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Analytic torsion of complete hyperbolic manifolds of finite volume

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Abstract

In this paper we define the analytic torsion for a complete oriented hyperbolic manifold of finite volume. It depends on a representation of the fundamental group. For manifolds of odd dimension, we study the asymptotic behavior of the analytic torsion with respect to certain sequences of representations obtained by restriction of irreducible representations of the group of isometries of the hyperbolic space to the fundamental group.

Keywords: Analytic torsion; Hyperbolic manifolds

1. Introduction

Let X be an oriented hyperbolic manifold of dimension d. Let $G = \mathrm{SO}_0(d,1)$, $K = \mathrm{SO}(d)$. Then there exists a discrete, torsion free subgroup $\Gamma \subset G$ such that $X = \Gamma \backslash \mathbb{H}^d$, where $\mathbb{H}^d \cong G/K$ is the d-dimensional hyperbolic space. First assume that X is compact. Let τ be an irreducible finite-dimensional representation of G. Restrict τ to Γ and let E_τ be the associated flat vector bundle over X. By [26] one can equip E_τ with a canonical metric, called admissible metric, which is unique up to scaling. Let $T_X(\tau)$ be the Ray–Singer analytic torsion with respect to the hyperbolic metric of X and the admissible metric in E_τ (see [37,32]). It was proved in [34], that for d even, $T_X(\tau) = 1$ for all representations τ as above.

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Now let d be odd, say $d = 2n + 1$. In [34] we introduced special sequences $\tau(m)$, $m \in \mathbb{N}$, of irreducible representations of G and we studied the asymptotic behavior of $T_X(\tau(m))$ as $m \to \infty$. The representations $\tau(m)$ are defined as follows. Fix natural numbers $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_{n+1}$. For $m \in \mathbb{N}$ let $\tau(m)$ be the finite-dimensional irreducible representation of G with highest weight $(\tau_1 + m, \dots, \tau_{n+1} + m)$ (see [12, p. 365]). By Weyl's dimension formula there exists a constant $C > 0$ such that

$$\dim\bigl(\tau(m)\bigr) = Cm^{\frac{n(n+1)}{2}} + O\bigl(m^{\frac{n(n+1)}{2}-1}\bigr), \quad m \to \infty. \tag{1.1}$$

One of the main results of [34] is the following asymptotic formula: There exists a constant $C(n) \neq 0$, which depends only on n, such that

$$\log T_X\bigl(\tau(m)\bigr) = C(n)\operatorname{vol}(X)m \cdot \dim\bigl(\tau(m)\bigr) + O\bigl(m^{\frac{n(n+1)}{2}}\bigr) \tag{1.2}$$

as $m \to \infty$. The 3-dimensional case was first treated in [33]. This result has been used in [25] to study the growth of torsion in the cohomology of arithmetic hyperbolic 3-manifolds.

The main goal of the present paper is to extend the results of [34] to complete oriented hyperbolic manifolds of finite volume. Let $\Gamma\backslash\mathbb{H}^d$ be such a manifold. To simplify some of the considerations we will assume that Γ satisfies the following condition: For every Γ-cuspidal parabolic subgroup $P = M_P A_P N_P$ of G we have

$$\Gamma \cap P = \Gamma \cap N_P. \tag{1.3}$$

We note that this condition is satisfied, if Γ is "neat", which means that the group generated by the eigenvalues of any $\gamma \in \Gamma$ contains no roots of unity $\neq 1$. We need (1.3) to eliminate some technical difficulties related to the Selberg trace formula.

The first problem is to define the analytic torsion for noncompact hyperbolic manifolds of finite volume. The Laplace operator $\Delta_p(\tau)$ on E_τ-valued p-forms has then a continuous spectrum and therefore, the heat operator $\exp(-t\Delta_p(\tau))$ is not trace class. So the usual zeta function regularization cannot be used to define the analytic torsion in this case. To overcome this problem we use a regularization of the trace of the heat operator which is similar to the b-trace of Melrose [27]. This kind of regularization was also used by Park [36] in the case of unitary representations of Γ.

The regularization of the trace of the heat operator is defined as follows. Chopping off the cusps at sufficiently high level $Y > Y_0$, we get a compact submanifold $X(Y) \subset X$ with boundary $\partial X(Y)$. Let $K^{p,\tau}(t, x, y)$ be the kernel of the heat operator $\exp(-t\Delta_p(\tau))$. Then it follows that there exists $\alpha(t) \in \mathbb{R}$ such that $\int_{X(Y)} \operatorname{tr} K^{p,\tau}(x, x, t)\,dx - \alpha(t)\log Y$ has a limit as $Y \to \infty$. Then we put

$$\operatorname{Tr}_{\mathrm{reg}}\bigl(e^{-t\Delta_p(\tau)}\bigr) := \lim_{Y\to\infty}\Biggl(\int\limits_{X(Y)} \operatorname{tr} K^{p,\tau}(t, x, x)\,dx - \alpha(t)\log Y\Biggr). \tag{1.4}$$

We note that one can also use relative traces as in [32] to regularize the trace of the heat operator. The methods are closely related.

It turns out that the right-hand side of (1.4) equals the spectral side of the Selberg trace formula applied to the heat operator $\exp(-t\Delta_p(\tau))$. Using the Selberg trace formula, it follows that

$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ has asymptotic expansions as $t \to +0$ and as $t \to \infty$. This permits to define the spectral zeta function. Let $(\tau_1, \ldots, \tau_{n+1})$ be the highest weight of τ. If $\tau_{n+1} \neq 0$, then it follows that $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ is exponentially decreasing as $t \to \infty$. In this case the definition of the zeta function is simplified. It is given by

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Tr}_{\mathrm{reg}}\bigl(e^{-t\Delta_p(\tau)}\bigr)\, dt. \tag{1.5}$$

The integral converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$. In analogy to the compact case we now define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ with respect to E_τ by

$$T_X(\tau) := \exp\left(\frac{1}{2}\sum_{p=1}^{d}(-1)^p p \frac{d}{ds}\zeta_p(s;\tau)|_{s=0}\right). \tag{1.6}$$

Again, the analytic torsion behaves quite differently in even and odd dimensions. We first consider the odd-dimensional case. The main result of this paper is the following theorem.

Theorem 1.1. *Let $X = \Gamma\backslash\mathbb{H}^{2n+1}$ be a $(2n+1)$-dimensional, complete, oriented, hyperbolic manifold of finite volume. Assume that Γ satisfies* (1.3). *There exists a constant $C(n) \neq 0$ which depends only on n, such that we have*

$$\log T_X\bigl(\tau(m)\bigr) = C(n)\,\mathrm{vol}(X) m \cdot \dim\bigl(\tau(m)\bigr) + O\bigl(m^{\frac{n(n+1)}{2}} \log m\bigr)$$

as $m \to \infty$.

This result generalizes (1.2) to the finite volume case. The constant $C(n)$ in Theorem 1.1 equals the constant $C(n)$ occurring in (1.2) and can be computed explicitly from the Plancherel polynomials. It equals

$$C(n) = (-1)^n \frac{\pi}{\mathrm{vol}(S^d)}, \tag{1.7}$$

where $\mathrm{vol}(S^d)$ is the Euclidean volume of the d-dimensional unit sphere, see [34, (2.24), (5.22)]. We also consider the L^2-torsion $T_X^{(2)}(\tau)$. Although X is noncompact, it can be defined as in the compact case [23]. It can be computed using the results of [34]. First of all, we show that there exists a polynomial $P_\tau(m)$ of degree $n(n+1)/2 + 1$ such that

$$\log T_X^{(2)}\bigl(\tau(m)\bigr) = \mathrm{vol}(X) P_\tau(m). \tag{1.8}$$

The polynomial is obtained from the Plancherel polynomials. Its leading term can be determined as in [34] and we obtain

$$\log T_X^{(2)}\bigl(\tau(m)\bigr) = C(n)\,\mathrm{vol}(X) m \cdot \dim\bigl(\tau(m)\bigr) + O\bigl(m^{\frac{n(n+1)}{2}}\bigr), \tag{1.9}$$

as $m \to \infty$. Compared with Theorem 1.1 we obtain the following theorem.

Theorem 1.2. *Let $X = \Gamma\backslash\mathbb{H}^{2n+1}$ be a $(2n+1)$-dimensional complete, oriented, hyperbolic manifold of finite volume. Assume that Γ satisfies* (1.3)*. Then we have*

$$\log T_X\bigl(\tau(m)\bigr) = \log T_X^{(2)}\bigl(\tau(m)\bigr) + O\bigl(m^{\frac{n(n+1)}{2}} \log m\bigr)$$

as $m \to \infty$.

Remark 1.3. If X has a spin structure, then $\Gamma \subset \mathrm{SO}_0(d,1)$ has a lift to $\mathrm{Spin}(d,1)$. In this case we may also assume that $\tau_1, \dots, \tau_{n+1}$ and m are in $\frac{1}{2}\mathbb{N}$. Then both Theorems 1.1 and 1.2 continue to hold.

Next we turn to the even-dimensional case. First recall that for a compact manifold of even dimension, the analytic torsion is always equal to 1 (see [37], [34, Proposition 1.7]). This is not true anymore in the noncompact case. Park [36, Theorem 1.4] has computed the analytic torsion of a unitary representation of Γ in even dimensions. His formula shows that in the noncompact case, the analytic torsion in even dimensions is not trivial in general. Nevertheless, the torsion has still a rather simple behavior as shown by the next proposition. For a hyperbolic manifold of finite volume X, denote by $\kappa(X)$ the number of cusps of X. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$ and let $\Lambda(G) \subset \mathfrak{h}_{\mathbb{C}}^*$ be the highest weight lattice. For $\lambda \in \Lambda(G)$ let τ_λ be the corresponding irreducible representation of G.

Proposition 1.4. *There exists a function $\Phi : \Lambda(G) \to \mathbb{R}$ such that for every even-dimensional complete oriented hyperbolic manifold X of finite volume one has*

$$\log T_X(\tau_\lambda) = \kappa(X)\Phi(\lambda), \quad \lambda \in \Lambda(G).$$

The function Φ can be described as follows. There is a distribution J which appears on the geometric side of the trace formula. It is of the form $J = \kappa(X) \cdot \tilde{J}$, where $\tilde{J}$ is defined in terms of weighted characters of principal series representations of G (see (6.13)). Let $k_t^\tau \in \mathcal{C}(G)$ be the function (1.10). There is $c > 0$ such that $\tilde{J}(k_t^\tau) = O(e^{-ct})$ as $t \to \infty$. Moreover $\tilde{J}(k_t^\tau)$ has an asymptotic expansion as $t \to 0$. Thus the Mellin transform $\mathcal{M}\tilde{J}(s;\tau)$ of $\tilde{J}(k_t^\tau)$ is defined for $\mathrm{Re}(s) \gg 0$ and admits a meromorphic extension to $\mathbb{C}$ which is regular at $s = 0$. Then we have

$$\Phi(\lambda) = \mathcal{M}\tilde{J}(0;\tau_\lambda)$$

for all highest weights $\lambda = (k_1, \dots, k_{n+1})$.

Next recall that for a compact manifold X, the analytic torsion equals the Reidemeister torsion (see [31]). This is the basis for the applications of the results of [33] to the cohomology of arithmetic hyperbolic 3-manifolds in [25]. Currently it is not known if there is an extension of the equality of analytic and Reidemeister torsion to the noncompact setting. This is an interesting problem and the present paper is a first step in this direction.

We shall now outline our method for the proof of our main result. For notational simplicity we will assume that X admits a spin structure. Then we take $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$ and $\Gamma \subset G$. Let $d = 2n+1$. We assume that the highest weight of τ satisfies $\tau_{n+1} \neq 0$. Let

$$K(t,\tau) := \sum_{p=0}^{2n+1} (-1)^p p \operatorname{Tr}_{\mathrm{reg}}\bigl(e^{-t\Delta_p(\tau)}\bigr).$$

By (1.5) and (1.6) we need to compute the finite part of the Mellin transform of $K(t,\tau)$ at 0. Let $\tilde{E}_\tau$ be the homogeneous vector bundle over $\tilde{X}=G/K$ associated to τ and let $\widetilde{\Delta}_p(\tau)$ be the Laplacian on $\tilde{E}_\tau$-valued p-forms on $\tilde{X}$. The heat operator $e^{-t\widetilde{\Delta}_p(\tau)}$ is a convolution operator with kernel $H_t^{\nu_p(\tau)}: G\to \operatorname{End}(\Lambda^p\mathfrak{p}^*\otimes V_\tau)$. Let $h_t^{\nu_p(\tau)}(g)=\operatorname{tr} H_t^{\nu_p(\tau)}(g)$, $g\in G$, and put

$$k_t^\tau=\sum_{p=1}^{d}(-1)^p p h_t^{\nu_p(\tau)}. \tag{1.10}$$

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. There exists an orthogonal R_Γ-invariant decomposition $L^2(\Gamma\backslash G)=L^2_d(\Gamma\backslash G)\oplus L^2_c(\Gamma\backslash G)$. The restriction R_Γ^d of R_Γ to $L^2_d(\Gamma\backslash G)$ decomposes into the orthogonal direct sum of irreducible unitary representations, each of which occurs with finite multiplicity. On the other hand, by the theory of Eisenstein series, the restriction R_Γ^c of R_Γ to $L^2_c(\Gamma\backslash G)$ is isomorphic to the direct integral over all tempered principal series representations of G. For $\phi\in L^2_d(\Gamma\backslash G)$ let

$$\big(R_\Gamma^d\big(k_t^\tau\big)\phi\big)(x):=\int_G k_t^\tau(g)\phi(xg)\,dg.$$

Then $R_\Gamma^d(k_t^\tau)$ is a trace class operator and the Selberg trace formula computes its trace. The right-hand side of the trace formula is the sum of terms associated to the continuous spectrum and orbital integrals associated to the various conjugacy classes of Γ. If we move the spectral terms to the left-hand side of the trace formula we end up with the spectral side $J_{\mathrm{spec}}(k_t^\tau)$ of the trace formula. The key fact is now that

$$K(t,\tau)=J_{\mathrm{spec}}\big(k_t^\tau\big).$$

By the Selberg trace formula, the spectral side equals the geometric side, that is, the sum of the orbital integrals. This leads to the following fundamental equality:

$$K(t,\tau)=I(t;\tau)+H(t;\tau)+T(t;\tau)+\mathcal{I}(t;\tau)+J(t;\tau), \tag{1.11}$$

where $I(t;\tau)$ is the contribution of the identity conjugacy class of Γ and $H(t;\tau)$ is the contribution of the hyperbolic conjugacy classes of Γ. Moreover, $T(t;\tau)$, $\mathcal{I}(t;\tau)$ and $J(t;\tau)$ are tempered distributions applied to k_t^τ which are constructed out of the parabolic conjugacy classes of Γ. Now we evaluate the Mellin transform of each term separately. Here an important simplification is obtained using a theorem of Kostant on Lie algebra cohomology.

Let $\mathcal{M}I(\tau)$ be the Mellin transform of $I(t;\tau)$ evaluated at 0. Then we show that

$$\log T_X^{(2)}(\tau)=\frac{1}{2}\mathcal{M}I(\tau).$$

Now consider the representations $\tau(m)$, $m\in\mathbb{N}$. Using the results of [34] we compute $\mathcal{M}I(\tau(m))$ and prove (1.8) and (1.9). Thus in order to prove our main result, we need to show that the Mellin transforms at 0 of all other terms are of lower order. It is easy to treat the hyperbolic term and the terms $T(t;\tau(m))$. The distribution $\mathcal{I}(t;\tau(m))$ is invariant and its Fourier transform was computed explicitly by Hoffmann [16]. Using his results we can estimate the Mellin transform of $\mathcal{I}(t;\tau(m))$ at 0. Finally, the distribution $J(t;\tau(m))$ is non-invariant. However it is described in terms of Knapp–Stein intertwining operators which are understood completely in our case. With this information its Mellin transform at 0 can also be estimated.

In [34] we have used a different method which does not rely on the trace formula. It would be interesting to generalize this method to the finite volume case. Especially the Fourier transform, which we use to deal with $\mathcal{I}(t;\tau(m))$, is a very heavy machinery and is not available in the higher rank case. Part of the arguments used in [34] go through in the finite volume case as well. The difficult part is to deal with the contribution of the parabolic terms.

This paper is organized as follows. In Section 2 we fix notations and collect some basic facts. In Section 3 we review some properties of the right regular representation of G on $L^2(\Gamma\backslash G)$. In Section 4 we introduce the locally invariant differential operators which act on locally homogeneous vector bundles over X. Section 5 is devoted to the regularized trace which we introduce there and relate it to the spectral side of the Selberg trace formula. In Section 6 we apply the Selberg trace formula which leads to (1.11). Furthermore, we study the Fourier transform of the distribution $\mathcal{I}$. Finally we derive an asymptotic expansion as $t\to 0$ for the regularized trace of the heat operator of a Bochner–Laplace operator. In 7 we introduce the analytic torsion. In Section 8 we express the test function k_t^τ as a combination of functions defined by the heat kernels of certain Bochner–Laplace operators. The results of this section are needed to deal with the Mellin transforms of the various terms on the right-hand side of (1.11). In Section 9 we study the L^2-torsion. In the final Section 10 we prove the main results.

This paper arose from the PhD thesis of the second author under the supervision of the first author.

2. Preliminaries

In this section we will establish some notation and recall some basic facts about representations of the involved Lie groups. As mentioned in the introduction, for simplicity we will assume that X admits a spin structure. For $d\in\mathbb{N}$, $d>1$ let $G:=\mathrm{Spin}(d,1)$. Recall that G is the universal covering group of $\mathrm{SO}_0(d,1)$. Let $K:=\mathrm{Spin}(d)$. Then K is a maximal compact subgroup of G. Put $\tilde{X}:=G/K$. Let

$$G=NAK$$

be the standard Iwasawa decomposition of G and let M be the centralizer of A in G. Then $M=\mathrm{Spin}(d-1)$. The Lie algebras of G, K, A, M and N will be denoted by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Define the standard Cartan involution $\theta:\mathfrak{g}\to\mathfrak{g}$ by

$$\theta(Y)=-Y^t,\quad Y\in\mathfrak{g}.$$

The lift of θ to G will be denoted by the same letter θ. Let

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$$

be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Let $x_0=eK\in\tilde{X}$. Then we have a canonical isomorphism

$$T_{x_0}\tilde{X}\cong\mathfrak{p}. \tag{2.1}$$

Define the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ by

$$\langle Y_1, Y_2\rangle := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \tag{2.2}$$

By (2.1) the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{p}$ defines an inner product on $T_{x_0}\tilde{X}$ and therefore an invariant metric on $\tilde{X}$. This metric has constant curvature -1. Then $\tilde{X}$, equipped with this metric, is isometric to the hyperbolic space $\mathbb{H}^d$.

2.1. Fix a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. Then

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{b}$$

is a Cartan subalgebra of $\mathfrak{g}$. We can identify $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}(d+1, \mathbb{C})$. Let $e_1 \in \mathfrak{a}^*$ be the positive restricted root defining $\mathfrak{n}$. Then for $d = 2n+1$, or $d = 2n+2$, we fix $e_2, \dots, e_{n+1} \in i\mathfrak{b}^*$ such that the positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ are chosen as in [18, pp. 684–685] for the root system D_{n+1} resp. B_{n+1}. We let $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ be the set of roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which do not vanish on $\mathfrak{a}_{\mathbb{C}}$. The positive roots $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ are chosen such that they are restrictions of elements from $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For $i = 1, \dots, n+1$ we let $H_i \in \mathfrak{h}_{\mathbb{C}}$ be such that $e_j(H_i) = \delta_{i,j}$, $j = 1, \dots, n+1$. For $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ there exists a unique $H'_\alpha \in \mathfrak{h}_{\mathbb{C}}$ such that $B(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_{\mathbb{C}}$. One has $\alpha(H'_\alpha) \neq 0$. We let

$$H_\alpha := \frac{2}{\alpha(H'_\alpha)} H'_\alpha.$$

One easily sees that

$$H_{\pm e_i \pm e_j} = \pm H_i \pm H_j. \tag{2.3}$$

For $j = 1, \dots, n+1$ let

$$\rho_j := \begin{cases} n+1-j, & G = \mathrm{Spin}(2n+1, 1); \\ n+3/2-j, & G = \mathrm{Spin}(2n+2, 1). \end{cases} \tag{2.4}$$

Then the half-sums of positive roots ρ_G and ρ_M, respectively, are given by

$$\rho_G := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha = \sum_{j=1}^{n+1} \rho_j e_j \tag{2.5}$$

and

$$\rho_M := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \alpha = \sum_{j=2}^{n+1} \rho_j e_j. \tag{2.6}$$

Let W_G be the Weyl-group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

2.2. Let $\mathbb{Z}[\frac{1}{2}]^j$ be the set of all $(k_1, \dots, k_j) \in \mathbb{Q}^j$ such that either all k_i are integers or all k_i are half-integers. Then the finite-dimensional irreducible representations $\tau \in \hat{G}$ of G are parametrized by their highest weights

$$\Lambda(\tau) = k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}; \quad k_1(\tau) \geqslant k_2(\tau) \geqslant \cdots \geqslant k_n(\tau) \geqslant \left|k_{n+1}(\tau)\right|, \tag{2.7}$$

if $G = \operatorname{Spin}(2n+1, 1)$ resp.

$$\Lambda(\tau) = k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}; \quad k_1(\tau) \geqslant k_2(\tau) \geqslant \cdots \geqslant k_n(\tau) \geqslant k_{n+1}(\tau) \geqslant 0, \tag{2.8}$$

if $G = \operatorname{Spin}(2n+2, 1)$, where $(k_1(\tau), \dots, k_{n+1}(\tau)) \in \mathbb{Z}[\frac{1}{2}]^{n+1}$.

Moreover, the finite-dimensional irreducible representations $\nu \in \hat{K}$ of K are parametrized by their highest weights

$$\Lambda(\nu) = k_2(\nu)e_2 + \cdots + k_{n+1}(\nu)e_{n+1}; \quad k_2(\nu) \geqslant k_3(\nu) \geqslant \cdots \geqslant k_n(\nu) \geqslant k_{n+1}(\nu) \geqslant 0, \tag{2.9}$$

if $G = \operatorname{Spin}(2n+1, 1)$ resp.

$$\Lambda(\nu) = k_1(\nu)e_1 + \cdots + k_{n+1}(\nu)e_{n+1}; \quad k_1(\nu) \geqslant k_2(\nu) \geqslant \cdots \geqslant k_n(\nu) \geqslant \left|k_{n+1}(\nu)\right|, \tag{2.10}$$

if $G = \operatorname{Spin}(2n+2, 1)$, where $(k_2(\nu), \dots, k_{n+1}(\nu)), (k_1(\nu), \dots, k_{n+1}(\nu)) \in \mathbb{Z}[\frac{1}{2}]^{n,n+1}$.

