stochastic processes and their

www.elsevier.com/locate/spa

Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces

Zdzisław Brzeźniak[a, *], Dariusz Gątarek[b]

[a] Department of Pure Mathematics, The University of Hull, Hull HU6 7RX, UK

[b] Systems Research Institute, 01-447 Warszawa, Newelska 6, Poland

Received 30 March 1998; received in revised form 19 March 1999; accepted 26 March 1999

Abstract

In this paper we study the existence and uniqueness of weak solutions of stochastic differential equations on Banach spaces. We also study the existence of invariant measures for the corresponding Markovian semigroups. Our main tool is the factorization of stochastic convolutions. We close the paper with some examples. © 1999 Elsevier Science B.V. All rights reserved.

MSC: Primary 60H15; 60G60; Secondary 60J25

Keywords: (Nonlinear) stochastic partial differential equations; Stochastic integration in Banach spaces; Dissipative mappings; Martingale solutions; Feller processes; Invariant measures

1. Introduction

The aim of this paper is twofold. Firstly, we study weak (martingale) solutions of stochastic evolution equations in Banach spaces. Secondly, we investigate existence of invariant measures for such equations. The general motivation for studying weak rather than strong solutions of stochastic equations in finite dimensions is that existence of weak solutions requires smaller degree of regularity of the coefficients than existence of strong solutions, see Ikeda and Watanabe (1981). This remains valid for equations in infinite-dimensional spaces. An important case, where there it is more natural to study martingale (rather than strong) solutions are stochastic Navier–Stokes equations, see Capiński and Cutland (1991) and Flandoli and Gątarek (1995), and references therein.

An extensive study of martingale solutions of stochastic evolution equations was initiated in the thesis of Viot (1976) in the Hilbert space setting. Some results of Viot (1976) were clarified and generalized in Metivier (1988). The paper by Dettweiler (1992) was the first (to our knowledge) to deal with martingale problems on Banach spaces. The model studied in Dettweiler (1992) is different from ours in that we allow unbounded coefficients.

* Corresponding author.

E-mail addresses: z.brzezniak@maths.hull.ac.uk (Z. Brzeźniak), gatarek@ibspan.waw.pl (D. Gątarek)

PII: S0304-4149(99)00034-4

In a vast literature on existence of invariant measures for stochastic evolution equations let us point out a paper by Manthey and Maslowski (1992) (see also references therein), where the authors study a problem similar to ours but with more stringent conditions. In particular, we do not assume that the diffusion coefficient is constant nor that the main linear part of the drift operator is symmetric. However, we do not study ergodic properties of the solutions as is done in Manthey and Maslowski (1992). It would be of some interest to study these in the framework of the present paper. One of the basic ingredients of our approach to existence of invariant measures is compactness which allows us to use a general scheme due to Krylov–Bogoluobov. A similar technique to the question of existence of an invariant measure for a single reaction–diffusion equation with Dirichlet boundary conditions was applied by Da Prato and Pardoux (1995). Let us point out that the unbounded linear part of the drift is the Laplacian with Dirichlet boundary condition. Our approach is more general as it allows to consider various problems (even systems) with mixed boundary conditions without making use of the specific forms of the fundamental solutions. Even when restricted to the case of Da Prato and Pardoux (1995) our methods of obtaining a priori bounds on moments of the solutions based on Burkholder inequality in Banach spaces is more transparent and elegant than the other one. Still another paper is the one by Mueller (1993), where the author uses in an ingenious way a coupling method for proving existence of invariant measures for a heat equation with space-time noise. Large deviation for an invariant measure in the case of globally Lipschitz coefficient is studied by Sowers (1992).

Related questions for infinite-dimensional stochastic differential equations but of completely different type have been studied by Albeverio and Röckner (1991) in the framework of Dirichlet forms.

We use two different techniques. On the one hand, we study some processes with values in Banach spaces by using various properties of dissipative mappings. On the other, we study other processes with values in Banach spaces by making use of Burkholder inequality. Thus, for the latter, we have to consider special types of Banach spaces: M-type 2 or (as they are also called) 2-uniformly smooth. In this paper we follow the general framework of integration in M-type 2 Banach spaces as introduced in Brzeźniak (1995, 1997) together with the factorization method as in Da Prato et al. (1987), Da Prato and Zabczyk (1992a) and Gątarek and Gołdys (1994). These tools seem to be very effective for our purpose. Another important case, of dissipative equations, is investigated in the present publication. Factorization approach to the problem of existence of martingale solutions of Hilbert-space-valued diffusions was introduced in Gątarek and Gołdys (1994): see also some applications: in control theory (Gątarek and Sobczyk, 1994) and for stochastic Navier–Stokes equation, see Capiński and Gątarek (1994). Stochastic reaction diffusion equations in L^p spaces are also studied by Peszat (1995).

Let us briefly present the content of our paper. In Section 2 we study the factorization operator in Banach spaces which proves to be a fractional power of some abstract parabolic operator, see also Brzeźniak (1996). We prove general smoothing and compactness properties of the operator in question. In Section 3 we prove general results on stochastic convolutions in M-type 2 Banach spaces. These results are applied to some examples in Section 3.2. The main results of the paper are in Section 4. We prove existence of weak solutions of nonlinear stochastic evolution equations in Banach spaces under general conditions: continuous and bounded diffusion coefficient and continuous drift coefficient with one-sided growth condition. This covers important examples of reaction–diffusion equations. In Section 5 we prove existence of invariant measures under the same assumptions as in the previous section but with an extra condition on the linear part: uniform asymptotic stability of the semigroup involved. In the last section, which is in some sense a continuation of Section 3.2, we show how the previous abstract results can be applied to a general stochastic reaction–diffusion equation with space-time white noise. In this respect our results are complimentary to those obtained recently by Bally et al. (1994). We close the section by showing how our results lead to solvability of stochastic Ginzburg–Landau equations with nonconstant diffusion and non-Lipschitz force. This generalizes the results from Funaki (1989) in the bounded domain case.

In Brzeźniak and Peszat (1999a,b) we continue the line of research developed here by studying SPDEs, contrary to the present paper, with all coefficients time dependent.

2. The parabolic operator Λ_T

Let us begin with a list of assumptions that will be frequently used throughout this and later sections. Whenever we use any of them this will be specifically written.

(H1a) X is an UMD Banach space.

(H1b) X is a type 2 Banach space.

(H2a) A is a positive operator in X, i.e. a densely defined and closed operator for which there exists $M > 0$ such that for $\lambda > 0$,

$$||(A+\lambda)^{-1}|| \leqslant \frac{M}{1+\lambda}, \quad \lambda > 0.$$

(H2b) $-A$ is a generator of an analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on X.

