \infty)$$

by Proposition 4.1. Let $t \to \infty$:

$$\Phi(s) = \psi(u, \Phi(s e^{-mu})).$$

Differentiate w.r.t. $u$ and put $u = 0$:

$$0 = \psi_1(0, \Phi(s)) - ms\Phi'(s) \psi_2(0, \Phi(s)).$$

By (1.9), (4.4), this gives

$$\Psi(\Phi(s)) = ms\Phi'(s),$$

proving the corollary. Now write

$$\Theta(s) = u, \quad s = \Phi(u).$$

Then $ds/du = \Phi'(u)$. Differentiating logarithmically,

$$\Theta'(s)/\Theta(s) = u^{-1} du/ds = 1/\left[ u ds/du \right],$$

(4.5) $\Theta(s)/\Theta'(s) = u ds/du = u\Phi'(s)$. 


That is,
\[ m\Theta(s)/\Theta'(s) = m\upsilon\Theta'(\upsilon) = \Psi(\Phi(\upsilon)) = \Psi(s) , \]
or
\[ \Psi(s) = m\Theta(s)/\Theta'(s) \]
which proves the first part of the theorem. By Proposition 4.1, (4.2) is equivalent to (4.2a); when either holds we may integrate to obtain the second part of the theorem; this completes the proof. □

Corollary 4.3 (and thus, essentially, Theorem 4.2) is due to Grey [13, Th.3].

Specialising Theorem 4.2 to the case of a (discrete state-space) Markov branching process with generating function \( h(s) = \sum h_n s^n \), \( h'(1) = \mu > 1 \) and rate \( \lambda \), we obtain:

**Proposition 4.4.** Write \( \theta(s) = \phi^{-1}(s) \); then
\[ \theta'(s)/\theta(s) = (\mu - 1)/(h(s) - s) \] , \( q < s \leq 1 \).

If \( \sum n \log nh_n \) converges, this may be integrated to give
\[ \theta(s) = \phi^{-1}(s) = (1 - s) \exp \left( \int_1^s \left[ \frac{\mu - 1}{h(x) - 1} + \frac{1}{1-x} \right] dx \right) \]
\[ q < s \leq 1 . \]

**Proof.** One has [1, III]
\[ \Psi(s) = \lambda \left[ 1 - e^{\mu h(e^{-s})} \right] . \]
Differentiating and putting \( s = 0, m = \lambda(\mu - 1) \). By Corollary 4.3,
\[ \Psi(\Phi(s)) = \Psi(- \log \phi(s)) = \lambda \left[ 1 - \frac{h(\phi(s))}{\phi(s)} \right] \]
\[ = ms\Phi'(s) = -ms\phi'(s)/\phi(s) , \]
or
\[ \lambda[\phi(s) - h(\phi(s))] = -ms\phi'(s) . \]
Put \( \phi(s) = u, s = \theta(u) \). Then \( \phi'(s) = du/ds = 1/\theta'(u) \). So
\[ \lambda(\mu - h(u)) = -\lambda(\mu - 1) \theta(u)/\theta'(u) \]
and the result follows. □
Proposition 4.4 was proved (essentially by this method) by Harris [17] (see also [31]).

One may use Proposition 4.4 to show that $\phi(s)$ satisfies the following functional equation:

**Proposition 4.5.** $\phi(s) = \int_0^\infty \lambda e^{-\lambda x} h(\phi(s e^{-\lambda(\mu-1)x})) \, dx$.

This is the specialisation to the Markov branching process of Doney's functional equation [7] for Crump–Mode (and in particular, Bellman–Harris) processes. In principle $\phi$ is uniquely determined by the functional equation (which, however, can hardly ever be solved).

The mean of $W$ is finite iff (4.2) holds; when (4.2) holds one can take $\mathcal{C} W = 1$ by suitable choice of $\rho$, and we shall do this.