Finally, the finite-dimensional irreducible representations $\sigma \in \hat{M}$ of M are parametrized by their highest weights

$$\Lambda(\sigma) = k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}; \quad k_2(\sigma) \geqslant k_3(\sigma) \geqslant \cdots \geqslant k_n(\sigma) \geqslant \left|k_{n+1}(\sigma)\right|, \tag{2.11}$$

if $G = \operatorname{Spin}(2n+1, 1)$ resp.

$$\Lambda(\sigma) = k_2(\sigma)e_1 + \cdots + k_{n+1}(\sigma)e_{n+1}; \quad k_2(\sigma) \geqslant \cdots \geqslant k_n(\sigma) \geqslant k_{n+1}(\sigma) \geqslant 0, \tag{2.12}$$

if $G = \operatorname{Spin}(2n+2, 1)$, where $(k_2(\sigma), \dots, k_{n+1}(\sigma)) \in \mathbb{Z}[\frac{1}{2}]^n$.

2.3. Let $d = 2n+1$. For $\tau \in \hat{G}$ let $\tau_\theta := \tau \circ \theta$. Let $\Lambda(\tau)$ denote the highest weight of τ as in (2.7). Then the highest weight $\Lambda(\tau_\theta)$ of τ_θ is given by

$$\Lambda(\tau_\theta) = k_1(\tau)e_1 + \cdots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}. \tag{2.13}$$

Let $\sigma \in \hat{M}$ with highest weight $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ as in (2.11). By the Weyl dimension formula [17, Theorem 4.48] we have

$$\begin{aligned}\dim(\sigma) &= \prod_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \frac{\langle \Lambda(\sigma) + \rho_M, \alpha \rangle}{\langle \rho_M, \alpha \rangle} \\ &= \prod_{i=2}^{n} \prod_{j=i+1}^{n+1} \frac{(k_i(\sigma) + \rho_i)^2 - (k_j(\sigma) + \rho_j)^2}{\rho_i^2 - \rho_j^2}. \end{aligned} \tag{2.14}$$

2.4. Let M' be the normalizer of A in K and let $W(A) = M'/M$ be the restricted Weyl-group. It has order two and it acts on the finite-dimensional representations of M as follows. Let $w_0 \in W(A)$ be the non-trivial element and let $m_0 \in M'$ be a representative of w_0. Given $\sigma \in \hat{M}$, the representation $w_0\sigma \in \hat{M}$ is defined by

$$w_0\sigma(m) = \sigma\big(m_0 m m_0^{-1}\big), \quad m \in M.$$

If $d = 2n+2$ one has $w_0\sigma \cong \sigma$ for every $\sigma \in \hat{M}$. Assume that $d = 2n+1$. Let $\Lambda(\sigma) = k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.11). Then the highest weight $\Lambda(w_0\sigma)$ of $w_0\sigma$ is given by

$$\Lambda(w_0\sigma) = k_2(\sigma)e_2 + \cdots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}. \tag{2.15}$$

2.5. Let $d = 2n+1$. Let $R(K)$ and $R(M)$ be the representation rings of K and M. Let $\iota : M \to K$ be the inclusion and let $\iota^* : R(K) \to R(M)$ be the induced map. If $R(M)^{W(A)}$ is the subring of $W(A)$-invariant elements of $R(M)$, then clearly ι^* maps $R(K)$ into $R(M)^{W(A)}$. The first part of the following proposition is due to Miatello and Vargas [29, Proposition 1]. The more precise statement is due to Bunke and Olbrich [3, Proposition 1.1].

Proposition 2.1. *The map ι is an isomorphism from $R(K)$ onto $R(M)^{W(A)}$. Explicitly, let $\sigma \in \hat{M}$ be of highest weight $\Lambda(\sigma)$ as in (2.11) and assume that $k_{n+1}(\sigma) \geqslant 0$. Then if $\nu(\sigma) \in R(K)$ is such that*

$$\iota^*\nu(\sigma) = \begin{cases} \sigma, & \sigma = w_0\sigma, \\ \sigma + w_0\sigma, & \sigma \neq w_0\sigma \end{cases}$$

one has

$$\nu(\sigma) = \sum_{\mu \in \{0,1\}^n} (-1)^{c(\mu)} \nu\big(\Lambda(\sigma) - \mu\big), \tag{2.16}$$

where the sum runs over all $\mu \in \{0,1\}^n$ such that $\Lambda(\sigma) - \mu$ is the highest weight of an irreducible representation $\nu(\Lambda(\sigma) - \mu)$ of K and $c(\mu) := \#\{1 \in \mu\}$.

Let $\sigma \in \hat{M}$ and assume that $\sigma \neq w_0\sigma$. Then by Proposition 2.1 there exist unique integers $m_\nu(\sigma) \in \{-1, 0, 1\}$, which are zero except for finitely many $\nu \in \hat{K}$, such that

$$\sigma + w_0\sigma = \sum_{\nu \in \hat{K}} m_\nu(\sigma) i^*(\nu). \tag{2.17}$$

2.6. Measures are normalized as follows. Every $a \in A$ can be written as $a = \exp\log a$, where $\log a \in \mathfrak{a}$ is unique. For $t \in \mathbb{R}$, we let $a(t) := \exp(tH_1)$. If $g \in G$, we define $n(g) \in N$, $H(g) \in \mathbb{R}$ and $\kappa(g) \in K$ by

$$g = n(g)\exp\big(H(g)e_1\big)\kappa(g).$$

Normalize the Haar measure on K such that K has volume 1. We let

$$\langle X, Y\rangle_\theta := -\frac{1}{2(d-1)} B\bigl(X, \theta(Y)\bigr). \tag{2.18}$$

We fix an isometric identification of $\mathbb{R}^{d-1}$ with $\mathfrak{n}$ with respect to the inner product $\langle\cdot,\cdot\rangle_\theta$. We give $\mathfrak{n}$ the measure induced from the Lebesgue measure under this identification. Moreover, we identify $\mathfrak{n}$ and N by the exponential map and we will denote by dn the Haar measure on N induced from the measure on $\mathfrak{n}$ under this identification. We normalize the Haar measure on G by setting

$$\int_G f(g)\, dg = \int_N \int_{\mathbb{R}} \int_K e^{-(d-1)t} f\bigl(na(t)k\bigr)\, dk\, dt\, dn. \tag{2.19}$$

The spaces $\tilde{X}$ and $\Gamma\backslash G$, Γ a discrete subgroup, will be equipped with the induced quotient-measure.

2.7. We parametrize the principal series as follows. Given $\sigma \in \hat{M}$ with $(\sigma, V_\sigma) \in \sigma$, let $\mathcal{H}^\sigma$ denote the space of measurable functions $f : K \to V_\sigma$ satisfying

$$f(mk) = \sigma(m) f(k), \quad \forall k \in K,\ \forall m \in M, \quad \text{and} \quad \int_K \bigl\| f(k)\bigr\|^2\, dk = \|f\|^2 < \infty.$$

Then for $\lambda \in \mathbb{C}$ and $f \in \mathcal{H}^\sigma$ let

$$\pi_{\sigma,\lambda}(g) f(k) := e^{(i\lambda + (d-1)/2)H(kg)} f\bigl(\kappa(kg)\bigr).$$

Recall that the representations $\pi_{\sigma,\lambda}$ are unitary iff $\lambda \in \mathbb{R}$. Moreover, for $\lambda \in \mathbb{R} - \{0\}$ and $\sigma \in \hat{M}$ the representations $\pi_{\sigma,\lambda}$ are irreducible and $\pi_{\sigma,\lambda}$ and $\pi_{\sigma',\lambda'}$, $\lambda, \lambda' \in \mathbb{C}$ are equivalent iff either $\sigma = \sigma'$, $\lambda = \lambda'$ or $\sigma' = w_0\sigma$, $\lambda' = -\lambda$. The restriction of $\pi_{\sigma,\lambda}$ to K coincides with the induced representation $\mathrm{Ind}_M^K(\sigma)$. Hence by Frobenius reciprocity [17, p. 208] for every $\nu \in \hat{K}$ one has

$$[\pi_{\sigma,\lambda} : \nu] = [\nu : \sigma]. \tag{2.20}$$

2.8. Assume that $d = 2n + 1$. For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\mu_\sigma(\lambda)$ be the Plancherel measure associated to $\pi_{\sigma,\lambda}$. Then, since $\mathrm{rk}(G) > \mathrm{rk}(K)$, $\mu_\sigma(\lambda)$ is a polynomial in λ of degree $2n$. Let $\langle\cdot,\cdot\rangle$ be the bilinear form defined by (2.2). Let $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ be the highest weight of σ as in (2.11). Then by Theorem 13.2 in [17] there exists a constant $c(n) \neq 0$ such that

$$\mu_\sigma(\lambda) = c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \frac{\langle i\lambda e_1 + \Lambda(\sigma) + \rho_M, \alpha\rangle}{\langle \rho_G, \alpha\rangle}.$$

For $z \in \mathbb{C}$ let

$$P_\sigma(z) = c(n) \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \frac{\langle z e_1 + \Lambda(\sigma) + \rho_M, \alpha\rangle}{\langle \rho_G, \alpha\rangle}. \tag{2.21}$$

One easily sees that

$$P_\sigma(z) = P_{w_0\sigma}(z). \tag{2.22}$$

3. The decomposition of the right regular representation

Let Γ be a discrete, torsion free subgroup of G with $\mathrm{vol}(\Gamma\backslash G) < \infty$. Let $\mathfrak{P}$ be a fixed set of representatives of Γ-nonequivalent proper cuspidal parabolic subgroups of G. Then $\mathfrak{P}$ is finite. Let $\kappa := \#\mathfrak{P}$. Without loss of generality we will assume that $P_0 := MAN \in \mathfrak{P}$. For every $P \in \mathfrak{P}$, there exists a $k_P \in K$ such that

$$P = N_P A_P M_P$$

with $N_P = k_P N k_P^{-1}$, $A_P = k_P A k_P^{-1}$, and $M_P = k_P M k_P^{-1}$. We let $k_{P_0} = 1$. We will assume that for each $P \in \mathfrak{P}$ one has

$$\Gamma \cap P = \Gamma \cap N_P. \tag{3.1}$$

Since N_P is abelian, we have $\Gamma \cap N_P\backslash N_P \cong T^{d-1}$, where T^{d-1} is the flat $(d-1)$-torus. For $P \in \mathfrak{P}$ let $a_P(t) := k_P a(t) k_P^{-1}$. If $g \in G$, we define $n_P(g) \in N_P$, $H_P(g) \in \mathbb{R}$ and $\kappa_P(g) \in K$ by

$$g = n_P(g) a_P\big(H_P(g)\big)\kappa_P(g). \tag{3.2}$$

For each $P \in \mathfrak{P}$ define

$$\iota_P : \mathbb{R}^+ \to A_P$$

by $\iota_P(t) := a_P(\log(t))$. For $Y > 0$, let

$$A_P^0[Y] := \iota_P(Y, \infty).$$

Then there exists $Y_0 > 0$ and, for every $Y \geqslant Y_0$, a compact connected subset $C(Y)$ of G such that in the sense of a disjoint union one has

$$G = \Gamma \cdot C(Y) \sqcup \bigsqcup_{P \in \mathfrak{P}} \Gamma \cdot N_P A_P^0[Y] K \tag{3.3}$$

and such that

$$\gamma \cdot N_P A_P^0[Y] K \cap N_P A_P^0[Y] K \neq \emptyset \quad \Leftrightarrow \quad \gamma \in \Gamma \cap N_P. \tag{3.4}$$

If for $Y \geqslant Y_0$ one lets

$$F_{P,Y} := A_P[Y] \times \Gamma \cap N_P\backslash N_P \cong [Y, \infty) \times \Gamma \cap N_P\backslash N_P, \tag{3.5}$$

it follows from (3.3) and (3.4) that there exists a compact manifold $X(Y)$ with smooth boundary such that X has a decomposition as

$$X = X(Y) \cup \bigsqcup_{P \in \mathfrak{P}} F_{P,Y} \tag{3.6}$$

with $X(Y) \cap F_{P,Y} = \partial X(Y) = \partial F_{P,Y}$ and $F_{P,Y} \cap F_{P',Y} = \emptyset$ if $P \neq P'$.

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$. We shall now describe some basic properties of R_Γ. The main references are [22,14,42]. There exists an orthogonal decomposition

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G) \tag{3.7}$$

of $L^2(\Gamma\backslash G)$ into closed R_Γ-invariant subspaces. The restriction of R_Γ to $L^2_d(\Gamma\backslash G)$ decomposes into the orthogonal direct sum of irreducible unitary representations of G and the multiplicity of each irreducible unitary representation of G in this decomposition is finite. On the other hand, by the theory of Eisenstein series, the restriction R^c_Γ of R_Γ to $L^2_c(\Gamma\backslash G)$ is isomorphic to the direct integral over all unitary principal series representations of G.

Next we recall the definition and some of the basic properties of the Eisenstein series. For $P = M_P A_p N_P \in \mathfrak{P}$ let $\mathcal{E}_P$ be the space of all functions on G which are measurable and left-invariant under $(\Gamma \cap P)N_P A_P$ and whose restriction to K is square-integrable. We turn $\mathcal{E}_P$ into a Hilbert space using the inner product

$$\langle \Phi, \Psi \rangle := \operatorname{vol}(\Gamma \cap N_P\backslash N_P) \int_K \Phi(k)\overline{\Psi}(k)\,dk.$$

For each $\lambda \in \mathbb{C}$ there is a representation $\pi_{P,\lambda}$ of G on $\mathcal{E}_P$, defined by

$$\big(\pi_{P,\lambda}(y)\Phi\big)(x) = e^{(\lambda+(d-1)/2)(H_P(xy))} e^{-(\lambda+(d-1)/2)(H_P(x))} \Phi(xy).$$

Given $\Phi \in \mathcal{E}_P$ and $\lambda \in \mathbb{C}$, put

$$\Phi_\lambda(x) = e^{(\lambda+(d-1)/2)H_P(x)} \Phi(x).$$

The action of the representation $\pi_{P,\lambda}$ is then given by

$$\big(\pi_{P,\lambda}(y)\Phi\big)_\lambda(x) = \Phi_\lambda(xy),$$

and $\pi_{P,\lambda}$ is unitary for $\lambda \in i\mathbb{R}$. Let $\mathcal{E}^0_P$ be the subspace of $\mathcal{E}_P$ consisting of all right K-finite and left $\mathfrak{Z}_M$-finite functions, where $\mathfrak{Z}_M$ denotes the center of the universal enveloping algebra of $\mathfrak{m}_\mathbb{C}$. For $\Phi \in \mathcal{E}^0_P$ and $\lambda \in \mathbb{C}$ the Eisenstein series $E(P, \Phi, \lambda, x)$ is defined by

$$E(P, \Phi, \lambda, x) = \sum_{\gamma \in \Gamma \cap P\backslash \Gamma} \Phi_\lambda(\gamma x).$$

It converges absolutely and uniformly on compact subsets of $\{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) > (d-1)/2\} \times G$, and it has a meromorphic extension to $\mathbb{C}$. Let $P' \in \mathfrak{P}$. The constant term $E_{P'}(P, \Phi, \lambda)$ of $E(P, \Phi, \lambda)$ along P' is defined by

$$E_{P'}(P, \Phi, \lambda, x) := \frac{1}{\operatorname{vol}(\Gamma \cap N_{P'} \backslash N_{P'})} \int\limits_{\Gamma \cap N_{P'} \backslash N_{P'}} E\big(P, \Phi, \lambda, n'x\big)\, dn'. \tag{3.8}$$

Let $W(A_P, A_{P'})$ be the set of all bijections $w : A_P \to A_{P'}$ for which there exists $x \in G$ such that $w(a) = xax^{-1}$, $a \in A_P$. Then one can identify $W(A_P, A_{P'})$ with $k_{P'} W(A) k_P^{-1}$. Thus $W(A_P, A_{P'})$ has order 2. We let $W(A_P, A_{P'})$ act on $\mathbb{C}$ as follows. For $w = k_{P'} k_P^{-1}$ and $\lambda \in \mathbb{C}$ we put $w\lambda := \lambda$. Let w_0 be the non-trivial element of $W(A)$. Then for $w = k_{P'} w_0 k_P^{-1}$ and $\lambda \in \mathbb{C}$ we put $w\lambda := -\lambda$. Then one has

$$E_{P'}(P, \Phi, \lambda, x) = \sum_{w \in W(A_P, A_{P'})} e^{(w\lambda + (d-1)/2)(H_{P'}(x))} \big(c_{P'|P}(w : \lambda)\Phi\big)(x), \tag{3.9}$$

where

$$c_{P'|P}(w : \lambda) : \mathcal{E}_P \to \mathcal{E}_{P'}$$

are linear maps which are meromorphic functions of $\lambda \in \mathbb{C}$. Put

$$\boldsymbol{\mathcal{E}} = \bigoplus_{P \in \mathfrak{P}} \mathcal{E}_P, \qquad \boldsymbol{\pi}_\lambda = \bigoplus_{P \in \mathfrak{P}} \pi_{P,\lambda}.$$

Then $\boldsymbol{\pi}_\lambda$ acts on $\boldsymbol{\mathcal{E}}$ as an induced representation. For $\boldsymbol{\Phi} = (\Phi_P) \in \boldsymbol{\mathcal{E}}$ and $\lambda \in \mathbb{C}$ put

$$E(\boldsymbol{\Phi}, \lambda, x) = \sum_{P \in \mathfrak{P}} E(P, \Phi_P, \lambda, x).$$

Let $\boldsymbol{\mathcal{E}}^0 = \bigoplus_{P \in \mathfrak{P}} \mathcal{E}_P^0$. Let w_0 be the non-trivial element of $W(A)$. Then the operators $c_{P'|P}(k'_P w_0 k_P^{-1} : \lambda)$ can be combined into a linear operator

$$\mathbf{C}(\lambda) : \boldsymbol{\mathcal{E}}^0 \to \boldsymbol{\mathcal{E}}^0,$$

which is a meromorphic function of λ.

The space $\boldsymbol{\mathcal{E}}^0$ decomposes into the direct sum of finite-dimensional subspaces as follows. Let $P = M_P A_P N_P$ be a Γ-cuspidal proper parabolic subgroup. For $\sigma_P \in \hat{M}_P$ and $\nu \in \hat{K}$ let $\mathcal{E}(\sigma_P, \nu)$ be the space of all continuous functions $\Phi : (\Gamma \cap P) A_P N_P \backslash G \to \mathbb{C}$ such that for all $x \in G$ the function $m \in M_P \mapsto \Phi(mx)$ belongs to the σ_P-isotypical subspace of the right regular representation of M and for all $x \in G$ the function $k \in K \mapsto \Phi(xk)$ belongs to the ν-isotypical subspace of the right regular representation of K. For $\sigma \in \hat{M}$ set

$$\boldsymbol{\mathcal{E}}(\sigma, \nu) := \bigoplus_{P \in \mathfrak{P}} \mathcal{E}(\sigma_P, \nu),$$

where $\sigma_P \in \hat{M}_P$ is obtained from σ by conjugation. Each $\boldsymbol{\mathcal{E}}(\sigma,\nu)$ is finite-dimensional. Furthermore, let

$$\boldsymbol{\mathcal{E}}(\sigma) := \bigoplus_{\nu\in\hat{K}} \boldsymbol{\mathcal{E}}(\sigma,\nu).$$

Then $\boldsymbol{\mathcal{E}}(\sigma)$ is invariant under $\boldsymbol{\pi}_\lambda$ and the restriction of $\boldsymbol{\pi}_\lambda$ to $\boldsymbol{\mathcal{E}}(\sigma)$ will be denoted by $\boldsymbol{\pi}_{\sigma,\lambda}$. Now consider an orbit $\vartheta \in W(A)\backslash\hat{M}$. Let $\vartheta = \{\sigma, w\sigma\}$. Put

$$\boldsymbol{\mathcal{E}}(\vartheta,\nu) := \begin{cases} \boldsymbol{\mathcal{E}}(\sigma,\nu), & w\sigma = \sigma, \\ \boldsymbol{\mathcal{E}}(\sigma,\nu) \oplus \boldsymbol{\mathcal{E}}(w\sigma,\nu), & w\sigma \neq \sigma. \end{cases}$$

Then it follows that

$$\boldsymbol{\mathcal{E}}^0 = \bigoplus_{\vartheta,\nu} \boldsymbol{\mathcal{E}}(\vartheta,\nu), \tag{3.10}$$

where ϑ runs over $W(A)\backslash\hat{M}$ and ν over $\hat{K}$. The operator $\mathbf{C}(\lambda)$ preserves this decomposition. For $\vartheta \in W(A)\backslash\hat{M}$, $\nu \in \hat{K}$ and $\lambda \in \mathbb{C}$ let

$$\mathbf{C}(\vartheta,\nu,\lambda) : \boldsymbol{\mathcal{E}}(\vartheta,\nu) \to \boldsymbol{\mathcal{E}}(\vartheta,\nu) \tag{3.11}$$

be the restriction of $\mathbf{C}(\lambda)$. We note that for $\vartheta = \{\sigma, w\sigma\}$, $\mathbf{C}(\vartheta,\nu,\lambda)$ maps $\boldsymbol{\mathcal{E}}(\sigma,\nu)$ into $\boldsymbol{\mathcal{E}}(w\sigma,\nu)$. We denote the corresponding operator by

$$\mathbf{C}(\sigma,\nu,\lambda) : \boldsymbol{\mathcal{E}}(\sigma,\nu) \to \boldsymbol{\mathcal{E}}(w\sigma,\nu). \tag{3.12}$$

Taking the direct sum with respect to $\nu \in \hat{K}$, we get operators

$$\mathbf{C}(\sigma,\lambda) : \boldsymbol{\mathcal{E}}(\sigma) \to \boldsymbol{\mathcal{E}}(w\sigma). \tag{3.13}$$

Next we recall the functional equations satisfied by E and $\mathbf{C}$. For $\boldsymbol{\Phi} \in \boldsymbol{\mathcal{E}}^0$ and $\lambda \in \mathbb{C}$ we have

$$E(\boldsymbol{\Phi},\lambda) = E\big(\mathbf{C}(\lambda)\boldsymbol{\Phi}, -\lambda\big), \tag{3.14}$$

and

$$\mathbf{C}(\lambda)\mathbf{C}(-\lambda) = \mathrm{Id}. \tag{3.15}$$

Furthermore, let $f \in C_c^\infty(G)$ be right K-finite. Then $\boldsymbol{\pi}_\lambda(f)$ acts on $\boldsymbol{\mathcal{E}}^0$ and we have

$$\mathbf{C}(\lambda)\boldsymbol{\pi}_\lambda(f) = \boldsymbol{\pi}_{-\lambda}(f)\mathbf{C}(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.16}$$

Thus $\mathbf{C}(\lambda)$ is an intertwining operator for the induced representation $\boldsymbol{\pi}_\lambda$.