(H3) There exist positive constants K and ϑ satisfying

$$\vartheta < \frac{\pi}{2} \tag{2.1}$$

such that

$$||A^{is}|| \leqslant K e^{\vartheta|s|}, \quad s \in \mathbb{R}. \tag{2.2}$$

Digression 1. A Banach space X is an UMD space iff there exist $\beta > 0$ and $p \in (1,\infty)$ such that for any X-valued martingale difference sequence $\eta = \{\eta_j\}_{j=1}^n$ and for any $\varepsilon \in \{-1,1\}^n$

$$\left\{\mathbb{E}\left|\sum_{i=1}^n \varepsilon_i \eta_i\right|_X^p\right\}^{1/p} \leqslant \beta \left\{\mathbb{E}\left|\sum_{i=1}^n \eta_i\right|_X^p\right\}^{1/p}. \tag{2.3}$$

The smallest constant β for which (2.3) holds will be denoted by $\beta_p(X)$. This definition is p independent, see Burkholder (1986).

A Banach space X is of type 2 iff there exists a constant $K > 0$ such that for any $x_1,\dots,x_n \in X$ and any symmetric i.i.d. random variables $\sigma_1,\dots,\sigma_n : \Omega \to \{-1,1\}$ the following holds:

$$\left\{\mathbb{E}\left|\sum_{i=1}^{n}\sigma_i x_i\right|^2\right\}^{1/2} \leqslant K\left\{\sum_{i=1}^{n}|x_i|^2\right\}^{1/2}.$$

The smallest number K for which the above holds is denoted by $K_2(X)$.

On the other hand, using the Kahane inequality which asserts that for any $r\in(0,\infty)$ there exist numbers $A_r, B_r>0$ such that for any Banach space Z, for all $z_1,\dots,z_n\in Z$ and for all $\sigma_1,\dots,\sigma_n:\Omega\to\{-1,1\}$ symmetric i.i.d. random variables

$$A_r\left\{\mathbb{E}\left|\sum_i\sigma_i z_i\right|^2\right\}^{1/2} \leqslant \left\{\mathbb{E}\left|\sum_i\sigma_i z_i\right|^r\right\}^{1/r} \leqslant B_r\left\{\mathbb{E}\left|\sum_i\sigma_i z_i\right|^2\right\}^{1/2},$$

one sees that X is of type 2 iff for any (some) $r\in(0,\infty)$ there exists $K'>0$ such that

$$\left\{\mathbb{E}\left|\sum_i\sigma_i x_i\right|^r\right\}^{1/r} \leqslant K'\left\{\sum_i|x_i|^2\right\}^{1/2} \tag{2.4}$$

for any $x_1,\dots,x_n\in X$ and any $\sigma_1,\dots,\sigma_n:\Omega\to\{-1,1\}$ symmetric i.i.d. random variables. The smallest constant K' is denoted by $K_{2,r}(X)$.

In this context, let us note that $K_{2,r}(X)\leqslant K_2(X)B_r$ and $K_2(X)=K_{2,2}(X)$.

Remark 2.1. If a linear operator A in Banach space X is positive then $-A$ is the generator of a $\mathscr{C}_0$ semigroup in X and one defines (see Triebel, 1978) the fractional powers A^α and $A^{-\alpha}$, $\alpha\in(0,1)$ of A as closures of operators given by the formulas

$$A^\alpha x=\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{\alpha-1}A(\lambda I+A)^{-1}x\,\mathrm{d}\lambda,\quad x\in D(A), \tag{2.5}$$

$$A^{-\alpha}x=\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty t^{-\alpha}(\lambda I+A)^{-1}x\,\mathrm{d}\lambda,\quad x\in X. \tag{2.6}$$

Note that $A^{-\alpha}$ is a bounded operator in X and the function $(0,1)\ni\alpha\mapsto A^{-\alpha}\in\mathscr{L}(X)$ extends to an analytic function A^{-z} on the open half-plane $\mathbb{C}^+=\{z\in\mathbb{C}\colon\ \mathrm{Re}\,z>0\}$. Moreover, if (for example) $-1<\mathrm{Re}\,z<1$, then for $x\in D(A)$,

$$A^{-z}x=\frac{\Gamma(2)}{\Gamma(1-z)\Gamma(1+z)}\int_0^\infty\lambda^{-z}A(A+\lambda I)^{-2}x\,\mathrm{d}\lambda.$$

Now the condition (H3) should be more clearly explained. It means, that the function A^{-z} has an extension to the imaginary line $\mathrm{i}\mathbb{R}$ and the inequality (2.2) holds (with ϑ satisfying (2.1)).

Most of the content of this section follows Brzeźniak (1997), but our approach to the compactness property in Theorem 2.6 is new. A result of this type has been first discovered and applied in Gątarek and Gołdys (1994), see Proposition 1 therein.

Let us note that if X is a UMD and type 2 Banach space, i.e. the assumptions (H1) are satisfied, then X is also an M-type 2 Banach space, see Brzeźniak (1996). It is known, see Prüss and Sohr (1990), that under the condition (H3), $-A$ is the generator of a uniformly bounded (!) analytic semigroup.

For fixed $q \in (1,\infty)$, $T \in (0,\infty]$ and a Banach space X, $H^{1,q}(0,T;X)$ is the Banach space of (classes of) functions $u \in L^q(0,T;X)$ whose weak derivative u' belongs to $L^q(0,T;X)$ as well. By $H_0^{1,q}(0,T;X)$ we will denote, not in agreement with the standard notation, the closure in $H^{1,q}(0,T;X)$ of the space[1] $\{u \in \mathscr{C}^\infty([0,T];X)\colon u(0)=0\}$. It is known that $H_0^{1,q}(0,T;X)$ equals to the subspace of $H^{1,q}(0,T;X)$ consisting of such u with $u(0)=0$. Note that due to the Sobolev imbedding theorem $u(0)$ is well defined.

For $\alpha \in (0,1)$ the Sobolev spaces $H^{\alpha,q}(0,T;X)$ and $H_0^{\alpha,q}(0,T;X)$ are defined by means of complex interpolation

$$H^{1,q}(0,T;X) = [L^q(0,T;X), H^{1,q}(0,T;X)]_\alpha, \tag{2.7}$$

$$H_0^{\alpha,q}(0,T;X) = [L^q(0,T;X), H_0^{1,q}(0,T;X)]_\alpha. \tag{2.8}$$

It is known that for $\alpha > 1/q$ the latter is equal to the space of all $u \in H^{\alpha,q}(0,T;X)$ for which $u(0)=0$ while for $\alpha \in (0,1/q)$ it equals $H^{\alpha,q}(0,T;X)$.

With q, T and X as above we set

$$B_T u = u', \quad u \in D(B_T), \tag{2.9}$$

$$D(B_T) = H_0^{1,q}(0,T;X). \tag{2.10}$$

It is also known, see Dore and Venni (1987) and Brzeźniak (1996), that the operator $-B_T$ generates a $\mathscr{C}_0$-semigroup $\{S(t)\}_{t\geqslant 0}$ on the Banach space $L^q(0,T;X) =: Y_T$

$$[S(t)u](r) = \begin{cases} u(r-t) & \text{if } 0\leqslant t\leqslant r, \\ 0 & \text{if } 0\leqslant r < t, \end{cases} \tag{2.11}$$

for $r \in [0,T]$ and $u \in L^q(0,T;X)$.