**Proposition 4.6.** We always have $c = 0$ in (4.3). Then $\Phi(\infty)$ is finite iff $Y$ is not a subordinator, and then

$$\Phi(\infty) = \mu(0, \infty) = \gamma.$$

**Proof.** $\Phi'(s) = s + \int_0^\infty x e^{-xs} \mu(dx)$, $\Phi'(s) = 1/\Theta'(\Phi(s))$. So $c = \Phi'(\infty) = 1/\Theta'(\Phi(\infty))$. If $c = \Phi'(\infty) > 0$, $\Phi(\infty) = \infty$, so $\Theta'(\infty) = 1/c < \infty$ and $\Theta(s) \sim s/c$ as $s \to \infty$. Then

$$s \Theta'(s)/\Theta(s) = ms/\Psi(s) \to 1 \quad (s \to \infty),$$

or

$$\Psi(s)/s \to m \quad (s \to \infty).$$

In particular, $\Psi(s) \to \infty$ as $s \to \infty$ and so $Y$ is a subordinator (or $\Psi(s)$ would be negative for large $s$). So $\int_0^\infty x \Pi(dx)$ converges, and we can write $\Psi(s)$ in the form

$$\Psi(s) = bs + \int_0^\infty (1 - e^{-xs}) \Pi(dx) = bs + \int_0^\infty e^{-xs} \Pi(x, \infty) \, dx.$$

Then $\Psi(s)/s \to b$ as $s \to \infty$, and so $b = m$. But then

$$m = \Psi'(0+) = b + \int_0^\infty x \Pi(dx) = m + \int_0^\infty x \Pi(dx).$$

Hence $\Pi$ must vanish, and so $Y$ is deterministic, which we have excluded. The contradiction shows that $c = 0$. 


We can now write

\[ \Phi(s) = \int_0^\infty (1 - e^{-xs}) \mu(dx). \]

So

\[ s\Phi'(s) = \int_0^\infty xs e^{-xs} \mu(dx). \]

If \( \Phi(\infty) = \mu(0, \infty) < \infty \), since \( xe^{-x} \) is bounded for \( x \geq 0 \) we can use dominated convergence to show that

\[ s\Phi'(s) \to 0, \quad s \to \infty. \]

Letting \( s \to \infty \) in \( \Psi(\Phi(s)) = ms\Phi'(s) \), we see that \( \Psi \) must have a positive zero, and so \( Y \) is not a subordinator. If \( \gamma \) is the positive zero of \( \Psi \), one then has \( \gamma = \Phi(\infty) = \mu(0, \infty) < \infty \).

If on the other hand \( \Phi(\infty) = \infty \), we see from \( \Psi(\Phi(s)) = ms\Phi'(s) \) and the fact that \( \Phi'(s) > 0 \) for all \( s \geq 0 \) that \( \Psi(s) \geq 0 \) for all \( s \geq 0 \). Thus \( Y \) is a subordinator; this completes the proof. \( \square \)

In view of Proposition 4.6 it is convenient to extend the definition of \( \gamma \) by writing \( \gamma = \infty \) in the subordinator case.

We give an important class of examples. If \( 0 < p \leq 1 \), let \( Y \) be a spectrally positive stable process of index \( 1 + p \) and \( \zeta \) \( Y(1) = m > 0 \); then \( \Psi \) is of the form

\[ \Psi(s) = ms - ms^{1+p}/c^p \]

for some \( c > 0 \). One verifies that

\[ \psi(t,s) = se^{mt}/[1 + (s/c)^p(e^{mpt} - 1)]^{1/p}, \]

\[ \Phi(s) = cs/(c^p + s^p)^{1/p}. \]

This class of examples was considered by Lamperti [24].

Notes. (1) In principle, Theorem 4.2 gives explicitly the class of possible limit distributions for \( W \) which can arise. The distribution function \( H \) of \( W \) which we seek is specified by its Lévy exponent \( \Phi \) after a Laplace transform. In order to exploit the infinite divisibility, we usually regard the distribution of \( W \) as being given by \( \Phi \), just as we usually regard the distribution of \( Y, Z \) as being given by \( \Psi \). Then we can pass from \( \Phi \) to \( \Theta \) by a functional inversion, and from \( \Theta \) to \( \Psi \) by a logarithmic differentiation, as in Theorem 4.2.
In no real sense, however, does this provide us with an explicit intrinsic description of the class of possible distributions of $H$ of $W$. Equally, it can hardly be regarded as enabling us to give the Lévy measure $\mu$ of $W$ explicitly in terms of the Lévy measure $\Pi$ of $Y, Z$.

