Clifford and Mackey theory for Weil representations of symplectic groups

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1. Introduction

For a semi-simple group scheme $G$ defined over $\mathbb{Z}$, let $X$ be an irreducible complex representation of the finite group $G(R)$ where $R$ is $\mathbb{Z}/p^i\mathbb{Z}$, or more generally $R = \mathcal{O}/P^i$ where $P$ is the maximal ideal of the ring $\mathcal{O}$ of integers of a local field. One can try to analyze $X$ using Clifford theory with respect to the congruence subgroup $\Gamma(P)$ which is the kernel of the natural map from $G(\mathcal{O}/P^i)$ to $G(\mathcal{O}/P^j)$, where $i$ is a positive integer less than $l$. This involves finding an irreducible constituent $Y$ of the restriction of $X$ to $\Gamma(P)$, finding the stabilizer $T$ in $G(\mathcal{O}/P^i)$ of $Y$, and expressing $X$ as the induced module $\text{ind}_{G(R)}^T Z$ where $Z$ is a $T$-module whose restriction to $\Gamma(P)$ is a direct sum of copies of $Y$. In general, it seems difficult to do this explicitly. In this paper, in the case that $G$ is the symplectic group and $X$ is an irreducible component of the Weil representation, we can indeed explicitly find $T$, $X$, and $Z$ for any congruence subgroup $\Gamma(P)$, assuming that the ring $R$ has odd characteristic.

For the most part we work in the generality that $R$ is a finite local commutative ring of odd characteristic, as in [CMS]. Let $W$ be a Weil representation of the symplectic group $\text{Sp}(V)$ where $V$ is a finite $R$-module admitting an $R$-bilinear alternating form $(,)$, We assume that the maximal ideal of $R$ is non-zero, and that $R$ admits a primitive complex linear character $\lambda$.

In our previous paper the symplectic group $\text{Sp}(V)$ and Heisenberg group $H(V)$ were only defined if the ambient form on $V$ was non-degenerate. The two constructions naturally extend to modules with degenerate forms, as do the notions of Schrödinger and Weil.

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representations. Given a non-zero totally isotropic \( \text{Sp}(V) \)-invariant submodule \( U \) of \( V \) we fix a Weil representation \( W' \) of \( \text{Sp}(U^\perp) \) associated to the Schrödinger representation \( S' \) of \( H(U^\perp) \). In Section 4 we construct a representation \( W \) of \( \text{Sp}(V) \) from the pair \( (W', S') \) which is readily seen to be Weil. Moreover, Mackey theory can be used to exhibit a decomposition

\[
W = \bigoplus_t \text{ind}^{\text{Sp}(V)}_{G_t} W_t,
\]

where \( t \) runs over a set of representatives of the orbits of \( \text{Sp}(V) \) acting on \( V/U^\perp \), \( G_t \) is the subgroup of \( \text{Sp}(V) \) fixing \( t \) modulo \( U^\perp \), and \( W_t \) is the representation of \( G_t \) given by

\[
W_t(g) = S'(\langle gt, t \rangle, gt - t)W'(g|U^\perp).
\]

This will be used in further work [CMS2] to derive a character formula for use in the study of character fields and Schur indices for these characters.

Each Weil module \( X \) has a canonical submodule \( \text{Bot} \). If the module \( V \) admits an element \( x \) for which \( \langle x, V \rangle = R \) then \( \text{Bot} \) is a proper submodule, hence the quotient module \( \text{Top} = X/\text{Bot} \) is non-zero. In Section 5 we show that \( \text{Top} \) is the sum of two irreducible \( \text{Sp}(V) \)-modules, \( \text{Top}^\pm \), both of which occur with multiplicity 1 in \( X \).

If \( X \) is constructed via the procedure of Section 4 then the module \( \text{Top} \) can be identified with a summand occurring in the decomposition (1.1). In this case the decomposition of \( \text{Top} \) into its irreducible components can be accomplished explicitly via Clifford theory with respect to congruence subgroups \( \Gamma(U) \). We show in Section 6 that the restriction of \( \text{Top}^\pm \) to \( \Gamma(U) \) involves the linear character \( \delta_k \) of \( \Gamma(U) \) defined by \( \gamma \mapsto \lambda(\langle yx, x \rangle) \). Moreover, the inertia group in \( \text{Sp}(V) \) of \( \delta_k \) is \( G_x \times \{1, \iota\} \), and \( W_t \) has two extensions, \( W^\pm_x \), to the group \( G_x \times \{1, \iota\} \), whose restriction to \( \Gamma(U) \) involves \( \delta_k \) and whose induction to \( \text{Sp}(V) \) is \( \text{Top}^\pm \). Clifford theory with respect to \( \Gamma(U^\perp) \) is somewhat harder. In Section 7 we show that the restriction of \( W_t \) to \( \Gamma(U^\perp) \) remains irreducible; Clifford theory for \( \text{Top}^\pm \) with respect to \( \Gamma(U^\perp) \) then follows. In Section 8 we present various characterizations of the submodule \( \text{Bot} \), as well as determine the kernels of the representations afforded by \( \text{Top}^\pm \) and \( \text{Top}^- \).

In Section 9 we return the case studied in [CMS], namely that in which \( R \) is principal and \( V \) is free, using the techniques developed here. All the irreducible submodules of the Weil module \( X \) can be recovered from our Mackey theory (1.1) via a simple refinement. As a result, we obtain explicit descriptions of the irreducible components of \( W \) in terms of the representations \( W' \) and \( S' \) introduced above. This extends the result of [CMS] obtained in the case of even nilpotency index. If the nilpotency index is even, one of the irreducible components of \( W \) has dimension one; if the nilpotency index is odd, two of the irreducible components of \( W \) are essentially Weil representations for the case that \( R \) is a finite field. All the other irreducible components of \( W \) can be analyzed using Clifford theory with respect to congruence subgroups. For the more general \( R \) and \( V \) considered here, we show by example that, in contrast to \( \text{Top} \), the structure of \( \text{Bot} \) can be quite complicated. We end the paper by establishing a connection between the classical theory of the field case and that of the general ring considered here in Section 10.
In the case $R = \mathcal{O}/P^l$ where $\mathcal{O}$ is the ring of integers of a (non-archimedean) local field $F$, the representations $T_\mathrm{op}^{\pm}$ of $\mathrm{Sp}(2n, R)$ arise naturally as $K$-types of a suitable Weil representation of $\mathrm{Sp}(2n, F)$, where $K = \mathrm{Sp}(2n, \mathcal{O})$. For example, if $W$ is the Weil representation of $\mathrm{Sp}(2n, F)$ associated to a character $\psi$ of $F$ with conductor $\mathcal{O}$, it is shown in [Pra] that the restriction of $W$ to $\mathrm{Sp}(2n, \mathcal{O})$ decomposes as the direct sum

$$W_0 \oplus \sum_{m \geq 1} \left( W_{2m}^+ \oplus W_{2m}^- \right)$$

where $W_0$ is trivial and $W_{2m}^\pm$ are irreducible representations of $\mathrm{Sp}(2n, \mathcal{O}/P^{2m})$. The representations $W_{2m}^\pm$ can be identified with our representations $T_\mathrm{op}^\pm$ for suitably chosen Weil representations of $\mathrm{Sp}(2n, \mathcal{O}/P^{2m})$. It can also be shown that $T_\mathrm{op}^\pm$ for $\mathrm{Sp}(2n, \mathcal{O}/P^l)$ for $l$ odd also arise in this manner, where the character $\psi$ of $F$ has conductor $P$. We also note that Weil representations $W$ for the symplectic group over $\mathcal{O}/P^l$ where $\mathcal{O}$ is a $p$-adic ring were first studied in [Yos], although [Yos] did not deal with the irreducible constituents of $W$.

2. Preliminaries

Let $R$ be a finite local commutative ring of odd characteristic, with maximal ideal $m$. Unless otherwise noted, we shall assume $m$ is non-zero. Let $V$ be a finite $R$-module endowed with an alternating $R$-bilinear form $\langle \cdot, \cdot \rangle$. Two groups are associated with the pair $(V, \langle \cdot, \cdot \rangle)$. The first is the symplectic group $\mathrm{Sp}(V)$ which is given by

$$\mathrm{Sp}(V) = \left\{ g \in \mathrm{GL}(V) \mid \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V \right\}.$$ 

In $\mathrm{Sp}(V)$ we have the central involution $i$ which maps $v$ to $-v$, $v \in V$. For $r \in R$ and $v \in V$, the symplectic transvection

$$\tau_{r,v} : w \mapsto w + r \langle w, v \rangle v, \quad w \in V,$$

also belongs to $\mathrm{Sp}(V)$.

The second group associated to the pair $(V, \langle \cdot, \cdot \rangle)$ is the Heisenberg group $H(V)$. The Heisenberg group $H(V)$ has $R \times V$ as its underlying set, with multiplication

$$(r, v)(s, w) = (r + s + \langle v, w \rangle, v + w), \quad (r, v), (s, w) \in R \times V.$$

The multiplicative identity is $(0, 0)$, and the element $(r, v)$ has the inverse $(-r, -v)$. An element $g$ of $\mathrm{Sp}(V)$ acts as an automorphism of $H(V)$ via the second factor:

$$g^*(r, v) = (r, gv), \quad (r, v) \in R \times V.$$

This gives us an action of $\mathrm{Sp}(V)$ on $H(V)$. 


Each submodule $U$ of $V$ inherits an alternating form from that of $V$, and as such we may form the groups $\text{Sp}(U)$ and $H(U)$. We note that $H(U)$ is a normal subgroup of $H(V)$. Given $U$ we have the submodule

$$U^\perp = \{ v \in V \mid \langle u, v \rangle = 0 \text{ for all } u \in U \}.$$  

We say that $U$ is totally isotropic if $U \subseteq U^\perp$.

The form $(,)$ on $V$ is said to be non-degenerate if $V^\perp = 0$. In general, the quotient module $V/V^\perp$ inherits a non-degenerate alternating form from that of $V$:

$$\langle v + V^\perp, w + V^\perp \rangle = \langle v, w \rangle, \quad v, w \in V.$$  

We set

$$\bar{V} = V/V^\perp$$  

and call it the non-degenerate quotient of $V$.

Let $\lambda$ be a complex linear character of the additive group $R^+$ of $R$. One may replace $R$ by factoring out the sum of all the ideals contained in the kernel of $\lambda$. Hence it is reasonable to assume that $\lambda$ is primitive, in the sense that its kernel does not contain a non-zero ideal of $R$. In this paper we shall assume the following hypothesis.

**Hypothesis H1.** $R^+$ admits a primitive linear character $\lambda$.

**Lemma 2.1.** Suppose that $[,]$ is a non-degenerate alternating or symmetric bilinear form on a finite $R$-module $M$. If $N$ is an $R$-submodule of $M$, then $N^{\perp\perp} = N$.

**Proof.** Let $\bar{N} = \text{Hom}(N, C^*)$ and $N^* = \text{Hom}_R(N, R)$. Consider the homomorphism $\alpha_N : N^* \to \bar{N}$ given by $\theta \mapsto \lambda \circ \theta$. If $\theta \in \ker \alpha_N$, then $\text{im} \theta$ is an ideal of $R$ contained in $\ker \lambda$, so $\theta = 0$ since $\lambda$ is primitive. Hence $\alpha_N$ is injective.