Now we come to the relation with the spectral resolution of R_Γ^c. For $P = M_P A_P N_P \in \mathfrak{P}$ let R_{M_P} denote the right regular representation of M_P on $L^2(M_P)$. Since M_P is compact, it decomposes discretely as

$$R_{M_P} = \bigoplus_{\sigma_P \in \hat{M}_P} d(\sigma_P)\sigma_P, \tag{3.17}$$

where $d(\sigma_P) = \dim(\sigma_P)$. For $\lambda \in \mathbb{C}$ let $\xi_\lambda : A_P \to \mathbb{C}$ be the quasi-character given by $\xi_\lambda(a_P(t)) := e^{t\lambda}$. Let $\operatorname{Ind}_P^G(R_{M_P}, \lambda)$ be the representation of G induced from $R_{M_P} \otimes \xi_{\lambda+(d-1)/2}$. Then we have

$$\pi_{P,\lambda} \cong \operatorname{Ind}_P^G(R_{M_P}, \lambda). \tag{3.18}$$

The theory of Eisenstein series implies that

$$R_\Gamma^c \cong \bigoplus_{P \in \mathfrak{P}} \int_{\mathbb{R}} \pi_{P,i\lambda}\, d\lambda = \int_{\mathbb{R}} \boldsymbol{\pi}_{i\lambda}\, d\lambda.$$

Using the decomposition (3.17), the induced representation decomposes correspondingly into the direct sum of principal series representations $\pi_{\sigma,\lambda}$. This gives the spectral resolution of R_Γ^c (see [42, Section 3]).

Now let α be a K-finite Schwarz function. Define an operator $R_\Gamma(\alpha)$ on $L^2(\Gamma\backslash G)$ by

$$R_\Gamma(\alpha)\phi(x) := \int_G \alpha(g)\phi(xg)\, dg, \quad \phi \in L^2(\Gamma\backslash G). \tag{3.19}$$

Then $R_\Gamma(\alpha)$ is an integral operator with smooth kernel $K_\alpha(x, y)$. Moreover, the decomposition of R_Γ in (3.7) induces a decomposition of the operator $R_\Gamma(\alpha)$ as

$$R_\Gamma(\alpha) = R_\Gamma^d(\alpha) \oplus R_\Gamma^c(\alpha).$$

It turns out that $R_\Gamma^c(\alpha)$ is again an integral operator with smooth kernel which can be computed explicitly in terms of Eisenstein series as follows. Let $\{e_n: n \in I\}$ be an orthonormal basis of $\boldsymbol{\mathcal{E}}$ which is adapted to the decomposition (3.10), i.e., each e_n belongs to some subspace $\boldsymbol{\mathcal{E}}(\vartheta, \nu)$. The following proposition is the main result about the spectral resolution of the kernel.

Proposition 3.1. *Let α be a K-finite function in $\mathcal{C}^1(G)$. Then $R_\Gamma^c(\alpha)$ is an integral operator with kernel $K_\alpha^c(x, y)$ given by*

$$K_\alpha^c(x, y) = \frac{1}{4\pi} \sum_{m,n \in I} \int_{\mathbb{R}} \langle \boldsymbol{\pi}_{i\lambda}(\alpha)e_m, e_n\rangle E(e_n, i\lambda, x)\overline{E(e_m, i\lambda, y)}\, d\lambda. \tag{3.20}$$

Furthermore, the kernel $K_\alpha^d = K_\alpha - K_\alpha^c$ is integrable over the diagonal, the operator $R_\Gamma^d(\alpha)$ is of trace class and its trace is given by

$$\operatorname{Tr}\big(R_\Gamma^d(\alpha)\big) = \int_{\Gamma\backslash G} K_\alpha^d(x, x)\, dx.$$

Proof. See [42, Theorem 4.7]. $\square$

The Eisenstein series are not square integrable. However, the truncated Eisenstein series, which are obtained by subtracting the constant terms in each cusp, are square integrable. Their inner product gives rise to the Maass–Selberg relations which we recall next.

Let $Y_0 > 0$ be such that (3.3) holds. Let $Y \geqslant Y_0$. For $P \in \mathfrak{P}$ let $\chi_{P,Y}$ be the characteristic function of $N_P A_P^0[Y]K \subset G$. Let $\Phi \in \boldsymbol{\mathcal{E}}^0$. For $Y \geqslant Y_0$ put

$$E^Y(\Phi,\lambda,x) := E(\Phi,\lambda,x) - \sum_{P\in\mathfrak{P}}\sum_{\gamma\in\Gamma\cap N_P\backslash\Gamma} \chi_{P,Y}(\gamma g)E_P(\Phi,\lambda,\gamma g),$$

where $E_P(\Phi,\lambda,x)$ is as in (3.8). By (3.4) at most one summand in this sum is not zero. By [14] the function $E^Y(\Phi,\lambda)$ belongs to $L^2(\Gamma\backslash G)$. Now we have the following proposition.

Proposition 3.2. *Let $\Phi,\Psi \in \boldsymbol{\mathcal{E}}^0$ and $\lambda \in \mathbb{C}$. Then one has*

$$\begin{aligned}&\int_{\Gamma\backslash G} E^Y(\Phi,i\lambda,x)\overline{E^Y}(\Psi,i\lambda,x)\,dx\\ &\quad = -\left\langle \mathbf{C}(-i\lambda)\frac{d}{dz}\mathbf{C}(i\lambda)\Phi,\Psi\right\rangle + 2\langle\Phi,\Psi\rangle\log Y + \frac{Y^{2i\lambda}}{2i\lambda}\bigl\langle\Phi,\mathbf{C}(i\lambda)\Psi\bigr\rangle - \frac{Y^{-2i\lambda}}{2i\lambda}\bigl\langle\mathbf{C}(i\lambda)\Phi,\Psi\bigr\rangle.\end{aligned}$$

At the end of this section, we remark that the space $L^2_d(\Gamma\backslash G)$ admits a further decomposition

$$L^2_d(\Gamma\backslash G) = L^2_{\mathrm{cusp}}(\Gamma\backslash G) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G). \tag{3.21}$$

Here $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ is the space spanned by the cusp forms, i.e. the square integrable functions f, which for all $P \in \mathfrak{P}$ satisfy

$$f_P^0(x) := \int_{\Gamma\cap N_P\backslash N_P} f(nx)\,dn = 0 \quad \text{for almost all } x \in G.$$

One does not know much about $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ and its size in general. On the other hand, let $\Phi \in \boldsymbol{\mathcal{E}}(\boldsymbol{\sigma},\nu)$. Let $s_0 \in (0,n]$ be a pole of $E(\Phi,s)$. Then the function $x \mapsto \operatorname{Res}|_{s=s_0}E(\Phi,s)$ is square integrable on $\Gamma\backslash G$ and $L^2_{\mathrm{res}}(\Gamma\backslash G)$ is spanned by all these residues of Eisenstein series.

4. Bochner Laplace operators

Regard G as a principal K-fibre bundle over $\tilde{X}$. By the invariance of $\mathfrak{p}$ under $\mathrm{Ad}(K)$, the assignment

$$T_g^{\mathrm{hor}} := \left\{\frac{d}{dt}\bigg|_{t=0} g\exp tX\colon\ X \in \mathfrak{p}\right\}$$

defines a horizontal distribution on G. This connection is called the canonical connection. Let ν be a finite-dimensional unitary representation of K on $(V_\nu,\langle\cdot,\cdot\rangle_\nu)$. Let

$$\tilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a G-invariant metric $\tilde{B}_\nu$ on $\tilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection. Then $\widetilde{\nabla}^\nu$ is G-invariant. Let

$$E_\nu := \Gamma\backslash(G \times_\nu V_\nu)$$

be the associated locally homogeneous bundle over X. Since $\tilde{B}_\nu$ and $\widetilde{\nabla}^\nu$ are G-invariant, they push down to a metric B_ν and a connection ∇^ν on E_ν. Let

$$C^\infty(G,\nu) := \bigl\{f : G \to V_\nu\colon\ f \in C^\infty,\ f(gk) = \nu\bigl(k^{-1}\bigr)f(g),\ \forall g \in G,\ \forall k \in K\bigr\}. \tag{4.1}$$

Let

$$C^\infty(\Gamma\backslash G,\nu) := \bigl\{f \in C^\infty(G,\nu)\colon\ f(\gamma g) = f(g),\ \forall g \in G,\ \forall \gamma \in \Gamma\bigr\}. \tag{4.2}$$

Let $C^\infty(X, E_\nu)$ denote the space of smooth sections of E_ν. Then there is a canonical isomorphism

$$A : C^\infty(X, E_\nu) \cong C^\infty(\Gamma\backslash G,\nu)$$

(see [28, p. 4]). There is also a corresponding isometry for the space $L^2(X, E_\nu)$ of L^2-sections of E_ν. For every $X \in \mathfrak{g}$, $g \in G$ and every $f \in C^\infty(X, E_\nu)$ one has

$$A\bigl(\nabla^\nu_{L(g)_* X} f\bigr)(g) = \frac{d}{dt}\bigg|_{t=0} Af(g \exp tX).$$

Let $\widetilde{\Delta}_\nu = \widetilde{\nabla}^{\nu *}\widetilde{\nabla}^\nu$ be the Bochner–Laplace operator of $\tilde{E}_\nu$. Since $\tilde{X}$ is complete, $\widetilde{\Delta}_\nu$ with domain the smooth compactly supported sections is essentially self-adjoint [5]. Its self-adjoint extension will be denoted by $\widetilde{\Delta}_\nu$ too. Let R be the regular representation of $Z(\mathfrak{g})$ on $C^\infty(G,\nu)$. Then by [28, Proposition 1.1] it follows that on $C^\infty(G,\nu)$ one has

$$\widetilde{\Delta}_\nu = -R(\Omega) + \nu(\Omega_K), \tag{4.3}$$

where Ω_K is the Casimir operator of $\mathfrak{k}$ with respect to the restriction of the normalized Killing form of $\mathfrak{g}$ to $\mathfrak{k}$. Let $\tilde{A}_\nu$ be the differential operator on E_ν which acts as $-R_\Gamma(\Omega)$ on $C^\infty(G,\nu)$. Then it follows from (4.3) that $\tilde{A}_\nu$ is bounded from below and is essentially self-adjoint. Its self-adjoint extension will be denoted by $\tilde{A}_\nu$ too. Let $e^{-t\tilde{A}_\nu}$ be the corresponding heat semigroup on $L^2(G,\nu)$, where $L^2(G,\nu)$ is defined analogously to (4.1). Then the same arguments as in [4, Section 1] imply that there exists a function

$$K_t^\nu \in C^\infty\bigl(G \times G, \operatorname{End}(V_\nu)\bigr), \tag{4.4}$$

with the following properties: $K_t^\nu(g,g')$ is symmetric in the G-variables, for each $g \in G$, the function $g' \mapsto K_t^\nu(g,g')$ belongs to $L^2(G, \mathrm{End}(V_\nu))$, it satisfies

$$K_t^\nu(gk, g'k') = \nu(k^{-1}) K_t^\nu(g,g') \nu(k'), \quad \forall g, g' \in G, \ \forall k, k' \in K$$

and it is the kernel of the heat operator, i.e.,

$$\left(e^{-t\tilde{A}_\nu}\phi\right)(g) = \int_G K_t^\nu(g,g')\phi(g')\,dg', \quad \forall \phi \in L^2(G,\nu).$$

Since Ω is G-invariant, K_t^ν is invariant under the diagonal action of G. Hence there exists a function

$$H_t^\nu : G \to \mathrm{End}(V_\nu)$$

which satisfies

$$H_t^\nu(k^{-1}gk') = \nu(k)^{-1} \circ H_t^\nu(g) \circ \nu(k'), \quad \forall k, k' \in K, \ \forall g \in G, \tag{4.5}$$

such that

$$K_t^\nu(g,g') = H_t^\nu(g^{-1}g'), \quad \forall g, g' \in G. \tag{4.6}$$

Thus one has

$$\left(e^{-t\tilde{A}_\nu}\phi\right)(g) = \int_G H_t^\nu(g^{-1}g')\phi(g')\,dg', \quad \phi \in L^2(G,\nu), \ g \in G. \tag{4.7}$$

By the arguments of [1, Proposition 2.4], H_t^ν belongs to all Harish-Chandra Schwartz spaces $(\mathcal{C}^q(G) \otimes \mathrm{End}(V_\nu))$, $q > 0$.

Now we pass to the quotient $X = \Gamma\backslash\tilde{X}$. Let $\Delta_\nu = \nabla^{\nu *}\nabla^\nu$ the closure of the Bochner–Laplace operator with domain the smooth compactly supported sections of E_ν. Then Δ_ν is self-adjoint and by (4.3) it induces the operator $-R_\Gamma(\Omega) + \nu(\Omega_K)$ on $C^\infty(\Gamma\backslash G, \nu)$. Thus if we let A_ν be the operator $-R_\Gamma(\Omega)$ on $C_c^\infty(\Gamma\backslash G, \nu)$, then A_ν is bounded from below and is essentially self-adjoint. The closure of A_ν will be denoted by A_ν too. Let e^{-tA_ν} be the heat semigroup of A_ν on $L^2(\Gamma\backslash G, \nu)$. Let

$$H^\nu(t; x, x') := \sum_{\gamma \in \Gamma} H_t^\nu(g^{-1}\gamma g'), \tag{4.8}$$

where $x, x' \in \Gamma\backslash G$, $x = \Gamma g$, $x' = \Gamma g'$. By [42, Chapter 4] this series converges absolutely and locally uniformly. It follows from (4.7) that

$$\left(e^{-tA_\nu}\phi\right)(x) = \int_{\Gamma\backslash G} H^\nu(t; x, x')\phi(x')\,dx', \quad \phi \in L^2(\Gamma\backslash G, \nu), \ x \in \Gamma\backslash G.$$

Put

$$h_t^\nu(g) := \operatorname{tr} H_t^\nu(g), \tag{4.9}$$

where tr denotes the trace in End V_ν. Define the operator $R_\Gamma(h_t^\nu)$ on $L^2(\Gamma\backslash G)$ as in (3.19). Then $R_\Gamma(h_t^\nu)$ is an integral operator on $L^2(\Gamma\backslash G)$, whose kernel is given by

$$h^\nu(t; x, x') := \operatorname{tr} H^\nu(t; x, x'). \tag{4.10}$$

We shall now compute the Fourier transform of h_t^ν. Let π be a unitary admissible representation of G on a Hilbert space $\mathcal{H}_\pi$. Let $\check{\nu}$ be the contragredient representation of ν and let $P_{\check{\nu}}(\pi)$ be the projection of $\mathcal{H}_\pi$ onto $\mathcal{H}_\pi^{\check{\nu}}$, the $\check{\nu}$-isotypical component of $\mathcal{H}_\pi$. By assumption $\mathcal{H}_\pi^{\check{\nu}}$ is finite-dimensional. Furthermore, it easily follows from (4.5) and the Schur orthogonality relations [18, Corollary 4.10] that

$$\pi(h_t^\nu) = P_{\check{\nu}}(\pi)\pi(h_t^\nu)P_{\check{\nu}}(\pi). \tag{4.11}$$

The restriction of $\pi(h_t^\nu)$ to $\mathcal{H}_\pi^{\check{\nu}}$ will be denoted by $\pi(h_t^\nu)$ too. Define a bounded operator $\tilde{\pi}(H_t^\nu)$ on $\mathcal{H}_\pi \otimes V_\nu$ by

$$\tilde{\pi}(H_t^\nu)(g) := \int_G \pi(g) \otimes H_t^\nu(g)\, dg. \tag{4.12}$$

Then relative to the splitting

$$\mathcal{H}_\pi \otimes V_\nu = (\mathcal{H}_\pi \otimes V_\nu)^K \oplus \big((\mathcal{H}_\pi \otimes V_\nu)^K\big)^\perp,$$

$\tilde{\pi}(H_t^\nu)$ has the form

$$\begin{pmatrix} \pi(H_t^\nu) & 0 \\ 0 & 0 \end{pmatrix},$$

where $\pi(H_t^\nu)$ acts on $(\mathcal{H}_\pi \otimes V_\nu)^K$. It follows as in [1, Corollary 2.2] that

$$\pi(H_t^\nu) = e^{t\pi(\Omega)}\, \mathrm{Id}, \tag{4.13}$$

where Id is the identity on $(\mathcal{H}_\pi \otimes V_\nu)^K$. Now let $A : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_K$. Then $A \circ \pi(h_t^\nu)$ is again a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi \otimes V_\nu$ by $\tilde{A} := A \otimes \mathrm{Id}$. Then by the same argument as in [1, Lemma 5.1] one has

$$\operatorname{Tr}\big(\tilde{A} \circ \tilde{\pi}(H_t^\nu)\big) = \operatorname{Tr}\big(A \circ \pi(h_t^\nu)\big). \tag{4.14}$$

Together with (4.13) we obtain

$$\operatorname{Tr}\big(A \circ \pi(h_t^\nu)\big) = e^{t\pi(\Omega)} \cdot \operatorname{Tr} \tilde{A}|_{(\mathcal{H}_\pi \otimes V_\nu)^K}. \tag{4.15}$$

Let $\pi \in \hat{G}$ and let Θ_π be its global character. Taking $A = \mathrm{Id}$ in (4.15), one obtains

$$\Theta_\pi\left(h_t^\nu\right) = e^{t\pi(\Omega)} \cdot \dim(\mathcal{H}_\pi \otimes V_\nu)^K = e^{t\pi(\Omega)} \cdot [\pi : \check{\nu}].$$

Now note that if d is odd, we have $\check{\nu} \cong \nu$ for every $\nu \in \hat{K}$ and if d is even we have $\check{\sigma} \cong \sigma$ for every $\sigma \in \hat{M}$, see for example [12, Section 3.2.5]. Thus, in any case we have $[\check{\nu} : \sigma] = [\nu : \sigma]$. Moreover, by [12, Theorems 8.1.3, 8.1.4] we have $[\nu : \sigma] \leqslant 1$ for all $\nu \in \hat{K}$ and all $\sigma \in \hat{M}$. Now consider the principal series representation $\pi_{\sigma,\lambda}$, where $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$. Let $\Theta_{\sigma,\lambda}$ be the global character of $\pi_{\sigma,\lambda}$. For all $\nu \in \hat{K}$ one has

$$[\pi_{\sigma,\lambda} : \nu] = [\nu : \sigma]$$

by Frobenius reciprocity [17, p. 208]. Hence for $[\nu : \sigma] \neq 0$ one has

$$\Theta_{\sigma,\lambda}\left(h_t^\nu\right) = e^{t\pi_{\sigma,\lambda}(\Omega)}$$

and one has $\Theta_{\sigma,\lambda}(h_t^\nu) = 0$ for $[\nu : \sigma] = 0$. The Casimir eigenvalue can be computed as follows. For $\sigma \in \hat{M}$ with highest weight given by (2.11) resp. (2.12), let

$$c(\sigma) := \sum_{j=2}^{n+1}\left(k_j(\sigma) + \rho_j\right)^2 - \sum_{j=1}^{n+1}\rho_j^2. \tag{4.16}$$

Then one has

$$\pi_{\sigma,\lambda}(\Omega) = -\lambda^2 + c(\sigma). \tag{4.17}$$

For $G = \mathrm{Spin}(2n + 1, 1)$ this was proved in [34, Corollary 2.4]. For $G = \mathrm{Spin}(2n + 2, 1)$, one can proceed in the same way. Thus we obtain the following proposition.

Proposition 4.1. *For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\Theta_{\sigma,\lambda}$ be the global character of $\pi_{\sigma,\lambda}$. Let $c(\sigma)$ be defined by* (4.16)*. Then one has*

$$\Theta_{\sigma,\lambda}\left(h_t^\nu\right) = e^{t(c(\sigma)-\lambda^2)}$$

for $[\nu : \sigma] \neq 0$ and $\Theta_{\sigma,\lambda}(h_t^\nu) = 0$ otherwise.

Finally, by (3.18), (4.17) also gives

$$\boldsymbol{\pi}_{\sigma,\lambda}(\Omega) = \lambda^2 + c(\sigma). \tag{4.18}$$

5. The regularized trace

In this section we define the regularized trace of the heat operator. The decomposition (3.7) induces a decomposition of $L^2(\Gamma\backslash G,\nu)\cong(L^2(\Gamma\backslash G,\nu)\otimes V_\nu)^K$ as

$$L^2(\Gamma\backslash G,\nu)=L^2_d(\Gamma\backslash G,\nu)\oplus L^2_c(\Gamma\backslash G,\nu). \tag{5.1}$$

Let A_ν be the operator induced by $-R_\Gamma(\Omega)$ on $C_c^\infty(\Gamma\backslash G)$. The decomposition (5.1) is invariant under A_ν in the sense of unbounded operators. Let A_ν^d denote the restriction of A_ν to $L^2_d(\Gamma\backslash G,\nu)$. Then the spectrum of A_ν^d is discrete. Let $\lambda_1\leqslant\lambda_2\leqslant\cdots$ be the sequence of eigenvalues of A_ν^d, counted with multiplicities. This sequence may be finite or infinite. For $\lambda\in[0,\infty)$ let

$$N(\lambda):=\#\{j\colon\ \lambda_j\leqslant\lambda\}$$

be the counting function of eigenvalues. By [30, Theorem 0.1] there exists $C>0$ such that

$$N(\lambda)\leqslant C\bigl(1+\lambda^{2d}\bigr) \tag{5.2}$$

for all $\lambda\geqslant 0$. In fact, in the present case, the exponent is $d/2$. This follows from an estimation of the counting function of the cuspidal eigenvalues, which can be obtained by adapting [7, Theorem I.1] and its proof to the case of a locally homogeneous vector bundle, and the fact that the residual spectrum is finite in the present case. Hence the sum $\sum_j e^{-t\lambda_j}$ converges for all $t>0$, the operator $e^{-tA_\nu^d}$ is of trace class and one has

$$\operatorname{Tr}\bigl(e^{-tA_\nu^d}\bigr)=\sum_j e^{-t\lambda_j}. \tag{5.3}$$

Let H_t^ν be the kernel of $e^{-t\tilde{A}_\nu}$ and let $h_t^\nu=\operatorname{tr}H_t^\nu$. Then h_t^ν belongs to $\mathcal{C}^1(G)$. Let $h^\nu(t;x,y)$ be the kernel of $R_\Gamma(h_t^\nu)$. By Proposition 3.1, the kernel $h_c^\nu(t;x,y)$ of $R_\Gamma^c(h_t^\nu)$ is given by

$$h_c^\nu(t;x,y)=\frac{1}{4\pi}\sum_{k,l}\int_{\mathbb{R}}\bigl\langle\boldsymbol{\pi}_{i\lambda}\bigl(h_t^\nu\bigr)e_l,e_k\bigr\rangle E(e_k,i\lambda,x)\overline{E}(e_l,i\lambda,y)\,d\lambda, \tag{5.4}$$

where $\{e_k\colon\ k\in I\}$ is an orthonormal basis of $\boldsymbol{\mathcal{E}}$ adapted to the decomposition (3.10). Let

$$h_d^\nu(t;x,y)=h^\nu(t;x,y)-h_c^\nu(t;x,y). \tag{5.5}$$

By the second part of Proposition 3.1, h_d^ν is the kernel of $R_\Gamma^d(h_t^\nu)$ and we have

$$\operatorname{Tr}\bigl(e^{-tA_\nu^d}\bigr)=\operatorname{Tr}\bigl(R_\Gamma^d\bigl(h_t^\nu\bigr)\bigr)=\int_{\Gamma\backslash G}h_d^\nu(t;x,x)\,dx. \tag{5.6}$$