It is interesting to compare this situation with that for simple branching processes. Here the determination of the class of distributions of $W$ is a major unsolved problem on which little progress has been made since Harris' original paper [16], although some interesting work has since been done in this area. All we know is that

(i) complete information on the distribution of $W$ is in principle contained in the functional equation for its Laplace transform (which, however, can hardly ever be solved); see [1]. The situation for CB-processes is similar; see Proposition 4.5.

(ii) the distribution of $W$ is absolutely continuous on $(0, \infty)$ with a continuous positive density satisfying a Lipschitz condition (see [1]; for the local limit theorem, see [91]). For simple branching processes, however, we have no criteria for deciding which densities can arise which are in any sense explicit even in principle. Thus the situation for simple branching processes is even less satisfactory than that for CB-processes, and Theorem 4.2 gains in interest in view of this.

(2) For Markov branching processes there is an alternative approach to Proposition 4.4 ([22]; see also [1, III.8]). Here one shows that

\begin{equation}
 f_t(s) = \mathcal{E} s^{Z(t)} = B(e^{-\alpha t} A(s))
\end{equation}

where if $q$ is the extinction probability [1, II.4], $\alpha = -u'(q)/\lambda$,

\begin{equation}
 [f_n(s) - q]^{1/[f_1(q)]^n} \to A(s) \quad (n \to \infty)
\end{equation}

and $B$ is the inverse function of $A$. This splits the dependence on $s$ and $t$ of $f_t(s)$ (and thus plays a role analogous to that of the spectral representations of Karlin–McGregor [21]). One then obtains

\begin{equation}
 \phi(s) = B(c/s^\alpha), \quad \phi^{-1}(s) = (c/A(s))^{1/\alpha}
\end{equation}

for some constant $c$. The proof is concluded by noting that

\begin{equation}
 u(s) = u'(q) A(s)/A'(s).
\end{equation}

Thus (4.6) is substantially equivalent to Proposition 4.4. This approach has been further exploited in unpublished work of Karlin [20], who obtained a local limit theorem and further properties of the density.
w(x) of W: for example, \( w(x^{1/\alpha})/x^{1/\alpha} \) extends to an entire function of order \( 1/\alpha \). Analogues of (4.6)–(4.9) hold more generally for CB-processes.

5. Left tail behaviour of \( W \)

We consider now the behaviour of the left tail of the random variable \( W \) in the supercritical case. Let \( W \) have distribution function \( H \) and Laplace–Stieltjes transform \( h(s) \).

**Theorem 5.1.** If \( Y \) is not a subordinator,
\[
\mathbb{P}\{W = 0\} = q = e^{-\gamma}.
\]
Write \( a = [-\Psi'(\gamma)]/m > 0 \); then
\[
\mathbb{P}\{0 < W \leq x\} \sim Cx^a \quad (x \to 0)
\]
for some positive constant \( C \), and
\[
\mu(x, \infty) \sim Cx^a/\gamma e^{-\gamma} \quad (x \to 0).
\]

**Proof.** We shall say that a function is \( \rho \)-varying (at infinity or zero) if it is regularly varying in Karamata's sense with index \( \rho \).

By (2.2) and Proposition 4.6,
\[
\mathbb{P}\{W = 0\} = H(\{0\}) = q = e^{-\gamma}, \quad \gamma = \Phi(\infty) = \mu(0, \infty).
\]

So
\[
\int_0^\infty e^{-sx} \, dH(x) = c^{-\gamma} + \int_0^\infty e^{-sx} \, dH(x) - \phi(s) = \exp\{-\Phi(s)\}.
\]

But we can write \( \mu = \gamma F \) where \( F \) is a probability distribution on \((0, \infty)\); let \( F \) have Laplace–Stieltjes transform \( \tilde{F}(s) \). Then
\[
\Phi(s) = \int_0^\infty (1 - e^{-xs}) \mu(dx) = \gamma - \gamma \tilde{F}(s).
\]

Thus
\[
\int_0^\infty e^{-sx} \, dH(x) = e^{-\gamma}(e^{\gamma \tilde{F}(s)} - 1) \sim \gamma e^{-\gamma \tilde{F}(s)} \quad (s \to \infty).
\]