Note also that the original norm on $D(B_T)$ (or on $H_0^{1,q}(0,T;X)$) (i.e. the one inherited from $H^{1,q}(0,T;X)$) is equivalent to the following one:

$$||u||^q = \int_0^T |u'(t)|^q \,\mathrm{d}t. \tag{2.12}$$

The spectral properties of the operators B_T for $T=\infty$ and T finite differs substantially. While the spectrum $\sigma(B_\infty)$ of B_∞ equals to the closed left half-plane $\{z \in \mathbb{C}\colon \operatorname{Re} z \leqslant 0\}$ the spectrum $\sigma(B_T)$ of B_T for T finite is empty. Moreover, $B_\infty + vI$ is positive for any $v\geqslant 0$ (or $\operatorname{Re} v \geqslant 0$ in the complex case) and, for $T<\infty$, $B_T + vI$ is positive for any $v \in \mathbb{R}$ (or $v \in \mathbb{C}$ in the complex case), see also below.

Finally let us recall, see Dore and Venni (1987) and Giga and Sohr (1991), that if X is a UMD Banach space and T is finite, B_T satisfies a condition similar to (2.2) but with the constant ϑ equal to $\pi/2$, i.e.

$$||B_T^{is}|| \leqslant C_T(1+s^2)\mathrm{e}^{\vartheta|s|}, \quad s \in \mathbb{R}. \tag{2.13}$$

However, one can easily show that the same holds for $B_T - vI$ for any $v \in \mathbb{C}$. The key observation is the following formula:

$$(B_T - vI)^{-1} = J_v B_T^{-1} J_v^{-1},$$

[1] For $T=\infty$ we consider $u : [0,\infty) \to X$.

where J_v is a linear isomorphism of $L^q(0,T;X)$ defined by $(J_v g)(s) = e^{vs}g(s)$. Therefore, with $R(B)$ denoting the range of operator B, we have, see Brzeźniak (1996),

$$R((B_T - vI)^{-\alpha}) := H_0^{\alpha,q}(0,T;X).$$

Define now a linear operator $\mathscr{A}_T$ by the formula

$$\begin{aligned} D(\mathscr{A}_T) &= \{u \in L^q(0,T;X) \text{ s.th. } Au \in L^q(0,T;X)\}, \\ \mathscr{A}_T u &:= \{[0,T] \ni t \mapsto A(u(t)) \in X\}. \end{aligned} \tag{2.14}$$

It is then easy to show, see Dore and Venni (1987), that if $A + vI$ satisfies the conditions (H2) and (H3) then $\mathscr{A}_T + vI$ satisfies them as well. Define finally the operator Λ_T by

$$\Lambda_T := B_T + \mathscr{A}_T, \tag{2.15}$$

$$D(\Lambda_T) := D(B_T) \cap D(\mathscr{A}_T). \tag{2.16}$$

If X is a UMD Banach space and $A + vI$, for some $v \geqslant 0$, satisfies the conditions (H2) and (H3) then, since $\Lambda_T = B_T - vI + \mathscr{A}_T + vI$, by Dore and Venni (1987) and Giga and Sohr (1991), Λ_T is a positive operator. In particular, Λ_T has a bounded inverse. The domain $D(\Lambda_T)$ of Λ_T endowed with a 'graph' norm

$$||u|| = \left\{ \int_0^T |u'(s)|^q \, ds + \int_0^T |Au(s)|^q \, ds \right\}^{1/p} \tag{2.17}$$

is a Banach space. This space will be frequently denoted by $H_0^{1,q}(0,T;X,A)$. Moreover, Λ_T satisfies the condition (H3). Therefore, compare with Brzeźniak (1996), we have

$$[L^q(0,T;X);\ D(\Lambda_T)]_\alpha = D(\Lambda_T^\alpha), \quad 0 < \alpha < 1. \tag{2.18}$$

We begin exposition of our results by presenting an explicit formula for the fractional power of the operator Λ_T, see Brzeźniak (1996) for the proof.

Proposition 2.2. *Assume that the conditions* (H1a), (H2) *are satisfied. Assume also that for some* $v \geqslant 0$, $A + vI$ *satisfies the condition* (H3).

Then, for $0 < \alpha \leqslant 1$, $\Lambda_T^{-\alpha}$ *is a bounded linear operator in* $L^q(0,T;X)$, *and for* $0 < \alpha < 1$,

$$\begin{aligned} (\Lambda_T^{-\alpha} f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-(t-s)A} f(s) \, ds, \\ & \qquad t \in (0,T), \ f \in L^q(0,T;X). \end{aligned} \tag{2.19}$$

Remark 2.3. The assumptions (H1b) and (H3) in Proposition 2.2 are to ensure that the operator Λ_T is closed. In fact, suppose that X is a Banach space and $-A$ is a generator of a $\mathscr{C}_0$ semigroup e^{-tA}, $t \geqslant 0$ on X. Let $P(t)$, $t \geqslant 0$ be a $\mathscr{C}_0$ semigroup on $Y_T = L^q(0,T;X)$ defined by $[P(t)f](r) = e^{-tA}(f(r))$, $r \in [0,T]$, $t \geqslant 0$. Then the semigroups $S(t)$ and $P(t)$ commute and the generator C of the $\mathscr{C}_0$ semigroup $Q(t) = P(t)S(t)$ is a positive operator and $C^{-\alpha} = \Lambda_{T,\alpha}$. Here the linear operator $\Lambda_{T,\alpha}$ is defined by the RHS of formula (2.19). Using just the Young inequality one can prove directly that $\Lambda_{T,\alpha}$ is bounded map in $L^q(0,T;X)$. With this notation, Proposition 2.2 states that if the conditions (H1a) and (H3) are satisfied, then $\Lambda_T^{-\alpha} = \Lambda_{T,\alpha}$. See Carroll (1999) for more details.

One can easily mimic the proof of Lemma 2 from Da Prato et al. (1987) (set up in a Hilbertian framework) to obtain the following generalization of that result.

Lemma 2.4. *Assume a Banach space X and a linear operator A satisfy the condition* (H2). *Suppose that the positive numbers α, β, δ satisfy*

$$0 < \beta < \alpha - \frac{1}{q} + \gamma - \delta. \tag{2.20}$$

Then, if $T \in (0,\infty)$ and $f \in L^q(0,T;D(A^\gamma))$, the function $u = \Lambda_{T,\alpha} f$ satisfies

$$u \in \mathscr{C}^\beta(0,T;D(A^\delta)). \tag{2.21}$$

Moreover, $\Lambda_{T,\alpha}$ is a bounded map in the above spaces.

If $T = \infty$ and the semigroup e^{-tA} is exponentially bounded, i.e. for some $a > 0$, $C > 0$

$$|e^{-tA}| \leqslant C e^{-at}, \quad t \geqslant 0, \tag{2.22}$$

then for any $f \in L^q(0,\infty;D(A^\gamma))$ the function $u = \Lambda_{\infty,\alpha} f$ belongs to $\mathscr{C}_b^\beta(0,\infty;D(A^\delta))$. Moreover, $\Lambda_{\infty,\alpha}$ is a bounded map in the above spaces.