Now the argument on p. 82 in [42] can be extended to $h_t^\nu \in \mathcal{C}(G)$ and one has

$$\int_{\mathbb{R}} \int_{\Gamma\backslash G} \left| \sum_{k,l} \langle \boldsymbol{\pi}_{i\lambda}\big(h_t^\nu\big) e_l, e_k \rangle E^Y(e_k, i\lambda, x) \overline{E}^Y(e_l, i\lambda, x) \right| dx\, d\lambda < \infty.$$

Thus one can apply Proposition 3.2 and interchange the order of integration. Let $\mathbf{C}(\sigma, \nu, \lambda)$ be the operator (3.12). Arguing now as in [42, pp. 82–84] and using Proposition 4.1 one obtains

$$\begin{aligned}\int_{X(Y)} h_c^\nu(t; x, x)\, dx = {}& \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} \frac{\operatorname{Tr}(\boldsymbol{\pi}_{\sigma,0}(h_t^\nu)\mathbf{C}(\sigma,\nu,0))}{4} + \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \left(\frac{\kappa e^{tc(\sigma)} \log Y \dim(\sigma)}{\sqrt{4\pi t}} \right.\\ & \left. - \frac{1}{4\pi} \int_{\mathbb{R}} \operatorname{Tr}\left(\boldsymbol{\pi}_{\sigma,i\lambda}\big(h_t^\nu\big) \mathbf{C}(\sigma,\nu,-i\lambda) \frac{d}{dz} \mathbf{C}(\sigma,\nu,i\lambda)\right) d\lambda \right) + o(1),\end{aligned}$$

as $Y \to \infty$. Now recall that the restriction of the representation $\boldsymbol{\pi}_{\sigma,i\lambda}$ to K is independent of the parameter λ. Let

$$\tilde{\boldsymbol{C}}(\sigma,\nu,\lambda) : \big(\boldsymbol{\mathcal{E}}(\sigma) \otimes V_\nu\big)^K \to \big(\boldsymbol{\mathcal{E}}(w\sigma) \otimes V_\nu\big)^K$$

be the restriction of $\boldsymbol{C}(\sigma,\lambda) \otimes \operatorname{Id}_{V_\nu}$ to $(\boldsymbol{\mathcal{E}}(\sigma)\otimes V_\nu)^K$, where $\boldsymbol{C}(\sigma,\lambda)$ is the operator (3.13). Using the intertwining property of $\boldsymbol{C}(\sigma,\lambda)$, Eqs. (4.15) and (4.18) one obtains

$$\begin{aligned}\int_{X(Y)} h_c^\nu(t; x, x)\, dx = {}& \sum_{\substack{\sigma\in\hat{M};\,\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)} \frac{\operatorname{Tr}(\tilde{\boldsymbol{C}}(\sigma,\nu,0))}{4} + \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \left(\frac{\kappa e^{tc(\sigma)} \log Y \dim(\sigma)}{\sqrt{4\pi t}} \right.\\ & \left. - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-t(\lambda^2-c(\sigma))} \operatorname{Tr}\left(\tilde{\boldsymbol{C}}(\sigma,\nu,-i\lambda) \frac{d}{dz} \tilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right) d\lambda \right) + o(1),\end{aligned}$$

as $Y \to \infty$. Thus together with (5.5), (5.6) we obtain

$$\begin{aligned}\int_{X(Y)} h^\nu(t; x, x)\, dx = {}& \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \frac{\kappa e^{tc(\sigma)} \dim(\sigma) \log Y}{\sqrt{4\pi t}} + \sum_j e^{-t\lambda_j}\\ & + \sum_{\substack{\sigma\in\hat{M};\,\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)} \frac{\operatorname{Tr}(\tilde{\boldsymbol{C}}(\sigma,\nu,0))}{4}\\ & - \frac{1}{4\pi} \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \int_{\mathbb{R}} e^{-t(\lambda^2-c(\sigma))} \operatorname{Tr}\left(\tilde{\boldsymbol{C}}(\sigma,\nu,-i\lambda) \frac{d}{dz} \tilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right) d\lambda\\ & + o(1) \qquad (5.7)\end{aligned}$$

as $Y \to \infty$. It follows that $\int_{X(Y)} \operatorname{tr} h^\nu(t;x,x)\,dx$ has an asymptotic expansion as $Y \to \infty$ and following [27], we take the constant coefficient as the definition of the regularized trace.

Definition 5.1. The regularized trace of e^{-tA_ν} is defined as

$$\operatorname{Tr}_{\mathrm{reg}}\bigl(e^{-tA_\nu}\bigr) = \operatorname{Tr}\bigl(e^{-tA_\nu^d}\bigr) + \sum_{\substack{\sigma\in\hat{M};\,\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)} \frac{\operatorname{Tr}(\tilde{\boldsymbol{C}}(\sigma,\nu,0))}{4} - \frac{1}{4\pi} \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \int_{\mathbb{R}} e^{-t(\lambda^2-c(\sigma))} \operatorname{Tr}\biggl(\tilde{\boldsymbol{C}}(\sigma,\nu,-i\lambda)\frac{d}{dz}\tilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\biggr)\,d\lambda. \tag{5.8}$$

Remark 5.2. The right-hand side of (5.8) equals the spectral side of the Selberg trace formula applied to $\exp(-tA_\nu)$. This follows from [42, Theorem 8.4].

Remark 5.3. There are slightly different methods to regularize the trace. One is to truncate the zero Fourier coefficients of $h^\nu(t;x,y)$ at level $Y \geqslant Y_0$. The resulting kernel $h_Y^\nu(t;x,y)$ is integrable over the diagonal. The integral $\int_X h_Y^\nu(t;x,x)\,dx$ depends on Y in a simple way. If one subtracts off the term which contains Y, one gets another definition of the regularized trace which is closely related to (5.8).

Remark 5.4. The definition of the regularized trace depends in a subtle way on a choice of the representatives $\mathfrak{P}$ of Γ-cuspidal proper parabolic subgroups of G since the terms in Eq. (5.8) involving the scattering matrices $\tilde{\boldsymbol{C}}$ depend on this choice. If one expresses the regularized traces as in Theorem 6.1 below, then its dependence on the set $\mathfrak{P}$ is incorporated in the constant $C(\Gamma)$ which occurs in the definition of the distribution T. This fact has been brought to our attention by Werner Hoffmann. However, it will follow immediately from our proof that our main result Theorem 1.1 is not affected by the choice of $\mathfrak{P}$ since the leading term $C(n)\operatorname{vol}(X)m\cdot\dim(\tau(m))$ in Theorem 1.1 is independent of the choice of $\mathfrak{P}$.

6. The trace formula

In this section we apply the Selberg trace formula to study the regularized trace of the heat operator e^{-tA_ν}. To begin with, we briefly recall the Selberg trace formula. First we introduce the distributions involved. Let α be a K-finite Schwartz function. Let

$$I(\alpha) := \operatorname{vol}(\Gamma\backslash G)\alpha(1).$$

By [15, Theorem 3], the Plancherel theorem can be applied to α. For groups of real rank one which do not possess a compact Cartan subgroup it is stated in [17, Theorem 13.2]. Thus if $P_\sigma(z)$ is as in Section 2.8, then for an odd-dimensional X one has

$$I(\alpha) = \operatorname{vol}(X) \sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} \int_{\mathbb{R}} P_\sigma(i\lambda)\Theta_{\sigma,\lambda}(\alpha)\,d\lambda, \tag{6.1}$$

where the sum is finite since α is K-finite. In even dimensions an additional contribution of the discrete series appears. Let Γ_{s} be the semi-simple elements of Γ and let $\mathrm{C}(\Gamma)_{\mathrm{s}}$ be the set of Γ-conjugacy classes $[\gamma]$, $\gamma\in\Gamma_{\mathrm{s}}$. Put

$$H(\alpha):=\int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma_{\mathrm{s}}-\{1\}}\alpha\bigl(x^{-1}\gamma x\bigr)\,dx.$$

By [42, Lemma 8.1] the integral converges absolutely. Its Fourier transform can be computed as follows. Since Γ is assumed to be torsion free, every non-trivial semi-simple element γ is conjugate to an element $m(\gamma)\exp\ell(\gamma)H_1$, $m(\gamma)\in M$. By [39, Lemma 6.6], $l(\gamma)>0$ is unique and $m(\gamma)$ is determined up to conjugacy in M. Moreover, $\ell(\gamma)$ is the length of the unique closed geodesic in X associated to $[\gamma]$. It follows that Γ_γ, the centralizer of γ in Γ, is infinite cyclic. Let γ_0 denote its generator which is semi-simple too. For $\gamma\in\mathrm{C}(\Gamma)_{\mathrm{s}}-\{[1]\}$ let $a_\gamma:=\exp\ell(\gamma)H_1$ and let

$$L(\gamma,\sigma):=\frac{\overline{\mathrm{Tr}(\sigma)(m_\gamma)}}{\det(\mathrm{Id}-\mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})}e^{-n\ell(\gamma)}. \tag{6.2}$$

Proceeding as in [39] and using [10, Eq. 4.6], one obtains

$$H(\alpha)=\sum_{\sigma\in\hat{M}}\sum_{[\gamma]\in\mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{l(\gamma_0)}{2\pi}L(\gamma,\sigma)\int_{-\infty}^{\infty}\Theta_{\sigma,\lambda}(\alpha)e^{-il(\gamma)\lambda}\,d\lambda, \tag{6.3}$$

where the sum is finite since α is K-finite.

Now let $P\in\mathfrak{P}$. For every $\eta\in\Gamma\cap N_P-\{1\}$ let $X_\eta:=\log\eta$. Write $\|\cdot\|$ for the norm induced on $\mathfrak{n}_P$ by the restriction of $\frac{1}{4n}B(\cdot,\theta\cdot)$. Then for $\mathrm{Re}(s)>0$ the Epstein-type zeta function ζ_P, defined by

$$\zeta_P(s):=\sum_{\eta\in\Gamma\cap N_P-\{1\}}\|X_\eta\|^{-2n(1+s)}, \tag{6.4}$$

converges and ζ_P has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 0. Let $C_P(\Gamma)$ be the constant term of ζ_P at $s=0$. Then put

$$T_P(\alpha):=\int_K\int_{N_P}\alpha\bigl(kn_Pk^{-1}\bigr)\,dn_P\,dk=\int_K\int_N\alpha\bigl(kn_0k^{-1}\bigr)\,dn_0\,dk,$$

$$T(\alpha):=\sum_{P\in\mathfrak{P}}C_P(\Gamma)\frac{\mathrm{vol}(\Gamma\cap N_P\backslash N_P)}{\mathrm{vol}(S^{2n-1})}T_P(\alpha),$$

$$T'_P(\alpha):=\int_K\int_{N_P}\alpha\bigl(kn_Pk^{-1}\bigr)\log\|\log n_P\|\,dn_P\,dk.$$

Then T and $T_{P'}$ are tempered distributions. The distribution T is invariant. Let

$$C(\Gamma) := \sum_{P\in\mathfrak{P}} C_P(\Gamma)\frac{\mathrm{vol}(\Gamma\cap N_P\backslash N_P)}{\mathrm{vol}(S^{2n-1})}.$$

Applying the Fourier inversion formula and the Peter–Weyl theorem to Eq. 10.21 in [17], one obtains the Fourier transform of T as

$$T(\alpha) = \sum_{\sigma\in\hat{M}} \dim(\sigma)\frac{1}{2\pi}C(\Gamma)\int_{\mathbb{R}} \Theta_{\sigma,\lambda}(\alpha)\,d\lambda. \tag{6.5}$$

The distributions T'_P are not invariant. However, they can be made invariant using the standard Knapp–Stein intertwining operators. These operators are defined as follows. Let $\bar{P}_0 := \bar{N}_0 A_0 M_0$ be the parabolic subgroup opposite to P_0. Let $\sigma \in \hat{M}$ and let $(\mathcal{H}^\sigma)^\infty$ be the subspace of C^∞-vectors in $\mathcal{H}^\sigma$. For $\Phi \in (\mathcal{H}^\sigma)^\infty$ and $\lambda \in \mathbb{C}$ define $\Phi_\lambda : G \to V_\sigma$ by

$$\Phi_\lambda(nak) := \Phi(k)e^{(i\lambda e_1+\rho)\log a}.$$

Then for $\mathrm{Im}(\lambda) < 0$ the integral

$$J_{\bar{P}_0|P_0}(\sigma,\lambda)(\Phi)(k) := \int_{\bar{N}} \Phi_\lambda(\bar{n}k)\,d\bar{n} \tag{6.6}$$

is convergent and $J_{\bar{P}_0|P_0}(\sigma,\lambda) : (\mathcal{H}^\sigma)^\infty \to (\mathcal{H}^\sigma)^\infty$ defines an intertwining operator between $\pi_{\sigma,\lambda}$ and $\pi_{\sigma,\lambda,\bar{P}_0}$, where $\pi_{\sigma,\lambda,\bar{P}_0}$ denotes the principal series representation associated to σ, λ and $\bar{P}_0$. As an operator-valued function, $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ has a meromorphic continuation to $\mathbb{C}$ (see [19]). Let $\nu \in \hat{K}$ be a K-type of $\pi_{\sigma,\lambda}$. Since $[\nu : \sigma] \leqslant 1$ for every $\nu \in \hat{K}$, it follows from Frobenius reciprocity and Schur's lemma that

$$J_{\bar{P}_0|P_0}(\sigma,\lambda)|_{(H^\sigma)^\nu} = c_\nu(\sigma:\lambda)\cdot \mathrm{Id}, \tag{6.7}$$

where $c_\nu(\sigma:\lambda) \in \mathbb{C}$. The function $z \mapsto c_\nu(\sigma:z)$ can be computed explicitly. Assume that $d = 2n+1$. Let $k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.11) and let $k_2(\nu)e_2+\cdots+k_{n+1}(\nu)e_{n+1}$ be the highest weight of ν as in (2.9). Then taking the different parametrization into account, it follows from Theorem 8.2 in [8] that there exists a constant $\alpha(n)$ depending on n such that

$$c_\nu(\sigma:z) = \alpha(n)\frac{\prod_{j=2}^{n+1}\Gamma(iz-k_j(\sigma)-\rho_j)\prod_{j=2}^{n+1}\Gamma(iz+k_j(\sigma)+\rho_j)}{\prod_{j=2}^{n+1}\Gamma(iz-k_j(\nu)-\rho_j)\prod_{j=2}^{n+1}\Gamma(iz+k_j(\nu)+\rho_j+1)}. \tag{6.8}$$

This formula implies that

$$c_\nu(\sigma:z)^{-1}\frac{d}{dz}c_\nu(\sigma:z) = \sum_{j=2}^{n+1}\sum_{|k_j(\sigma)|<l\leqslant k_j(\nu)}\frac{i}{iz-l-\rho_j} - \sum_{j=2}^{n+1}\sum_{l=|k_j(\sigma)|}^{k_j(\nu)}\frac{i}{iz+l+\rho_j}. \tag{6.9}$$

Next let $d=2n+2$. Let $k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.12) and let $k_1(\nu)e_1+\cdots+k_{n+1}(\nu)e_{n+1}$ be the highest weight of ν as in (2.10). Then by [8, Theorem 8.2], there exists a constant $\alpha(n)$ depending only on n such that

$$c_\nu(\sigma:z)=\alpha(n)\frac{\Gamma(2iz)\prod_{j=2}^{n+1}\Gamma(iz-k_j(\sigma)-\rho_j)\prod_{j=2}^{n+1}\Gamma(iz+k_j(\sigma)+\rho_j)}{2^{2iz}\prod_{j=1}^{n+1}\Gamma(iz-k_j(\nu)-\rho_j+1)\prod_{j=1}^{n+1}\Gamma(iz+k_j(\nu)+\rho_j)}. \tag{6.10}$$

Eqs. (6.8) and (6.10) imply that $J_{\bar{P}_0|P_0}(\sigma,\lambda)$ has no poles on $\mathbb{R}-\{0\}$ and is invertible there and that $J_{\bar{P}_0|P_0}(\sigma,z)^{-1}$ is defined as a meromorphic function of z. It follows that the weighted character

$$\operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma,z)^{-1}\frac{d}{dz}J_{\bar{P}_0|P_0}(\sigma,z)\pi_{\sigma,z}(\alpha)\right) \tag{6.11}$$

is regular for $z\in\mathbb{R}-\{0\}$. Let $\epsilon>0$ be sufficiently small. Let H_ϵ be the half-circle from $-\epsilon$ to ϵ in the lower half-plane, oriented counter-clockwise. Let D_ϵ be the path which is the union of $(-\infty,-\epsilon]$, H_ϵ and $[\epsilon,\infty)$. Using (6.8), (6.10) and the fact that the matrix coefficients of $\pi_{\sigma,z}(\alpha)$ are rapidly decreasing, it follows that (6.11) is integrable over D_ϵ. Let

$$J_\sigma(\alpha):=\frac{\kappa\dim\sigma}{4\pi i}\int_{D_\epsilon}\operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma,z)^{-1}\frac{d}{dz}J_{\bar{P}_0|P_0}(\sigma,z)\pi_{\sigma,z}(\alpha)\right)dz. \tag{6.12}$$

The change of contour is only necessary if $J_{\bar{P}_0|P_0}(\sigma,s)$ has a pole at 0. Let

$$J(\alpha):=-\sum_{\sigma\in\hat{M}}J_\sigma(\alpha). \tag{6.13}$$

Using [12, Section 3.2.5], [12, Theorems 8.1.3, 8.1.4], (6.8) and (6.10) it is easy to see that $[\check{\nu}:\sigma]=[\nu:\sigma]$ and $c_{\check{\nu}}(\sigma:z)=c_\nu(\sigma:z)$ for all $\nu\in\hat{K}$ and all $\sigma\in\hat{M}$. Thus by (4.11) and Proposition 4.1 one has

$$J\left(h_t^\nu\right)=-\frac{\kappa}{4\pi i}\sum_{\sigma\in\hat{M}}[\nu:\sigma]\dim(\sigma)\int_{D_\epsilon}e^{-t(z^2-c(\sigma))}c_\nu(\sigma:z)^{-1}\frac{d}{dz}c_\nu(\sigma:z)\,dz. \tag{6.14}$$

For notational convenience, if $\nu\in\hat{K}$ and $\sigma\in\hat{M}$ with $[\nu:\sigma]=0$ we let $c_\nu(\sigma:z):=0$. Now we define a distribution $\mathcal{I}$ by

$$\mathcal{I}(\alpha):=\sum_{P\in\mathfrak{P}}T'_P(\alpha)-J(\alpha). \tag{6.15}$$

We claim that $\mathcal{I}$ is an invariant distribution. This can be seen as follows. Using the formula for $J_M(m,\alpha)$ on p. 92 of [16], we get $J_{M_P}(1,\alpha)=T'_P(\alpha)$. Next using the formula for the invariant distribution $I_P(m,\alpha)$ on p. 93 of [16] and formula (8) of [16], it follows that

$$I_P(1,\alpha) = T'_P(\alpha) + \sum_{\sigma\in\hat{M}_0} \frac{\dim(\sigma)}{4\pi i} \int_{D_\epsilon} \operatorname{Tr}\left(J_{\bar{P}_0|P_0}(\sigma,z)^{-1} \frac{d}{dz} J_{\bar{P}_0|P_0}(\sigma,z)\pi_{\sigma,z}(\alpha)\right) dz.$$

Adding over $P \in \mathfrak{P}$, we get

$$\sum_{P\in\mathfrak{P}} I_P(1,\alpha) = \mathcal{I}(\alpha) - J(\alpha),$$

which proves our claim.

Theorem 6.1. *With the above notations, one has*

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-tA_\nu}\right) = I\left(h_t^\nu\right) + H\left(h_t^\nu\right) + T\left(\tilde{h}_t^\nu\right) + \mathcal{I}\left(h_t^\nu\right) + J\left(h_t^\nu\right).$$

Proof. By (5.8), $\operatorname{Tr}_{\mathrm{reg}}(e^{-tA_\nu})$ is the difference of $\operatorname{Tr}(e^{-tA_\nu^d})$ and the terms in the trace formula which are associated to the continuous spectrum. These are the last two terms in the trace formula [42, Theorem 8.4]. Using [42, Theorem 8.4], the theorem on p. 299 in [35], and taking our normalization of measures into account, we obtain the claimed equality. □

The Fourier transform of the distribution $\mathcal{I}$ was computed in [16]. We shall now state his result. For $\sigma \in \hat{M}$ with highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ and $\lambda \in \mathbb{R}$ define $\lambda_\sigma \in (\mathfrak{h})^*_{\mathbb{C}}$ by

$$\lambda_\sigma := i\lambda e_1 + \sum_{j=2}^{n+1}\left(k_j(\sigma) + \rho_j\right)e_j. \tag{6.16}$$

Let $S(\mathfrak{b}_{\mathbb{C}})$ be the symmetric algebra of $\mathfrak{b}_{\mathbb{C}}$. Define $\Pi \in S(\mathfrak{b}_{\mathbb{C}})$ by

$$\Pi := \prod_{\alpha\in\Delta^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{b}_{\mathbb{C}})} H_\alpha. \tag{6.17}$$

The restriction of the Killing form to $\mathfrak{h}_{\mathbb{C}}$ defines a non-degenerate symmetric bilinear form. We will identify $\mathfrak{h}^*_{\mathbb{C}}$ with $\mathfrak{h}_{\mathbb{C}}$ via this form and denote the induced symmetric bilinear form on $\mathfrak{h}^*_{\mathbb{C}}$ by $\langle\cdot,\cdot\rangle$. Then for $\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ we denote by $s_\alpha : \mathfrak{h}^*_{\mathbb{C}} \to \mathfrak{h}^*_{\mathbb{C}}$ the reflection $s_\alpha(x) = x - 2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$. Now the Fourier transform of $\mathcal{I}$ is computed as follows.

Theorem 6.2. *For every K-finite $\alpha \in C^2(G)$ one has*

$$\mathcal{I}(\alpha) = \frac{\kappa}{4\pi} \sum_{\sigma\in\hat{M}} \int_{\mathbb{R}} \Omega(\check{\sigma},-\lambda)\Theta_{\sigma,\lambda}(\alpha)\, d\lambda,$$

where

$$\Omega(\sigma,\lambda) := -2\dim(\sigma)\gamma - \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)}\big(\psi\big(1+\lambda_\sigma(H_\alpha)\big)+\psi\big(1-\lambda_\sigma(H_\alpha)\big)\big).$$

Here ψ denotes the digamma function and γ denotes the Euler–Mascheroni constant. Moreover $\check{\sigma}$ denotes the contragredient representation of σ and Π is as in (6.17).

Proof. This follows from [16, Theorems 5, 6], [16, Corollary on p. 96]. Here we use that for d even and $\pi \in \hat{G}_d$, the discrete series of G, the term $|D_G(a)|^{1/2}\Theta_{\check{\pi}}(a)$ occurring in [16, Theorem 5] vanishes for $a = 1$. This can be seen as follows. By the formula for the character of the discrete series [17, Theorem 12.7], [41, Theorem 10.1.1.1], one needs to show that $\sum_{w\in W_K}\det(w) = 0$. This has been established in the proof of Lemma 5 in [6]. □

For the applications we have in mind, we shall now transform the functions $\Omega(\lambda,\sigma)$ a bit. In the rest of this section we assume that $d = \dim(X)$ is odd, $d = 2n+1$. We start with the following elementary lemma.