Let $\beta : M \to M^*$ be the map $m \mapsto [m, ]$. Let

$$r_1 : M^* \to N^*, \quad r_2 : \bar{M} \to \bar{N}$$  

be given by restriction to $N$. We have the commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\beta} & M^* \xrightarrow{\alpha_M} \bar{M} \\
& \downarrow{r_1} & \downarrow{r_2} \\
N^* & \xrightarrow{\alpha_N} & \bar{N}.
\end{array}$$  

On the top horizontal line, $\beta$ is injective by non-degeneracy, and $\alpha_M$ is injective, from the previous paragraph. Since $|M| = |\bar{M}|$, then both $\beta$ and $\alpha_M$ are isomorphisms. The map $r_2$ is onto, since $C$ is divisible. So the composite $\phi = r_2 \alpha_M \beta = \alpha_N r_1 \beta$ is onto.
αN is injective, ker φ = ker(r₁β) = N⁺. Thus |M| = |N⁺||[N]| = |N⁺||N|. Starting with the submodule N⁺ instead of N, we get |M| = |N⁺⁺||N⁺|. Hence |N| = |N⁺⁺|. Since N ⊆ N⁺⁺, then N = N⁺⁺, as desired. □

3. Schrödinger and Weil representations

Let (V, ⟨, ⟩) be as in the preceding section. Under the assumption that ⟨, ⟩ is non-degenerate, Section 2 of [CMS] establishes the existence of an irreducible character χₐ of H(V) given by

\[ χₐ(h) = \begin{cases} \sqrt{|V|} \lambda(r), & \text{if } h = (r, 0); \\ 0, & \text{otherwise.} \end{cases} \]

This is called the Schrödinger character of H(V) of type λ.

In general, the canonical projection of V onto the non-degenerate quotient V gives rise to a surjective homomorphism H(V) → H(V). The Schrödinger character χₐ of H(V) of type λ is taken to be the inflation of that of H(V). The character χₐ is thus seen to be given by the following definition.

Definition 3.1. The Schrödinger character χₐ of H(V) of type λ is the character

\[ χₐ(h) = \begin{cases} \sqrt{|V| : V⁺} \lambda(r), & \text{if } h = (r, v) \in H(V⁺); \\ 0, & \text{otherwise.} \end{cases} \]

Observe that if U is totally isotropic then χₐ is the linear character given by (r, u) → λ(r), for (r, u) ∈ H(U).

Proposition 3.3. Suppose that the form ⟨, ⟩ on V is non-degenerate and U is a totally isotropic submodule of V.

\[ X(U) = \text{inv}_{(0, U)} X. \]
(1) There is a $CH(V)$-isomorphism

$$X \cong \text{ind}^{H(V)}_{H(U^\perp)} X(U).$$

(2) The module $X(U)$ affords a Schrödinger representation of $H(U^\perp)$ of type $\lambda$.

Proof. Observing that $H(U^\perp)$ coincides with the subgroup $A(U^\perp)$ introduced in [CMS], part (1) is seen to be contained in [CMS, Proposition 4.1]. To prove (2) we use the uniqueness of the Schrödinger character $\chi_{U^\perp}$ of Lemma 3.2. Take an element $(r,u) \in H(U^\perp)$. Since $U^\perp \cap U = U$ by Lemma 2.1, then $u \in U$. So $(r,u)$ acts on $X(U)$ via the scalar $\lambda(r)$, since $X(U) = \text{inv}(0,U) X$. Then (2) follows from Lemma 3.2. $\square$

The symplectic group $\text{Sp}(V)$ acts on $H(V)$, and from [CMS, Section 3] we know there is a representation of $\mathbb{W} : \text{Sp}(V) \to \text{GL}(X)$ with the property that

$$\mathbb{W}(g) S(h) \mathbb{W}(g)^{-1} = \mathbb{W}(g h), \quad g \in \text{Sp}(V), \quad h \in H(V).$$

Let $W$ be the inflation of $\mathbb{W}$ to a representation of $\text{Sp}(V)$. Then $W$ satisfies the following definition.

Definition 3.4. Let $S : H(V) \to \text{GL}(X)$ be a Schrödinger representation of type $\lambda$. A Weil representation of type $\lambda$ is a group homomorphism $W : \text{Sp}(V) \to \text{GL}(X)$, such that for all $g \in \text{Sp}(V), h \in H(V)$

$$W(g) S(h) W(g)^{-1} = S(g h).$$

As $S$ is irreducible, Schur’s Lemma ensures that $W$ is unique up to multiplication by a linear character of $\text{Sp}(V)$. Note that there exists a Weil representation of $\text{Sp}(V)$ which is trivial on the kernel of the canonical map $\text{Sp}(V) \to \text{Sp}(V)$.

We let $C^V$ denote the permutation module for $C\text{Sp}(V)$ on the finite set $V$.

Proposition 3.5. The map

$$v + V^\perp \mapsto S(0,v), \quad v \in V,$$

extends to an isomorphism of $C\text{Sp}(V)$-modules $C^V \cong \text{End}_C(X)$.

Proof. This follows from [CMS, Theorem 4.5]. $\square$

Corollary 3.6. If $T$ is a transversal for $V^\perp$ in $V$ then $\{S(0,t) \mid t \in T\}$ is a basis of $\text{End}_C(X)$.

Proof. The set $\{t + V \mid t \in T\}$ is a $C$-basis of $C^V$. $\square$
4. Mackey theory

For the rest of this paper, we shall assume, unless otherwise stated, the following hypothesis.

**Hypothesis H2.** The alternating form \( \langle , \rangle \) on \( V \) is non-degenerate.

Let \( U \) be a fixed non-zero totally isotropic \( \text{Sp}(V) \)-invariant submodule of \( V \). Then \( U^\perp \) is also \( \text{Sp}(V) \)-invariant, and the restriction of the action to \( U^\perp \) yields a homomorphism

\[
r_{U^\perp} : \text{Sp}(V) \rightarrow \text{Sp}(U^\perp).
\]

The image of \( g \) in \( \text{Sp}(V) \) under the map \( r_{U^\perp} \) shall be denoted \( g|_{U^\perp} \).

Let \( S' \) be a Schrödinger representation of \( H(U^\perp) \) of type \( \lambda \), with ambient module \( Z \), and \( W' \) be an associated Weil representation of \( \text{Sp}(U^\perp) \). Then \( Z \) is also a module for \( \text{Sp}(V) \) via the composition

\[
\text{Sp}(V) \xrightarrow{r_{U^\perp}} \text{Sp}(U^\perp) \xrightarrow{W'} \text{GL}(Z).
\]

Thus \( Z \) is a module for the semi-direct product \( H(U^\perp) \rtimes \text{Sp}(V) \). As \( H(U^\perp) \rtimes \text{Sp}(V) \) is a subgroup of \( H(V) \rtimes \text{Sp}(V) \), we can form the induced module

\[
X = \text{ind}_{H(U^\perp)\rtimes\text{Sp}(V)}^{H(V)\rtimes\text{Sp}(V)} Z. \tag{4.1}
\]

The restriction of \( X \) to \( H(V) \) is \( \text{ind}_{H(U^\perp)\rtimes\text{Sp}(V)}^{H(V)\rtimes\text{Sp}(V)} Z \), which is a Schrödinger module for \( H(V) \) by Proposition 3.3. Hence \( X \) is a Weil module for \( \text{Sp}(V) \). We let \( W \) denote the Weil representation of \( \text{Sp}(V) \) acting on \( X \).

We shall use Mackey theory with respect to the subgroups \( \text{Sp}(V) \) and \( H(U^\perp) \rtimes \text{Sp}(V) \) of \( H(V) \rtimes \text{Sp}(V) \). Denote a typical element of \( H(V) \rtimes \text{Sp}(V) \) by a triple \((r, v, g)\) where \( r \in R, v \in V \), and \( g \in \text{Sp}(V) \). The element \((0, v, 1)\) shall be denoted \( a_v \). If \( T \) is a set of representatives of the \( \text{Sp}(V) \)-orbits on \( V/U^\perp \) then \( \{a_v \mid t \in T\} \) is a set of representatives for the double cosets \( \text{Sp}(V) \backslash H(V) \rtimes \text{Sp}(V) / H(U^\perp) \rtimes \text{Sp}(V) \). If \( g \in \text{Sp}(V) \), we have

\[
a_t^{-1}(0, 0, g)a_t = \langle gt, t \rangle, gt - t, g \rangle.
\]

We define

\[
G_t = \{a_t \mid H(U^\perp) \rtimes \text{Sp}(V) \cap \text{Sp}(V) \}.
\]

It follows that

\[
G_t = \{g \in \text{Sp}(V) \mid gt \equiv t \text{ mod } U^\perp\}, \quad t \in T. \tag{4.2}
\]
Hence, if \( t \in T \), then the representation of \( G_t \) acting on \( Z_t = a^t Z \) is given by
\[
W_t(g) = S'(\langle gt, t \rangle, gt - t)W'(g|_{U^\perp}).
\] (4.3)

We apply Mackey’s Subgroup Theorem [CR, 10.13] to the subgroups \( \text{Sp}(V) \) and \( H(U^\perp) \times \text{Sp}(V) \) of \( H(V) \times \text{Sp}(V) \), yielding the following result.

**Theorem 4.1.** Let \( T \subset V \) be a set of representatives for the orbits of \( \text{Sp}(V) \) acting on \( V/U^\perp \). Let \( X \) be the module given by (4.1). Then
\[
\text{res}_{\text{Sp}(V)} X \cong \bigoplus_{t \in T} \text{ind}_{\text{Sp}(V)} G_t Z_t.
\]

We refine this decomposition with respect to the action of \( t \). For \( t \in T \), define
\[
X_t = \text{ind}_{\text{Sp}(V)} G_t Z_t, \quad X_t^\pm = \{ x \in X_t \mid \iota x = \pm x \}.
\]

Assume that \( t \notin U^\perp \); then \( t \notin G_t \) since 2 is a unit of \( R \). Extend the action of the \( G_t \) on the module \( Z_t \) to an action of \( G_t \times \{1, \iota\} \) on the module which we shall call \( Z_t^\pm \) by having \( \iota \) action as \( \pm 1 \). Then
\[
\text{ind}_{G_t \times \{1, \iota\}} G_t Z_t \cong Z_t^+ \oplus Z_t^-.
\]

By transitivity of induction,
\[
X_t \cong \text{ind}_{G_t \times \{1, \iota\}} G_t Z_t^+ \oplus \text{ind}_{G_t \times \{1, \iota\}} G_t Z_t^-.
\]

Since \( \iota \) acts on \( \text{ind}_{G_t \times \{1, \iota\}} G_t Z_t^\pm \) as \( \pm 1 \), we have proved the following result.

**Proposition 4.2.** If \( t \notin U^\perp \), there are isomorphisms of \( C\text{Sp}(V) \)-modules
\[
X_t^\pm \cong \text{ind}_{G_t \times \{1, \iota\}} G_t Z_t^\pm.
\]

In particular, neither of the eigenspaces \( X_t^\pm \) is 0. If \( t \in U^\perp \), it may occur that one of the \( \pm 1 \)-eigenspaces of \( \iota \) on \( X_t \) is 0 (see Example 9.4).

5. Two irreducible constituents of the Weil module

An element \( x \in V \) is said to be **primitive** if \( \langle x, V \rangle = R \). Let \( \mathcal{P}(V) \) denote the set of primitive elements of \( V \). For the remainder of this paper we shall assume the following hypothesis.

**Hypothesis H3.** \( \mathcal{P}(V) \) is non-empty.
The goal of this section is to determine two canonical irreducible constituents of the Weil module.

**Lemma 5.1.** The annihilator of \( m \) is the unique minimal ideal of \( R \).

**Proof.** Apply Lemma 2.1 to the case where \( M = R \) and \( [ , ] \) is the multiplication map of \( R \). We deduce that taking annihilators is an involution of the lattice of all ideals of \( R \).

Let \( s \) be a minimal ideal of \( R \). Then the annihilator of \( s \), denoted by \( \text{ann} \, s \), is a maximal ideal, hence is \( m \). By our opening remark, \( s = \text{ann}(\text{ann} \, s) = \text{ann} \, m \), so the only minimal ideal is \( \text{ann} \, m \). \( \blacksquare \)

The unique minimal ideal of \( R \) shall be denoted by \( s \).

