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# On the geometric conservation law for unsteady flow simulations on moving mesh

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## Abstract

When using the dynamic mesh method to deal with moving boundary problems, the Geometric Conservation Law (GCL) must be considered carefully. This paper reviews the study on GCL problem of dynamic grid in finite volume framework based on Arbitrary Lagrangian-Eulerian (ALE) methods. Several common approaches to satisfy discretized geometric conservation law (DGCL) are studied and simplified to a uniform form. The uniform flow and the isentropic vortex are tested to validate the geometric conservation property and the temporal-accuracy. Numerical results illustrate that although the violating of GCL does not pollute the original time-accuracy of numerical schemes for the governing equations, it may introduce extra artificial errors. When the “Adding Source Term” methods are adopted, the temporal accuracy order of face velocity scheme must match the temporal accuracy order of numerical schemes for the governing equations, otherwise it will pollute the original temporal accuracy.

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## 1. Introduction

When using the dynamic mesh method in finite volume based on Arbitrary Lagrangian-Eulerian (ALE) methods, the Geometric Conservation Law (GCL) must be satisfied. The issue of GCL came from the conception of Free-stream Preserving Property and firstly realized and studied by Pulliam & Steger [1] in 1978. Then the concept was

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well-defined by Thomas & Lambard [2,3]. Actually, the concept of GCL consists of two meanings: the Surface Conservation Law (SCL) and the Volume Conservation Law (VCL). SCL implies that the surface of a discretized computational cell should keep enclosing during the mesh movement, finite volume methods can satisfy SCL automatically.

VCL concerns on the volume conservation on moving mesh. It is the focus we concern in this paper. Actually, many studies have illustrated the importance of GCL for time-integration schemes on moving mesh. Thomas and Lombard[3] implied that an extra error would be brought into the flow field if the numerical scheme does not satisfy GCL, it may result in erroneous flutter boundary predictions [4,5]. Therefore, the respecting of GCL is a fundamental condition for any time-integration scheme on moving mesh. The suggestion by Thomas and Lombard [3] is to calculate the Jacobian-matrix  $J$  with a conservative way in finite difference method. In a similar way, Liu et al.[6] introduced a “revised volume” of the moving cell to preserve the conservation of volume on dynamic unstructured mesh. Farhat et al. developed serial GCL-compliant schemes for finite volume method [4,5,7,8,9]. Then Wang et al. [10] and Zhang et al. [11-14] introduced a simplified method to preserve the conservation of “sweeping-volume” of each individual surface of the control volume. On the contrary, Schulz & Kallinderis [15] decomposed the time derivative terms in the governing equations to satisfy the GCL.

In this paper, several common GCL-compliant finite volume approaches on dynamic mesh are studied and classified. These approaches are classified into two categories: the “Adding Source Term” methods and the “Geometry Correction” methods. Then, both categories of methods are simplified to a uniform form. Finally, the uniform flow and the isentropic vortex are tested to validate the geometric conservation property and the time-accuracy.

## 2. GCL-compliant approaches

### 2.1. GCL and the discretized form

For simplicity, the Euler equation is considered here. The integral form of the governing equation on moving mesh can be written as follow:

$$\frac{d}{dt} \int_V Q dV = R(Q), \quad R = - \int_S (F(Q) - Q \cdot \overline{vg} \cdot \vec{n}) dS \tag{1}$$

in which  $S$  is the surface surrounding the control volume  $V$ ,  $Q$  represents conservative variables,  $F(Q)$  represents the in viscid flux,  $\overline{vg}$  and  $\vec{n}$  represent the velocity and the outward normal vector of  $S$ .

The Uniform Flow Preservation must be satisfied for any computational grids, which means Eq.(2) should be compliant for uniform flow:

$$\frac{dV}{dt} = \int_S \overline{vg} \cdot \vec{n} dS \tag{2}$$

Equation (2) also means the variation of  $V$  must equal to the integration for the normal velocity of the faces. However, the semi-discretized form of Eq.(2) may not satisfy the original equation exactly, due to the geometric variables such as the velocity/outward normal/area of  $S$  do not match the volume  $V$  in semi-discretized.