Lemma 6.3. *One has*

$$\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{a}_\mathbb{C})}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_M)} = 2\dim\sigma.$$

Proof. This is proved in [16, p. 95] but can also be seen as follows. Let $\xi \in \mathfrak{b}_\mathbb{C}^*$, $\xi = \xi_2e_2 + \cdots + \xi_{n+1}e_{n+1}$. Then it follows from (2.3) that

$$\Pi(\xi) = \prod_{2\leqslant i<j\leqslant n+1}(\xi_i-\xi_j)(\xi_i+\xi_j). \tag{6.18}$$

If τ is a permutation of $\{2,\dots,n+1\}$ and

$$\xi_\tau := \xi_2e_{\tau(2)} + \cdots + \xi_{n+1}e_{\tau(n+1)}$$

it follows from (6.18) that

$$\Pi(\xi_\tau) = \pm\Pi(\xi). \tag{6.19}$$

Write $\Lambda(\sigma) + \rho_M = \xi_2e_2 + \cdots + \xi_{n+1}e_{n+1}$. Let λ_σ be as in (6.16). Then if $\alpha = e_1 \pm e_j$, one has

$$s_\alpha(\lambda_\sigma) = \mp\xi_je_1 + \xi_2e_2 + \cdots + \xi_{j-1}e_{j-1} \mp i\lambda e_j + \xi_{j+1}e_{j+1} + \cdots + \xi_{n+1}e_{n+1}. \tag{6.20}$$

Using (2.15) and (6.18) it follows that

$$\Pi\big(s_{e_1+e_j}(\lambda_\sigma)\big) = \Pi\big(s_{e_1-e_j}(\lambda_\sigma)\big); \qquad \Pi\big(s_{e_1+e_j}(\lambda_\sigma)\big) = \Pi\big(s_{e_1+e_j}(\lambda_{w_0\sigma})\big). \tag{6.21}$$

Thus by (2.14) and (6.18) for $\alpha = e_1 \pm e_j$ one gets

$$\frac{\Pi(s_\alpha(\lambda_\sigma))}{\Pi(\rho_M)} = \frac{(-1)^j}{\Pi(\rho_M)} \prod_{\substack{2\leqslant k<l\leqslant n+1\\ k,l\neq j}} \left(\xi_k^2 - \xi_l^2\right) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \left(-\lambda^2 - \xi_p^2\right)$$

$$= \frac{1}{\Pi(\rho_M)} \prod_{2\leqslant k<l\leqslant n+1} \left(\xi_k^2 - \xi_l^2\right) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2}$$

$$= \dim(\sigma) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2}. \tag{6.22}$$

Now as in [34, Lemma 5.6] one has

$$\sum_{j=2}^{n+1} \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - \xi_p^2}{\xi_j^2 - \xi_p^2} = 1$$

for every λ. This proves the lemma. □

For $j = 2, \dots, n+1$ and $\lambda \in \mathbb{C}$ let

$$P_j(\sigma, \lambda) := \frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}. \tag{6.23}$$

Then if σ is of highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$ as in (2.11) it follows from (6.22) that

$$P_j(\sigma, \lambda) = \dim(\sigma) \prod_{\substack{p=2\\ p\neq j}}^{n+1} \frac{-\lambda^2 - (k_p(\sigma) + \rho_p)^2}{(k_j(\sigma) + \rho_j)^2 - (k_p(\sigma) + \rho_p)^2}. \tag{6.24}$$

In particular $P_j(\sigma, \lambda)$ is an even polynomial in λ of degree $2n - 2$.

Proposition 6.4. *Let $\sigma \in \hat{M}$ be of highest weight $k_2(\sigma)e_2 + \cdots + k_{n+1}(\sigma)e_{n+1}$. Assume that all $k_j(\sigma)$ are integral and that $k_{n+1}(\sigma) > 0$. Let the notation be as in Theorem 6.2. Then one has*

$$\Omega(\sigma, \lambda) = \Omega(w_0\sigma, \lambda); \qquad \Omega(\sigma, \lambda) = \Omega(\check{\sigma}, -\lambda).$$

Moreover one can write

$$\Omega(\sigma, \lambda) = \Omega_1(\sigma, \lambda) + \Omega_2(\sigma, \lambda),$$

where $\Omega_1(\sigma,\lambda)$ and $\Omega_2(\sigma,\lambda)$ are defined as follows. Let $m_0 := |k_{n+1}(\sigma)| - 1$. Then one puts

$$\Omega_1(\sigma,\lambda) := -\dim(\sigma)\left(2\gamma + \psi(1+i\lambda) + \psi(1-i\lambda) + \sum_{1 \leqslant l \leqslant m_0} \frac{2l}{l^2+\lambda^2}\right).$$

Furthermore for every j let $P_j(\sigma,\lambda)$ be as in (6.23). For $m_0 \leqslant l \leqslant k_j(\sigma)+\rho_j$ define an even polynomial $Q_{j,l}(\sigma,\lambda)$ by

$$Q_{j,l}(\sigma,\lambda) := \frac{P_j(\sigma,\lambda) - P_j(\sigma,il)}{l+i\lambda} + \frac{P_j(\sigma,\lambda) - P_j(\sigma,il)}{l-i\lambda}. \tag{6.25}$$

Then

$$\begin{aligned}\Omega_2(\sigma,\lambda) :=& -\sum_{j=2}^{n+1} \sum_{m_0<l<k_j(\sigma)+\rho_j} P_j(\sigma,il)\frac{2l}{\lambda^2+l^2} - \sum_{j=2}^{n+1} \dim(\sigma)\frac{k_j(\sigma)+\rho_j}{(k_j(\sigma)+\rho_j)^2+\lambda^2} \\ & - \sum_{j=2}^{n+1} \sum_{m_0<l<k_j(\sigma)+\rho_j} Q_{j,l}(\sigma,\lambda) - \frac{1}{2} \sum_{\substack{l=k_j(\sigma)+\rho_j \\ 2\leqslant j\leqslant n+1}} Q_{j,l}(\sigma,\lambda).\end{aligned}$$

Finally, if $k_{n+1}(\sigma) < 0$, one puts $\Omega_1(\sigma,\lambda) = \Omega_1(w_0\sigma,\lambda)$, $\Omega_2(\sigma,\lambda) = \Omega_2(w_0\sigma,\lambda)$.

Proof. Let $j \in \{2,\dots,n+1\}$. We have

$$\lambda_\sigma(H_{e_1\pm e_j}) = i\lambda \pm \big(k_j(\sigma)+\rho_j\big). \tag{6.26}$$

Now recall that $\rho_{n+1} = 0$ and that the highest weight of $w_0\sigma$ is given by $k_2(\sigma)e_2 + \cdots + k_n(\sigma)e_n - k_{n+1}(\sigma)e_{n+1}$. Moreover recall that for $M = \mathrm{Spin}(n)$ one has $\check{\sigma} \cong \sigma$ if n is odd and $\check{\sigma} \cong w_0\sigma$ if n is even. Thus (6.21) and (6.26) imply that $\Omega(\lambda,\sigma) = \Omega(\lambda,w_0\sigma)$ and $\Omega(\lambda,\sigma) = \Omega(-\lambda,\check{\sigma})$. Moreover, using $\psi(z+1) = \frac{1}{z} + \psi(z)$, (6.21) and (6.26) we obtain

$$\begin{aligned}&\frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}\big(\psi\big(1+\lambda_\sigma(H_{e_1+e_j})\big) + \psi\big(1-\lambda_\sigma(H_{e_1+e_j})\big)\big) \\ &\quad + \frac{\Pi(s_{e_1-e_j}\lambda_\sigma)}{\Pi(\rho_M)}\big(\psi\big(1+\lambda_\sigma(H_{e_1-e_j})\big) + \psi\big(1-\lambda_\sigma(H_{e_1-e_j})\big)\big) \\ &= 2\frac{\Pi(s_{e_1+e_j}\lambda_\sigma)}{\Pi(\rho_M)}\Bigg(\psi(1+i\lambda) + \psi(1-i\lambda) + \sum_{1\leqslant l\leqslant m_0} \frac{2l}{l^2+\lambda^2} \\ &\quad + \sum_{m_0<l<k_j(\sigma)+\rho_j} \frac{2l}{l^2+\lambda^2} + \frac{(k_j(\sigma)+\rho_j)}{(k_j(\sigma)+\rho_j)^2+\lambda^2}\Bigg).\end{aligned}$$

Using Lemma 6.3 and (6.21) we obtain

$$\Omega(\sigma,\lambda) = \Omega_1(\sigma,\lambda) - \sum_{j=2}^{n+1} P_j(\sigma,\lambda)\left(\sum_{m_0<l<k_j(\sigma)+\rho_j} \frac{2l}{l^2+\lambda^2} + \frac{(k_j(\sigma)+\rho_j)}{(k_j(\sigma)+\rho_j)^2+\lambda^2}\right).$$

Since $P_j(\sigma,\lambda)$ is an even polynomial in λ, for every $j=2,\dots,n+1$ and every l with $m_0 \leqslant l \leqslant |k_j(\sigma)|+\rho_j$ we can write

$$\frac{P_j(\sigma,\lambda)l}{l^2+\lambda^2} = \frac{1}{2}Q_{j,l}(\sigma,\lambda) + P_j(\sigma,il)\frac{l}{l^2+\lambda^2}.$$

Using (6.24) it follows that

$$P_j\bigl(\sigma, i\bigl(k_j(\sigma)+\rho_j\bigr)\bigr) = \dim(\sigma).$$

This implies the proposition. □

Remark 6.5. There is a similar formula for $\sigma \in \hat{M}$ with half-integral weight.

In order to define the analytic torsion, we need to know that the regularized trace of $e^{-t\Delta_p(\tau)}$ admits an asymptotic expansion as $t \to +0$. We establish this in general for the operators e^{-tA_ν}. To begin with, we prove some auxiliary lemmas.

Lemma 6.6. *Let $\phi_1(t) := \int_{\mathbb{R}} e^{-t\lambda^2}\frac{1}{\lambda^2+c^2}\,d\lambda$. Then there exist $a_j \in \mathbb{C}$ such that*

$$\phi_1(t) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j}{2}}$$

as $t \to 0$.

Proof. We have

$$\phi_1(t) = e^{tc^2}\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+c^2)}}{\lambda^2+c^2}\,d\lambda.$$

One has

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+c^2)}}{\lambda^2+c^2}\,d\lambda = -\frac{\sqrt{\pi}}{\sqrt{t}}.$$

Thus one has

$$\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+c^2)}}{\lambda^2+c^2}\,d\lambda = C + \sqrt{\pi t}.$$

Writing e^{tc^2} as a power series, the proposition follows. □

Lemma 6.7. *Let $\phi_2(t) := \int_{\mathbb{R}} e^{-t\lambda^2}\psi(1+i\lambda)\,d\lambda$. Then there exist complex coefficients a_j, b_j, c_j such that as $t \to 0$, there is an asymptotic expansion*

$$\phi_2(t) \sim \sum_{j=0}^{\infty} a_j t^{j-1/2} + \sum_{j=0}^{\infty} b_j t^{j-1/2} \log t + \sum_{j=0}^{\infty} c_j t^j.$$

Proof. The asymptotic behavior of the Laplace transform at 0 of functions which admit suitable asymptotic expansions at infinity has been treated in [13].

Recall that

$$\psi(z+1) = \log z + \frac{1}{2z} - \sum_{k=1}^{N} \frac{B_{2k}}{2k} \cdot \frac{1}{z^{2k}} + R_N(z), \quad N \in \mathbb{N}, \tag{6.27}$$

where B_i are the Bernoulli numbers and

$$R_N(z) = O\big(z^{-2N-2}\big), \quad z \to \infty$$

uniformly on sectors $-\pi + \delta < \arg(z) < \pi - \delta$. Consider

$$\phi_2^+(t) := \int_0^{\infty} e^{-t\lambda^2}\psi(1+i\lambda)\,d\lambda.$$

Let χ be the characteristic function of $[1,\infty)$. Define a function

$$g(\lambda) := \psi(1+i\lambda) - \log(i\lambda) - \frac{\chi(\lambda)}{2i\lambda}$$

and define a function

$$h(\lambda) := \frac{g(\sqrt{\lambda})}{2\sqrt{\lambda}}.$$

Then by (6.27) there is an asymptotic expansion

$$h(\lambda) \sim \sum_{k=1}^{\infty} a_k \lambda^{-k-1/2}, \quad \lambda \to \infty. \tag{6.28}$$

First define

$$\psi_2^+(t) := \int_0^{\infty} e^{-t\lambda^2} g(\lambda)\,d\lambda = \int_0^{\infty} e^{-t\lambda} h(\lambda)\,d\lambda.$$

Then by (6.28) and [13, Corollary 5.2] one obtains

$$\psi_2^+(t) \sim \sum_{k=0}^{\infty} a'_k t^{k+1/2} + \sum_{k=0}^{\infty} c'_k t^k$$

for complex a'_k, c'_k. Next we have

$$\int_0^{\infty} e^{-t\lambda^2} \log\lambda \, d\lambda = t^{-1/2} \int_0^{\infty} e^{-\lambda^2} \log\lambda \, d\lambda - \frac{\sqrt{\pi}}{4} t^{-1/2} \log t.$$

Finally we have

$$\int_1^{\infty} e^{-t\lambda^2} \lambda^{-1} \, d\lambda = \int_{\sqrt{t}}^{1} e^{-\lambda^2} \lambda^{-1} \, d\lambda + \int_1^{\infty} e^{-\lambda^2} \lambda^{-1} \, d\lambda = \int_{\sqrt{t}}^{1} \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k-1}}{k!} \, d\lambda + C$$
$$= -\log\sqrt{t} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k! 2k} + C'.$$

Putting everything together, we obtain the desired asymptotic expansion for ϕ_2^+. For the integral over $(-\infty, 0]$ we proceed similarly.

Alternatively, one can also proceed as in [21, pp. 156–157, 165–166]. The methods of [13] and [21] are closely related. □

Lemma 6.8. *Let $P(z) := \sum_{j=0}^{N} a_j z^{2j}$ be an even polynomial. Then there exist $a'_j \in \mathbb{C}$ such that*

$$\int_{\mathbb{R}} e^{-t\lambda^2} P(\lambda) \, d\lambda = \sum_{j=0}^{N} a'_j t^{-j-\frac{1}{2}}.$$

Proof. This follows by a change of variables. □

Proposition 6.9. *Assume that* $\dim(X)$ *is odd. There exist coefficients a_j, b_j, c_j, $j \in \mathbb{N}$, such that one has*

$$\operatorname{Tr}_{\mathrm{reg}}\big(e^{-tA_\nu}\big) \sim \sum_{j=0}^{\infty} a_j t^{j-\frac{d}{2}} + \sum_{j=0}^{\infty} b_j t^{j-\frac{1}{2}} \log t + \sum_{j=0}^{\infty} c_j t^j$$

as $t \to +0$.

Proof. We use Theorem 6.1 and derive an asymptotic expansion of each term on the right-hand side. We can always ignore additional factors of the form e^{-tc}, $c > 0$ by expanding this term in a power series. The term $I(h_t^\nu)$ has the desired asymptotic expansion by Proposition 4.1, Eq. (6.1) and Lemma 6.8. Secondly, using [11, Proposition 5.4] one obtains $H(h_t^\nu) = O(e^{-\frac{c}{t}})$ for a constant $c>0$. By Proposition 4.1 and Eq. (6.5), the term $T(h_t^\nu)$ has an asymptotic expansion starting with $t^{-\frac{1}{2}}$. For every $\sigma \in \hat{M}$ with $[\nu:\sigma]\neq 0$ we write $\Omega(\lambda,\sigma)$ as in Proposition 6.4. Then by Propositions 4.1, 6.4 together with Remark 6.5, Lemmas 6.6, 6.7 and 6.8 it follows that the term $\mathcal{I}(h_t^\nu)$ has the claimed asymptotic expansion in t. The term $J(h_t^\nu)$ has the claimed asymptotic expansion by Eq. (6.14) and Lemma 6.6. □

Remark 6.10. The proposition remains true in even dimensions. The proof, however, would require more work due to the discrete series. This is not needed for our purpose.

7. The analytic torsion

Let τ be an irreducible finite-dimensional representation of G on V_τ. Let E'_τ be the flat vector bundle associated to the restriction of τ to Γ. Then E'_τ is canonically isomorphic to the locally homogeneous vector bundle E_τ associated to $\tau|_K$. By [26], there exists an inner product $\langle\cdot,\cdot\rangle$ on V_τ such that

(1) $\langle\tau(Y)u,v\rangle=-\langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{k}$, $u,v\in V_\tau$,
(2) $\langle\tau(Y)u,v\rangle=\langle u,\tau(Y)v\rangle$ for all $Y\in\mathfrak{p}$, $u,v\in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on E_τ which will be called admissible too. Let $\Lambda^p(E_\tau)$ be the bundle of E_τ-valued p-forms on X. Let

$$\nu_p(\tau):=\Lambda^p\operatorname{Ad}^*\otimes\tau:K\to\operatorname{GL}\bigl(\Lambda^p\mathfrak{p}^*\otimes V_\tau\bigr). \tag{7.1}$$

There is a canonical isomorphism

$$\Lambda^p(E_\tau)\cong\Gamma\backslash\bigl(G\times_{\nu_p(\tau)}\bigl(\Lambda^p\mathfrak{p}^*\otimes V_\tau\bigr)\bigr). \tag{7.2}$$

If $\Lambda^p(X,E_\tau)$ are the smooth E_τ-valued p-forms on X, the isomorphism (7.2) induces an isomorphism

$$\Lambda^p(X,E_\tau)\cong C^\infty\bigl(\Gamma\backslash G,\nu_p(\tau)\bigr). \tag{7.3}$$

A corresponding isomorphism also holds for the L^2-spaces. Let $\Delta_p(\tau)$ be the Hodge–Laplacian on $\Lambda^p(X,E_\tau)$ with respect to the admissible inner product. By (6.9) in [26], on $C^\infty(\Gamma\backslash G,\nu_p(\tau))$ one has

$$\Delta_p(\tau)=-\Omega+\tau(\Omega)\operatorname{Id}. \tag{7.4}$$

If $\Lambda(\tau)=k_1(\tau)e_1+\cdots+k_{n+1}(\tau)e_{n+1}$ is the highest weight of τ, we have

$$\tau(\Omega)=\sum_{j=1}^{n+1}\bigl(k_j(\tau)+\rho_j\bigr)^2-\sum_{j=1}^{n+1}\rho_j^2. \tag{7.5}$$

For $G = \mathrm{Spin}(2n+1,1)$ this was proved in [34, Section 2]. For $G = \mathrm{Spin}(2n+2,1)$, one can proceed in the same way. Let $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues of $\Delta_p(\tau)$. By (7.4) and (5.8) we have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{reg}}\big(e^{-t\Delta_p(\tau)}\big) = \sum_j e^{-t\lambda_j} + \sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\frac{\mathrm{Tr}(\tilde{\boldsymbol{C}}(\sigma,\nu_p(\tau),0))}{4}\\ -\frac{1}{4\pi}\sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))}\\ \cdot\int_{\mathbb{R}} e^{-t\lambda^2}\,\mathrm{Tr}\Big(\tilde{\boldsymbol{C}}\big(\sigma,\nu_p(\tau),-i\lambda\big)\frac{d}{dz}\tilde{\boldsymbol{C}}\big(\sigma,\nu_p(\tau),i\lambda\big)\Big)\,d\lambda.\end{aligned} \tag{7.6}$$

Let

$$K(t,\tau) := \sum_{p=0}^{d}(-1)^p p\,\mathrm{Tr}_{\mathrm{reg}}\big(e^{-t\Delta_p(\tau)}\big). \tag{7.7}$$

Then the analytic torsion is defined in terms of the Mellin transform of $K(t,\tau)$. For every $p = 0,\dots,d$, let $\nu_p(\tau)$ be the representation (7.1) and let $h_t^{\nu_p(\tau)}$ be defined by (4.9). Put

$$k_t^\tau := e^{-t\tau(\Omega)}\sum_{p=0}^{d}(-1)^p p h_t^{\nu_p(\tau)}. \tag{7.8}$$

By Theorem 6.1 we have

$$K(t,\tau) = I\big(k_t^\tau\big) + H\big(k_t^\tau\big) + T\big(k_t^\tau\big) + \mathcal{I}\big(k_t^\tau\big) + J\big(k_t^\tau\big). \tag{7.9}$$

This equality will be used in Section 10 to study the Mellin transform of $K(t,\tau)$.

To define the analytic torsion, we need to determine the asymptotic behavior of the regularized trace of $e^{-t\Delta_p(\tau)}$ as $t\to\infty$. To begin with we estimate the exponential factors occurring on the right-hand side of (7.6).

Lemma 7.1.

(1) *Let $G = \mathrm{Spin}(2n+2,1)$. Let τ be an irreducible representation of G. Then*

$$\tau(\Omega) - c(\sigma) \geqslant \frac{1}{4}$$

for all $\sigma\in\hat{M}$ with $[\nu_p(\tau):\sigma]\neq 0$.

(2) *Let $G = \mathrm{Spin}(2n+1,1)$. Let τ be an irreducible representation of G with highest weight $\tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ as in (2.7). Then*

$$\tau(\Omega) - c(\sigma) \geqslant \tau_{n+1}^2$$

for all $\sigma \in \hat{M}$ with $[\nu_p(\tau) : \sigma] \neq 0$. Moreover assume that $\sigma \in \hat{M}$ is such that $[\nu_p(\tau) : \sigma] \neq 0$ and such that $\sigma = w_0 \sigma$. Then one has

$$\tau(\Omega) - c(\sigma) \geqslant (\tau_n + 1)^2 + \tau_{n+1}^2 \geqslant 1 + \tau_{n+1}^2.$$

Proof. For $p = 0, \ldots, d$ let

$$\nu_p := \Lambda^p \operatorname{Ad}_{\mathfrak{p}}^* : K \to \mathrm{GL}\big(\Lambda^p \mathfrak{p}^*\big).$$

Recall that $\nu_p(\tau) = \tau|_K \otimes \nu_p$. Let $\nu \in \hat{K}$ with $[\nu_p(\tau) : \nu] \neq 0$. Then by [17, Proposition 9.72], there exists $\nu' \in \hat{K}$ with $[\tau : \nu'] \neq 0$ of highest weight $\Lambda(\nu') \in \mathfrak{b}_{\mathbb{C}}^*$ and $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ which is a weight of ν_p such that the highest weight $\Lambda(\nu)$ of ν is given by $\mu + \Lambda(\nu')$. Now let $\nu' \in \hat{K}$ be such that $[\tau : \nu'] \neq 0$. Let $\Lambda(\nu')$ be the highest weight of ν' as in (2.9) resp. (2.10). Then by [12, Theorems 8.1.3, 8.1.4] we have

$$\tau_{j-1} \geqslant k_j\big(\nu'\big) \geqslant 0, \quad j = 2, \ldots, n+1,$$

if $d = 2n+1$ and

$$\tau_j \geqslant \big|k_j\big(\nu'\big)\big|, \quad j = 1, \ldots, n+1,$$

if $d = 2n+2$. Moreover, every weight $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ of ν_p is given as

$$\mu = \pm e_{j_1} \pm \cdots \pm e_{j_p}, \quad j_1 < j_2 < \cdots < j_p \leqslant n+1.$$

Thus, if $\nu \in \hat{K}$ is such that $[\nu_p(\tau) : \nu] \neq 0$, the highest weight $\Lambda(\nu)$ of ν, given as in (2.9) resp. (2.10), satisfies

$$\tau_{j-1} + 1 \geqslant k_j(\nu) \geqslant 0, \quad j \in \{2, \ldots, n+1\},$$

if $d = 2n+1$ and

$$\tau_j + 1 \geqslant \big|k_j(\nu)\big| \geqslant 0, \quad j \in \{1, \ldots, n+1\},$$

if $d = 2n+2$. Let $\sigma \in \hat{M}$ be such that $[\nu_p(\tau) : \sigma] \neq 0$. Then using [12, Theorems 8.1.3, 8.1.4] it follows that

$$\tau_{j-1} + 1 \geqslant \big|k_j(\sigma)\big|$$

for every $j \in \{2, \dots, n+1\}$, where the $k_j(\sigma)$ are as in (2.11) resp. (2.12). Furthermore note that by (2.4) we have $\rho_{j-1} = \rho_j + 1$. Using (7.5) and (4.16) we get

$$c(\sigma) = \sum_{j=2}^{n+1} \big(k_j(\sigma) + \rho_j\big)^2 - \sum_{j=1}^{n+1} \rho_j^2 \leqslant \sum_{j=2}^{n+1} (\tau_{j-1} + \rho_{j-1})^2 - \sum_{j=1}^{n+1} \rho_j^2 = \tau(\Omega) - (\tau_{n+1} + \rho_{n+1})^2.$$

If $G = \mathrm{Spin}(2n+2, 2)$, we have $\rho_{n+1} = 1/2$ and $\tau_{n+1} \geqslant 0$. If $G = \mathrm{Spin}(2n+1, 1)$, we have $\rho_{n+1} = 0$. Thus item (1) and the first statement of item (2) are proved.