**Lemma 5.2.** The set \( \mathcal{P}(V) \) is an \( \text{Sp}(V) \)-orbit of \( V \).

**Proof.** It is clear that \( \mathcal{P}(V) \) is \( \text{Sp}(V) \)-invariant. Let \( x \) and \( y \) be elements of \( \mathcal{P}(V) \). If \( r = \langle x, y \rangle \) is a unit of \( R \) then the symplectic transvection \( \tau_{r^{-1}, x-y} \) maps \( x \) to

\[
x = \langle x, y \rangle^{-1} \langle x, y \rangle (x - y) = y.
\]

In general, there exist \( x', y' \in V \) such that \( \langle x, x' \rangle, \langle y, y' \rangle \in R^\ast \). We want to find \( z \in V \) such that both \( \langle x, z \rangle, \langle z, y \rangle \in R^\ast \), because then from the previous paragraph there exist \( g_1 \) and \( g_2 \) in \( \text{Sp}(V) \) such that \( g_1 x = z \) and \( g_2 z = y \), whence \( g = g_2 g_1 \in \text{Sp}(V) \) maps \( x \) to \( y \).

If \( \langle x, y' \rangle \in R^\ast \) then we take \( z = y' \) and, if \( \langle x', y \rangle \in R^\ast \), we take \( z = x' \). The remaining alternative is that both \( \langle x, y' \rangle \) and \( \langle y, x' \rangle \) are in \( m \), in which case we take \( z = x' + y' \). \( \blacksquare \)

Let \( i \) be an ideal of \( R \). Write \( iV \) for the submodule of \( V \) generated by the elements \( rv \) with \( r \in i \) and \( v \in V \), and let

\[
V(i) = \{ v \in V \mid \langle v, V \rangle \subseteq i \}.
\]

Note that \( iV \subseteq V(i) \), and both are \( \text{Sp}(V) \)-invariant. Furthermore, the non-degeneracy of \( \langle , \rangle \) ensures that \( V(i) \) is annihilated by \( \text{ann} \, i \).

**Lemma 5.3.** The set \( sV \setminus \{0\} \) is an \( \text{Sp}(V) \)-orbit of \( V \).

**Proof.** Let \( s \) be a non-zero element of \( s \). Since \( s \) is a minimal ideal, then \( s = Rs \). Thus any non-zero element \( v \) of \( sV \) can be written in the form \( v = sx \) for some \( x \in V \), and \( x \) must be primitive, since \( s \) annihilates \( V(m) \). Now consider two non-zero elements \( sx_1 \) and \( sx_2 \) of \( sV \), with \( x_1 \) and \( x_2 \in \mathcal{P}(V) \). In virtue of Lemma 5.2, there exists \( g \in \text{Sp}(V) \) such that \( x_2 = gx_1 \), hence \( sx_2 = g(sx_1) \). \( \blacksquare \)

**Lemma 5.4.** (1) For an ideal \( i \) of \( R \), \( (iV)^\perp = V(\text{ann} \, i) \) and \( V(\text{ann} \, i)^\perp = iV \).

(2) \( (sV)^\perp = V(m) \) and \( V(m)^\perp = sV \).
Proof. For an element \( v \in V \),
\[
\begin{align*}
v \in V(\text{ann}i) & \iff \langle v, V \rangle \subseteq \text{ann} i \iff \langle v, V \rangle i = 0 \iff \langle v, iV \rangle = 0 \\
& \iff v \in (iV) \perp.
\end{align*}
\]
This proves \((iV) \perp = V(\text{ann}i)\). From Lemma 2.1, we then get \( V(\text{ann}i) \perp = iV \). This proves (1); (2) is then immediate. \( \square \)

Lemma 5.5. The set \( \mathcal{P}(V) \) generates \( V \) as an abelian group.

Proof. Given \( v \in V \), then either \( v \in \mathcal{P}(V) \) or \( v \in V(m) \). In the latter case, let \( x \in \mathcal{P}(V) \) and write \( v = x + (v - x) \). As \( x \) and \( v - x \) belong to \( \mathcal{P}(V) \), the result follows. \( \square \)

Lemma 5.6. The unique maximal \( \text{Sp}(V) \)-invariant submodule of \( V \) is \( V(m) \) and the unique minimal \( \text{Sp}(V) \)-invariant submodule of \( V \) is \( sV \).

Proof. Let \( U \) be an \( \text{Sp}(V) \)-invariant submodule of \( V \). Suppose first that \( U \neq V \). In view of Lemmas 5.2 and 5.5, the module \( U \) cannot contain any element of \( \mathcal{P}(V) \), whence \( U \subseteq V(m) \). Suppose next that \( U \neq (0) \). As \( \langle , \rangle \) is non-degenerate we have \( U \perp \neq V \), so \( U \perp \subseteq V(m) \) by the first case. Passing to the orthogonal modules, Lemma 5.4(2) shows \( sV \subseteq U \). \( \square \)

In light of Lemma 5.4(2), we have
\[
\overline{V(m)} = V(m)/V(m) \perp = V(m)/sV.
\]

Lemma 5.7. The number of \( \text{Sp}(V) \)-orbits in \( V(m) \) exceeds by one the number of \( \text{Sp}(V) \)-orbits in \( \overline{V(m)} \). In symbols:
\[
|\text{Sp}(V) \setminus V(m)| = |\text{Sp}(V) \setminus \overline{V(m)}| + 1.
\]

Proof. Consider the natural map \( f : \text{Sp}(V) \setminus V(m) \to \text{Sp}(V) \setminus \overline{V(m)} \) that sends the \( \text{Sp}(V) \)-orbit of each \( v \in V(m) \) to the \( \text{Sp}(V) \)-orbit of \( v + sV \).

We know that \( sV \setminus [0] \) is an \( \text{Sp}(V) \)-orbit, and clearly so is \([0]\). Moreover, these orbits have the same image under \( f \). We claim that \( f \) is one-to-one on the remaining orbits of \( V(m) \). Indeed, let \( v, w \not\in sV \) and suppose that \( v + sV \) and \( w + sV \) are in the same \( \text{Sp}(V) \)-orbit. Then \( w \) lies in the same \( \text{Sp}(V) \)-orbit as \( v + u \) for some \( u \in sV \). Since \( v \not\in sV \) and \( sV = V(m) \perp \), there must exist \( v' \in V(m) \) with \( \langle v, v' \rangle \neq 0 \).

Write \( u = sx \) with \( s \in s \) and \( x \in \mathcal{P}(V) \). Choosing \( a \in R \) such that \( \langle v, av' + x \rangle \neq 0 \), the uniqueness of \( s \) ensures the existence of \( r \in R \) such that \( r(v, av' + x) = s \). Then
\[
\tau_{r, av' + x} v = v + r\langle v, av' + x \rangle (av' + x) = v + s(av' + x) = v + u,
\]
the last identity following from the fact that \( s \) annihilates \( V(m) \). We deduce that \( v \) is \( \text{Sp}(V) \)-equivalent to \( v + u \), hence it is also \( \text{Sp}(V) \)-equivalent to \( w \), as claimed.
As only two elements of Sp(V)\V(m) have the same image under f and this map is clearly surjective, the result follows. □

Let X be a Weil module for Sp(V). If U is a totally isotropic Sp(V)-invariant submodule of V then Sp(V) normalizes the subgroup (0, U) of H(V), hence the set X(U) is an Sp(V)-submodule of X.

By the bottom and top layers of X we mean the Sp(V)-modules

\[ \text{Bot} = X(sV) \quad \text{and} \quad \text{Top} = X/\text{Bot}. \]

Let us denote by \( \text{Top}^\pm \) the ±1-eigenspaces of \( \iota \) acting on Top. Write \( \Omega \) for the character afforded by \( X \) and, given any Sp(V)-submodule \( Y \) of \( X \), let \( \Omega_Y \) denote the character afforded by \( Y \).

**Theorem 5.8.** The eigenspaces \( \text{Top}^+ \) and \( \text{Top}^- \) are irreducible Sp(V)-modules. Each has degree equal to \( (\sqrt{V} - \sqrt{[V(m) : sV]})/2 \) and occurs with multiplicity one in \( X \).

**Proof.** From Proposition 3.5,

\[ CV \cong \text{End}_C(X). \] (5.1)

By Proposition 3.3, \( X(sV) \) is isomorphic to the Schrödinger module for \( H((sV)^\perp) \). The latter group is precisely \( H(V(m)) \), by Lemma 5.4(2), hence Proposition 3.5 asserts

\[ CV(m) \cong \text{End}_C(X(sV)) = \text{End}_C(Bot). \] (5.2)

Taking Sp(V)-invariants and comparing dimensions, in Eqs. (5.1) and (5.2), we deduce

\[ |\text{Sp}(V)\backslash V| = (\Omega, \Omega) \quad \text{and} \quad |\text{Sp}(V)\backslash V(m)| = (\Omega_{\text{Bot}}, \Omega_{\text{Bot}}). \]

In virtue of Lemmas 5.2 and 5.7, we have

\[ |\text{Sp}(V)\backslash V| = |\text{Sp}(V)\backslash V(m)| + 1 = (|\text{Sp}(V)\backslash V(m)| + 1) + 1 = |\text{Sp}(V)\backslash V(m)| + 2. \]

We conclude that

\[ (\Omega_{\text{Top}}, \Omega_{\text{Top}}) + 2(\Omega_{\text{Top}}, \Omega_{\text{Bot}}) = (\Omega, \Omega) - (\Omega_{\text{Bot}}, \Omega_{\text{Bot}}) \]

\[ = |\text{Sp}(V)\backslash V| - |\text{Sp}(V)\backslash V(m)| = 2. \]

Since \( \text{Top}^+ \) and \( \text{Top}^- \) are non-zero Sp(V)-submodules of \( X \) with the given degrees [CMS, Lemma 4.4], the result follows. □

Now suppose that the Weil module \( X \) under consideration was constructed using the procedure describe in Section 4. From Theorem 4.1, \( X \) has a decomposition

\[ X \cong \bigoplus_{t \in T} X_t. \]
where $T$ is a set of representatives of the orbits of $V/U^\perp$ under the action of $\text{Sp}(V)$. The $\text{Sp}(V)$-invariant submodule $U^\perp$ is proper, since $\langle , \rangle$ in non-degenerate, so Lemmas 5.2 and 5.6 ensure that the set $T \cap \mathcal{P}(V)$ contains precisely one element, which we denote by $x$.