Remark 2.5. For a Banach space Y, $\mathscr{C}_b^\beta(0,\infty;Y)$ denotes a set of all continuous and bounded functions $u : [0,\infty) \to Y$ such that

$$||u||_{\mathscr{C}^\beta(0,\infty;Y)} := \sup_{t \geqslant 0} |u(t)| + \sup_{0 \leqslant s < t < \infty} \frac{|u(t) - u(s)|}{|t-s|^\beta} \tag{2.23}$$

is finite. $\mathscr{C}_b^\beta(0,\infty;Y)$ endowed with a norm $||\cdot||_{\mathscr{C}^\beta(0,\infty;\,Y)}$ is a Banach space.

Our main result in this section is the following.

Theorem 2.6. *Assume that X is an UMD Banach space and a operator A satisfying the condition* (H2) *is such that $A + \nu I$, for some $\nu \geqslant 0$, satisfies* (H3). *We suppose also that $(A+\nu I)^{-1}$ is a compact operator in X (i.e. the imbedding $D(A) = D(A+\nu I) \hookrightarrow X$ is compact). Then, for any finite T and $\alpha \in (0,1]$, the fractional power operator $\Lambda_T^{-\alpha} : L^q(0,T;X) \to L^q(0,T;X)$ is compact.*

Proof. We begin with $\alpha = 1$. The compactness of Λ_T^{-1} is equivalent to compactness of the imbedding $D(\Lambda_T) \hookrightarrow L^q(0,T;X)$. Since X, as being an UMD Banach space is also reflexive, the last compactness follows directly from Theorem 2.1 in Chapter III of Temam (1977).

To prove compactness of $\Lambda_T^{-\alpha}$ for $\alpha \in (0,1)$ one can follow many different roots, we choose the one based on interpolation. As above $\Lambda_T^{-\alpha}$ is compact iff the imbedding $D(\Lambda_T^\alpha) \hookrightarrow Y_T$ is compact. Since $Y_T = L^q(0,T;X)$ is an UMD Banach space (see Burkholder, 1986) the latter is compact as follows from Theorem 9 in Cwikel and Kalton (1993) by using (2.18) and compactness of the imbedding $D(\Lambda_T) \hookrightarrow Y_T$. □

Remark 2.7. (i) The second part of the above proof can be summarized by saying that if Λ_T^{-1} is compact then also $\Lambda_T^{-\alpha}$ is compact for $\alpha \in (0,1)$.

(ii) The conditions (H1a) and (H3) simplify many technical arguments and make the proofs more transparent. However, if one simply assumes that X is any separable Banach space and A satisfies the condition (H2) with A^{-1} being compact, then the operator $\Lambda_{T,\alpha}$, see Remark 2.3, is compact. For a proof of a related result in a Hilbert space framework, see Gątarek and Gołdys (1994, Proposition 1).

Theorem 2.6 in conjunction with Lemma 2.4 yields the following.

Corollary 2.8. *Supposing that the assumptions of Theorem* 2.6 *are satisfied and that the nonnegative numbers* α,β,δ *satisfy the following condition*:

$$0\leqslant\beta+\delta<\alpha-\frac{1}{q}, \tag{2.24}$$

$\Lambda_T^{-\alpha}$ *is a compact map from* $L^q(0,T;X)$ *into* $\mathscr{C}^\beta(0,T;D(A^\delta))$. *In particular*, *if* $\alpha>1/q$, *the map* $\Lambda_T^{-\alpha}$: $L^q(0,T;X)\to\mathscr{C}(0,T;X)$ *is compact*.

Proof. Let $\alpha'\in(0,\alpha)$ be such that the condition (2.24) holds with α being replaced by α'. Then, by Lemma 2.4 and Proposition 2.2, $\Lambda_T^{-\alpha'}$ is a bounded linear map from $L^q(0,T;X)$ into $\mathscr{C}^\beta(0,T;D(A^\delta))$. On the other hand, as $\alpha-\alpha'>0$, by Theorem 2.6, $\Lambda_T^{-(\alpha-\alpha')}$ is compact in $L^q(0,T;X)$. This together with the semigroup property $\Lambda_T^{-\alpha}=\Lambda_T^{-\alpha'}\Lambda_T^{-(\alpha-\alpha')}$ concludes the proof. □

3. Stochastic preliminaries

3.1. Regularity properties of stochastic convolution

Let us begin with an assumption that we will assume from now on in the whole paper.

(H4) H is a separable Hilbert space, and $W(t)$, $t\geqslant0$ is an H-cylindrical Wiener process on $(\Omega,\mathscr{F},\mathbb{P})$ with respect to the filtration $(\mathscr{F}_t)_{t\geqslant0}$.

Remark 3.1. $W(t)$, $t\geqslant0$ is an H-cylindrical Wiener process on $(\Omega,\mathscr{F},\mathbb{P})$ with respect to the filtration $(\mathscr{F}_t)_{t\geqslant0}$ iff, see Da Prato and Zabczyk (1992a) and Brzeźniak (1997) or Brzeźniak and Peszat (1999a), it is a family of bounded linear operators acting from H into $L^2(\Omega,\mathscr{F},\mathbb{P})$ such that

(i) for all $t\geqslant0$, $h_1,h_2\in H$, $\mathbb{E}W(t)h_1W(t)h_2=t\langle h_1,h_2\rangle_H$,
(ii) for each $h\in H$, $W(t)h$, $t\geqslant0$ is a real-valued Wiener process adapted to the family $(\mathscr{F}_t)$.

Suppose that E is a real separable Banach space such that H is densily and continuously imbedded into H. Denote by $i:H\hookrightarrow E$ the natural imbedding. If $i:H\hookrightarrow E$ is an abstract Wiener space (AWS), see Kuo (1975), and $w(t)$, $t\geqslant0$, is the canonical E-valued Wiener process on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, then one can define a family of linear operators from E^* into $L^2(\Omega,\mathscr{F},\mathbb{P})$,

$$W(t)h=\langle h,w(t)\rangle, \quad h\in E^*,\ t\geqslant 0. \tag{3.1}$$

Since $W(t)$ is a continuous linear map from E^* with topology induced by H (recall that $E^*\subseteq H^*\cong H\subseteq E$) $W(t)$ extends in a unique way to a linear bounded map from H into $L^2(\Omega,\mathscr{F},\mathbb{P})$ such that the conditions (i) and (ii) above are satisfied.