Now assume that $G = \mathrm{Spin}(2n+1, 1)$. Assume that σ additionally satisfies $\sigma = w_0\sigma$. This is equivalent to $k_{n+1}(\sigma) = 0$ by (2.15). Thus since $\rho_{n+1} = 0$, $\rho_n = 1$ we get

$$c(\sigma) = \sum_{j=2}^{n} \big(k_j(\sigma) + \rho_j\big)^2 - \sum_{j=1}^{n+1} \rho_j^2 \leqslant \sum_{j=2}^{n} (\tau_{j-1} + \rho_{j-1})^2 - \sum_{j=1}^{n+1} \rho_j^2 = \tau(\Omega) - (\tau_n + 1)^2 - \tau_{n+1}^2.$$

Finally by (2.7) we have $\tau_n \geqslant 0$. This proves the lemma. $\square$

The next two lemmas are also needed to determine the behavior of the regularized trace as $t \to \infty$.

Lemma 7.2. *There is an asymptotic expansion*

$$\int_{\mathbb{R}} e^{-t\lambda^2} \operatorname{Tr}\Big(\tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), -i\lambda\big) \frac{d}{dz} \tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), i\lambda\big)\Big)\, d\lambda \sim \sum_{j=1}^{\infty} b_j t^{-j/2}$$

as $t \to \infty$.

Proof. Since $\tilde{\boldsymbol{C}}(\sigma : \nu_p(\tau) : i\lambda)$ is analytic near $\lambda = 0$, we have a power series expansion

$$\operatorname{Tr}\Big(\tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), -i\lambda\big) \frac{d}{dz} \tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), i\lambda\big)\Big) = \sum_{j=0}^{\infty} a_j \lambda^j$$

which converges for $|\lambda| \leqslant 2\varepsilon$. Hence we get an asymptotic expansion

$$\int_{-\varepsilon}^{\varepsilon} e^{-t\lambda^2} \operatorname{Tr}\Big(\tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), -i\lambda\big) \frac{d}{dz} \tilde{\boldsymbol{C}}\big(\sigma, \nu_p(\tau), i\lambda\big)\Big)\, d\lambda \sim \sum_{j=1}^{\infty} b_j t^{-j/2}.$$

The integral over $(-\infty, -\varepsilon/2] \cup [\varepsilon/2, \infty)$ is exponentially decreasing. This proves the lemma. $\square$

Lemma 7.3. *Let* $G = \mathrm{Spin}(2n+1, 1)$. *Let* $\tau \in \hat{G}$ *and assume that* $\tau \neq \tau_\theta$. *For* $p \in \{0, \dots, d\}$ *let* $\lambda_0 \in \mathbb{R}^+$ *be an eigenvalue of* $\Delta_p(\tau)$. *Then one has* $\lambda_0 \geqslant 1/4$.

Proof. If $\tau \neq \tau_\theta$ one has $|\tau_{n+1}| \geqslant 1/2$. Let $\hat{G}$ be the unitary dual of G. Recall that if λ_0 is an eigenvalue of $\Delta_p(\tau)$, there exists a $\pi \in \hat{G}$ with $[\pi : \check{\nu}_p(\tau)] = [\pi : \nu_p(\tau)] \neq 0$ such that

$$\lambda_0 = -\pi(\Omega) + \tau(\Omega).$$

Since $\mathrm{rk}(G) > \mathrm{rk}(K)$, it follows from [17, Theorem 8.54] and [38, Corollary 6.2] that $\hat{G}$ consist of the unitary principal series representations $\pi_{\sigma,\lambda}$, $\sigma \in \hat{M}$, $\lambda \in \mathbb{R}$ and the complementary series representations $\pi^c_{\sigma,\lambda}$, $\sigma \in \hat{M}$, $\lambda \in \mathbb{R}$. First consider a unitary principal series representation $\pi_{\sigma,\lambda}$. Then by Frobenius reciprocity [17, p. 208], $[\pi_{\sigma,\lambda} : \nu_p(\tau)]$ is non-zero iff $[\nu_p(\tau) : \sigma]$ is non-zero. Thus together with (4.17) and Lemma 7.1, for every $\lambda \in \mathbb{R}$ one has

$$-\pi_{\sigma,\lambda}(\Omega) + \tau(\Omega) = -c(\sigma) + \lambda^2 + \tau(\Omega) \geqslant 1/4.$$

Next consider a complementary series representation $\pi^c_{\sigma,\lambda}$. Again it follows from Frobenius reciprocity that $[\pi_{\sigma,\lambda} : \nu_p(\tau)]$ is non-zero iff $[\nu_p(\tau) : \sigma]$ is non-zero. Moreover by [19, Propositions 49, 53], if $\pi^c_{\sigma,\lambda}$ belongs to the complementary series one has $\sigma = w_0\sigma$ and $0 < \lambda < 1$. Recall that by (4.17) one has

$$\pi^c_{\sigma,\lambda}(\Omega) = c(\sigma) + \lambda^2.$$

Thus together with Lemma 7.1 one gets

$$-\pi^c_{\sigma,\lambda}(\Omega) + \tau(\Omega) = -c(\sigma) - \lambda^2 + \tau(\Omega) \geqslant \tau^2_{n+1} \geqslant 1/4. \qquad \square$$

We are now ready to introduce the analytic torsion. We distinguish between the odd- and even-dimensional case. The reason is that the even-dimensional case can be treated in a more elementary way.

First assume that $d = 2n+1$. Let $h_p(\tau) := \dim(\ker \Delta_p(\tau) \cap L^2)$. Using (7.6), Lemmas 7.1 and 7.2, it follows that there is an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{reg}}\big(e^{-t\Delta_p(\tau)}\big) \sim h_p(\tau) + \sum_{j=1}^{\infty} c_j t^{-j/2}, \quad t \to \infty. \tag{7.10}$$

On the other hand, by Proposition 6.9, $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ has also an asymptotic expansion as $t \to 0$. Thus we can define the spectral zeta function by

$$\begin{aligned}\zeta_p(s;\tau) := {} & \frac{1}{\Gamma(s)} \int_0^1 t^{s-1}\big(\mathrm{Tr}_{\mathrm{reg}}\big(e^{-t\Delta_p(\tau)}\big) - h_p(\tau)\big)\,dt \\ & + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1}\big(\mathrm{Tr}_{\mathrm{reg}}\big(e^{-t\Delta_p(\tau)}\big) - h_p(\tau)\big)\,dt.\end{aligned} \tag{7.11}$$

By Proposition 6.9, the first integral on the right converges in the half-plane $\mathrm{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. By (7.10), the second integral converges in the half-plane $\operatorname{Re}(s) < 1/2$ and also admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$.

Now assume that $\tau \neq \tau_\theta$. This is equivalent to $\tau_{n+1} \neq 0$. Then by (2.7) and Lemma 7.1 we have $\tau(\Omega) - c(\sigma) \geqslant 1/4$ for all $\sigma \in \hat{M}$ with $[\nu_p(\tau) : \sigma] \neq 0$ and $p = 0, \dots, d$. Furthermore by Lemma 7.3 we have $\ker(\Delta_p(\tau) \cap L^2) = 0$, $p = 0, \dots, d$. By (7.6) it follows that there exist $C, c > 0$ such that for all $p = 0, \dots, d$:

$$\operatorname{Tr}_{\mathrm{reg}}\bigl(e^{-t\Delta_p(\tau)}\bigr) \leqslant Ce^{-ct}, \quad t \geqslant 1. \tag{7.12}$$

Using Proposition 6.9, it follows that $\zeta_p(s; \tau)$ can be defined as in the compact case by

$$\zeta_p(s; \tau) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}_{\mathrm{reg}}\bigl(e^{-t\Delta_p(\tau)}\bigr)\, dt. \tag{7.13}$$

The integral converges absolutely and uniformly on compact subsets of $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. We define the regularized determinant of $\Delta_p(\tau)$ as in the compact case by

$$\det \Delta_p(\tau) := \exp\biggl(-\frac{d}{ds}\zeta_p(s; \tau)|_{s=0}\biggr). \tag{7.14}$$

In analogy to the compact case we now define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ associated to the flat bundle E_τ, equipped with the admissible metric, by

$$T_X(\tau) := \prod_{p=0}^{d} \det \Delta_p(\tau)^{(-1)^{p+1} p/2}. \tag{7.15}$$

Let $K(t, \tau)$ be defined by (7.7). If $\tau \ncong \tau_\theta$, then $K(t, \tau) = O(e^{-ct})$ as $t \to \infty$, and the analytic torsion is given by

$$\log T_X(\tau) = \frac{1}{2} \frac{d}{ds}\bigg|_{s=0} \biggl(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t, \tau)\, dt\biggr), \tag{7.16}$$

where the right-hand side is defined near $s = 0$ by analytic continuation.

Now assume that $d = 2n + 2$. We use (7.16) as the definition of $T_X(\tau)$. Let $h_p(\tau) := \dim(\ker \Delta_p(\tau) \cap L^2)$ and let

$$h(\tau) := \sum_{p=0}^{d} (-1)^p p h_p(\tau).$$

Then it follows from (7.6) and Lemma 7.1 that there exists a constant $c > 0$ such that

$$K(t, \tau) - h(\tau) = O\bigl(e^{-ct}\bigr), \quad t \to \infty. \tag{7.17}$$

Next we use (7.9) to determine the short-time asymptotics of $K(t,\tau)$ and to prove Proposition 1.4. To compute the terms on the right-hand side of (7.9), we note that by [34, Lemma 4.1] we have

$$\Theta_{\sigma,\lambda}(k_t^\tau)=0, \quad \forall\sigma\in\hat{M},\ \lambda\in\mathbb{R}. \tag{7.18}$$

This result immediately implies $H(k_t^\tau)=0$ by (6.3), $T(k_t^\tau)=0$ by (6.5), and $\mathcal{I}(k_t^\tau)=0$ by Theorem 6.2. The identity contribution is given by

$$I(k_t^\tau)=\operatorname{vol}(X)k_t^\tau(1).$$

Since k_t^τ is a K-finite function in $\mathcal{C}(G)$, the Plancherel theorem can be applied to k_t^τ by [15, Theorem 3]. Thus by [17, Theorem 13.5] and (7.18) we have

$$k_t^\tau(1)=\sum_{\pi\in\hat{G}_d}a(\pi)\Theta_\pi(k_t^\tau),$$

where $\hat{G}_d$ denotes the discrete series and $a(\pi)\in\mathbb{C}$. Since k_t^τ is K-finite, the sum is finite. In [34, Section 5] it was shown that for each $\pi\in\hat{G}_d$, $\Theta_\pi(k_t^\tau)$ is independent of $t>0$. This implies that $I(k_t^\tau)$ is independent of t. Summarizing, it follows from (7.9) that there exists $c(\tau)\in\mathbb{C}$ such that

$$K(t,\tau)=c(\tau)+J(k_t^\tau). \tag{7.19}$$

Next we investigate $J(k_t^\tau)$. Using (7.8) and (6.14), we have

$$J(k_t^\tau)=-\frac{\kappa(X)}{4\pi i}\sum_{p=1}^{d}(-1)^p p\sum_{\substack{\nu\in\hat{K}\\ [\nu_p(\tau):\nu]\neq 0}}\sum_{\sigma\in\hat{M}}[\nu:\sigma]\dim(\sigma)e^{-t(\tau(\Omega)-c(\sigma))}$$
$$\cdot\int_{D_\epsilon}e^{-tz^2}c_\nu(\sigma:z)^{-1}\frac{d}{dz}c_\nu(\sigma:z)\,dz. \tag{7.20}$$

Thus by Lemma 7.1 one has

$$J(k_t^\tau)=O(e^{-ct}),\quad t\to\infty$$

for some constant $c>0$. Using (7.17) and (7.19) it follows that $c(\tau)=h(\tau)$ and we get

$$K(t,\tau)-h(\tau)=J(k_t^\tau). \tag{7.21}$$

For the short-time asymptotics of $K(t,\tau)$, we use Eq. (6.10), Lemmas 6.6, 6.7 and (7.21). This implies that there exist $a_j,b_j\in\mathbb{C}$ such that

$$K(t,\tau)\sim\sum_{j=0}^\infty a_jt^{j-1/2}+\sum_{j=0}^\infty b_jt^{j-1/2}\log t+\sum_{j=0}^\infty c_jt^j$$

as $t \to 0$. Together with (7.17) it follows that the integral

$$\int_0^\infty t^{s-1}\bigl(K(t,\tau)-h(\tau)\bigr)\,dt$$

converges for $\operatorname{Re}(s) \gg 0$ and admits a meromorphic continuation to $s \in \mathbb{C}$ with at most a simple pole at $s=0$. Then in analogy with (7.16), we define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ of E_τ with respect to the admissible metric by

$$T_X(\tau) = \exp\left(\frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\bigl(K(t,\tau)-h(\tau)\bigr)\,dt\right)\bigg|_{s=0}\right).$$

Let $\tau = \tau_\lambda$ be an irreducible finite-dimensional representation of G with highest weight $\lambda \in \Lambda(G)$. Using (7.20) it follows that there exists a function $\psi : \mathbb{R}^+ \times \Lambda(G) \to \mathbb{R}$ such that

$$J\bigl(k_t^{\tau_\lambda}\bigr) = \kappa(X)\psi(t,\lambda)$$

for all even-dimensional X and $\lambda \in \Lambda(G)$. For $\lambda \in \Lambda(G)$ let

$$\Phi(\lambda) := \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \psi(t,\lambda)t^{s-1}\,dt\right)\bigg|_{s=0},$$

where the value at $s=0$ is defined by analytic continuation. Then by the definition of $T_X(\tau)$ we have

$$\log T_X(\tau_\lambda) = \kappa(X)\Phi(\lambda)$$

for all even-dimensional X and $\lambda \in \Lambda(G)$. This proves Proposition 1.4.

8. Virtual heat kernels

In order to deal with the Mellin transform of the terms on the right-hand side of (6.1) we express k_t^τ in terms of certain auxiliary heat kernels which are easier to handle. These functions first occurred in [3] in a different context. To begin with, we need some preparation. In this section we assume that $d = 2n+1$.

Let $\tau \in \hat{G}$ and let $\Lambda(\tau) = \tau_1 e_1 + \cdots + \tau_{n+1} e_{n+1}$ be its highest weight. For $w \in W$ let $l(w)$ denote its length with respect to the simple roots which define the positive roots above. Let

$$W^1 := \bigl\{w \in W_G:\ w^{-1}\alpha > 0,\ \forall \alpha \in \Delta(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})\bigr\}.$$

Let V_τ be the representation space of τ. For $k = 0,\ldots,2n$ let $H^k(\mathfrak{n}, V_\tau)$ be the cohomology of $\mathfrak{n}$ with coefficients in V_τ. Then $H^k(\mathfrak{n}, V_\tau)$ is an MA module. For $w \in W^1$ let $V_{\tau(w)}$ be the

MA module of highest weight $w(\Lambda(\tau)+\rho_G)-\rho_G$. By a theorem of Kostant (see [40, Theorem 2.5.1.3]), it follows that as MA-modules one has

$$H^k(\mathfrak{n};V_\tau)\cong\sum_{\substack{w\in W^1\\ l(w)=k}}V_{\tau(w)}.$$

Using the Poincaré principle [20, (7.2.3)], we get

$$\sum_{k=0}^{2n}(-1)^k\Lambda^k\mathfrak{n}^*\otimes V_\tau=\sum_{w\in W^1}(-1)^{l(w)}V_{\tau(w)} \tag{8.1}$$

as MA-modules.

For $w\in W^1$ let $\sigma_{\tau,w}$ be the representation of M with highest weight

$$\Lambda(\sigma_{\tau,w}):=w\big(\Lambda(\tau)+\rho_G\big)\big|_{\mathfrak{b}_{\mathbb{C}}}-\rho_M \tag{8.2}$$

and let $\lambda_{\tau,w}\in\mathbb{C}$ such that

$$w\big(\Lambda(\tau)+\rho_G\big)\big|_{\mathfrak{a}_{\mathbb{C}}}=\lambda_{\tau,w}e_1. \tag{8.3}$$

For $k=0,\dots,n$ let

$$\lambda_{\tau,k}=\tau_{k+1}+n-k \tag{8.4}$$

and let $\sigma_{\tau,k}$ be the representation of M with highest weight

$$\Lambda_{\sigma_{\tau,k}}:=(\tau_1+1)e_2+\cdots+(\tau_k+1)e_{k+1}+\tau_{k+2}e_{k+2}+\cdots+\tau_{n+1}e_{n+1}. \tag{8.5}$$

Then by the computations in [2, Chapter VI.3] one has

$$\begin{aligned}\big\{\big(\lambda_{\tau,w},\sigma_{\tau,w},l(w)\big):\ w\in W^1\big\}&=\big\{(\lambda_{\tau,k},\sigma_{\tau,k},k):\ k=0,\dots,n\big\}\\&\quad\sqcup\big\{(-\lambda_{\tau,k},w_0\sigma_{\tau,k},2n-k):\ k=0,\dots,n\big\}.\end{aligned} \tag{8.6}$$

We will also need the following lemma.

Lemma 8.1. *For every $w\in W^1$ one has*

$$\tau(\Omega)=\lambda_{\tau,w}^2+c(\sigma_{\tau,w}).$$

Proof. See [34, Proposition 2.7]. □

Fix $\sigma \in \hat{M}$ and assume that $\sigma \neq w_0\sigma$. For $\nu \in \hat{K}$ let $m_\nu(\sigma) \in \{-1, 0, 1\}$ be defined by (2.17). Let H_t^ν be the kernel of $e^{-t\tilde{A}_\nu}$ as in (4.7) and let $h_t^\nu := \operatorname{tr} H_t^\nu$. Put

$$h_t^\sigma(g) := e^{-tc(\sigma)} \sum_{\substack{\nu \\ m_\nu(\sigma)\neq 0}} m_\nu(\sigma) h_t^\nu(g). \tag{8.7}$$

Proposition 8.2. *For $k = 0, \ldots, n$ let $\sigma_{\tau,k}$ and $\lambda_{\tau,k}$ be as in (8.6). Then one has*

$$k_t^\tau = \sum_{k=0}^{n} (-1)^{k+1} e^{-t\lambda_{\tau,k}^2} h_t^{\sigma_{\tau,k}}.$$

Proof. It is easy to see that as M-modules $\mathfrak{p}$ and $\mathfrak{a} \oplus \mathfrak{n}$ are equivalent. Thus in the sense of M-modules one has

$$\sum_{p=0}^{d} (-1)^p p \Lambda^p \mathfrak{p}^* = \sum_{p=0}^{d} (-1)^p p \big(\Lambda^p \mathfrak{n}^* + \Lambda^{p-1} \mathfrak{n}^*\big) = \sum_{p=0}^{d-1} (-1)^{p+1} \Lambda^p \mathfrak{n}^*. \tag{8.8}$$

Let $i^* : R(K) \to R(M)^{W(A)}$ be the restriction map. Then it follows from (8.8), (8.1) and (8.6) that we have

$$\sum_{p=0}^{d} (-1)^p p\, i^*\big(\nu_p(\tau)\big) = \sum_{k=0}^{n} (-1)^{k+1} (\sigma_{\tau,k} + w_0\sigma_{\tau,k}). \tag{8.9}$$

Since $\tau \neq \tau_\theta$ we have $\sigma_{\tau,k} \neq w_0\sigma_{\tau,k}$ for all k by (2.13), (2.15) and (8.5). Thus as in (2.17) we can write

$$\sigma_{\tau,k} + w_0\sigma_{\tau,k} = \sum_{\nu \in \hat{K}} m_\nu(\sigma_{\tau,k}) i^*(\nu).$$

Moreover, the restriction map i^* is injective. Therefore the following equality holds in $R(K)$:

$$\sum_{p=0}^{d} (-1)^p p \nu_p(\tau) = \sum_{k=0}^{n} (-1)^{k+1} \sum_{\nu \in \hat{K}} m_\nu(\sigma_{\tau,k}) \nu.$$

Since $R(K)$ is a free abelian group generated by the representations $\nu \in \hat{K}$, it follows that for every $\nu \in \hat{K}$ one has

$$\sum_{p=0}^{d} (-1)^p p \big[\nu_p(\tau) : \nu\big] = \sum_{k=0}^{n} (-1)^{k+1} m_\nu(\sigma_{\tau,k}). \tag{8.10}$$

Moreover let us remark that if ν, ν_1, ν_2 are finite-dimensional unitary representations of K with $\nu = \nu_1 \oplus \nu_2$ one has

$$h_t^\nu = h_t^{\nu_1} + h_t^{\nu_2}. \tag{8.11}$$

Thus we obtain

$$
\begin{aligned}
k_t^\tau = \sum_{p=0}^{d}(-1)^p p e^{-t\tau(\Omega)} h_t^{\nu_p(\tau)} &= \sum_{p=0}^{d}(-1)^p p \sum_{\nu\in\hat{K}}\big[\nu_p(\tau):\nu\big] e^{-t\tau(\Omega)} h_t^{\nu} \\
&= \sum_{\nu\in\hat{K}}\sum_{p=0}^{d}(-1)^p p\big[\nu_p(\tau):\nu\big] e^{-t\tau(\Omega)} h_t^{\nu} \\
&= \sum_{\nu\in\hat{K}}\sum_{k=0}^{n}(-1)^{k+1} m_\nu(\sigma_{\tau,k}) e^{-t(\tau(\Omega))} h_t^{\nu} \qquad (+)\\
&= \sum_{k=0}^{n}(-1)^{k+1}\sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau,k}) e^{-t(\tau(\Omega))} h_t^{\nu} \\
&= \sum_{k=0}^{n}(-1)^{k+1}\sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau,k}) e^{-t(\lambda_{\tau,k}^2+c(\sigma_{\tau,k}))} h_t^{\nu} \qquad (++)\\
&= \sum_{k=0}^{n}(-1)^{k+1} e^{-t\lambda_{\tau,k}^2} h_t^{\sigma_{\tau,k}}. \qquad (+++)
\end{aligned}
$$