**Theorem 5.9.** The bottom and top layers of $X$ can be described as follows:

$$\text{Bot} \cong \bigoplus_{t \in T \setminus \{x\}} X_t \quad \text{and} \quad \text{Top} \cong X_x.$$ 

**Proof.** If $T'$ is an $U^\perp$-transversal of $V$ then $\{a_w \mid w \in T'\}$ is an $H(U^\perp) \rtimes \text{Sp}(V)$-transversal of $H(V) \rtimes \text{Sp}(V)$. We may then write

$$X = \bigoplus_{w \in T'} a_w Z = \left( \bigoplus_{w \in T' \cap \mathcal{P}(V)} a_w Z \right) \oplus \left( \bigoplus_{w \in T' \cap V(\mathfrak{m})} a_w Z \right).$$

For $t \in T$, denote the double coset

$$D_t = \text{Sp}(V)a_t \left( H \left( U_{\perp} \right) \right) \rtimes \text{Sp}(V).$$

The proof of Mackey’s Theorem tells us that

$$X_t \cong \bigoplus_{a_w \in (0, T' \cup D_t)} a_w Z.$$ 

From Lemma 5.2, all elements of $\mathcal{P}(V)$ are in the same $\text{Sp}(V)$-orbit, hence are in the same double coset, namely $D_x$. Therefore

$$\bigoplus_{w \in T' \cap \mathcal{P}(V)} a_w Z \cong X_x \quad \text{and} \quad \bigoplus_{w \in T' \cap V(\mathfrak{m})} a_w Z \cong \bigoplus_{t \in T \setminus \{x\}} X_t.$$ 

Each $a_w Z$ is a module for the normal subgroup $H(sV)$ of $H(V)$; in particular, $a_w Z$ is a module for the subgroup $(0, sV)$ of $H(sV)$. We compute the action of $(0, sV)$ on $a_w Z$. Given $w \in T'$ and $u \in sV$, we have, for $z \in Z$,

$$(0, u)a_w z = a_w (2 \langle u, w \rangle, u)z.$$ 

From Proposition 3.3, $Z$ is a Schrödinger module for $H(U^\perp)$, and from Lemma 5.6, $(0, sV) \subseteq (0, U)$. Thus $u \in U_{\perp}^\perp$, so $2 \langle u, w \rangle, u)z = \lambda(2 \langle u, w \rangle)z$, according to Definition 3.1 of the Schrödinger representation. We conclude that $(0, u)$ acts on $a_w Z$ via multiplication by $\lambda(2 \langle u, w \rangle)$. But

$$\langle sV, w \rangle = \begin{cases} (0) & \text{if } w \in V(\mathfrak{m}); \\ \mathfrak{s} & \text{if } w \in \mathcal{P}(V). \end{cases}$$
Thus \((0, sV)\) acts trivially on \(a_wZ\) if \(w \in V(m)\), and since \(\lambda\) is primitive, \((0, sV)\) acts on \(a_wZ\) via multiplication by a non-trivial character if \(w \in P(V)\). We deduce

\[
\text{Bot} = \text{inv}_{(0, sV)} X = \bigoplus_{w \in T \cap V(m)} a_wZ \cong \bigoplus_{r \in T \setminus \{s\}} X_r
\]

and hence

\[
\text{Top} \cong X_s. \quad \Box
\]

6. Clifford theory

Let \(U\) be an \(\text{Sp}(V)\)-invariant submodule of \(V\). The congruence subgroup associated with \(U\) is

\[
\Gamma(U) = \{ \gamma \in \text{Sp}(V) \mid \gamma v \equiv v \mod U, \text{ for all } v \in V \}.
\]

As the name implies, \(\Gamma(U)\) is a subgroup of \(\text{Sp}(V)\); in fact it is a normal subgroup. In this section we analyze \(\text{Top}^\pm\) using Clifford theory with respect \(\Gamma(U)\), where \(U\) is a totally isotropic \(\text{Sp}(V)\)-invariant submodule of \(V\).

We first have some preliminary results which hold for any \(\text{Sp}(V)\)-invariant submodule \(U\) of \(V\).

**Lemma 6.1.** For an \(\text{Sp}(V)\)-invariant submodule \(U\) of \(V\), we have

\[
\langle U, V \rangle V \subseteq U.
\]

**Proof.** Let \(x \in P(V)\). We first claim that

\[
\langle U, V \rangle = \langle U, x \rangle. \quad (6.1)
\]

The inclusion \(\langle U, x \rangle \subseteq \langle U, V \rangle\) is clear. For the reverse inclusion, suppose that \(u \in U\) and \(v \in V\). By Lemma 5.5, there exist \(x_1, \ldots, x_k \in P(V)\) such that \(v = \sum_{i=1}^k x_i\), therefore

\[
\langle u, v \rangle = \sum_{i=1}^k \langle u, x_i \rangle \in \sum_{i=1}^k \langle U, x_i \rangle.
\]

For each \(x_i\), Lemma 5.2 gives us \(g_i \in \text{Sp}(V)\) such that \(x_i = g_ix\). Then

\[
\langle U, x_i \rangle = \langle U, g_ix \rangle = \langle g_iU, g_ix \rangle = \langle U, x_i \rangle,
\]

since \(U\) is \(\text{Sp}(V)\)-invariant, hence \(\langle u, v \rangle \in \langle U, x \rangle\). As \(u\) and \(v\) were arbitrary, \((6.1)\) follows.

Let \(r \in \langle U, V \rangle\). In light of Eq. \((6.1)\), there exists \(u \in U\) such that \(r = \langle u, x \rangle\), whence

\[
r x = \tau_{1,x} u - u \in U
\]
since $U$ is $\text{Sp}(V)$-invariant. Since $x$ is an arbitrary element of $\mathcal{P}(V)$, Lemma 5.5 allows us to deduce $rV \subseteq U$. We conclude $\langle U, V \rangle V \subseteq U$. □

If $u$ and $w$ are elements of $V$ with $\langle u, w \rangle = 0$, then $\text{Sp}(V)$ contains the transformation

$$
\gamma_{u, w} : v \mapsto v + \langle v, u \rangle w + \langle v, w \rangle u, \quad v \in V.
$$

(6.2)

**Lemma 6.2.** The congruence subgroup $\Gamma(U)$ contains all the symplectic transvections $\tau_{r, v}$ with $r \in \langle U, V \rangle$, $v \in V$. If $u \in U$ and $w \in V$ are such that $\langle u, w \rangle = 0$ then $\gamma_{u, w}$ is in $\Gamma(U)$.

**Proof.** If $r \in \langle U, V \rangle$ then Lemma 6.1 shows $\tau_{r, v}w - w = r(w, v)v \in U$. If $u \in U$ then the same lemma shows $\langle v, u \rangle w \in U$ for all $v \in V$, whence $\gamma_{u, w} \in \Gamma(U)$ is seen to be an immediate consequence of definition (6.2). □

**Lemma 6.3.** The congruence subgroup $\Gamma(U)$ acts trivially on $U^\perp$.

**Proof.** Suppose that $\gamma \in \Gamma(U)$ and $w \in U^\perp$. Then for any $v \in V$ we have

$$
\langle \gamma w - w, v \rangle = \langle \gamma w, v \rangle - \langle w, v \rangle = \langle w, \gamma^{-1}v \rangle - \langle w, v \rangle = \langle w, \gamma^{-1}v - v \rangle.
$$

We have $\gamma^{-1}v - v \in U$ since $\gamma \in \Gamma(U)$, hence $\langle w, \gamma^{-1}v - v \rangle = 0$ because $w \in U^\perp$. Thus $\langle \gamma w - w, v \rangle = 0$ for all $v \in V$, so $\gamma w - w = 0$ by non-degeneracy of $\langle , \rangle$. □

**Lemma 6.4.** If $V = V_1 \oplus V_2$ is an orthogonal decomposition then

$$
U = U \cap V_1 \oplus U \cap V_2.
$$

**Proof.** The automorphism $g$ of $V$ defined by the conditions

$$
g|_{V_1} = 1_{V_1} \text{ and } g|_{V_2} = -1_{V_2}
$$

is symplectic. If $u \in U$ then $(u + gu)/2$ and $(u - gu)/2$ both belong to $U$, since $U$ is $\text{Sp}(V)$-invariant. On the other hand, it is clear that $(u + gu)/2$ belongs to $V_1$ and $(u - gu)/2$ belongs to $V_2$. The result thus follows by observing that

$$
u = \frac{u + gu}{2} + \frac{u - gu}{2}.
$$

A hyperbolic plane $P$ is an $R$-submodule of $V$ spanned by a pair of elements $x$ and $y$ with $\langle x, y \rangle = 1$. It is clear that $P$ is free, with basis $\{x, y\}$. The restriction of $\langle , \rangle$ to $P$ is non-degenerate and one has an orthogonal decomposition

$$
V = P \oplus P^\perp.
$$

For the rest of this section, as well as Sections 7 and 8, we fix a non-zero totally isotropic $\text{Sp}(V)$-invariant submodule $U$ of $V$. As in Section 4, let $S'$ be a Schrödinger
representation of $H(U^\perp)$ of type $\lambda$, with ambient module $Z$, and let $W'$ be an associated Weil representation of $\text{Sp}(U^\perp)$. We let $X$ be the Weil module of type $\lambda$ for $\text{Sp}(V)$ constructed from the pair $(S', W')$ via the procedure of Section 4.

The group $G_t = \{ g \in \text{Sp}(V) \mid gt \equiv t \mod U^\perp \}$ contains the congruence subgroup $\Gamma(U^\perp)$, and hence also contains $\Gamma(U)$ since $\langle U, U \rangle = 0$. Let $W_t$ be the representation of $G_t$ given by Eq. (4.3).

**Proposition 6.5.** For $\gamma \in \Gamma(U)$, $W_t(\gamma)$ acts on $Z$ as

$$W_t(\gamma) = \lambda \left( \langle \gamma t, t \rangle \right) 1_Z.$$

**Proof.** By definition of $\Gamma(U)$, we have $\gamma t - t \in U$. Since $U^\perp = U$ by Lemma 2.1, Definition 3.1 of the Schrödinger character shows that $S'(\langle \gamma t, t \rangle, \gamma t - t) = \lambda \left( \langle \gamma t, t \rangle \right) 1_Z$. Lemma 6.3 shows that $\gamma |_{U^\perp}$ is trivial, whence a fortiori $W'(\gamma |_{U^\perp}) = 1_Z$. Therefore (4.3) yields

$$W_t(\gamma) = S'(\langle \gamma t, t \rangle, \gamma t - t) W'(\gamma |_{U^\perp}) = \lambda \left( \langle \gamma t, t \rangle \right) 1_Z.$$ 

Since $W_t$ is a representation of $G_t$, the map

$$\delta_t : \gamma \mapsto \lambda \left( \langle \gamma t, t \rangle \right), \quad \gamma \in \Gamma(U), \quad (6.3)$$

is a complex linear character of the normal subgroup $\Gamma(U)$ of $\text{Sp}(V)$. We note that

$$\delta_t = \delta_{gt}, \quad \text{for all } g \in \text{Sp}(V).$$

**Theorem 6.6.** For a primitive element $x \in V$, the inertia group of the character $\delta_x$ of $\Gamma(U)$ is $G_x \times \{ 1, i \}$.

**Proof.** Set $i = \langle U, V \rangle$ and $j = \text{ann } i$. Note that $i$ is non-zero since $U$ is non-zero and $(\cdot, \cdot)$ is non-degenerate, whence $j$ is a proper $R$-ideal.

Let $g \in G_x$. Since $\Gamma(U)$ is normal, Proposition 6.5 allows us to deduce that $\delta_x$ is an irreducible constituent of the restriction of $\delta W_x$ to $\Gamma(U)$. On the other hand, $W_x$ is a representation of $G_x$, hence $\delta W_x$ is similar to $W_x$. It follows that $\delta_x$ is an irreducible constituent of $W_x$, whence Proposition 6.5 allows us to deduce $\delta_x = \delta_x$. We conclude that $G_x$ stabilizes $\delta_x$. The central involution $i$ trivially stabilizes $\delta_x$.