Now
\[
\Theta'(s)/\Theta(s) = m/\Psi(s), \quad \Psi(\gamma - \epsilon) \sim [-\Psi'(\gamma)] \epsilon \quad (\epsilon \to 0).
\]
But
\[(5.1) \quad \Theta'(\Phi(s))/\Theta(\Phi(s)) = 1/s\Phi'(s) = m/\Psi(\Phi(s)) .\]

so
\[s\Phi'(s) = \Psi(\Phi(s))/m = \Psi(\gamma - \gamma \tilde{F}(s))/m\]

\[\sim [ -\Psi'(\gamma)/m ] \cdot \gamma \tilde{F}(s) = a\gamma \tilde{F}(s)\]
as \(s \to \infty\). Since \(\Phi'(s) = -\gamma \tilde{F}'(s)\), this gives

\[s\tilde{F}'(s)/\tilde{F}(s) \to -a \quad (s \to \infty)\]

and so \(\tilde{F}(s)\) is \(-a\)-varying at infinity. Thus by Karamata's theorem
\[P\{0 < W \leq x\}\] is \(a\)-varying at zero. It remains to show that the slowly varying functions occurring are asymptotically constant.

Write \(\Theta(\gamma - \gamma u) = G(u)\). Then since \(\Phi(s) = \gamma - \gamma \tilde{F}(s)\) and \(\Theta, \Phi\) are inverse, \(\tilde{F}, G\) are inverse. Since \(\tilde{F}\) is \(-a\)-varying at infinity \((a > 0)\), \(G\) is \(-1/a\)-varying at zero. Put \(G(s) = L(1/s)/s^{1/a}\) with \(L\) slowly varying at infinity; then differentiating logarithmically and using Theorem 4.2 we obtain

\[sL'(s)/L(s) + 1/a = \gamma m/s\Psi(\gamma - s^{-1}\gamma) .\]

Expanding the right-hand side and using \(a = -\Psi'(\gamma)/m\), we obtain

\[sL'(s)/L(s) = \gamma m\Psi''(\gamma)/2a^2s + \mathcal{O}(1/s^2) \to 0 \quad (s \to \infty) .\]

Thus the slowly varying function \(L\) may be written in the form

\[L(s) = \exp\{ \int_1^s e(u) \, du/u \}\]

where

\[e(s) \sim \gamma m\Psi''(\gamma)/2a^2s \quad (s \to \infty) .\]

Thus \(\int_1^\infty e(u) \, du/u\) converges, and \(L(s)\) converges (to \(L(\infty)\), say) as \(s \to \infty\).

So \(G(s) \sim c_1/s^{1/a} \quad (s \to 0)\), \(\tilde{F}(s) \sim c_2/s^a \quad (s \to \infty)\), and \(P\{0 < W \leq x\} \sim c_3 x^a \quad (x \to 0)\) for suitable constants \(c_i > 0\), as required. \(\square\)

In the subordinator case when extinction is impossible, three types of left tail behaviour of \(W\) can arise. We begin with

**Theorem 5.2.** If \(Y\) is a subordinator with zero drift and finite Lévy meas-
Proof. We have
\[ \Psi(s) = \int_0^\infty (1 - e^{-xs}) \Pi(dx), \quad \int_0^\infty x \Pi(dx) = m, \quad \Pi(0, \infty) = \alpha m. \]

Since \( \Psi(s) \to \alpha m \) as \( s \to \infty \),
\[ s\Theta'(s)/\Theta(s) = ms/\Psi(s) \sim s/\alpha \quad (s \to \infty). \]

But since \( \Theta, \Phi \) are inverse, \( \Theta'(\Phi(s))\Phi'(s) = 1 \). So
\[ \Phi(s)/s\Phi'(s) = \Phi(s)\Theta'(\Phi(s))/\Theta(\Phi(s)) \sim \Phi(s)/\alpha \quad (s \to \infty), \]
or
\[ s\Phi'(s) \to \alpha \quad (s \to \infty). \]