Conversely, if $W(t)$, $t\geqslant 0$, is an H-cylindrical Wiener process with respect to the filtration $(\mathscr{F}_t)_{t\geqslant 0}$ and $i:H\to E$ is an AWS then there exists an E-valued Wiener process $w(t)$, $t\geqslant 0$ such that the condition (3.1) is satisfied. In fact, it is enough to take a continuous version of the process $\tilde{w}(t):=\sum_k W(t)(e_k)e_k$, $t\geqslant 0$, for some orthonormal basis $\{e_k\}_k$ of H. In Brzeźniak (1997) we studied Itô integrals with respect to the canonical E-valued Wiener process which, as we observed therein, played only an auxiliary role. In this paper we consider cylindrical Wiener processes and Itô integrals with respect to them. This is motivated by many reasons. Firstly, it is more canonical, as we avoid using any auxiliary objects.

Secondly, as explained above, any cylindrical Wiener process generates a true Wiener process. The third reason is related to an Itô integral. For this we need to recall some notation introduced in Brzeźniak (1997). We begin with

Definition 3.2. For separable Hilbert and Banach spaces H and X we put

$$M(H,X):=\{L:H\to X: L \text{ is linear bounded and } \gamma\text{-radonifying}\}. \tag{3.2}$$

Thus, a bounded linear operator $L:H\to X$ belongs to $M(H,X)$ iff the image $L(\gamma_H)$ of the canonical finitely additive Gaussian function γ_H on H by L is σ-additive on the algebra of cylindrical sets in X. By v_L we will denote the unique extension of $L(\gamma_H)$ to the Borel σ-algebra $B(x)$.

For $L\in M(H,X)$ we put

$$||L||_{M(H,X)}:=\left\{\int_X |x|^2\,\mathrm{d}v_L(x)\right\}^{1/2}. \tag{3.3}$$

In view of the Landau–Shepp–Fernique Theorem, $||L||$ is a finite number. It is known, see Neidhardt (1978), that $M(H,X)$ is a separable Banach space.

Given a Banach S, usually S is one of the spaces X, $M(H,X)$ or $\mathscr{L}(E,X)$, by $\mathscr{N}(a,b;S)$ we denote the space of (equivalence classes of) functions $\xi:[a,b)\times\Omega\to S$ which are *progressively measurable*.

For $q\in[1,\infty)$, we set

$$\mathscr{N}^q(a,b;S)=\left\{\xi\in\mathscr{N}(a,b;S)\colon \int_a^b|\xi(s)|^q\,\mathrm{d}s<\infty \text{ a.s.}\right\}, \tag{3.4}$$

$$\mathscr{M}^q(a,b;S)=\left\{\xi\in\mathscr{N}(a,b;S)\colon \mathbb{E}\int_a^b|\xi(s)|^q\,\mathrm{d}s<\infty\right\}. \tag{3.5}$$

Let $\mathscr{N}_{\mathrm{step}}(a,b;S)$ be the space of all $\xi\in\mathscr{N}(a,b;S)$ for which there exists a partition $a=t_0<t_1<\cdots<t_n=b$ such that $\xi(t)=\xi(t_k)$ for $t\in[t_k,t_{k+1})$. We put $\mathscr{M}^q_{\mathrm{step}}=\mathscr{M}^q\cap\mathscr{N}_{\mathrm{step}}$. Note that $\mathscr{M}^q(a,b;S)$ is a closed subspace of $L^q([a,b]\times\Omega;S)$. Assume that X

satisfies the assumptions (H1), thus in particular, X is an M-type 2 Banach space. Then for any $\xi \in \mathscr{M}(0,T;M(H,X))$ there exists a continuous X-valued process, denoted by $x(t) := \int_0^t \xi \, \mathrm{d}w(s)$, $0 \leqslant t \leqslant T$, such that if $\xi = 1_{[c,d)} f$, $f \in L^2(\Omega, \mathscr{F}_c, \mathscr{L}(E,X))$, $x(t) = f(w(d \wedge t) - w(c \wedge t))$. Moreover, the following Burkholder-type inequality holds, see Dettweiler (1991). For any $r \in (1,\infty)$ there exists a constant C depending only on X and r (and so independent from ξ) such that for each $\xi \mathscr{M}^2(0,T;M(H,X))$,

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} \left| \int_0^t \xi(s) \, \mathrm{d}w(s) \right|^r \leqslant \left(\frac{r}{r-1} \right)^r C_r(X) \mathbb{E} \left(\left\{ \int_0^T ||\xi(s)||^2_{M(H,X)} \, \mathrm{d}s \right\}^{r/2} \right). \tag{3.6}$$

The above recalled definition of the Itô integral involves an auxiliary Banach space E such that i: $H \hookrightarrow E$ is an AWS. The third reason for considering cylindrical Wiener processes is described below. If two true Wiener processes (E and respectively $\tilde{E}$-valued) $w(t)$ and $\tilde{w}(t)$ correspond to the same H-cylindrical Wiener process $W(t)$, then for any $\xi \in \mathscr{M}(0,T;M(H,X))$ the processes $x(t) := \int_0^t \xi(s) \, \mathrm{d}w(s)$ and $x(t) := \int_0^t \xi(s) \, \mathrm{d}\tilde{w}(s)$ are modifications of each other. Let us also point out that in Brzeźniak and Peszat (1999) we define an Itô integral without involving any auxiliary space E.

By $(X, D(A))_{1/2,2}$ we shall denote the real interpolation space with parameters (1/2, 2) between $D(A)$ and X, see Triebel (1978).

Given $\eta \in \mathscr{M}^2(0,T;\ M(H,X))$ and $x_0 \in L^2(\Omega, \mathscr{F}_0, X)$ a process $x \in \mathscr{M}^2(0,T;X)$ is called a (mild) solution to

$$\begin{aligned} &\mathrm{d}x(t) + Ax(t) \, \mathrm{d}t = \eta(t) \, \mathrm{d}W(t), \quad t \geqslant 0, \\ &x(0) = x_0, \end{aligned} \tag{3.7}$$

iff for all $t \in [0,T]$, a.s.

$$x(t) = \mathrm{e}^{-At} x_0 + \int_0^t \mathrm{e}^{-(t-s)A} \eta(s) \, \mathrm{d}W(s). \tag{3.8}$$

Now, we want to recall a couple of sufficient conditions for the stochastic Itô integral in (3.8) to make sense. The most general one is the following:

$$\mathrm{e}^{-(t-\cdot)A} \eta(\cdot) \in \mathscr{M}^2(0,t;M(H,X)), \quad t \in [0,T]. \tag{3.9}$$

Another one is presented in the theorem below.

Theorem 3.3. *Denote $V = (X, D(A))_{1/2,2}$ and assume that η is an operator-valued process such that $A^{-1}\eta \in \mathscr{M}^2(0,T;M(H,V))$. Then, for almost all $t \in (0,T)$, the Itô integral*

$$v(t) = \int_0^t \mathrm{e}^{-(t-s)A} \eta(s) \, \mathrm{d}W(s) \tag{3.10}$$

exists and the following inequality holds:

$$\mathbb{E} \int_0^T |v(t)|^2 \, \mathrm{d}t \leqslant C_2(X) \int_0^T ||A^{-1}\eta(s)||^2_{M(H,V)} \, \mathrm{d}s. \tag{3.11}$$

Proof of Theorem 3.3. We begin with a technical lemma.