Conversely, suppose that $g \in \text{Sp}(V)$ stabilizes $\delta_x$. The definition of $\delta_x$ shows that

$$\langle y g x, g x \rangle - \langle y x, x \rangle \in \ker \lambda \quad \text{for all } y \in \Gamma(U).$$

For $v \in V$ and $r \in i$, the symplectic transvection $\tau_{r,v}$ belongs to $\Gamma(U)$ by Lemma 6.2. Then

$$\langle \tau_{r,v} g x, g x \rangle - \langle \tau_{r,v} x, x \rangle = \langle g x + r (g x, v) v, g x \rangle - \langle x + r (x, v) v, x \rangle = r (-(g x, v)^2 + (x, v)^2) \in \ker \lambda.$$
Since \( r \) is an arbitrary element of \( i \), the primitivity of \( \lambda \) forces

\[
(gx, v)^2 - (x, v)^2 \in j = \text{ann } i, \quad \text{for all } v \in V. \tag{6.4}
\]

Let \( v_1 \) and \( v_2 \) be in \( V \), and take \( v = v_1 + v_2 \) in (6.4). Since 2 is a unit of \( R \), we get

\[
(gx, v_1)(gx, v_2) \equiv (x, v_1)(x, v_2) \mod j, \quad \text{for all } v_1, v_2 \in V. \tag{6.5}
\]

Since \( x \) is primitive, there exists \( y \in V \) such that \( \langle x, y \rangle = 1 \). It follows from (6.4) that

\[
(gx, y)^2 - 1 \in j.
\]

The unit group \( R^* \) of \( R \), modulo the odd order subgroup \( 1 + m \), is isomorphic to the cyclic group \( (R/m)^* \), so \(-1\) is the unique element of order 2 in \( R \). The same is true for \( R/j \). Then

\[
\langle gx, y \rangle \equiv \epsilon \mod j,
\]

where \( \epsilon = \pm 1 \). Hence if one takes \( v_2 = y \) in (6.5), one obtains \( \langle gx, v_1 \rangle \equiv \epsilon \langle x, v_1 \rangle \mod j \), or equivalently,

\[
\langle gx - \epsilon x, v_1 \rangle \in j, \quad \text{for all } v_1 \in V.
\]

We conclude that

\[
gx - \epsilon x \in V(j) = (iV)^\perp. \tag{6.7}
\]

Next we shall show that \( gx - \epsilon x \in (P^\perp \cap U)^\perp \). Take \( u \in P^\perp \cap U \). Then

\[
\langle yu, gx \rangle - \langle yu, x \rangle = \langle gx + (gx, u)y + (gx, y)u, gx \rangle - \langle x + (x, u)y + (x, y)u, x \rangle
\]

\[
= (gx, u)(y, gx) + (gx, y)(u, gx) \quad \text{(since } \langle u, x \rangle = 0\text{)}
\]

\[
= -2\langle gx, u \rangle \langle gx, y \rangle \in \ker \lambda.
\]

Since \( u \) was arbitrary, we deduce that

\[
2\langle gx, y \rangle \langle gx, P^\perp \cap U \rangle \subseteq \ker \lambda.
\]

From (6.6), since \( j \) is a proper ideal of the local ring \( R \), it follows that \( \langle gx, y \rangle \) is a unit of \( R \). As 2 is also a unit of \( R \), then the primitivity of \( \lambda \) implies that \( \langle gx, P^\perp \cap U \rangle = 0 \); that is, \( gx \in (P^\perp \cap U)^\perp \). Since \( x \in P \), then \( x \) is also in \( (P^\perp \cap U)^\perp \), hence

\[
gx - \epsilon x \in (P^\perp \cap U)^\perp. \tag{6.8}
\]
From Lemma 6.4, $U = (P \cap U) + (P^\perp \cap U)$, hence from (6.7) and (6.8) we see that 
\[ gx - \epsilon x \in U^\perp. \]

We conclude that $g \in G_x \times \{1, \iota\}$.

\[ \blacksquare \]

Now, from Clifford’s Theorem [CR, 11.1] we have the following result.

**Theorem 6.7.** Let $U$ be a non-zero totally isotropic $\text{Sp}(V)$-invariant submodule of $V$ and let $x$ be a primitive element of $V$. Then

1. The character $\delta_x$ is a constituent of the restriction of $\text{Top}^\pm$ to the normal subgroup $\Gamma(U)$.
2. The inertia group of $\delta_x$ in $\text{Sp}(V)$ is $G_x \times \{1, \iota\}$.
3. $W_x^\pm$ is an irreducible representation of $G_x \times \{1, \iota\}$ whose restriction to $\Gamma(U)$ is a multiple of $\delta_x$.
4. $\text{Top}^\pm$ is isomorphic to $\text{ind}_{G_x \times \{1, \iota\}}^{\text{Sp}(V)} W_x^\pm$.

### 7. More Clifford theory

In this section we analyze $\text{Top}^\pm$ using Clifford theory with respect to the normal subgroup $\Gamma(U^\perp)$, where $U$ is a non-zero totally isotropic $\text{Sp}(V)$-invariant submodule of $V$. This is somewhat more difficult than using the normal subgroup $\Gamma(U)$, since it was not hard to see that the restriction of $W_x$ to $\Gamma(U)$ is a multiple of the linear character $\delta_x$. Here we shall show that the restriction of $W_x$ to $\Gamma(U^\perp)$ remains irreducible.

**Theorem 7.1.** Let $U$ be a non-zero $\text{Sp}(V)$-invariant totally isotropic submodule of $V$ and $x$ be a primitive element of $V$. Let $W_x$ be the representation of $G_x$ afforded by $Z$, given by Eq. (4.3). Then the restriction to $\Gamma(U^\perp)$ of $W_x$ remains irreducible.

**Proof.** We shall show that the commuting ring of $W_x(\Gamma(U^\perp))$ consists entirely of scalar operators. Indeed, let $L$ be a linear operator in $\text{End}_C(Z)$ that commutes with all $W_x(\gamma)$, $\gamma \in \Gamma(U^\perp)$. As

\[ S'(\gamma x, x, \gamma x - x) = S'(\gamma x, x, 0) S'(0, \gamma x - x) = \lambda((\gamma x, x)) S'(0, \gamma x - x), \]

the operator $L$ commutes with all

\[ S'(0, \gamma x - x) W'(\gamma|_{U^\perp}), \quad \gamma \in \Gamma(U^\perp). \]

Let $T$ be a transversal of $U$ in $U^\perp$. In virtue of Corollary 3.6, the set $\{S'(0, t) \mid t \in T\}$ is a $C$-basis of $\text{End}_C(Z)$. We may thus write

\[ L = \sum_{t \in T} a_t S'(0, t) \]
for unique elements $\alpha_t \in \mathbf{C}$, whence
\[
\left( \sum_{t \in T} \alpha_t S'(0, t) \right) S'(0, \gamma x - x) W'(\gamma |_{\mathbb{U}^\perp}) = S'(0, \gamma x - x) W'(\gamma |_{\mathbb{U}^\perp}) \left( \sum_{t \in T} \alpha_t S'(0, t) \right).
\]

Using the definition of a Weil representation, the right-hand side can be written as
\[
S'(0, \gamma x - x) \left( \sum_{t \in T} \alpha_t S'(0, \gamma t) \right) W'(\gamma |_{\mathbb{U}^\perp}).
\]

As $W'(\gamma |_{\mathbb{U}^\perp})$ is invertible, we deduce that
\[
\left( \sum_{t \in T} \alpha_t S'(0, t) \right) S'(0, \gamma x - x) = \left( \sum_{t \in T} \alpha_t S'(0, \gamma t) \right) S'(0, \gamma x - x).
\]

As $S'(0, \gamma x - x)$ is invertible, we deduce that
\[
\sum_{t \in T} \alpha_t S'(0, t) = \sum_{t \in T} \alpha_t \lambda(2\langle \gamma x - x, \gamma t \rangle) S'(0, \gamma t). \tag{7.1}
\]

For $t \in T$ and $g \in \text{Sp}(V)$, we have $gt = t' + u$ for some unique $t' \in T, u \in \mathbb{U}$. Write
\[
g \cdot t = t'.
\]

This gives us an action of $\text{Sp}(V)$ on $T$ (equivalent to the action of $\text{Sp}(V)$ on $\overline{\mathbb{U}^\perp}$). Since $S'(0, u)$ is the identity for $u \in \mathbb{U}$,
\[
\sum_{t \in T} \alpha_t S'(0, t) = \sum_{t \in T} \alpha_{\gamma \cdot t} S'(0, \gamma t).
\]

Then from Eq. (7.1) we obtain
\[
\sum_{t \in T} \alpha_{\gamma \cdot t} S'(0, \gamma t) = \sum_{t \in T} \alpha_t \lambda(2\langle \gamma x - x, \gamma t \rangle) S'(0, \gamma t).
\]

This gives us
\[
\alpha_{\gamma \cdot t} = \lambda(2\langle \gamma x - x, \gamma t \rangle) \alpha_t, \quad t \in T, \gamma \in \Gamma(\mathbb{U}^\perp). \tag{7.2}
\]
Since \( \lambda \) is primitive, the unique minimal ideal \( s \) is not contained in \( \ker \lambda \), so there exists an element \( s \in s \) such that \( \lambda(s) \neq 1 \). We will presently show that if \( t \in T, t \notin U \), there exists \( \gamma \in \Gamma(U^{\perp}) \) enjoying the following properties.

(i) \( \gamma \cdot t = t \), i.e. \( \gamma t \equiv t \mod U \).
(ii) \( 2\langle yx - x, \gamma t \rangle = s \).

Applying (7.2) to the pair \( (t, \gamma) \), we deduce:

\[
\alpha_t = \alpha_{\gamma \cdot t} = \lambda(2\langle yx - x, \gamma t \rangle) \alpha_t = \lambda(s) \alpha_t, \]

Since \( \lambda(s) \neq 1 \), then \( \alpha_t = 0 \). Therefore, if \( u \) is the unique element of \( T \cap U \), we conclude that

\[
L = \alpha_u S'(0, u) = \alpha_u 1_Z.
\]

Since \( L \) was arbitrary, we deduce that

\[
\text{End}_{\Gamma(U^{\perp})} Z = C1_Z,
\]
as claimed.

We now proceed with the proof of the existence of \( \gamma \) in \( \Gamma(U^{\perp}) \) enjoying the properties (i) and (ii) listed above. In fact, the element \( \gamma \) will be seen to satisfy the following two conditions.

(i') \( \gamma t \equiv t \mod sV \).
(ii') \( 2\langle yx - x, t \rangle = s \).

The condition (i') implies (i), since \( sV \) is a submodule of \( U \). In the presence of (i), the conditions (ii) and (ii') are equivalent. Indeed, note that

\[
\langle yx - x, \gamma t \rangle = \langle yx - x, \gamma t - t \rangle + \langle yx - x, t \rangle = \langle yx - x, t \rangle,
\]
since \( yx - x \in U^{\perp} \) by definition of \( \Gamma(U^{\perp}) \), and \( \gamma t - t \in U \) by (i).

As in Section 6, set \( i = \langle U, V \rangle \) and \( j = \text{ann} \ i \). We observe that \( i \) is non-zero, hence \( j \) is proper. From Lemma 6.1, we have \( iV \subseteq U \subseteq V(i) \). Applying Lemma 5.4(1), we deduce that \( jV \subseteq U^{\perp} \subseteq V(j) \), whence

\[
j = \langle U^{\perp}, V \rangle. \quad (7.3)
\]

Non-degeneracy of \( \langle , \rangle \) thus allows us to deduce

\[
iU^{\perp} = 0. \quad (7.4)
\]

Fix \( y \in V \) such that \( \langle x, y \rangle = 1 \) and let \( P \) be the hyperbolic plane generated by the pair \( x \) and \( y \). Note that the identity (7.3) ensures that \( \langle y, t \rangle \) and \( \langle x, t \rangle \) belong to \( j \). We consider three cases.
(a) \( \langle y, t \rangle \in j \setminus i \).
(b) \( \langle y, t \rangle \in i \) and \( \langle x, t \rangle \in j \setminus i \).
(c) \( \langle y, t \rangle \) and \( \langle x, t \rangle \) both belong to \( i \).

**Case (a).** The given hypothesis ensures that \( j \langle y, t \rangle \) is a non-zero \( R \)-ideal, hence \( s \subseteq j \langle y, t \rangle \).

In particular, there exists \( r \in j \) such that

\[
s = 2r \langle y, t \rangle.
\]

In light of the identity (7.3), Lemma 6.2 asserts that \( r_{r, y} \in \Gamma(U^\perp) \). We calculate:

\[
\tau_{r, y} t = t + r(t, y)y = t - 2^{-1}sy \equiv t \mod sV.
\]

Moreover,

\[
2(\tau_{r, y}x - x, t) = 2r \langle y, t \rangle = s.
\]

In this case one may take \( \gamma = r_{r, y} \).