Since \( \phi(s) = \exp \{-\Phi(s)\} \), we thus have
\[ s\phi'(s)/\phi(s) = -s\Phi'(s) \to -\alpha \quad (s \to \infty). \]
So \( \phi(s) \) is \(-\alpha\)-varying at infinity, and \( \mathbb{P}\{W < x\} \) is \( \alpha \)-varying at zero, as required. In particular, we have
\[ \Phi(s) \sim \alpha \log s. \]
\[ \int_0^\infty e^{-xs} \mu(x, \infty) \, dx \sim \frac{\alpha (\log s)/s}{s \to \infty} \]
and so
\[ \mu(x, \infty) \sim \alpha \log \left( \frac{1}{x} \right) \quad (x \to 0). \]

Now write \( \phi(s) = L(s)/s^\alpha \). Then
\[
\Phi(s) = -\log \phi(s) = \alpha \log s - \log L(s),
\]
\[ s\Phi'(s) - \alpha = -sL'(s)/L(s) \to 0 \quad (s \to \infty). \]

So the slowly varying function \( L \) may be written in the form \( L(s) = \exp \left\{ \int_0^s e(u) \, du/u \right\} \) with \( e(\cdot) \to 0 \) at infinity (see, e.g. [10, p. 282]), and has \( e \)-function
\[ e(s) = sL'(s)/L(s) = \alpha - s\Phi'(s). \]

Now by (5.1)
\[ s\Phi'(s) = \Psi(\Phi(s))/m, \]
so
\[ \alpha - s\Phi'(s) = m^{-1}(\alpha m - \Psi(\Phi(s))) = m^{-1} \int_0^\infty e^{-x\Phi(s)} \Pi(dx); \]
\[ me(s) = \int_0^\infty e^{-x\Phi(s)} \Pi(dx). \]

Now \( L \) is asymptotically constant iff \( \int_1^\infty e(s) \, ds/s \) converges, that is,
\[
\int_1^\infty \int_0^\infty e^{-x\Phi(s)} \Pi(dx) \, ds/s = \int_0^\infty \Pi(dx) \int_1^\infty \left[ \phi(s) \right]^x \, ds/s < \infty.
\]

Since \( L(s) \) is bounded between \( s^{\pm \epsilon} \) for all \( \epsilon > 0 \) and large enough \( s \),
\[ \int_1^\infty \left[ L(s) \right]^x \, ds/s^{1+\alpha x} \]
is bounded between \( 1/x(\alpha \pm \epsilon) \), and the result follows.

**Theorem 5.4.** When \( Y \) is a subordinator with drift \( a > 0 \),
\[ -\log P\{W < x\} \sim L(1/x)/x^a/(m-a) \quad (x \to 0) \]
for some function \( L \) varying slowly at infinity.

**Corollary 5.5.** \( \mu(x, \infty) \sim M(1/x)/x^a/m \quad (x \to 0) \) for some function \( M \) varying slowly at infinity.
Proof. Since $\Psi(s) = as + \int_0^\infty (1 - e^{-xs}) \Pi(dx)$,

$$\Psi'(0+) = m = a + \int_0^\infty x \Pi(dx),$$

so $m > a$ (as $Y$ is not deterministic). So

$$s\Theta'(s)/\Theta(s) = ms/\Psi(s) \to m/a, \quad s \to \infty.$$ 

Thus $\Theta(s)$ is $(m/a)$-varying at infinity, and so $\Phi(s)$ is $(a/m)$-varying at infinity. The theorem and corollary follow from [6, Th.4]. □

Letting $\alpha \to \infty$ in Theorem 5.2, one sees that if $Y$ is a subordinator with zero drift and infinite Lévy measure, $P\{W \leq x\}$ is smaller than any power. That $-\log P\{W \leq x\}$ is slowly varying (as indicated by letting $a \to 0$ in Theorem 5.4) follows from [6, Th.4, Cor.1].

**Theorem 5.6.** If $Y$ is a subordinator with zero drift and infinite Lévy measure,

$$-\log P\{W \leq x\} \sim L^*(1/x), \quad x \to 0,$$

where $L^*$ is a slowly varying function of the form

$$L^*(x) = \int_1^x L(u) \, du/u$$

with $L$ slowly varying.