For simplicity, First Order Backward Differentiation Formula (BDF1) is taken for example in the whole paper except when indicated specially. Equation (2) can be semi-discretized to:

$$\frac{V^{n+1} - V^n}{\Delta t} = \sum_{i=1}^{nf} \overline{vg}_i^{n+1} \cdot \vec{n}_i^{n+1} S_i^{n+1} = \sum_{i=1}^{nf} vgn_i^{n+1} S_i^{n+1} \tag{3}$$

in which subscript represents different faces of the control volume and superscript represents different time layers.  $\Delta t$  represents the time step.  $nf$  indicates the number of faces of the control volume.  $vgn = \overline{vg} \cdot \vec{n}$  is the normal velocity of the face.

Equation (3) is the DGCL equation for BDF1. The mismatching of geometric variables will result in some artificial errors, and then pollute the flow field or degenerate the stability. In order to eliminate this part of error, CFD practitioners have developed various approaches. These approaches can be classified into two categories: the first one is the “Adding Source Term” methods which eliminate the artificial error integrally by introducing source

term, satisfying DGCL in the sense of monolithic volume; and the second one is the “Geometry Correction” methods which constrain geometric variables to satisfy the “Sweeping Volume Conservation Law (SVCL)”.

2.2. The “Adding Source Term” methods

The semi-discretization of Eq.(2) will introduce an extra error  $E$  :

$$E = \frac{dV}{dt} - \int_S \overline{v}g \cdot \vec{n}dS \tag{4}$$

in which the superscript “ $\hat{\phantom{x}}$ ” represents that it is calculated by any specified numerical scheme. To eliminate the extra error, a source term is introduced into the governing equations:

$$\frac{d}{dt} \int_V QdV - R(Q) = GCL_{source} = Q \cdot E \tag{5}$$

If the time derivative terms in Eq.(5) are discretized with the same temporal scheme, and the geometric variables in both sides are calculated by the same approach, the uniform flow will be preserved automatically.

2.2.1. The decomposing approaches by Schulz et al.[15]

The main idea of Schulz’s approach is to decompose the time derivative term in the governing equation. Therefore, the derivative of cell volume and the derivative of flow variables with respect to time can be decomposed as:

$$\frac{d(\overline{Q}V)}{dt} = \overline{Q} \frac{dV}{dt} + V \frac{d\overline{Q}}{dt} \tag{6}$$

in which the superscript “ $\overline{\phantom{x}}$ ” represents the average value in the control volume. Thus Eq. (1) can be rewritten as:

$$V \frac{d\overline{Q}}{dt} = R(Q) - \overline{Q} \int_S \overline{v}gndS \tag{7}$$

And the the semi-discretized form is:

$$V^{n+1} \frac{\overline{Q}^{n+1} - \overline{Q}^n}{\Delta t} = R^{n+1} - \overline{Q}^{n+1} \sum_{i=1}^{nf} \overline{v}g n_i^{n+1} S_i^{n+1} \tag{8}$$

It can be rewritten into the style of Eq.(5):

$$\frac{V^{n+1} \overline{Q}^{n+1} - V^n \overline{Q}^n}{\Delta t} - R^{n+1} = GCL_{source}, \quad GCL_{source} = \overline{Q}^n \frac{V^{n+1} - V^n}{\Delta t} - \overline{Q}^{n+1} \sum_{i=1}^{nf} \overline{v}g n_i^{n+1} S_i^{n+1} \tag{9}$$

2.2.2. The “revised volume” approach by Liu et al.[6]

Liu et al.[6] introduced the revised volume approach during time marching to respect DGCL in finite volume method. Thomas & Lambard’s approach[2,3] is similar but in the frame of finite difference method.