**Case (b).** The given hypothesis ensures that \( j \langle x, t \rangle \) is a non-zero \( R \)-ideal, whence there exists \( r \in j \) such that

\[
s = 2r \langle x, t \rangle.
\]

Observing that \( 1 + r \) is an invertible element of \( R \), since \( j \) is proper, the transformation

\[
\gamma: v \mapsto v + r\langle v, y \rangle x + r(1 + r)^{-1}\langle v, x \rangle y, \quad v \in V,
\]

is well-defined. It is clearly \( R \)-linear and an elementary calculation shows it is symplectic. Since \( r \in j \), the identity (7.3) and Lemma 6.1 show that \( \gamma \in \Gamma(U^\perp) \). We calculate:

\[
\gamma t = t + r(t, y)x + r(1 + r)^{-1}(t, x)y
= t + r(1 + r)^{-1}(t, x)y \quad (\text{since } r \in j \text{ annihilates } i),
= t - 2^{-1}(1 + r)^{-1}xy \equiv t \mod sV.
\]

Moreover,

\[
2(\gamma x - x, t) = 2\langle x, t \rangle = s.
\]

**Case (c).** Since \( U = U^\perp \), the fact \( t \in U^\perp \setminus U \) ensures the existence of an element \( v \in U^\perp \) such that \( \langle v, t \rangle \neq 0 \). Thus \( s \subseteq R\langle v, t \rangle \); whence there exists a scalar multiple \( w \) of \( v \) with

\[
s = 2\langle w, t \rangle.
\]
Set

\[ z = w - \langle w, y \rangle x - \langle x, w \rangle y. \]

When combined with Lemma 6.1, the fact \( w \in U^\perp \) implies \( z \in U^\perp \). Moreover, \( \langle z, y \rangle = 0 \) by construction, hence Lemma 6.2 asserts that \( y_{z, y} \) belongs to \( \Gamma(U^\perp) \).

Since \( \langle x, t \rangle \) and \( \langle y, t \rangle \) belong to \( i \) and \( w \) belongs to \( U^\perp \), the identity (7.3) yields

\[ \langle z, t \rangle = \langle w, t \rangle - \langle w, y \rangle \langle x, t \rangle - \langle x, w \rangle \langle y, t \rangle = \langle w, t \rangle. \]

Observing that \( \langle t, y \rangle \) annihilates \( z \in U^\perp \), by (7.4) we deduce:

\[ y_{z, y}(t) = t + \langle t, z \rangle y + \langle t, y \rangle z = t - \langle z, t \rangle y = t - \langle w, t \rangle y = t - 2^{-1} s y \mod s V. \]

Moreover, since \( \langle x, z \rangle = 0 \) by construction,

\[ 2\langle y_{z, y}(x) - x, t \rangle = 2\langle z, t \rangle = 2\langle w, t \rangle = s. \]

In this case one may take \( \gamma = y_{z, y} \).

We have now shown the existence of \( \gamma \in \Gamma(U^\perp) \) satisfying properties (i) and (ii) in all three cases; so the proof of Theorem 7.1 is complete. \( \square \)

Clifford theory in this situation is presented by the following theorem.

**Theorem 7.2.** Let \( U \) be a non-zero totally isotropic \( \text{Sp}(V) \)-invariant submodule of \( V \) and let \( x \) be a primitive element of \( V \).

1. The restriction of \( \text{Top}^\pm \) to the normal subgroup \( \Gamma(U^\perp) \) has \( \text{res}_{\Gamma(U^\perp)} W_x \) as an irreducible constituent.
2. The inertia group in \( \text{Sp}(V) \) of \( \text{res}_{\Gamma(U^\perp)} W_x \) is \( G_x \times \{1, \iota\} \).
3. \( W_x^\pm \) is an irreducible representation of \( G_x \times [1, t] \) whose restriction to \( \Gamma(U) \) is a multiple of \( b_x \).
4. \( \text{Top}^\pm \) is isomorphic to \( \text{ind}_{G_x \times [1, t]}^{\text{Sp}(V)} W_x^\pm \).

8. Characterization of \( \text{Bot} \) and faithfulness of \( \text{Top} \)

We recall that the \( \text{Sp}(V) \)-submodule \( \text{Bot} \) was introduced as the set of invariants of the subgroup \( (0, sV) \) of \( H(V) \). Subsequently, we were able to identify \( \text{Bot} \) as a sum of terms appearing in the Mackey decomposition of the Weil module (Theorem 5.9). We shall here discuss the characterization of \( \text{Bot} \) as invariants of normal subgroups of \( \text{Sp}(V) \). We also show that the representations afforded by \( \text{Top}^\pm \) have small kernels.
We first generalize the notion of congruence subgroup introduced in Section 6. Let \( U_1 \) and \( U_2 \) be \( \text{Sp}(V) \)-invariant submodules of \( V \) such that \( U_1 \subseteq U_2 \). Define

\[
\Gamma(U_1, U_2) = \{ g \in \text{Sp}(V) \mid gv \equiv v \mod U_1, \text{ for all } v \in U_2 \}.
\]

This is a normal subgroup of \( \text{Sp}(V) \). Note that \( \Gamma(U_1, V) = \Gamma(U_1) \).

In the case when \( R \) is principal and \( V \) is free, denoting by \( l \) the index of nilpotency of the maximal ideal \( m \), it was proved in [CMS] that \( \text{Bot} \subseteq \text{inv} \Gamma(m^{-2})X \), provided the rank of \( V \) was greater than 2 or the residue-class field of \( R \) was not equal to \( F_3 \). It is not hard to see that in this case \( \Gamma(m^{-2}) = \Gamma(sV, V(m)) \). This partly motivates the next theorem. Its statement uses the notation of Section 4, in particular, \( T \) is a set of representatives of \( \text{Sp}(V) \) acting on \( V/U_\perp \), where \( U \) is a non-zero totally isotropic \( \text{Sp}(V) \)-invariant submodule of \( V \) and \( W' \) is a Weil representation of \( \text{Sp}(U_\perp) \).

**Theorem 8.1.** The module \( \text{Bot} \) has the following three characterizations.

1. \( \text{Bot} = \text{inv}_{\{0,sV\}}X \).
2. \( \text{Bot} = \bigoplus_{t \in T \cap V(m)} X_t \).
3. \( \text{Bot} = \text{inv} \Gamma(sV) X \).

Furthermore, if the Weil representation \( W' \) used in the construction of \( X \) is assumed to be the inflation of a Weil representation of \( \text{Sp}(U_\perp) \) then

4. \( \text{Bot} = \text{inv} \Gamma(sV, V(m)) X \).

**Proof.** Part (1) is the definition and (2) is contained in Theorem 5.9.

Observing that \( \Gamma(sV) \subseteq \Gamma(U) \), Proposition 6.5 shows that \( \Gamma(sV) \) acts on the module \( Z_t \) via multiplication by the linear character:

\[
\delta_t : \gamma \mapsto \lambda(\langle gt, t \rangle), \quad \gamma \in \Gamma(sV).
\]

Suppose that \( t \in V(m) \). In virtue of Lemma 6.3, \( t \) is a fixed point of \( \Gamma(sV) \) and thus \( \delta_t \) is immediately seen to be trivial. Since \( \Gamma(sV) \) is a normal subgroup of \( \text{Sp}(V) \) contained in \( G_t \), the definition of an induced module allows us to conclude that the action of \( \Gamma(sV) \) on \( X_t = \text{ind}_{G_t}^{\text{Sp}(V)} Z_t \) is trivial.

On the other hand, let \( x \) be the primitive element of \( T \). If we fix an element \( y \) of \( V \) with \( \langle x, y \rangle = 1 \), then evaluation of \( \delta_x \) at the symplectic transvections \( tr_y \) with \( r \in s \) show that

\[
\delta_x(\Gamma(sV)) = \lambda(s) \neq 1;
\]

that is, the restriction of \( \delta_x \) to \( \Gamma(sV) \) is non-trivial. In this case, the definition of an induced module shows that the trivial representation of \( \Gamma(sV) \) fails to occur in \( X_x = \text{ind}_{G_x}^{\text{Sp}(V)} Z_x \). The preceding observations allow us to deduce (3) from (2) and the Mackey decomposition provided by Theorem 4.1.
From the definition, it is clear that $\Gamma(sV)$ is a subgroup of $\Gamma(sV, V(m))$, whence

$$\text{inv}_{\Gamma(sV, V(m))} X \subseteq \text{inv}_{\Gamma(sV)} X.$$  

Now, if $t \in V(m)$ then $\Gamma(sV, V(m))$ is a subgroup of $G_t$, in virtue of the inclusion $sV \subseteq U^\perp$. Therefore $\Gamma(sV, V(m))$ acts on $Z_t$ via the representation $W_t$ given by the formula (4.3). We will presently show that this action is trivial if $W'$ is the inflation of a Weil representation of $\text{Sp}(U^\perp)$. It thus follows, for reasons identical to those given in the case of $\Gamma(sV)$ above, that $\Gamma(sV, V(m))$ acts trivially on $X_t$. Since $t$ was arbitrary, we deduce

$$\bigoplus_{t \in T/V(m)} X_t \subseteq \text{inv}_{\Gamma(sV, V(m))} X,$$

whence (4) is seen to follow from (2) and (3).

We now proceed with the proof that the restriction of $W_t$ to $\Gamma(sV, V(m))$ is trivial. First recall that $U = U^\perp\perp$ by Lemma 2.1. If $\gamma \in \Gamma(sV, V(m))$ then $\gamma t - t \in sV \subseteq U$. As $sV = V(m)^{\perp}$ by Lemma 5.4(2), it follows that $\langle \gamma t, t \rangle = \langle \gamma t - t, t \rangle = 0$. Definition 3.1 of the Schrödinger character thus yields

$$S'(\langle \gamma t, t \rangle, \gamma t - t) = S'(0, \gamma t - t) = 1_{Z_t}.$$  

Moreover, $\Gamma(sV, V(m))$ is a subgroup of $\Gamma(U, U^\perp)$ since $sV \subseteq U \subseteq U^\perp \subseteq V(m)$. The image of the latter group under the restriction map $\Gamma(U, U^\perp)$ is readily seen to lie in the kernel of the canonical homomorphism from $\text{Sp}(U^\perp)$ to $\text{Sp}(U^\perp)$, so our choice of representation $W'$ allows to deduce

$$W'(\gamma|_{U^\perp}) = 1_{Z_t}.$$  

In light of the preceding two identities, (4.3) yields

$$W_t(\gamma) = S'(\langle \gamma t, t \rangle, \gamma t - t)W'(\gamma|_{U^\perp}) = 1_{Z_t},$$

as claimed. \qed

We conclude this section with a descriptions of the kernels of the representations afforded by the modules $\text{Top}^\pm$.

**Theorem 8.2.** Let $K^\pm$ be the kernel of the representation of $\text{Sp}(V)$ afforded by $\text{Top}^\pm$. Then $K^- = \{1_V\}$ and $K^+ = \{1_V, \iota\}$.

**Proof.** The proof is divided into two steps.

**Step 1.** $K^+ \subseteq (\bigcap_{v \in \mathcal{P}(V)} \ker \delta_v) \times \{1_V, \iota\}$ and $K^- \subseteq \bigcap_{v \in \mathcal{P}(V)} \ker \delta_v$.

To see this, let $g \in K^\pm$. We know that $\text{Top}^\pm \cong \text{ind}_{G_v \times \{1_V, \iota\}}^G Z_v^\pm$ for all $v \in \mathcal{P}(V)$. Thus
We claim that
\[ g \in \bigcap_{v \in \mathcal{P}(V)} (G_v \times \{1, \iota\}). \]

Otherwise, from (a) there exist \( v \) and \( w \) in \( \mathcal{P}(V) \) such that
\[ gv \equiv v \mod U^\perp \quad \text{and} \quad gw \equiv -w \mod U^\perp. \]

From \((v + w) - (v - w) = 2v \in \mathcal{P}(V)\) we see that \( v + u \) is primitive for at least one \( u \) belonging to \([w, -w]\). As \( g \in G_{v+u} \times \{1, \iota\}, \) we have \( g(v + u) \equiv \pm(v + u) \mod U^\perp. \) But by above, \( g(v + u) \equiv v - u \mod U^\perp. \) It follows that \( v - u \equiv \pm(v + u) \mod U^\perp, \) whence one of \( 2v \) or \( 2u \) belongs to \( U^\perp, \) contradicting Lemma 5.6. This contradiction establishes our claim.