**Corollary 5.7.** $\mu(x, \infty) = O(1/L(1/x)), x \to 0$.

**Note 5.8.** Together, Theorems 5.1, 5.2, 5.4 and 5.5 give complete information on the behaviour of the left-tail of $W$. It is interesting to compare these results with the corresponding results for simple branching processes $\{Z_n\}$ with generating function $f(s)$, $\mu = f'(s) > 1$. The extinction probability $q$ is the largest root of $f(q) = q$. One can pass from the case $q > 0$ to the case $q = 0$ by writing

$$g(s) = [f(q + (1-q)s) - q] / (1-q).$$

This allows us to deal with the cases of positive and zero extinction probability together (Harris [16]), in contrast to the situation here where Theorems 5.1 and 5.2 need separate treatment. Then Theorems 5.1 and 5.2 correspond to Theorem 3.3 of Harris [16] (see also Dubuc [8, Th.22]).
in which \( p_0 + p_1 > 0 \) (the minimum family size is 0 or 1). Instead of a regularly varying function, Harris obtains a power \( s^\gamma \) times a function \( L \) which, while not slowly varying in general, is of the form

\[
L(s) = L_1(s) + L_2(s)
\]

with \( L_1 \) multiplicatively periodic with period \( \mu \),

\[
L_1(\mu s) = L_1(s)
\]

and

\[
L_2(s) = o(1/s^\gamma).
\]

Theorems 5.4 and 5.5 correspond to the simple branching-process situation in which the minimum family size is at least 2. In this case Harris [16, Remark c below Th.3.3] observed that the left tail of \( W \) is exponentially small; see also Dubuc [8, Th.23].

We point out that a density version of Theorem 5.2 for the (discrete state-space) Markov branching process was proved by Karlin [20]; see [1, p.131].

6. Right tail behaviour

We pass now to the right tails of \( W, Z(t), \Pi \) and \( \mu \). A detailed comparison of the right-tails of \( W \) and \( Z(1) \) is given in [5, Theorem 4]; see also Grey [13, §5]. The results and methods are analogous to those of [4] for simple branching processes: (using the infinite divisibility of \( Z(1) \) it is easy to pass from \( Z(1) \) to \( Z(t) \) for any \( t > 0 \)). In our present notation, two typical results of this type are given below; it is easy to give similar elaborations of the other results.

**Theorem 6.1.** If \( n = 2, 3, \ldots \) the following statements are equivalent:

1. \( \mathbb{E} \left( [Z(t)]^n \right) < \infty \) for some (equivalently, for all) \( t > 0 \),
2. \( \mathbb{E} W^n < \infty \),
3. \( \int_0^\infty x^n \Pi(dx) < \infty \),
4. \( \int_0^\infty x^n \mu(dx) < \infty \).
Theorem 6.2. If \(\alpha > 1\) is not an integer and \(L\) varies slowly at infinity, the following statements are equivalent:

\[
(6.5) \quad \Pr\{Z(t) \geq x\} \sim (e^{\alpha mt} - e^{mt}) L(x)/x^\alpha \quad (x \to \infty)
\]

for some (equivalently, for all) \(t > 0\),

\[
(6.6) \quad \Pr\{W \geq x\} \sim L(x)/x^\alpha \quad (x \to \infty),
\]

\[
(6.7) \quad \Pi(x, \infty) \sim m(\alpha - 1) L(x)/x^\alpha \quad (x \to \infty),
\]

\[
\mu(x, \infty) \sim L(x)/x^\alpha \quad (x \to \infty).
\]

In Theorem 6.2, for instance, the equivalence of (6.5) and (6.6) follows as in [4], that of (6.6) and (6.8) as in Feller [11], and that of (6.7) and (6.8) by passing to Laplace transforms.

One can show that the characteristic function \(\psi\) of \(W\) can never be entire. If the tail of \(\Pi\) is exponentially small (in particular, if \(\Pi\) has compact support), the characteristic function \(\psi\) of \(W\) converges in a half-plane containing the origin in its interior. Analogous behaviour in the simple branching-process case arises when \(f(s)\) is geometric and \(W\) is exponential (this being one of the very few cases in which the distribution of \(W\) can actually be found explicitly).