Lemma 3.4. *For $x \in X$*

$$\int_0^\infty |e^{-A\tau}x|^2 \, d\tau = |A^{-1}x|_V^2.$$

Proof of Lemma 3.4. If $x \in X$ then $e^{-tA}x = Ae^{-tA}A^{-1}x$. Therefore,

$$\int_0^\infty |e^{-tA}x|^2 \, dt = \int_0^\infty |Ae^{-tA}A^{-1}x|^2 \, dx = |A^{-1}x|_V^2. \qquad \square$$

The Fubini theorem yields

$$\mathbb{E}\int_0^T \int_0^t ||e^{-(t-s)A}\eta(s)||^2_{M(H,X)} \, ds \, dt = \int_0^T \int_s^t ||e^{-(t-s)A}\eta(s)||^2_{M(H,X)} \, dt \, ds$$

$$\leqslant \int_0^T \int_0^\infty |e^{-A\tau}\eta(s)|^2_{M(H,X)} \, d\tau \, ds. \tag{3.12}$$

Taking into account Lemma 3.4 from which it follows that

$$\int_0^\infty ||e^{-A\tau}\eta(s)||^2_{M(H,X)} \, d\tau = ||A^{-1}\eta(s)||^2_{M(H,V)},$$

we infer that

$$\mathbb{E}\int_0^T \int_0^t ||e^{-(t-s)A}\eta(s)||^2_{M(H,X)} \, ds \, dt \leqslant \int_0^T ||A^{-1}\eta(s)||^2_{M(H,(X,V))} \, ds, \tag{3.13}$$

which concludes the proof of the Theorem 3.3. $\square$

The process $v(t)$ defined above by the convolution type formula should be seen as a candidate for a solution to the problem (3.7). As the following result shows, under some additional assumptions, $v(t)$ is indeed a strong solution. For a similar result see Lemma 4.5 in Brzeźniak (1995).

Lemma 3.5. *Assume that $u_0 \in L^2(\Omega, \mathscr{F}_0, \mathbb{P}, V)$, $g \in \mathscr{M}^2(0, T, M(H, V))$ and that, for some $\bar{\gamma} > 0$, $f \in \mathscr{M}^2(0, T, D(A^{\bar{\gamma}}))$. Then the following conditions are equivalent*:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-r)A} g(r) \, dW(r) + \int_0^t e^{-(t-r)A} f(r) \, dr, \quad \text{a.s., for } t \leqslant T, \tag{3.14}$$

$$u(t) + \int_0^t Au(s) \, ds = u_0 + \int_0^t g(s) \, dw(s) + \int_0^t f(s) \, ds, \quad \text{a.s., for } t \leqslant T. \tag{3.15}$$

The following result is a slight modification of Theorem 3.2 from Brzeźniak (1997).

Theorem 3.6. *Assume that a Banach space X satisfies the condition* (H1) *and a linear operator A satisfies condition* (H2b) *and, for some $\nu \geqslant 0$, the operator $A + \nu I$ satisfies the condition* (H2a). *Assume that $q \geqslant 2$, $T \in (0, \infty)$ and $\alpha, \sigma \in [0, \frac{1}{2})$ satisfy*

$$\alpha + \sigma < \tfrac{1}{2}. \tag{3.16}$$

Assume also that the stochastic process η *is such that*

$$A^{-\sigma}\eta \in \mathscr{M}^q(0,T;M(H,X)). \tag{3.17}$$

Let the process $v(t)$ *be given by Theorem* 3.3, *then, there exists a stochastic process* $\tilde{v}(t)$, $t \in [0,T]$, *such that*

$$\tilde{v}(t) = \int_0^t \mathrm{e}^{-(t-s)A}\eta(s)\,\mathrm{d}W(s), \quad a.s., \ for\ each\ t \in [0,T], \tag{3.18}$$

and which satisfies the following conditions:

$$\tilde{v}(\cdot,\omega) \in [L^q(0,T;X);H^{1,q}(0,T;X,A)]_\alpha, \quad a.s.\ in\ \omega \in \Omega, \tag{3.19}$$

$$\mathbb{E}||\tilde{v}||^q_{\alpha,T} \leqslant CT^{((1/2)-\alpha-\sigma)q}\,\mathbb{E}\int_0^T ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,\mathrm{d}s \tag{3.20}$$

for some constant C *independent of* η *and* T *but* (*possibly*) *depending on* α, p, X *and* A. *Here* $||\cdot||_{\alpha,T}$ *denotes the norm in the interpolation space* $[L^q(0,T;X);H^{1,q}(0,T;X,A)]_\alpha$.

The idea of the proof is taken from Da Prato et al. (1987) and Da Prato and Zabczyk (1992a) (where only the Hilbert space case is considered) and is an extension of the modification used by the author in Brzeźniak (1996). An important ingredient of the proof of the above result is the following lemma. It is of particular interest (as it will be used in the sequel independently of Theorem 3.6) and so it is stated and proven below.

Lemma 3.7. *Assume that a Banach space* X *satisfies the conditions* (H1) *and a linear operator* A *satisfies condition* (H2b) *and, for some* $v \geqslant 0$, *the operator* $A + vI$ *satisfies the condition* (H2a). *Assume that* $q \geqslant 2$, $T \in (0,\infty)$, $\alpha,\delta,\sigma \geqslant 0$ *satisfy* $\alpha+\delta+\sigma < \frac{1}{2}$. *Assume that the stochastic process* η *satisfies* (3.17).

Then there exists a constant $C > 0$ *such that if the process* $y(t)$ *is defined by*

$$y(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-(t-s)A}\eta(s)\,\mathrm{d}W(s), \quad t \in [0,T], \tag{3.21}$$

then

$$||y||_{L^q(\Omega\times[0,T];D(A^\delta))} \leqslant C\,T^{(1/2)-\alpha-\delta-\sigma}||A^{-\sigma}\eta||_{L^q(\Omega\times[0,T];\,M(H;X))}. \tag{3.22}$$

In particular, $y \in L^q(0,T;D(A^\delta))$ *a.s.*

Proof of Lemma 3.7. Assume without loss of generality that $v=0$. Then the Burkholder inequality, see Brzeźniak (1996), gives, for $t \in [0,T]$

$$\mathbb{E}|A^\delta y(t)|^q \leqslant c_p\mathbb{E}\left\{\int_0^t (t-s)^{-2\alpha}||A^\delta \mathrm{e}^{-(t-s)A}\eta(s)||^2_{M(H,X)}\,\mathrm{d}s\right\}^{q/2}. \tag{3.23}$$

Hence,

$$\mathbb{E}\int_0^T |A^\delta y(t)|^q\,\mathrm{d}t \leqslant c_p\mathbb{E}\int_0^T \left\{\int_0^t (t-s)^{-2\alpha}||A^\delta \mathrm{e}^{-(t-s)A}\eta(s)||^2_{M(H,X)}\,\mathrm{d}s\right\}^{q/2}\mathrm{d}t. \tag{3.24}$$