The revised volume  $\tilde{V}^{n+1}$  is introduced:

$$\tilde{V}^{n+1} = V^n + \sum_{i=1}^{nf} \overline{v}g n_i^{n+1} S_i^{n+1} \tag{10}$$

Then the governing equations can be semi-discretized as:

$$\frac{\overline{Q}^{n+1} \tilde{V}^{n+1} - \overline{Q}^n V^n}{\Delta t} = R^{n+1} \tag{11}$$

It can also be reform into the style of Eq.(5):

$$\frac{V^{n+1}\bar{Q}^{n+1} - V^n\bar{Q}^n}{\Delta t} - R^{n+1} = GCLsource, \quad GCLsource = \bar{Q}^{n+1} \frac{V^{n+1} - V^n}{\Delta t} - \bar{Q}^{n+1} \sum_{i=1}^{nf} vgn_i^{n+1} S_i^{n+1} \quad (12)$$

### 2.3. The “Geometry Correction” methods

The normal velocity of the face in Wang [10] & Zhang’s approach [11-14] is revised. It requires that the “sweeping volume” calculated by numerical schemes of each face is equal to the actual volume, namely satisfying the SVCL:

$$vgn_i^{n+1} = vgn_i^{n+1} = \frac{\Delta V_i^n}{S_i^{n+1} \Delta t} \quad (13)$$

in which  $\Delta V_i^n$  represents the volume swept by the  $i$ -th face between the time layer  $n$  and  $n+1$ . The DGCL is satisfied automatically when Eq.(13) is adopted. The approach by Farhat is very similar, but the velocity/outward normal/area of each face are all constrained simultaneously (please refer to Refs.[4-9] for more details).

### 2.4. Uniform form of scheme satisfy the DGCL

The two categories can be simplified into a uniform form:

$$\frac{V^{n+1}\bar{Q}^{n+1} - V^n\bar{Q}^n}{\Delta t} - R^{n+1} = GCLsource, \quad GCLsource = \bar{Q}^\theta \frac{V^{n+1} - V^n}{\Delta t} - \bar{Q}^{n+1} \sum_{i=1}^{nf} vgn_i^{n+1} S_i^{n+1} \quad (14)$$

The normal velocity of the face in  $R$  takes the same value as  $vgn$ .

For the “Adding Source Term” methods, the normal velocity of the face usually calculated by the linear-multi-step methods using the face center position of different time layers according to the temporal discretization. Coefficients  $\theta$  is related to the approaches, please refer to Eqs.(9) and (12).

For the “Geometry Correction” methods, for example the Wang&Zhang’s approach, the normal velocity of the face is calculated by Eq.(13). Besides, since the satisfaction of SVCL, the  $GCLsource$  is equal to 0.

It should be mentioned that, even though the two categories approaches can be simplified into a uniform form, their basic idea are different from each other.

## 3. Numerical experiment and validation

All the tests are carried out using the CFD solver, HyperFLOW [16,17], which is developed by the authors’ group. HyperFLOW is a multi-disciplinary platform for CFD research and engineering applications. The basic numerical method is the cell-centered, second-order accuracy, finite volume method for Reynolds-averaged Navier-Stokes equations on hybrid grids. Both structured and unstructured solver can run simultaneously in this platform on hybrid grids. The inviscid flux is calculated across the cell interfaces using Roe’s flux-splitting scheme. The dual-time stepping approach [18] with LU-SGS implicit sub-iteration is adopted for temporal advancing. The computational domain is a square region from -10 to 10 in both x and y directions, and is filled with uniform Cartesian grids.

### 3.1. The uniform flow

“Free Stream Preservation Property” is the fundamental condition for any time-integral scheme on moving mesh, which means that no disturbance should be introduced into the uniform flow by numerical scheme. To test the influence of moving mesh, the grid nodes oscillate around their original positions with amplitude of  $-0.5L$  and  $0.5L$  randomly in each time step ( $L$  is the spatial size of a Cartesian cell).

Wang & Zhang’s approach (Method1), the “revised volume” approach by Liu et al. (Method2) and the decomposing approaches by Schulz et al. (Method3) are compared for this case. Moreover, a GCL-non-compliant

approach (Method4) is also tested, in which no source term is introduced, and the geometric variables of each face are not constrained (just like these in Method2&3) either.

Both the BDF1 and BDF2 time schemes are tested. Table 1 show the errors of density obtained by these four approaches. Obviously the three GCL-compliant approaches can preserve the uniform flow very well. However, Method4 introduces some artificial error into the flow field, and the density contour shows that the uniform flow field has been greatly polluted (Fig. 1). These results illustrate that the GCL must be satisfied to avoid artificial errors introducing into the flow field.