As an example, consider the (discrete state-space) Markov branching process with \(f(s)\) geometric with parameter \(p\), \(f(s) = p/(1 - qs)\) (here \(q = 1 - p > p\)). If \(q/p = a > 1\), Proposition 4.4 gives

\[
\phi^{-1}(1 - z) = z/(1 - (a/(a - 1))z)^a.
\]

Hence

\[
\phi(s) = 1 + \sum_{n=1}^{\infty} \frac{(-s)^n}{n} \left(\frac{na}{n + 1}\right) \left(\frac{a}{a - 1}\right)^{n-1},
\]

which has radius of convergence \((a - 1)^a/a^{a+1}\). The methods of [6] give the weak tail-estimate

\[
-\log \Pr\{W \geq x\} \sim x(a - 1)^a/a^{a+1} \quad (x \to \infty).
\]

Note 6.3. There remains one important result on the simple branching process which has no analogue in this context. It was shown by Harris [16, Theorem 3.4] that if the maximum family size is \(d\) (\(f(s)\) is a polynomial of degree \(d\)) the moment-generating function of \(W\) is entire (and
thus so is the characteristic function). If \( \rho \) is defined by \( \mu^\rho = d \), one has
\[
\log \psi(s) = s^\rho L(s), \quad L(s) = L_1(s) + L_2(s)
\]
where \( L_1 \) is multiplicatively periodic with period \( \mu \) and \( L_2(s) = O(1/s^\rho) \) as \( s \to \infty \) (see also Dubuc [8]; note the parallel with the situation discussed at the end of Section 5). In the CB-case, each \( Z(t) \) for \( t > 0 \) has unbounded support (except in the trivial deterministic case), and \( W \) cannot exhibit tail-behaviour of this type.

7. A limit theorem for the critical case

In this section we restrict attention to the critical case \( m = 0 \).

A particularly important CB-process is that for which the corresponding spectrally positive Lévy process is centred and stable with index \( 1 + p, 0 < p < 1 \). We write
\[
(7.1) \quad \psi_\infty(s) = -\beta s^{1+p}/2p, \quad \beta > 0, \quad 0 < p < 1
\]
for the exponent of this stable process, which we call \( Y_\infty \). If \( Z_\infty \) is the corresponding CB-process, \( Z_\infty \) has exponent \( \psi_\infty(t,s) \) where (Lamperti [24])
\[
(7.2) \quad \psi_\infty(t,s) = s/(1 + \frac{1}{4} t\beta s^p)^{1/p}.
\]

By domain-of-attraction theory, \( Y_\infty \) is the only non-degenerate limit process obtainable from a spectrally positive \( Y \) by norming and passage to the limit (if \( Y \) has finite means, and the asymmetric Cauchy process is excluded). The random time-change enables us to translate this into a statement on CB-processes.

Let \( b(\lambda) \) be continuous and strictly increasing with \( b(\lambda) \to \infty \) as \( \lambda \to \infty \). Then
\[
(7.3) \quad \mathbb{E}\exp\{-sZ(\lambda t)/b(\lambda) \mid Z(0) = b(\lambda)\} = \exp\{-b(\lambda) \psi(\lambda t, s/b(\lambda))\} = \exp\{-\psi_\lambda(t, s)\}
\]
say. Thus \( \psi_\lambda \) is the exponent of a CB-process, which we call \( Z_\lambda \). Write \( \psi_\lambda(s) = \lambda b(\lambda) \psi(s/b(\lambda)) \). Then \( \psi_\lambda \) is the exponent of the process \( Y_\lambda \) corresponding to \( Z_\lambda \) under the random time-change, and
\[
(7.4) \quad \int_s^t du/\psi_\lambda(u) = t
\]
Theorem 7.1. The following statements are equivalent:

\[(7.5)\quad Y_\lambda \to Y_\infty \quad (\lambda \to \infty) \text{ weakly under } J_1,\]
\[(7.6)\quad Z_\lambda \to Z_\infty \quad (\lambda \to \infty) \text{ weakly under } J_1,\]
\[(7.7)\quad \Psi_\lambda(s) \to \Psi_\infty(s) \quad (\lambda \to \infty) \quad (s \geq 0),\]
\[(7.8)\quad \psi_\lambda(t,s) \to \psi_\infty(t,s) \quad (\lambda \to \infty) \quad (t,s \geq 0).\]

Proof. The equivalence of (7.5) and (7.7) is a standard result for Levy processes (see, e.g. [3, Th. 10]); when (7.5), (7.7) hold \(b(\lambda)\) is regularly varying with index \((1 - p)/p\), and the convergence in (7.7) is uniform on compact \(s\)-sets.