In view of Baxendale (1976) the condition (3.36) implies that

$$\begin{aligned}||A^{\delta}\mathrm{e}^{-\tau A}\eta(s)||_{M(H,X)} &= ||A^{\delta}A^{\sigma}\mathrm{e}^{-\tau A}A^{-\sigma}\eta(s)||_{M(H,X)}\\ &\leqslant C|A^{\delta+\sigma}\mathrm{e}^{-\tau A}|_{\mathscr{L}(X)}||A^{-\sigma}\eta(s)||_{M(H,X)}\\ &\leqslant C\tau^{-(\delta+\sigma)}||A^{-\sigma}\eta(s)||_{M(H,X)}\end{aligned}$$

for some constant $C>0$. Therefore,

$$\mathbb{E}\int_0^T |A^{\delta}y(t)|^q\,\mathrm{d}t \leqslant \mathbb{E}\int_0^T \left\{\int_0^t (t-s)^{-2(\alpha+\delta+\sigma)}||A^{-\sigma}\eta(s)||^2_{M(H,X)}\,\mathrm{d}s\right\}^{q/2}\mathrm{d}t. \quad (3.25)$$

Since the RHS of (3.25) equals to $||h_1 * h_2||^{q/2}_{L^{q/2}(0,T)}$, where $h_1(s)=1_{(0,T]}(s)s^{-2(\alpha+\delta+\sigma)}$, $h_2(s)=1_{(0,T]}(s)||A^{-\sigma}\eta(s)||^2_{M(H,X)}$, by applying pathwise Young's inequality we get

$$\begin{aligned}&\int_0^T \left\{\int_0^t (t-s)^{-2(\alpha+\delta+\sigma)}||A^{-\sigma}\eta(s)||_{M(H,X)}\mathrm{d}s\right\}^{q/2}\mathrm{d}t\\ &\quad\leqslant c_{q,\sigma}(1-2(\alpha+\delta+\sigma))^{-q/2}T^{2((1/2)-(\alpha+\delta+\sigma))}\int_0^T ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,\mathrm{d}s. \quad (3.26)\end{aligned}$$

Indeed, we have

$$|h_1|_{L^1(0,T)} = \int_0^T s^{-2(\alpha+\delta+\sigma)}|A^{\sigma}\mathrm{e}^{-sA}|^2\,\mathrm{d}s = (1-2(\alpha+\delta+\sigma))^{-1}T^{1-2(\alpha+\delta+\sigma)},$$

$$|h_2|_{L^{q/2}(0,T)} = \left(\int_0^T ||A^{-\sigma}\eta(s)||^{2q/2}_{M(H,X)}\,\mathrm{d}s\right)^{2/q} = ||A^{-\sigma}\eta||^2_{\mathscr{M}^q(0,T;M(H,X))}.$$

Taking expectation of (3.26) in view of (3.25) yields

$$\begin{aligned}\mathbb{E}\int_0^T |A^{\delta}y(t)|^q\,\mathrm{d}t &\leqslant c_{q,\sigma}(1-2(\alpha+\delta+\sigma))^{-q/2}T^{[(1/2)-(\alpha+\delta+\sigma)]q}\\ &\quad\times\mathbb{E}\int_0^T ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,\mathrm{d}s, \quad (3.27)\end{aligned}$$

which proves (3.22). □

Corollary 3.8. *Under the assumptions of Theorem* 3.6 *the following holds*:

$$v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{e}^{-(t-s)A}y(s)\,\mathrm{d}s, \quad \textit{a.s., for each } t\in[0,T], \quad (3.28)$$

where $y(t)$ *is defined by* (3.21).

Remark 3.9. A special case of Theorem 3.6 is when the operator A is positive and the semigroup e^{-tA} is exponentially bounded.

Then with $T=\infty$ one can show that the process $y(t)$ defined by (3.21) satisfies

$$\mathbb{E}\int_0^{\infty} |A^{\delta}y(t)|^q\,\mathrm{d}t \leqslant c_{q,\sigma}(1-2(\alpha+\delta+\sigma))^{-q/2}\int_0^{\infty} ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,\mathrm{d}s. \quad (3.29)$$

The only modification of the proof of Theorem 3.6 lies in choosing the auxiliary function h_1 of the form $h_1(s)=s^{-2(\alpha+\delta+\sigma)}e^{-as}$, $s\in[0,\infty)$. Since obviously $|h_1|_{L^1(0,\infty)}<\infty$ one may infer that there exists a modification $\tilde{v}(t)$ of the process $v(t)$ satisfying

$$\begin{aligned}&\tilde{v}(\cdot,\omega)\in[L^q(0,\infty;X);H^{1,q}(0,\infty;X,A)]_\alpha,\quad \text{a.s. in } \omega\in\Omega,\\ &\mathbb{E}||\tilde{v}||^q\leqslant C\mathbb{E}\int_0^\infty ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,ds.\end{aligned}\tag{3.30}$$

In particular, under the assumptions of Corollary 3.10 below, using Lemma 2.4 and still in the case $T=\infty$, one can see that $\tilde{v}$ satisfies (3.34) and the following version of (3.35). If $\beta<\alpha-1/q$ then

$$\mathbb{E}||\tilde{v}||^q_{\mathscr{C}^\beta(0,\infty;\,D(A^\delta))}\leqslant C\mathbb{E}\int_0^\infty ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,ds.\tag{3.31}$$

In the very special but important case of $\beta=0$, if $\delta+\sigma+1/q<1/2$ one gets

$$\mathbb{E}\sup_{t\geqslant 0}||\tilde{v}(t)||^q_{D(A^\delta)}\leqslant C\int_0^\infty ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,ds.\tag{3.32}$$

Lemma 2.4 in conjunction with Theorem 3.6 (applied for $\gamma=0$) yields the following.

Corollary 3.10. *Suppose that not only the assumptions of Theorem* 3.6 *are satisfied but also that the positive numbers β,δ satisfy*

$$\beta+\delta+\sigma<\frac{1}{2}-\frac{1}{q}.\tag{3.33}$$

Then there exists a stochastic process $\tilde{v}(t)$, $t\in[0,T]$, a modification of $\int_0^t\eta(s)\,dW(s)$, such that

$$\tilde{v}(\cdot,\omega)\in\mathscr{C}^\beta(0,T;D(A^\delta)),\quad a.s.\ in\ \omega\in\Omega,\tag{3.34}$$

$$\mathbb{E}||\tilde{v}||^q_{\mathscr{C}^\beta(0,T;\,D(A^\delta))}\leqslant C_T\mathbb{E}\int_0^T ||A^{-\sigma}\eta(s)||^q_{M(H,X)}\,ds.\tag{3.35}$$

Proof. It is sufficient to choose $\alpha>\beta+1/q$ satisfying $\alpha+\delta+\sigma<\frac{1}{2}$ and then to apply the previous results. □

Remark 3.11. If for some $\sigma\in[0,\frac{1}{2})$ satisfying (3.16),

$$A^{-\sigma}\in M(H,X),\tag{3.36}$$

i.e. $A^{-\sigma}$ extends to a bounded linear map from H to X that is radonifying, then all the previous results hold true with the condition (3.17) being replaced by

$$\eta\in\mathscr{M}^q(0,T;\mathscr{L}(H))\tag{3.37}$$

and with replacing $||A^{-\sigma}\eta(s)||_{M(H,X)}$ in formulae (3.20) and (3.29)–(3.32) by $|\eta(s)|_{\mathscr{L}(X)}$. Indeed, in view of Baxendale (1976), $||A^{-\sigma}\eta(s)||_{M(H,X)}\leqslant|\eta(s)|_{\mathscr{L}(X)}$ as long as (3.36) holds.