As \( \mathcal{P}(V) \) generates \( V, \) the definitions of \( G_v \) and \( \Gamma(U^\perp) \) yield
\[ \bigcap_{v \in \mathcal{P}(V)} G_v = \Gamma(U^\perp). \]

Accordingly we may write \( g = \iota^\epsilon \gamma, \) where \( \gamma \in \Gamma(U^\perp) \) and \( \epsilon \) is 0 or 1. Now from (b) we see that, for all \( v \in \mathcal{P}(V), \)
\[ 1_Z = W^\pm_v (\iota^\epsilon \gamma) = (\pm1)^\epsilon S'(\langle \gamma v, v \rangle, \gamma v - v)W'(\gamma|U^\perp). \]

In particular, for all \( v, w \in \mathcal{P}(V), \)
\[ (\pm1)^\epsilon S'(\langle \gamma v, v \rangle, \gamma v - v)W'(\gamma|U^\perp) = (\pm1)^\epsilon S'(\langle \gamma w, w \rangle, \gamma w - w)W'(\gamma|U^\perp), \]

which simplifies to
\[ S'(\langle \gamma(v - w), v + w \rangle - 2\langle v, w \rangle, \gamma(v - w) - (v - w) = 1_Z, \quad \text{for all } v \text{ and } w \in \mathcal{P}(V). \]

Using Definition 3.1 of the Schrödinger character and the fact \( U = U^\perp \perp, \) the last identity allows us to deduce
\[ \gamma(v - w) \equiv v - w \mod U. \]

Let \( u \in V \) be arbitrary. If \( u \) is primitive, take \( v = 2u \) and \( w = u; \) otherwise take \( v \) to be primitive and \( w = v - u. \) It follows that \( \gamma u \equiv u \mod U \) for all \( u \in V; \) that is, \( \gamma \in \Gamma(U). \)
Since $\gamma \in \Gamma(U)$, Proposition 6.5 and (b) allow us to conclude that

$$1_Z = W^\pm_v (t^\epsilon \gamma) = (\pm 1)^\epsilon \delta_v (\gamma) 1_Z, \quad \text{for all } v \in \mathcal{P}(V).$$

The order of $\delta_v (\gamma) = \lambda (\langle \gamma v, v \rangle)$ is odd, since it necessarily divides the characteristic of $R$. The last identity therefore allows us to deduce

$$\delta_v (\gamma) = 1, \quad \text{for all } v \in \mathcal{P}(V),$$

and $\epsilon = 0$ if the minus sign prevails in $\pm 1$. This completes the proof of the first step.

**Step 2.** $\bigcap_{v \in \mathcal{P}(V)} \ker \delta_v = \{ 1_V \}$.

Suppose that $\gamma \in \Gamma(U)$ is in the kernel of $\delta_v$ for all $v \in \mathcal{P}(V)$. Let $v$ and $w$ be arbitrary primitive elements of $V$. Note that

$$\langle \gamma (v + w), v + w \rangle = \langle \gamma v, v \rangle + \langle \gamma w, w \rangle + \langle \gamma v, w \rangle + \langle \gamma w, v \rangle.$$

As above, $v + u$ is primitive for some $u \in \{ w, -w \}$. It follows that $\langle \gamma v, u \rangle + \langle \gamma u, v \rangle \in \ker \lambda$ or, equivalently, $\langle \gamma v, u \rangle - \langle v, \gamma u \rangle \in \ker \lambda$. Since $\ker \lambda$ is a subgroup of $R^+$, the conclusion is also valid for $-u$. We infer that

$$\langle \gamma v, w \rangle - \langle v, \gamma w \rangle \in \ker \lambda. \quad (8.1)$$

Now

$$\langle \gamma v, w \rangle = \langle (\gamma v - v) + v, (w - \gamma w) + \gamma w \rangle$$

$$= \langle \gamma v - v, \gamma w \rangle + \langle v, w - \gamma w \rangle + \langle v, \gamma w \rangle$$

$$= \langle v, w \rangle - \langle v, \gamma w \rangle + \langle v, w \rangle - \langle v, \gamma w \rangle + \langle v, \gamma w \rangle$$

$$= 2 \langle v, w \rangle - \langle v, \gamma w \rangle,$$

so

$$\langle \gamma v, w \rangle + \langle v, \gamma w \rangle = 2 \langle v, w \rangle. \quad (8.2)$$

Adding (8.1) and (8.2) yields

$$2 \langle \gamma v - v, w \rangle = 2 \langle \gamma v, w \rangle - 2 \langle v, w \rangle \in \ker \lambda.$$

Since $v$ and $w$ are arbitrary and $\mathcal{P}(V)$ generates $V$, we deduce

$$\langle \gamma v - v, V \rangle = 2 \langle \gamma v - v, V \rangle \subseteq \ker \lambda, \quad \text{for all } v \in \mathcal{P}(V).$$

As $\langle \gamma v - v, V \rangle$ is an ideal of $R$ and $\lambda$ is primitive, we infer

$$\langle \gamma v - v, V \rangle = (0), \quad \text{for all } v \in \mathcal{P}(V).$$
But \( \langle , \rangle \) is non-degenerate, so for all \( v \in \mathcal{P}(V) \) we have \( \gamma v = v \), that is, \( \gamma v = v \). Since \( \mathcal{P}(V) \) generates \( V \), we finally obtain \( \gamma = 1_V \). This completes the proof of the second step and demonstrates the theorem.

9. Principal \( R \) revisited

We now revisit the case in which the ring \( R \) is principal and the module \( V \) is free. As well, at the end of this section, we give an example to show that \( \text{Bot} \) can be badly behaved if \( R \) is not principal.

In [CMS] it was observed that the Weil representations \( W \) of \( \text{Sp}(V) \) were multiplicity free. We shall presently see that this result is a consequence of the refined Mackey decomposition provided by Theorem 4.1 and Proposition 4.2. Moreover, the last two results provide explicit descriptions of the irreducible constituents of \( W \) which extend the results of [CMS] obtained in the case the nilpotency degree of \( m \) is even.

Let \( l \) denote the nilpotency degree of \( m = R\pi \). Recall that \( l > 0 \) since \( R \) is not a field. The residue-class field of \( R \) shall be denoted \( F \). The module \( V \) is assumed to be free of rank \( 2n \). The remaining notation is as in Section 4. The given hypotheses on \( R \) and \( V \) ensure that each element of \( V \) is a scalar multiple of a primitive element. If \( x \) is a fixed primitive element \( V \) then, since \( \text{Sp}(V) \) acts transitively on \( \mathcal{P}(V) \) by Lemma 5.2, a set of representatives of the \( \text{Sp}(V) \)-orbits of \( V \) is given by

\[
\{ t_j = \pi^j x \mid 0 \leq j \leq l \}.
\]

Proposition 3.5 thus allows one to deduce

\[
(\Omega, \Omega)_{\text{Sp}(V)} = l + 1. \tag{9.1}
\]

For a rational number \( r \), let \( \lceil r \rceil \) denote the smallest integer \( m \geq r \) and \( \lfloor r \rfloor \) denote the largest integer \( m \leq r \). Let \( U = m^{\lfloor l/2 \rfloor} V \), which is the unique maximal totally isotropic \( \text{Sp}(V) \)-invariant submodule of \( V \), with \( U^\perp = m^{\lceil l/2 \rceil} V \). We observe that

\[
U^\perp \cong \begin{cases} 0, & \text{if } l \text{ is even;} \\ V/mV & \text{if } l \text{ is odd.} \end{cases}
\]

In both cases the canonical map

\[
\text{Sp}(V) \to \text{Sp}(U^\perp) \to \text{Sp}(U^\perp)
\]

is surjective. The \( \text{Sp}(V) \)-orbits of \( V/U^\perp \) has

\[
T = \{ t_j \mid 0 \leq j \leq \lfloor l/2 \rfloor \}
\]

as a set of representatives, with

\[
G_j = G_{t_j} = \{ g \in \text{Sp}(V) \mid g x = x \mod m^{\lceil l/2 \rceil - j} V \}, \quad 0 \leq j \leq \lfloor l/2 \rfloor.
\]
Let $X$ be the Weil module given by (4.1). We shall write $X_j$ and $Z_j$ in place of $X_{tj}$ and $Z_{tj}$, respectively. Theorem 4.1 provides the $\text{Sp}(V)$-decomposition

$$X \cong \bigoplus_{j=0}^{[l/2]} X_j.$$  

If $j < [l/2]$ then Proposition 4.2 asserts that $X_j$ decomposes as the sum of the two non-zero submodules $X_j^\pm$. The remaining summand $X_{[l/2]}$ affords the representation $W_{[l/2]} = W' \circ r_U^\perp$. In the case $l$ is even, the representation $W'$ has degree 1 since $U = U^\perp$, hence $X_{[l/2]}$ is irreducible. In the case $l$ is odd, the representation $W'$ is essentially a Weil representation defined over the residue-class field $F$ of $R$ (see Section 10). As such the two eigenspaces $X_{[l/2]}^\pm$ are non-zero. In either case we obtain a decomposition of $X$ into a sum of $l + 1$ non-zero submodules. When combined with the identity (9.1), the following result, which first appeared as Theorem 5.4 of [CMS], is obtained.

**Theorem 9.1.** If $0 \leq j < [l/2]$ then $X_j^\pm$ are irreducible $\text{Sp}(V)$-modules. In the case $l$ is even, $X_{[l/2]}$ affords a linear character of $\text{Sp}(V)$ which is trivial except possibly the case $n = 2$ and $F = F_3$. In the case $l$ is odd, $X_{[l/2]}$ affords the inflation of a Weil representation of $\text{Sp}(V/mV)$. In the latter case, $X_{[l/2]}^+$ and $X_{[l/2]}^-$ are irreducible $\text{Sp}(V)$-modules.

The discussion of Section 4 shows that each of the representations $X_j$, $0 \leq j < [l/2]$, are imprimitive. Indeed, if $W_j = W_{tj}$ is the representation of $G_j$ given by (4.3) then we have the following theorem.

**Theorem 9.2.** If $0 \leq j < [l/2]$ then

$$X_j^\pm = \text{ind}_{G_j \times \{1, \iota\}}^{\text{Sp}(V)} Z_j^\pm,$$

where $Z_j^\pm = Z$ affords the extension $W_j^\pm$ of $W_j$ given by

$$W_j^\pm(i): x \mapsto \pm x.$$

If $l$ is even then the module $Z$ has degree 1. As a result, each of the modules $X_j^\pm$, $0 \leq j < [l/2]$, is seen to be monomial. Since $X_{[l/2]}$ has degree 1, we conclude that, in the case $l$ is even, all the irreducible constituents of the Weil module $X$ are monomial, a result first proved in [CMS, Section 6]. On the other hand, Theorem 9.2 is equally applicable to the case of odd $l$, and it provides explicit descriptions of all but two of the irreducible $\text{Sp}(V)$-submodules of $X$ in terms of the representations $S'$ and $W'$. The remaining two representations are essentially the irreducible constituents of a Weil representation defined
over the residue-class field $F$ (see Section 10 for further details), hence are fairly well understood.