If (7.7) holds, choose \(\lambda_n \to \infty\), and let \(\phi(t,s)\) be any limit-point of \(\psi_{\lambda_n}(t,s)\). Letting \(n \to \infty\) through a suitable subsequence and using (7.4),

\[
\int_s^\infty \frac{du}{\Psi_\infty(u)} = t = \int_s^\infty \frac{du}{\Psi_\infty(u)}.
\]

Thus

\[
\int_{\phi(t,s)}^\infty \frac{du}{\Psi_\infty(u)} = 0
\]

and since the integrand is of constant sign (negative, as \(Z_\infty\) is critical), this gives \(\phi(t,s) = \psi_\infty(t,s)\). As the limit is independent of the sequence \(\lambda_n\) chosen, (7.8) follows.

By the Markov property, the one-dimensional distributions of a CB-process \(Z\) are obtained from its transition probabilities by iteration. These are specified through (1.5) by the exponent \(\psi(t,s)\), which in turn is specified by (and specifies) the one-dimensional distributions. Thus (7.8) holds if and only if the finite-dimensional distributions of \(Z_\lambda\) converge to those of \(Z_\infty\). That this is equivalent to (7.6) follows by slightly adapting a result of Grimvall ([14, Th. 5.5]; see also [151]).

Consider now the processes of integration, functional inversion and function composition leading from \(Z\) to \(Y\) via \(J\) and \(I\) in (1.4). That integration is \(J_1\)-continuous follows from the definition of the \(J_1\)-topology and the continuity of translation \(f(x) \to f(x+h)\) in the \(L_1\)-norm;
hence \( J \) depends \( J_1 \)-continuously on \( Z \). It was shown by Whitt [35] that the first-passage functional is continuous under the Skorohod \( M_1 \)-topology [33] (which is weaker than the \( J_1 \)-topology); hence \( I \) depends \( M_1 \)-continuously on \( J \) (and through \( J \), on \( Z \)). Also, \( I(t) = \int_0^t \frac{du}{Y(u)} \) is continuous and strictly increasing, while \( Z \) has right-continuous paths with left limits. It follows from this by a result of Whitt [36] that \( Y(t) = Z(I(t)) \) depends \( J_1 \)-continuously (and so also \( M_1 \)-continuously) on \( (Z, I) \) (and hence on \( Z \)). By the continuous-mapping theorem of weak-convergence theory [2], (7.6) thus implies that \( Y_\lambda \rightarrow Y_\infty \) weakly under \( M_1 \). This in turn implies convergence of finite-dimensional distributions (Skorohod [33]), whence (7.7) and (7.5); the proof is complete.

When \( p = 1 \), \( Y_\infty \) is a Brownian motion and \( Z_\infty \) is a diffusion with generator \( \frac{1}{2} \beta x \frac{d^2}{dx^2} \).

**Note 7.2.** It was shown by Lamperti [26] (see also Kawazu–Watanabe [23]) that the CB-processes which can arise as limits of conditioned simple branching processes of the form \( \{ Z_{nt} / b_n \mid Z_0 = b_n \} \) are precisely the processes \( Z_\infty \). Grimvall’s result [14] strengthens the convergence here from that of finite-dimensional distributions to \( J_1 \)-convergence. Theorem 7.1 is simply the continuous-state analogue of this result. We point out that Theorem 7.1 corresponds to [3, Th. 10] under the random time-change.

We shall not discuss sub-critical CB-processes, referring only to the work of Seneta and Vere-Jones [29, 30] and Grey [13]. We shall also not discuss continuous-state branching processes with immigration (CBI-processes), for which we refer to Kawazu–Watanabe [23] and Pinsky [27].

**References**