Here the second equality in the first line follows from (8.11), (+) is (8.10), (++) follows from Lemma 8.1 and (+++) follows from (8.7). □

Finally we compute the Fourier transform of h_t^σ, $\sigma \in \hat{M}$. Using (2.17) and Proposition 4.1, it follows that for a principal series representation $\pi_{\sigma',\lambda}$, $\lambda \in \mathbb{R}$ we have

$$
\Theta_{\sigma',\lambda}\big(h_t^\sigma\big) = e^{-t\lambda^2} \quad \text{for } \sigma' \in \{\sigma, w_0\sigma\}; \qquad \Theta_{\sigma',\lambda}\big(h_t^\sigma\big) = 0, \quad \text{otherwise}. \tag{8.12}
$$

9. L^2-torsion

In this section we briefly discuss the L^2-torsion $T_X^{(2)}(\tau)$. We assume that $d = 2n+1$. For the trivial representation, the L^2-torsion of complete hyperbolic manifolds of finite volume has been studied in [24]. Although X is not compact, the L^2-torsion can be defined as in the compact case [23]. This follows from the fact that $\tilde{X}$ is homogeneous. We assume that the highest weight of τ satisfies $\tau_{n+1} \neq 0$. Let $\widetilde{\Delta}_p(\tau)$ be the Laplace operator on $\tilde{E}_\tau$-valued p-forms on $\tilde{X}$. By (7.4) the kernel of $e^{-t\widetilde{\Delta}_p(\tau)}$ is given by $e^{-t\tau(\Omega)} H_t^{\nu_p(\tau)}$ where $H_t^{\nu_p(\tau)}$ is the kernel of the operator induced by $-\Omega$ in the homogeneous bundle attached to $\nu_p(\tau)$ (see (4.6)). Then the Γ-trace of $e^{-t\widetilde{\Delta}_p(\tau)}$ (see [23] for its definition) is given by

$$
\mathrm{Tr}_\Gamma\big(e^{-t\widetilde{\Delta}_p(\tau)}\big) = \mathrm{vol}(X) e^{-t\tau(\Omega)} h_t^{\nu_p(\tau)}(1). \tag{9.1}
$$

Applying the Plancherel theorem to $h_t^{\nu_p(\tau)}(1)$ and using Proposition 4.1, we get

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) = \operatorname{vol}(X) \sum_{\substack{\sigma\in\hat{M}\\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))} \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)\, d\lambda. \tag{9.2}$$

Since $P_\sigma(z)$ is an even polynomial of degree $d-1$, we get an asymptotic expansion

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) \sim \sum_{k=0}^{\infty} a_j t^{j-d/2}, \quad t\to 0. \tag{9.3}$$

Since we are assuming that the highest weight of τ satisfies $\tau_{n+1}\neq 0$, it follows from Lemma 7.1 and (9.2) there exists $c>0$ such that

$$\operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) = O\left(e^{-ct}\right) \tag{9.4}$$

as $t\to\infty$. Therefore the Mellin transform

$$\int_0^\infty \operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) t^{s-1}\, dt$$

converges absolutely and uniformly on compact subsets of $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$. Moreover, since the asymptotic expansion (9.3) has no constant term, the Mellin transform is regular at $s=0$. So we can define the L^2-torsion $T_X^{(2)}(\tau)\in\mathbb{R}^+$ by

$$\log T_X^{(2)}(\tau) = \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\sum_{p=1}^{d}(-1)^p p \int_{\mathbb{R}} \operatorname{Tr}_\Gamma\left(e^{-t\widetilde{\Delta}_p(\tau)}\right) t^{s-1}\, dt\right)\bigg|_{s=0}. \tag{9.5}$$

Now recall that the contribution of the identity $I(k_t^\tau)$ to the right-hand side of (7.9) is given by

$$I(t,\tau) := \operatorname{vol}(X) k_t^\tau(1).$$

Let

$$\mathcal{M}I(s,\tau) := \int_0^\infty I(t,\tau) t^{s-1}\, dt$$

be the Mellin transform. Using (7.8) and the considerations above, it follows that the integral converges for $\operatorname{Re}(s) > d/2$ and has a meromorphic extension to $\mathbb{C}$ which is regular at $s=0$. Let $\mathcal{M}I(\tau)$ be its value at $s=0$. Then by (7.8), (9.1), and (9.5) we have

$$\log T_X^{(2)}(\tau) = \frac{1}{2}\mathcal{M}I(\tau). \tag{9.6}$$

Our next goal is to compute $\mathcal{M}I(\tau)$. Let $\sigma_{\tau,k}$ and $\lambda_{\tau,k}$, $k = 0, \dots, n$, be defined by (8.4) and (8.5), respectively. Then for every k we have $\sigma_{\tau,k} \neq w_0\sigma_{\tau,k}$. Let $P_{\sigma_{\tau},k}$ be the Plancherel polynomial. Using Proposition 8.2, the Plancherel theorem, (8.12) and (2.22), we obtain

$$I(t,\tau) = 2\operatorname{vol}(X)\sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda_{\tau,k}^2}\int_{\mathbb{R}} e^{-t\lambda^2}P_{\sigma_\tau,k}(i\lambda)\,d\lambda. \tag{9.7}$$

To evaluate the Mellin transform of $I(t,\tau)$ at $s = 0$, we use the following elementary lemma.

Lemma 9.1. *Let P be an even polynomial. Let $c > 0$ and $\sigma \in \hat{M}$. For $\operatorname{Re}(s) > \frac{d}{2}$ let*

$$E(s) := \int_0^\infty t^{s-1}e^{-tc^2}\int_{\mathbb{R}} e^{-t\lambda^2}P(i\lambda)\,d\lambda\,dt.$$

Then $E(s)$ has a meromorphic continuation to $\mathbb{C}$. Moreover $E(s)$ is regular at zero and

$$E(0) = -2\pi\int_0^c P(\lambda)\,d\lambda.$$

Proof. This follows from Lemmas 2 and 3 in [9]. □

We have $\lambda_{\tau,k} > 0$ for every k. Applying Lemma 9.1 to the right-hand side of (9.7) we obtain

$$\mathcal{M}I(\tau) = 4\pi\operatorname{vol}(X)\sum_{k=0}^{n}(-1)^k\int_0^{\lambda_{\tau,k}} P_{\sigma_{\tau,k}}(\lambda)\,d\lambda.$$

Together with (9.6) we get the following proposition.

Proposition 9.2. *Let τ be such that $\tau_{n+1} \neq 0$. Then we have*

$$\log T_X^{(2)}(\tau) = 2\pi\operatorname{vol}(X)\sum_{k=0}^{n}(-1)^k\int_0^{\lambda_{\tau,k}} P_{\sigma_{\tau,k}}(\lambda)\,d\lambda.$$

10. Proof of the main results

In this section we assume that $d = \dim(X)$ is odd. Let $d = 2n+1$. We fix natural numbers $\tau_1, \dots, \tau_{n+1}$ with $\tau_1 \geqslant \tau_2 \geqslant \dots \geqslant \tau_{n+1}$. For $m \in \mathbb{N}$ we let $\tau(m)$ be the representation of G with highest weight $(m+\tau_1)e_1 + \dots + (m+\tau_{n+1})e_{n+1}$. Then $\tau(m)$ satisfies $\tau(m)\circ\theta \not\approx \tau$. Hence the analytic torsion $T_X(\tau(m))$ is defined by (7.16).

Our goal is to study the asymptotic behavior of $\log T_X(\tau(m))$ as $m \to \infty$. To begin with, for $k \in \{0, \dots, n\}$ let $\lambda_{\tau(m),k} \in \mathbb{R}$ and $\sigma_{\tau(m),k} \in \hat{M}$ with highest weight $\Lambda(\sigma_{\tau(m),k})$ be defined as in (8.4) resp. (8.5). One has

$$\begin{aligned}\Lambda(\sigma_{\tau(m),k}) = {} & (m+\tau_1+1)e_2 + \cdots + (m+\tau_k+1)e_{k+1} \\ & + (m+\tau_{k+2})e_{k+2} + \cdots + (m+\tau_{n+1})e_{n+1}\end{aligned} \tag{10.1}$$

and

$$\lambda_{\tau(m),k} = m + \tau_{k+1} + n - k. \tag{10.2}$$

We use the decomposition (7.9) of $K(t, \tau(m))$ and study the Mellin transform of each term on the right-hand side separately. First we consider the identity contribution which is given by

$$I\bigl(t, \tau(m)\bigr) := \operatorname{vol}(X) k_t^{\tau(m)}(1).$$

Its Mellin transform $\mathcal{M}I(\tau(m))$ has been computed in the previous section and the contribution to $\log T_X(\tau(m))$ equals

$$\frac{1}{2}\mathcal{M}I\bigl(\tau(m)\bigr) = \log T_X^{(2)}\bigl(\tau(m)\bigr).$$

In order to study the asymptotic behavior of $\log T_X^{(2)}(\tau(m))$ as $m \to \infty$, we use Proposition 9.2. Let

$$P_\tau(m) := 2\pi \sum_{k=0}^{n} (-1)^k \int_0^{\lambda_{\tau(m),k}} P_{\sigma_{\tau(m),k}}(\lambda)\, d\lambda.$$

Using (10.2) and the explicit form of the Plancherel polynomial $P_{\sigma_{\tau(m),k}}(\lambda)$, it follows that $P_\tau(m)$ is a polynomial in m of degree $n(n+1)/2 + 1$. The coefficient of the leading power has been determined at the end of Section 5 of [34]. Let $C(n)$ be constant given by (1.7). Combining the results above with the computations of the leading coefficient of $P_\tau(m)$ in [34], we get

Proposition 10.1. *We have*

$$\log T_X^{(2)}\bigl(\tau(m)\bigr) = C(n)\operatorname{vol}(X) m \dim \tau(m) + O\bigl(m^{\frac{n(n+1)}{2}}\bigr),$$

as $m \to \infty$.

Thus to prove our main results we have to show that the Mellin transform of the terms in (7.9) which are different from the identity contribution is of lower order as $m \to \infty$. We begin with the contribution of the hyperbolic term to the analytic torsion. For $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ and $\sigma \in \hat{M}$ let $L(\gamma, \sigma)$ be defined by (6.2). Put

$$L_{\mathrm{sym}}(\gamma; \sigma) := L(\gamma; \sigma) + L(\gamma; w_0\sigma). \tag{10.3}$$

Using (6.3), Proposition 8.2 and (8.12), it follows that the hyperbolic contribution is given by

$$H\big(t,\tau(m)\big) := \sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda_{\tau(m),k}^2}\sum_{[\gamma]\in C(\Gamma)_s-[1]}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L_{\mathrm{sym}}(\gamma;\sigma_{\tau(m),k})\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}}. \tag{10.4}$$

In order to study the Mellin transform of $H(t,\tau(m))$, we use the following proposition.

Proposition 10.2. *Let $\lambda > \sqrt{2}n$ and $\sigma \in \hat{M}$. For every $s \in \mathbb{C}$ the integral*

$$G(s,\lambda;\sigma) := \int_0^\infty t^{s-1}e^{-t\lambda^2}\sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}}\,dt \tag{10.5}$$

converges absolutely and is an entire function of s. There exists a constant C_0 which is independent of σ and λ such that

$$\big|G(0,\lambda;\sigma)\big| \leqslant C_0\dim(\sigma). \tag{10.6}$$

Proof. Let

$$f(t) := \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L(\gamma;\sigma)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}}.$$

We have

$$\big|f(t)\big| \leqslant \dim(\sigma)\sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L(\gamma;1)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}},$$

where 1 stands for the trivial representation of M. Now let Δ_0 be the Laplace operator acting on $C^\infty(X)$ and let Δ_0^d be its restriction to the point spectrum. Then the right-hand side is exactly the hyperbolic contribution to the Selberg trace formula for $\mathrm{Tr}(e^{-t\Delta_0^d})$. So we can apply the trace formula to estimate the right-hand side. Denote the trivial representation of K by 1 too. Then if we apply the trace formula [42, Theorems 8.4, 9.3] and use Eq. (4.16), Proposition 4.1, Eqs. (6.1) and (6.3), it follows that there exist constants $c_1(\Gamma)$, $c_2(\Gamma)$ such that

$$\begin{aligned} &e^{-tn^2}\sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\frac{\ell(\gamma)}{n_\Gamma(\gamma)}L(\gamma;1)\frac{e^{-\ell(\gamma)^2/4t}}{(4\pi t)^{\frac{1}{2}}} \\ &\quad= \mathrm{Tr}\big(e^{-t\Delta_0^d}\big) - \int_{\mathbb{R}} e^{-t(\lambda^2+n^2)}\operatorname{vol}(X)P_1(i\lambda)\,d\lambda \\ &\qquad- \int_{\mathbb{R}} e^{-t(\lambda^2+n^2)}\bigg(\psi(1+i\lambda)+c_2(\Gamma)+\mathrm{Tr}\bigg(\tilde{\boldsymbol{C}}(1,1,-i\lambda)\frac{d}{dz}\tilde{\boldsymbol{C}}(1,1,i\lambda)\bigg)\bigg)\,d\lambda \\ &\qquad+ c_1(\Gamma)e^{-tn^2}. \end{aligned}$$

The right-hand side of this equation is bounded for $t \geqslant 1$. Thus there exists a constant C_1 which is independent of σ such that

$$\left|f(t)\right| \leqslant C_1 \dim(\sigma)\, e^{tn^2}, \quad t \geqslant 1. \tag{10.7}$$

For $\lambda > n$ and $s \in \mathbb{C}$ put

$$G_0(s,\lambda;\sigma) := \int_1^\infty t^{s-1} e^{-t\lambda^2} f(t)\, dt.$$

Then it follows from (10.7) that $G_0(s,\lambda;\sigma)$ is an entire function of s and that for $\lambda > \sqrt{2}n$ we can estimate

$$\left|G_0(0,\lambda;\sigma)\right| \leqslant \int_1^\infty t^{-1} e^{-t\lambda^2} \left|f(t)\right| dt \leqslant C_1 \dim(\sigma)\, e^{-\frac{\lambda^2}{4}}, \quad \lambda > \sqrt{2}n. \tag{10.8}$$

Next we consider the integral from 0 to 1. To begin with, we need to estimate $L(\gamma,\sigma)$. By [11, Proposition 5.4] there exist a constant $C_2 > 0$ such that for $R > 0$ one has

$$\#\left\{[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}}\colon \ell(\gamma) \leqslant R\right\} \leqslant C_2 e^{2nR}. \tag{10.9}$$

Thus if we let

$$c := \min\left\{\ell(\gamma)\colon [\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]\right\} \tag{10.10}$$

we have $c > 0$. Moreover one has

$$\det\left(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}\right) \geqslant \left(1 - e^{-\ell(\gamma)}\right)^n.$$

Hence there exists a constant C_3 such that for all $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ one has

$$\frac{1}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} \leqslant C_3.$$

It follows that there exists a constant C_4 which is independent of σ such that for every $[\gamma] \in \mathrm{C}(\Gamma)_{\mathrm{s}} - [1]$ one has

$$\frac{\ell(\gamma)}{n_\Gamma(\gamma)}\left|L(\gamma;\sigma)\right| \leqslant \frac{\dim(\sigma)\ell(\gamma)e^{-n\ell(\gamma)}}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})} \leqslant C_4 \dim(\sigma). \tag{10.11}$$

Using (10.9) and (10.11), it follows that there exist constants $c_1 > 0$, $C_5 > 0$ which are independent of σ such that

$$\left|f(t)\right| \leqslant C_5 \dim(\sigma) e^{-c_1/t}, \quad 0 < t \leqslant 1. \tag{10.12}$$

For $\lambda \geqslant 0$ and $s \in \mathbb{C}$ put

$$G_1(s,\lambda;\sigma) = \int_0^1 t^{s-1} e^{-t\lambda^2} f(t)\,dt.$$

By (10.12), $G_1(s,\lambda;\sigma)$ is an entire function of s and its value at zero can be estimated as follows

$$\left|G_1(0,\lambda;\sigma)\right| \leqslant \int_0^1 t^{-1} e^{-t\lambda^2} \left|f(t)\right| dt \leqslant C_6 \dim(\sigma) \int_0^1 e^{-t\lambda^2} e^{-\frac{c_1}{2t}}\,dt \leqslant C_6 \dim(\sigma).$$

Together with (10.8) the proposition follows. □

Now let $m > \sqrt{2}n$. Then by (10.2) one has $\lambda_{\tau(m),k} > \sqrt{2}n$ for every k. Thus by (10.4) and Proposition 10.2 the integral

$$\mathcal{M}H\big(s,\tau(m)\big) := \int_0^\infty t^{s-1} H\big(t,\tau(m)\big)\,dt$$

converges absolutely and uniformly on compact subsets of $\mathbb{C}$ and defines an entire function of s. Denote by $\mathcal{M}H(\tau(m))$ its value at zero. It can be estimated as follows.

Proposition 10.3. *There exists a constant C such that for every $m > \sqrt{2}n$ one has*

$$\left|\mathcal{M}H\big(\tau(m)\big)\right| \leqslant C m^{\frac{n(n-1)}{2}}.$$

Proof. By (2.14) and (10.1) there exists a constant C such that for every $m \in \mathbb{N}$ one has

$$\dim(\sigma_{\tau(m),k}) \leqslant C m^{\frac{n(n-1)}{2}}. \tag{10.13}$$

The proposition follows from Proposition 10.2. □

The contribution of the distribution T can be treated without difficulty.

Proposition 10.4. *For* $\operatorname{Re}(s) \gg 0$ *let*

$$\mathcal{M}T\big(s,\tau(m)\big) := \int_0^\infty t^{s-1} T\big(k_t^{\tau(m)}\big)\,dt.$$

Then $\mathcal{M}T(s,\tau(m))$ has a meromorphic continuation to $\mathbb{C}$ and is regular at $s=0$. Let $\mathcal{M}T(\tau(m))$ denote its value at $s=0$. Then there exists a constant C which is independent of m such that

$$\left|\mathcal{M}T\big(\tau(m)\big)\right| \leqslant C m^{\frac{n(n+1)}{2}}.$$

Proof. By Proposition 8.2, Eqs. (6.5) and (8.12) we have

$$\mathcal{M}T\big(s,\tau(m)\big)=\frac{C(\Gamma)}{2\sqrt{\pi}}\sum_{k=0}^{n}(-1)^{k+1}\dim(\sigma_{\tau(m),k})(\lambda_{\tau(m),k})^{-2s+1}\Gamma\left(s-\frac{1}{2}\right).$$

The proposition follows from (10.2) and (10.13). □

To treat the remaining terms, we need the following two auxiliary lemmas.

Lemma 10.5. *For $c\in(0,\infty)$, $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$, $j\in[0,\infty)$ let*

$$\zeta_c(s):=\frac{1}{\pi}\int_0^\infty t^{s-1}e^{-tc^2}\int_{D_\epsilon}\frac{e^{-tz^2}}{iz+j}\,dz\,dt,$$

where D_ϵ is the same contour as in (6.12). Then $\zeta_c(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at 0. Moreover, one has

$$\frac{d}{ds}\bigg|_{s=0}\frac{\zeta_c(s)}{\Gamma(s)}=-2\log(c+j).$$

Proof. The statement about the convergence of the integral and the meromorphic continuation follows from Lemma 6.6 and standard methods. Note that

$$\int_{D_\epsilon}\frac{e^{-tz^2}}{iz}\,dz=\frac{1}{2}\int_{|z|=\epsilon}\frac{e^{-tz^2}}{iz}\,dz=\pi.$$

Hence, for $j=0$ we have

$$\zeta_c(s)=c^{-2s}\Gamma(s)$$

and the claim follows in this case. Assume that $j>0$. Then one has

$$\zeta_c(s)=\frac{j}{\pi}\int_0^\infty t^{s-1}e^{-tc^2}\int_{\mathbb{R}}\frac{e^{-t\lambda^2}}{\lambda^2+j^2}\,d\lambda\,dt. \tag{10.14}$$

For $\mathrm{Re}(z^2)>0$, $\mathrm{Re}(z)>0$ define a function $\zeta(z,s)$ by

$$\zeta(z,s):=\frac{j}{\pi}\int_0^\infty t^{s-1}e^{-tz^2}\int_{\mathbb{R}}\frac{e^{-t\lambda^2}}{\lambda^2+j^2}\,d\lambda\,dt.$$

Then it is easy to see that $\zeta(z,s)$ is holomorphic in z. Let

$$\phi(z,s) := \frac{j}{\pi}\int_0^\infty t^{s-1}\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+z^2)}}{\lambda^2+j^2}\,d\lambda\,dt - \frac{j}{\pi}\int_0^\infty t^{s-1}\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+j^2)}}{\lambda^2+j^2}\,d\lambda\,dt.$$

Then, since $e^{-tz^2} - e^{-tj^2} = O(t)$, $t \to 0$, the integral converges for $\operatorname{Re}(s) > -1$. One has

$$\frac{d}{dz}\phi(z,0) = -\frac{2jz}{\pi}\int_0^\infty\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+z^2)}}{\lambda^2+j^2}\,d\lambda\,dt = \frac{-2}{z+j}.$$

Since $\phi(j,0) = 0$, one has

$$\phi(z,0) = -2\log(z+j) + 2\log 2j. \tag{10.15}$$

On the other hand, one has

$$\zeta(j,s) = \frac{j}{\pi s}\int_0^\infty \left(\frac{d}{dt}t^s\right)\int_{\mathbb{R}} \frac{e^{-t(\lambda^2+j^2)}}{\lambda^2+j^2}\,d\lambda\,dt = \frac{j^{-2s}}{\sqrt{\pi}\,s}\Gamma\left(s+\frac{1}{2}\right).$$

Hence for $s \to 0$ one has

$$\zeta(j,s) = \frac{1}{s} - 2\log j + \frac{\Gamma'(\frac{1}{2})}{\sqrt{\pi}} + O(s) = \frac{1}{s} - 2\log j + \psi\left(\frac{1}{2}\right) + O(s).$$