Table 1. Results of Uniform Flow by different schemes.

Methods	BDF1		BDF2	
	$L_1$ error of density	$L_2$ error of density	$L_1$ error of density	$L_2$ error of density
1	$1.89 \times 10^{-16}$	$2.57 \times 10^{-16}$	$1.02 \times 10^{-15}$	$1.34 \times 10^{-15}$
2	$2.14 \times 10^{-16}$	$2.78 \times 10^{-16}$	$3.41 \times 10^{-16}$	$4.44 \times 10^{-15}$
3	$2.31 \times 10^{-16}$	$3.05 \times 10^{-16}$	$3.57 \times 10^{-16}$	$4.67 \times 10^{-16}$
4	$8.04 \times 10^{-16}$	$1.04 \times 10^{-1}$	$8.93 \times 10^{-2}$	$1.13 \times 10^{-1}$

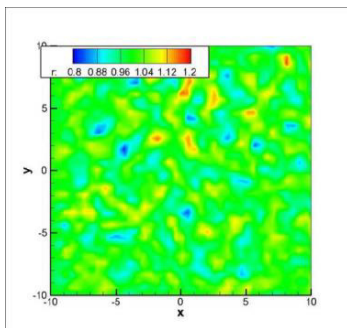


Fig. 1. Density contours obtained by GCL-non-compliant approach

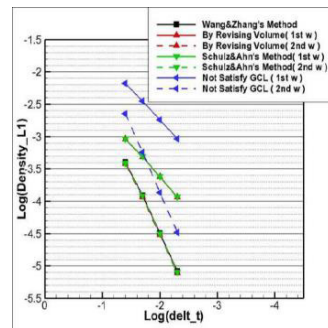


Fig. 2. Density errors obtained by different approaches

### 3.2. The isentropic vortex tests

To investigate the performance of these approaches for non-uniform flows, the isentropic vortex case are tested to study the temporal-accuracy. A relatively dense Cartesian mesh with the size of  $320 \times 320$  is adopted to reduce the influence of the spatial discretization. Five different non-dimensional time-steps are selected: 0.04/0.02/0.01/0.005/0.002. The temporal discretization method is BDF2, and all the four methods mentioned above are carried out. Moreover, when methods 2/3/4 are adopted, the normal velocity of the faces are calculated by the 1<sup>st</sup> order and the 2<sup>nd</sup> order approaches, respectively. Therefore, we get 7 different methods here.

Numerical results (Fig. 2) show that all the methods can reach 1<sup>st</sup> or 2<sup>nd</sup> order temporal-accuracy. The methods which adopt 1<sup>st</sup> order normal velocity present 1<sup>st</sup> order temporal-accuracy, while the ones adopt 2<sup>nd</sup> order normal velocity reach 2<sup>nd</sup> order temporal-accuracy. In addition, the absolute errors of GCL-non-compliant methods are evidently larger than those of the GCL-compliant methods. The fact illustrates that the violating of GCL does not pollute the original time-accuracy of numerical schemes for the governing equation, but it may introduce extra artificial errors. When the “Adding Source Term” methods are adopted, the temporal accuracy order of face velocity scheme must match with the temporal accuracy order of the whole numerical scheme, otherwise it will pollute the original temporal accuracy.

#### 4. Conclusions

In this paper, several common GCL-compliant approaches are studied and simplified into a uniform form. Two typical cases are tested to validate the geometric conservation property and the temporal-accuracy. Numerical results illustrate that although the violating of GCL does not pollute the original time-accuracy of numerical schemes for the governing equations, it may introduce extra artificial errors. Therefore, the GCL must be satisfied for unsteady flow simulations on moving grids. When the “Adding Source Term” methods are adopted, the time accuracy depends greatly on the numerical scheme of face velocity, which should be dealt with cautiously.

However, although the GCL problem has been discussed for a long time in literature, there are still some problems to be studied further, such as: Is the “Uniform Flow Preservation” the necessary and sufficient condition of GCL problems? What is the practical performance of different GCL-compliant approaches for complex unsteady flows? Which one of the two categories is more preferable? We will study these issues in the future.

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