We can also apply Clifford theory to the modules $X_j$ for $0 < j < l/2$. Define the function $\delta_j$ on $\text{Sp}(V)$ by

$$\delta_j(\gamma) = \lambda(\langle \gamma t_j, t_j \rangle) = \lambda(\pi^{2j}(yx, x)).$$

This is a linear character of $\Gamma(m^k V)$ if $k \geq [(l - 2j)/2]$, and is the trivial character if $k \geq l - 2j$. Suppose that $[(l - 2j)/2] \leq k < l - 2j$. Define

$$G_{j,k} = \{ g \in \text{Sp}(V) \mid gx \equiv x \mod m^{l-2j-k} V \},$$

$$Z_{j,k} = \text{ind}_{G_j}^{G_{j,k}} Z_j, \quad Z_{j,k}^\pm = \text{ind}_{G_j \times \{1, \iota\}}^{G_{j,k} \times \{1, \iota\}} Z_j^\pm.$$

Note that, in this notation,

$$G_j = G_{j,[(l-2j)/2]}.$$

**Theorem 9.3.** Suppose that $[(l - 2j)/2] \leq k < l - 2j$. Then

1. The character $\delta_j$ is a constituent of the restriction of $X_j$ to $\Gamma(m^k V)$.
2. The inertia group of $\delta_j$, viewed as a linear character of $\Gamma(m^k V)$, is $G_{j,k} \times \{1, \iota\}$.
3. $Z_{j,k}^\pm$ affords an irreducible representation of $G_{j,k} \times \{1, \iota\}$ whose restriction to $\Gamma(m^k V)$ is a multiple of $\delta_j$.
4. $X_j^\pm \cong \text{ind}_{G_{j,k} \times \{1, \iota\}}^{:\text{Sp}(V)} Z_{j,k}^\pm$.

Moreover if $1 \leq k \leq [(l - 2j)/2]$, the restriction of $Z_{j,k}$ to $\Gamma(m^k)$ remains irreducible.

**Proof.** The proof of (1) is similar to one of Proposition 6.5, and of (2), to Theorem 6.6. Mackey theory of Section 4 gives us

$$X_j \cong \text{ind}_{G_j}^{\text{Sp}(V)} Z_j.$$

Transitivity of induction yields

$$X_j \cong \text{ind}_{G_{j,k}}^{\text{Sp}(V)} Z_{j,k},$$

from which (3) and (4) follow. The proof of the last statement is analogous to the proof of Theorem 7.1. □

Theorem 5.3 of [CMS] states that $X(m^l - 1) V$ affords a Weil representation of the group $\text{Sp}(V)/\Gamma(m^l - 2 V)$, which is isomorphic to the symplectic group $\text{Sp}(V/m^l - 2 V)$. This can
be explained in the present context as follows. Let $U = \mathfrak{m}^{l-1}V$; then $U^\perp = \mathfrak{m}V$. The composition

$$\text{Sp}(V) \to \text{Sp}(U^\perp) \to \text{Sp}(U^\perp / U)$$

is surjective, and $U^\perp / U = \mathfrak{m}V / \mathfrak{m}^{l-1}V \cong V / \mathfrak{m}^{l-2}V$.

If $R$ is not principal, then the structure of $\text{Bot}$ can be quite complicated, as shown by the following example.

**Example 9.4.** Let $F$ be a finite field of odd characteristic, and set

$$R = F[S, T] / (S^2, T^2) = F[s, t].$$

$R$ is a finite local ring with maximal ideal $\mathfrak{m} = (s, t)$ and unique minimal ideal $\mathfrak{s} = (st)$. If $\lambda_0$ is a non-trivial linear character of $F^+$ then the map

$$a + bs + ct + dst \mapsto \lambda_0(d), \quad a, b, c, d \in F,$$

is a primitive linear character $\lambda$ of $R$.

Let $V$ be a free $R$-module of rank $2n$ admitting a non-degenerate alternating bilinear form. We observe that the quotient module $V' = V / \mathfrak{m}$ is naturally endowed with a non-degenerate alternating $F$-alternating form and the canonical projection $V \to V'$ induces a surjective homomorphism $\text{Sp}(V) \to \text{Sp}(V')$.

The module $V$ has precisely two maximal totally isotropic $\text{Sp}(V)$-invariant submodules, namely $U_1 = sV$ and $U_2 = tV$ with $U_i^\perp = U_i$ in both cases. We choose to work with $U_1$.

Since $U_1^\perp = U_1$, the representation $S'$ is the linear character

$$(r, u) \mapsto \lambda(r), \quad (r, u) \in H(U_1),$$

while $W'$ is a linear character of $\text{Sp}(U^\perp)$, which we shall assume is trivial. If $x$ is a primitive element of $V$ then

$$T = \{x, tx, 0\}$$

is a set of representatives of the $\text{Sp}(V)$-orbits of $V / U_1$. If $X$ is the Weil module constructed from the pair $(S', W')$ then Theorem 4.1 provides the decomposition

$$X = X_x \oplus X_{tx} \oplus X_0.$$

The decomposition of $X_x$ is handled in complete generality in Section 5, while $X_0$ is readily seen to afford the trivial representation of $\text{Sp}(V)$.

It remains to consider the summand $X_{tx}$. In light of the descriptions of $S'$ and $W'$ given above and the fact that $t^2 = 0$, Eq. (4.3) shows that $W_{tx}$ is the trivial representation of $G_{tx}$. We deduce that $X_{tx}$ is the permutation module associated with the permutation...
representation of \( \text{Sp}(V) \) on the coset space \( \text{Sp}(V)/Gtx \). The latter set is readily seen to be \( \text{Sp}(V) \)-isomorphic to the set \( V' \setminus \{0\} \) via the map

\[
g \mapsto gx + mV, \quad g \in \text{Sp}(V).
\]

We deduce that the representation of \( \text{Sp}(V) \) afforded by \( Xtx \) is equivalent to the inflation of the permutation representation of \( \text{Sp}(V') \) acting on the non-zero elements of \( V' \).

A calculation of the permutation rank of the last representation yields

\[
(\Omega_{Xtx}, \Omega_{Xtx})_{\text{Sp}(V)} = \begin{cases} 
2(q - 1), & \text{if } n = 1; \\
2q - 1, & \text{if } n > 0.
\end{cases}
\]

In the case \( n = 1 \), the module \( Xtx \) admits \( (q + 5)/2 \) non-equivalent irreducible submodules. Of these all but four occur with multiplicity 2, with the others occurring with multiplicity 1. In the case \( n > 1 \), the module \( Xtx \) has \( (q + 7)/2 \) non-equivalent irreducible submodules. Here \( (q - 3)/2 \) occur with multiplicity 2 and the remaining 5 each occur with multiplicity 1 (for further details in the case \( n > 1 \), see [ZS, Section 4]).

In case of a free module defined over a principal ring, it was observed in [CMS] that the Weil representations are multiplicity free. Moreover, the number of irreducible constituents depended only on the nilpotency degree of the maximal ideal; in particular, they were independent of the order of the residue-class field. The preceding example shows that the Weil representations defined over general local rings admitting primitive linear characters fail to enjoy the stated properties.

10. Reduction to \( R \) a field

In this section we observe a connection between the classical theory of Weil representations defined over finite fields and the more general theory we have developed in this paper.

The notation and hypotheses are as introduced in Section 2; it is assumed here that the form \( \langle , \rangle \) is non-degenerate. Furthermore, \( F \) shall be used to denote the residue-class field of \( R \), and the unique simple ideal of \( R \) shall be denoted \( s \). In Section 4 we provided a construction of a Weil module \( X \) for \( \text{Sp}(V) \) that started from a totally isotropic submodule \( U \), a Weil representation \( W' \) of \( \text{Sp}(U^\perp) \), and the Schrödinger representation of \( H(U^\perp) \). Moreover, Theorem 4.1 asserts the existence of a decomposition of \( X \) as a sum of submodules which are readily constructed from the pair \( (W', S') \). By definition, \( S' \) is the inflation of the Schrödinger representation \( S \) of \( H(U^\perp) \), while the remarks proceeding Definition 3.4 allow us to assume that \( W' \) is the inflation of a Weil representation \( \overline{W} \) of \( \text{Sp}(U^\perp) \). The summands in the decomposition alluded to above can thus be expressed in terms of the pair \( (\overline{W}, S) \).

We shall presently show that if \( U \) is maximal then \( \overline{S} \) and \( \overline{W} \) are essentially Schrödinger and Weil representations, respectively, defined over the residue-class field \( F \). Our starting point is the following result.
Lemma 10.1. If $U$ is a maximal totally isotropic $\text{Sp}(V)$-invariant submodule of $V$ then

1. $m \subseteq (U : U^\perp) = \{r \in R \mid rU^\perp \subseteq U\}$; and
2. $(U^\perp, U^\perp) \subseteq s$.

Proof. If (1) fails to hold, let $j$ be a minimal proper $R$-ideal with $j \not\subseteq (U : U^\perp)$ and consider the submodule

$$U' = U + jU^\perp.$$

Observing that $j$ is non-zero, Nakayama’s Lemma asserts that $j^2$ is properly contained in $j$, whence $j^2U^\perp \subseteq U$. When combined with the bilinearity of $(,)$, the last inclusion allows us to deduce that $U'$ is totally isotropic. Furthermore, $U'$ is $\text{Sp}(V)$-invariant, being the sum of $\text{Sp}(V)$-invariant modules, and properly contains $U$ by definition of $j$. This contradicts the maximality of $U$.

In light of (1), the bilinearity of $(,)\theta$ shows that

$$m(U^\perp, U^\perp) = \langle mU^\perp, U^\perp \rangle \subseteq \langle U, U^\perp \rangle = 0,$$

whence (2) follows from the uniqueness of $s$. ✷

Since $U = U^{\perp\perp}$ by Lemma 2.1, the first part of the preceding lemma shows that $U^\perp$ can be viewed as a vector space over the residue-class field $F$. Furthermore, if $\theta : F \to s$ is a fixed $R$-isomorphism then the second part of the same lemma shows that

$$(,)_\theta = \theta^{-1} \circ (,)$$

is a well-defined non-degenerate alternating $F$-bilinear form on $U^\perp$. The symplectic groups constructed from the pairs $(U^\perp, (,))$ and $(U^\perp, (,)\theta)$ are equal when viewed as subgroups of the automorphism group of $U^\perp$, and as for such the notation $\text{Sp}(U^\perp)$ will be used for both. We can also form the Heisenberg groups $H(U^\perp, R)$ and $H(U^\perp, F)$. The homomorphism $\theta$ induces an embedding

$$\theta_\sigma : H(U^\perp, F) \to H(U^\perp, R),$$

$$(x, w) \mapsto (\theta(x), w).$$

The map $\theta_\sigma$ commutes with the actions of $\text{Sp}(U^\perp)$.

Let $\overline{S}$ and $\overline{W}$ be as above. We consider first the representation $\overline{S} \circ \theta_\sigma$. A calculation of its character shows that $\overline{S} \circ \theta_\sigma$ is the Schrödinger representation of $H(U^\perp, F)$ of type $\lambda_\sigma = \lambda \circ \theta$, where $\lambda$ is the type of $\overline{S}$. Note that $\lambda_\sigma$ is a non-trivial linear character of $F^+$, since $s$ is a non-zero $R$-ideal and $\lambda$ is primitive. Thus $\overline{S}$ is seen to be a rather simple
extension of a Schrödinger representation defined over the residue field. Moreover, the fact that \( \theta_* \) commutes with the action of \( \text{Sp}(U^\perp) \) ensures that

\[
\bar{W}(g) \circ \theta_*(h) \bar{W}(g)^{-1} = \bar{S} \circ \theta_*(gh), \quad g \in \text{Sp}(U^\perp), \quad h \in H(U^\perp, F).
\]

It follows that \( \bar{W} \) can equally be described as a Weil representation of \( \text{Sp}(U^\perp) \) of type \( \lambda_\theta \).

In light of the preceding discussion, we conclude that the Weil module \( X \) admits a decomposition as a sum of submodules which are easily constructed from the Schrödinger and Weil representations defined over the residue-class field \( F \) of \( R \). As noted in the introduction, the Schrödinger and Weil representations defined over finite fields are fairly well-understood, so the decomposition of \( X \) obtained is concrete. As such it may be possible to exploit this decomposition in the investigation of the irreducible components of \( X \).

References