Together with (10.15) this gives for $s \to 0$:

$$\begin{aligned}\zeta(z,s) &= \frac{1}{s} - 2\log j + \psi\left(\frac{1}{2}\right) - 2\log(z+j) + 2\log 2j + O(s)\\ &= \frac{1}{s} - 2\log(z+j) - \gamma + O(s),\end{aligned}$$

where we used $\psi(\frac{1}{2}) = -2\log 2 - \gamma$. Since for $s \to 0$ one has

$$\frac{1}{\Gamma(s)} = s + \gamma s^2 + O\big(s^3\big), \tag{10.16}$$

the proposition follows. □

Lemma 10.6. *Let $c \in \mathbb{R}^+$, $s \in \mathbb{C}$,* $\operatorname{Re}(s) > 1/2$. *Define*

$$\tilde{\zeta}_c(s) := \frac{1}{\pi}\int_0^\infty t^{s-1}e^{-tc^2}\int_{\mathbb{R}} e^{-t\lambda^2}\psi(1+i\lambda)\,d\lambda\,dt.$$

Then $\tilde{\zeta}_c(s)$ has a meromorphic continuation to $s \in \mathbb{C}$ with at most a simple pole at $s = 0$. Moreover there exists a constant $C(\psi)$ which is independent of c such that

$$\frac{d}{ds}\bigg|_{s=0} \frac{\tilde{\zeta}_c(s)}{\Gamma(s)} = -2\log\Gamma(1+c) + C(\psi).$$

Proof. The convergence of the integral and the statement about the meromorphic continuation follow from Lemma 6.7 and standard methods. Fix $c_0 \in \mathbb{R}^+$. Since $e^{-tz^2} - e^{-tc_0^2} = O(t)$ as $t \to 0$, it follows from Lemma 6.7 that the integral

$$\tilde{\phi}_c(s,z) := \int_0^\infty t^{s-1} \int_{\mathbb{R}} \left(e^{-t(\lambda^2+z^2)} - e^{-t(\lambda^2+z_0^2)}\right)\psi(1+i\lambda)\,d\lambda\,dt$$

converges for $\mathrm{Re}(s) > -\frac{1}{2}$ and is holomorphic in $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, $\mathrm{Re}(z^2) > 0$. One has

$$\frac{\partial}{\partial z}\tilde{\phi}_c(0,z) = -2z\int_{\mathbb{R}} \frac{\psi(1+i\lambda)}{\lambda^2+z^2}\,d\lambda = -2\pi\psi(1+z).$$

This proves the lemma. $\square$

Next we treat the contribution of the distribution $\mathcal{I}$ to the analytic torsion. By Theorem 6.2, Propositions 6.4, 8.2 and (8.12) we have

$$\mathcal{I}\left(k_t^{\tau(m)}\right) = \frac{\kappa}{2\pi}\sum_{k=0}^{n}(-1)^{k+1}e^{-t\lambda_{\tau(m),k}^2}\int_{\mathbb{R}} \Omega(\sigma_{\tau(m),k},\lambda)e^{-t\lambda^2}\,d\lambda. \tag{10.17}$$

By Proposition 6.4 we have the decomposition

$$\Omega(\sigma_{\tau(m),k},\lambda) = \Omega_1(\sigma_{\tau(m),k},\lambda) + \Omega_2(\sigma_{\tau(m),k},\lambda).$$

Using the description of Ω_1 and Ω_2 together with Lemmas 6.6, 6.7 and 6.8, it follows that $\mathcal{I}(k_t^{\tau(m)})$ admits an asymptotic expansion

$$\mathcal{I}\left(k_t^{\tau(m)}\right) \sim \sum_{k=0}^\infty a_k t^{k-(d-2)/2} + \sum_{k=0}^\infty b_k t^{k-1/2}\log t + c_0$$

as $t \to 0$. Moreover, since $\lambda_{\tau(m),k} > m$ for every k, it follows that $\mathcal{I}(k_t^{\tau(m)}) = O(e^{-tm^2})$ as $m \to \infty$. Thus for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > (d-2)/2$ the integral

$$\mathcal{MI}\left(s;\tau(m)\right) := \int_0^\infty t^{s-1}\mathcal{I}\left(k_t^{\tau(m)}\right)dt$$

converges and has a meromorphic continuation to $\mathbb{C}$ with at most a simple pole at $s=0$. Let

$$\mathcal{MI}\bigl(\tau(m)\bigr) := \frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{MI}(s;\tau(m))}{\Gamma(s)}.$$

Next we will estimate $\mathcal{MI}(\tau(m))$ as $m\to\infty$. To this end we establish some auxiliary lemmas.

Lemma 10.7. *There exists a constant C such that for every m one has*

$$\sum_{k=0}^{n}(-1)^k \dim(\sigma_{\tau(m),k})\bigl(\log\Gamma\bigl(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k}\bigr)+\gamma\lambda_{\tau(m),k}+C(\psi)\bigr)$$
$$\leqslant Cm^{\frac{n(n+1)}{2}}, \tag{10.18}$$

where $C(\psi)$ is as in Lemma 10.6.

Proof. By (8.6) and (8.1) one has

$$2\sum_{k=0}^{n}(-1)^k \dim(\sigma_{\tau(m),k}) = \dim(\tau)\sum_{p=0}^{2n}(-1)^p \dim\Lambda^p\mathfrak{n}^* = 0. \tag{10.19}$$

Thus the sum on the left-hand side of (10.18) equals

$$\sum_{k=0}^{n}(-1)^k \dim(\sigma_{\tau(m),k})\biggl(\log\frac{\Gamma(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k})}{\Gamma(2m)}+\gamma\lambda_{\tau(m),k}\biggr).$$

It follows from (10.2) that there exists a constant C which is independent of m such that

$$\log\frac{\Gamma(k_{n+1}(\sigma_{\tau(m),k})+\lambda_{\tau(m),k})}{\Gamma(2m)} \leqslant C\log m.$$

Using (10.2) and (10.13) the proposition is proved. □

The next two lemmas are concerned with the polynomials $P_j(\sigma,\lambda)$, $j=2,\dots,n+1$, which are defined by (6.23).

Lemma 10.8. *Let $k\in\{0,\dots,n\}$ and let $j\in\{2,\dots,n+1\}$. Then there exists a constant C such that for every m one has*

$$\bigl|P_j(\sigma_{\tau(m),k},\lambda)\bigr| \leqslant Cm^{\frac{(n-1)(n-2)}{2}}\sum_{i=0}^{2(n-1)}\bigl(1+|\lambda|\bigr)^i m^{2(n-1)-i} \tag{10.20}$$

and such that

$$\left|\frac{d}{d\lambda}P_j(\sigma_{\tau(m),k},\lambda)\right| \leqslant Cm^{\frac{(n-1)(n-2)}{2}} \sum_{i=0}^{2(n-1)-1} \bigl(1+|\lambda|\bigr)^i m^{2(n-1)-i} \tag{10.21}$$

for all $\lambda \in \mathbb{C}$.

Proof. If we use the explicit formula (6.24) for the polynomials $P_j(\sigma,\lambda)$, combined with (10.1) and (10.13), the lemma follows. □

Lemma 10.9. *Let* $k \in \{0,\dots,n\}$ *and let* $j \in \{2,\dots,n+1\}$. *For* $l \in \mathbb{N}$ *with* $m \leqslant l \leqslant k_j(\sigma_{\tau(m),k}) + \rho_j$ *let the even polynomial* $Q_{j,l}(\sigma_{\tau(m),k},\lambda)$ *be defined by* (6.25). *Then there exists a constant* C *such that for every* m *one has*

$$\left|\int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)\,d\lambda\right| \leqslant Cm^{\frac{n(n+1)}{2}}.$$

Proof. By (6.25) we have

$$Q_{j,l}(\sigma_{\tau(m),k}, i\lambda) = \frac{P_j(\sigma_{\tau(m),k}, i\lambda) - P_j(\sigma_{\tau(m),k}, il)}{l-\lambda} + \frac{P_j(\sigma_{\tau(m),k}, i\lambda) - P_j(\sigma_{\tau(m),k}, il)}{l+\lambda}.$$

Using the fact that $P_j(\sigma,z)$ is an even polynomial, together with Eqs. (10.2) and (10.21), we obtain

$$\int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)\,d\lambda \leqslant 2\lambda_{\tau(m),k} \max_{|\xi|\leqslant l+\lambda_{\tau(m),k}} \left|\frac{d}{d\lambda}\right|_{\lambda=\xi} P_j(\sigma,i\lambda)\right| \leqslant Cm^{\frac{n(n+1)}{2}}. \qquad \square$$

Now we can estimate $\mathcal{MI}(\tau(m))$ as $m \to \infty$.

Proposition 10.10. *There exists a constant* C *such that for every* m *one has*

$$\bigl|\mathcal{MI}\bigl(\tau(m)\bigr)\bigr| \leqslant Cm^{\frac{n(n+1)}{2}}.$$

Proof. For $k \in \{0,\dots,n\}$ let

$$\mathcal{MI}(s;\sigma_{\tau(m),k}) = \int_0^\infty t^{s-1} e^{-t\lambda^2_{\tau(m),k}} \mathcal{I}\bigl(h_t^{\sigma_{\tau(m),k}}\bigr)\,dt.$$

As in the case of $\mathcal{MI}(s;\tau(m))$ it follows that the integral converges for $\mathrm{Re}(s) > (d-2)/2$ and admits a meromorphic continuation to $\mathbb{C}$ with at most a simple pole at $s=0$. By Proposition 8.2 we have

$$\mathcal{MI}\bigl(\tau(m)\bigr) = \sum_{k=0}^n (-1)^{k+1} \frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{MI}(s;\sigma_{\tau(m),k})}{\Gamma(s)}.$$

Let $k \in \{0, \dots, n\}$. Using Proposition 6.4, Lemma 10.5 together with (10.14), Lemma 10.6 and Lemma 9.1 we obtain

$$\begin{aligned}
&\frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{MI}(s;\sigma_{\tau(m),k})}{\Gamma(s)} \\
&\quad = 2\kappa \dim(\sigma_{\tau(m),k})\bigl(\log\Gamma\bigl(k_{n+1}(\sigma_{\tau(m),k}) + \lambda_{\tau(m),k}\bigr) + \gamma\lambda_{\tau(m),k} + C(\psi)\bigr) \\
&\qquad + \kappa\sum_{j=2}^{n+1}\sum_{\substack{k_{n+1}(\sigma_{\tau(m),k})\leqslant l \\ <k_j(\sigma_{\tau(m),k})+\rho_j}}\left(2P_j(\sigma_{\tau(m),k}, il)\log(l+\lambda_{\tau(m),k}) + \int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)\,d\lambda\right) \\
&\qquad + \kappa\sum_{\substack{j=2 \\ l=k_j(\sigma_{\tau(m),k})+\rho_j}}^{n+1}\left(\dim(\sigma_{\tau(m),k})\log(l+\lambda_{\tau(m),k}) + \frac{1}{2}\int_0^{\lambda_{\tau(m),k}} Q_{j,l}(\sigma_{\tau(m),k}, i\lambda)\,d\lambda\right).
\end{aligned}$$

By (10.1) we have $m \leqslant k_{n+1}(\sigma_{\tau(m),k})$ and $k_j(\sigma_{\tau(m),k}) + \rho_j \leqslant m + \tau_1 + n + 1$ for every $j = 2, \dots, n+1$, and by (10.2) we have $0 \leqslant \lambda_{\tau(m),k} \leqslant m + \tau_1 + n$. Thus if we apply Lemmas 10.7, 10.8, 10.9 and (10.13), the proposition follows. $\square$

Finally we consider the asymptotic behavior of the contribution of the non-invariant distribution J to $\log T_X(\tau(m))$. For $k \in \{0, \dots, n\}$ let $h_t^{\sigma_{\tau(m),k}}$ be as in (8.7), and for $\nu \in \hat{K}$ let

$$m_\nu(\sigma_{\tau(m),k}) \in \{-1, 0, 1\}$$

be defined by (2.17). By (6.14) we have

$$\begin{aligned}
J\bigl(h_t^{\sigma_{\tau(m),k}}\bigr) &= e^{-tc(\sigma_{\tau(m),k})}\sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau(m),k}) J\bigl(h_t^\nu\bigr) \\
&= -\frac{\kappa e^{-tc(\sigma_{\tau(m),k})}}{4\pi i}\sum_{\sigma\in\hat{M}}\dim(\sigma)\sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau(m),k})[\nu:\sigma] \\
&\quad \times \int_{D_\epsilon} c_\nu(\sigma:z)^{-1}\frac{d}{dz}c_\nu(\sigma:z)e^{-t(z^2-c(\sigma))}\,dz.
\end{aligned} \tag{10.22}$$

To continue with the investigation of the right-hand side, we need the following lemma.

Lemma 10.11. *Let $k = 0, \dots, n$. For $\sigma \in \hat{M}$ let*

$$f_{k,m}(z,\sigma) := \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau(m),k})[\nu:\sigma] c_\nu(\sigma:z)^{-1}\frac{d}{dz}c_\nu(\sigma:z). \tag{10.23}$$

Then one has

$$f_{k,m}(z,\sigma)=\sum_{\nu\in\hat{K}}m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]\sum_{j=2}^{n+1}\Bigg(\sum_{\substack{m\leqslant l\leqslant k_j(\nu)\\ |k_j(\sigma)|<l}}\frac{i}{iz-l-\rho_j}-\sum_{\substack{m\leqslant l\leqslant k_j(\nu)\\ |k_j(\sigma)|\leqslant l}}\frac{i}{iz+l+\rho_j}\Bigg).$$

Proof. By Proposition 2.1 and Eq. (10.1), it follows that for every $\nu\in\hat{K}$ with $m_\nu(\sigma_{\tau(m),k})\neq 0$ and every $j=2,\dots,n+1$ we have

$$m-1\leqslant k_j(\nu),$$

where $(k_2(\nu),\dots,k_{n+1}(\nu))$ is the highest weight of ν. Thus, using (6.9) one can write

$$\begin{aligned}f_{k,m}(z,\sigma)=&\sum_{\nu\in\hat{K}}m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]\\ &\cdot\sum_{j=2}^{n+1}\Bigg(\sum_{\substack{m\leqslant l\leqslant k_j(\nu)\\ |k_j(\sigma)|<l}}\frac{i}{iz-l-\rho_j}-\sum_{\substack{m\leqslant l\leqslant k_j(\nu)\\ |k_j(\sigma)|\leqslant l}}\frac{i}{iz+l+\rho_j}\Bigg)\\ &+\sum_{\nu\in\hat{K}}m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]\\ &\cdot\sum_{j=2}^{n+1}\Bigg(\sum_{\substack{l=1\\ l>|k_j(\sigma)|}}^{m-1}\frac{i}{iz-l-\rho_j}-\sum_{\substack{l=0\\ l\geqslant|k_j(\sigma)|}}^{m-1}\frac{i}{iz+l+\rho_j}\Bigg).\end{aligned}\tag{10.24}$$

Now if $\sigma=\sigma_{\tau(m),k}$ or $\sigma=w_0\sigma_{\tau(m),k}$ the sum in the second row of (10.24) is zero by (10.1) and (2.15). On the other hand, assume that $\sigma\neq\sigma_{\tau(m),k}$, $\sigma\neq w_0\sigma_{\tau(m),k}$. Since $R(M)$ is the free abelian group generated by $\sigma\in\hat{M}$, it follows from (2.17) that

$$\sum_{\nu\in\hat{K}}m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]=0.$$

Thus, in this case the sum in the second row of (10.24) is zero too. This proves the proposition. □

Proposition 10.12. *For* $s\in\mathbb{C}$, $\mathrm{Re}(s)>0$ *let*

$$\mathcal{M}J(s;\sigma_{\tau(m),k}):=\int_0^\infty t^{s-1}e^{-t\lambda^2_{\tau(m),k}}J\big(h_t^{\sigma_{\tau(m),k}}\big)\,dt.$$

Then $\mathcal{M}J(s;\sigma_{\tau(m),k})$ *has a meromorphic continuation to* $\mathbb{C}$ *with at most a simple pole at* 0 *and we have*

$$\frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{M}J(s;\sigma_{\tau(m),k})}{\Gamma(s)} = -\kappa \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]\dim(\sigma)$$
$$\cdot \sum_{j=2}^{n+1} \sum_{\substack{m\leqslant l\leqslant k_j(\nu)\\ l>|k_j(\sigma)|}} \log\Bigl(\sqrt{\lambda^2_{\tau(m),k}+c(\sigma_{\tau(m),k})-c(\sigma)}+l+\rho_j\Bigr)$$
$$-\frac{\kappa}{2} \sum_{\sigma\in\hat{M}} \sum_{\nu\in\hat{K}} m_\nu(\sigma_{\tau(m),k})[\nu:\sigma]\dim(\sigma)$$
$$\cdot \sum_{\substack{j=2\\ |k_j(\sigma)|\geqslant m}}^{n+1} \log\Bigl(\sqrt{\lambda^2_{\tau(m),k}+c(\sigma_{\tau(m),k})-c(\sigma)}+\bigl|k_j(\sigma)\bigr|+\rho_j\Bigr).$$

Proof. Let $\sigma\in\hat{M}$. By (2.16) the highest weights of $\nu\in\hat{K}$ with $m_\nu(\sigma_{\tau(m),k})\neq 0$ are of the form $\Lambda(\sigma_{\tau(m),k})-\mu$, where $\mu\in\{0,1\}^n$. Now assume that also $[\nu:\sigma]\neq 0$. Then by [12, Theorem 8.1.4] we have $k_j(\sigma_{\tau(m),k})\geqslant k_j(\sigma)$. Hence if $\sigma\in\hat{M}$ is such that $[\nu:\sigma]m_\nu(\sigma_{\tau(m),k})\neq 0$ for some $\nu\in\hat{K}$, it follows from (4.16) that

$$c(\sigma_{\tau(m),k})-c(\sigma)\geqslant 0. \tag{10.25}$$

Thus the proposition follows from Lemma 10.5, Eq. (10.22) and Lemma 10.11. □

Proposition 10.13. *Let $k\in\{0,\dots,n\}$. There exists a constant C such that for every m one has*

$$\left|\frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{M}J(s;\sigma_{\tau(m),k})}{\Gamma(s)}\right| \leqslant Cm^{\frac{n(n+1)}{2}}\log m.$$

Proof. Let $\nu\in\hat{K}$ such that $m_\nu(\sigma_{\tau(m),k})\neq 0$. Let $\sigma\in\hat{M}$ such that $[\nu:\sigma]\neq 0$. Then (10.25) holds as shown in the proof of the previous proposition. Hence

$$m\leqslant\sqrt{\lambda^2_{\tau(m),k}+c(\sigma_{\tau(m),k})-c(\sigma)}\leqslant\sqrt{\lambda^2_{\tau(m),k}+c(\sigma_{\tau(m),k})}.$$

By (4.16), (10.1) and (10.2) there exists a constant C_1 which is independent of ν and σ such that for every m we have

$$m\leqslant\sqrt{\lambda^2_{\tau(m),k}+c(\sigma_{\tau(m),k})-c(\sigma)}\leqslant C_1 m.$$

For $\nu\in\hat{K}$ as above, it follows from (2.16) and (10.1) that for every $j\in\{2,\dots,n+1\}$ one has

$$k_j(\nu)\leqslant m+\tau_1+1.$$

Thus there exists a constant C_2 which is independent of ν and σ such that for every m we have

$$\sum_{j=2}^{n+1} \sum_{m \leqslant l \leqslant k_j(\nu)} \left| \log\left(\sqrt{\lambda^2_{\tau(m),k} + c(\sigma_{\tau(m),k}) - c(\sigma)} + l + \rho_j \right) \right| \leqslant C_2 \log m.$$

By Proposition 10.12 it follows that there exists a constant C_3 such that for every $m \in \mathbb{N}$ we have

$$\begin{aligned} \left| \frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{M}J(s; \sigma_{\tau(m),k})}{\Gamma(s)} \right| &\leqslant C_3 \log m \sum_{\nu \in \hat{K}} \left| m_\nu(\sigma_{\tau(m),k}) \right| \sum_{\sigma \in \hat{M}} [\nu : \sigma] \dim(\sigma) \\ &= C_3 \log m \sum_{\nu \in \hat{K}} \left| m_\nu(\sigma_{\tau(m),k}) \right| \dim(\nu). \end{aligned}$$

Now by (2.16) the number of $\nu \in \hat{K}$ with $m_\nu(\sigma_{\tau(m),k}) \neq 0$ is bounded by 2^n and one has $|m_\nu(\sigma_{\tau(m),k})| \leqslant 1$ for every $\nu \in \hat{K}$. Let $\Lambda(\nu) \in \mathfrak{b}^*_{\mathbb{C}}$ be the highest weight of ν as in (2.9). Then by Weyl's dimension formula [17, Theorem 4.48] we have

$$\begin{aligned} \dim(\nu) &= \prod_{\alpha \in \Delta^+(\mathfrak{k}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \frac{\langle \Lambda(\nu) + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \\ &= \prod_{i=2}^{n+1} \left(k_i(\nu) + \rho_i + 1/2 \right) \prod_{j=i+1}^{n+1} \frac{(k_i(\nu) + \rho_i + 1/2)^2 - (k_j(\nu) + \rho_j + 1/2)^2}{(\rho_i + 1/2)^2 - (\rho_j + 1/2)^2}. \end{aligned} \tag{10.26}$$

By (2.16) the highest weights of $\nu \in \hat{K}$ with $m_\nu(\sigma_{\tau(m),k}) \neq 0$ are of the form $\Lambda(\sigma_{\tau(m),k}) - \mu$, where $\mu \in \{0, 1\}^n$. Using (10.1) it follows that there exists $C_4 > 0$, which is independent of m, such that for each $\nu \in \hat{K}$ with $m_\nu(\sigma_{\tau(m),k}) \neq 0$ one has

$$\dim(\nu) \leqslant C_4 m^{\frac{n(n+1)}{2}}.$$

This proves the proposition. □

Summarizing, we have proved the following proposition.

Proposition 10.14. *For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$ the integral*

$$\mathcal{M}J(s; \tau(m)) := \int_0^\infty t^{s-1} J\left(k_t^{\tau(m)}\right) dt$$

converges and $\mathcal{M}J(s; \tau(m))$ admits a meromorphic continuation to $\mathbb{C}$ with at most simple a pole at 0. Let

$$\mathcal{M}J(\tau(m)) := \frac{d}{ds}\bigg|_{s=0} \frac{\mathcal{M}J(s; \tau(m))}{\Gamma(s)}.$$

Then there exists a constant C such that for every $m \in \mathbb{N}$ one has

$$\left|\mathcal{M}J\big(\tau(m)\big)\right| \leqslant Cm^{\frac{n(n+1)}{2}} \log m.$$

Proof. By Proposition 8.2 one has

$$\mathcal{M}J\big(s;\tau(m)\big) = \sum_{k=0}^{n}(-1)^{k+1}\mathcal{M}J(s;\sigma_{\tau(m),k}).$$

The proposition follows from Propositions 10.12 and 10.13. □

Now by Eqs. (7.16), (7.9) and Proposition 8.2 we have

$$\log T_X\big(\tau(m)\big) = \frac{1}{2}\big(\mathcal{M}I\big(\tau(m)\big) + \mathcal{M}H\big(\tau(m)\big) + \mathcal{M}T\big(\tau(m)\big) + \mathcal{M}\mathcal{I}\big(\tau(m)\big) + \mathcal{M}J\big(\tau(m)\big)\big).$$

Combining Eq. (9.6) and Propositions 10.1, 10.3, 10.4, 10.10 and 10.14, Theorems 1.1 and 1.2 follow.

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