3.2. Examples

Assume that $\mathcal{O}$ is an open and bounded interval in $\mathbb{R}$ and that $r \in [0, \frac{1}{2})$ and $p \in [2, \infty)$. Let $H = L^2(\mathcal{O})$, $X = H^{r,p}(\mathcal{O})$ and $A = A_{p,r} = -\Delta$ with $D(A_{p,r}) = H^{2+r,p}(\mathcal{O}) \cap H_0^{1+r,p}(\mathcal{O})$. Note that $-A$ is the Laplace operator with Dirichlet boundary conditions. It is well known that the Banach space $H^{r,p}(\mathcal{O})$ and the operator $A_{p,r}$ satisfy all the assumptions (H1)–(H3) from Section 2, see Triebel (1978). The operator $A_{p,0}$ will be also denoted by A_p.

As a preliminary step we will prove the following simple but crucial Lemma.

Lemma 3.12. *In the framework described above, if the number σ satisfies*

$$\sigma > \frac{1}{4} + \frac{r}{2}, \tag{3.38}$$

then $A^{-\sigma} \in M(H,X)$, i.e. the condition (3.36) *is satisfied. In particular, there exists $\sigma \in (0, \frac{1}{2})$ for which the condition* (3.36) *holds true.*

Proof. It is known, see for example Brzeźniak (1996), that the imbedding map $i_\varepsilon : H_0^{1,2}(\mathcal{O}) \to H_0^{\varepsilon,p}(\mathcal{O})$ is radonifying if $\varepsilon < \frac{1}{2}$. Since by (3.38) $r + 1 - 2\sigma < \frac{1}{2}$,

$$A_{p,r}^{-\sigma} = A_{p,r}^{1/2-\sigma} i_{r+1-2\sigma} A_2^{-1/2}$$

and the maps $A_2^{-1/2} : H \to H_0^{1,2}$ and $A_{p,r}^{1/2-\sigma} : H_0^{r+1-2\sigma,q}(\mathcal{O}) \to H^{r,p} = X$ are bounded, the result follows by using Baxendale (1976).

The second part of the lemma follows as well since $r < \frac{1}{2}$. □

Remark 3.13. One can consider even more spatially irregular Wiener processes by considering the Hilbert space $H = H^{\theta,2}(\mathcal{O})$ instead of $H = L^2(\mathcal{O})$. The smaller the θ the more spatially irregular the Wiener process is. The condition (3.38) takes the form $\sigma + \theta/2 > r/2 + \frac{1}{4}$. In particular, if $\sigma + \theta/2 > \frac{1}{4}$ then $A_p^{-\sigma} \in M(H^{\theta,2}(\mathcal{O}); L^p(\mathcal{O}))$ and, by the Sobolev imbedding theorem, $A_p^{-\sigma} \in M(H^{\theta,2}(\mathcal{O}); \mathscr{C}(\bar{\mathcal{O}}))$.

In the multidimensional case the situation is similar but details are different. Suppose that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary and $A = A_p = -\Delta$ with Dirichlet boundary conditions in $X = L^p(\mathcal{O})$, i.e. $D(A_p) = H^{2,p}(\mathcal{O}) \cap H_0^{r,p}(\mathcal{O})$. Then $A_p^{-\sigma} \in M(H^{\theta,2}(\mathcal{O}); L^p(\mathcal{O}))$ if $\sigma + \theta/2 > d/4$. Hence one can find $\sigma < \frac{1}{2}$ for which $A_p^{-\sigma}$ is γ-radonifying iff $\theta > d/2 - 1$.

The main result of this subsection is given below. To prove it, to apply Theorem 3.6 and Corollary 3.10 in the framework as outlined above.

Theorem 3.14. *Assume that the stochastic process η belongs to $\mathscr{M}^2(0, T; \mathscr{L}(H))$ for some $T \in (0, \infty]$.*

(i) *If $r + \frac{1}{p} < \frac{1}{2}$, then, for almost all $t \in (0, T)$, the Itô integral*

$$v(t) = \int_0^t e^{-(t-s)A} \eta(s)\, dW(s) \tag{3.39}$$

exists and, with some positive constant C independent of η, the following inequality holds:

$$\mathbb{E}\int_0^T |v(t)|^2_{B^r_{p,2}(\mathcal{O})}\,dt \leqslant C\int_0^T |\eta(s)|^2_{\mathscr{L}(H)}\,ds, \tag{3.40}$$

where $B^r_{p,2}$ are the Besov spaces, see Triebel (1978).

(ii) *Assume in addition that the numbers $q\geqslant 2$ and β,δ satisfy the following condition:*

$$\beta+\delta+\frac{1}{q}<\frac{1}{4}. \tag{3.41}$$

Then, if the process η belongs to $\mathscr{M}^q(0,T;\mathscr{L}(H))$, there exists a modification $\tilde{v}(t)$, $t\in[0,T]$ of the stochastic process $v(t)$, such that, for some constant C_T,

$$\tilde{v}(\cdot,\omega)\in\mathscr{C}^\beta(0,T;H_0^{2\delta,p}(0,1)),\quad a.s.\ in\ \omega\in\Omega, \tag{3.42}$$

$$\mathbb{E}||\tilde{v}||^q_{\mathscr{C}^\beta(0,T;H_0^{2\delta,p}(0,1))}\leqslant C_T\mathbb{E}\int_0^T |\eta(s)|^q_{\mathscr{L}(H)}\,ds. \tag{3.43}$$

(iii) *If the numbers $q\geqslant 2$ and $\kappa,\beta\geqslant 0$ satisfy the following inequality:*

$$\beta+\frac{\kappa}{2}+\frac{1}{q}<\frac{1}{4} \tag{3.44}$$

and the process η belongs to $\mathscr{M}^q(0,T;\mathscr{L}(H))$, then there exists a modification $\tilde{v}(t)$ of the stochastic process $v(t)$ such that

$$\tilde{v}(\cdot,\omega)\in\mathscr{C}^\beta(0,T;\mathscr{C}_0^\kappa(0,1)),\quad a.s.\ in\ \omega\in\Omega, \tag{3.45}$$

$$\mathbb{E}||\tilde{v}||^q_{\mathscr{C}^\beta(0,T;\mathscr{C}_0^\kappa(0,1))}\leqslant C_T\mathbb{E}\int_0^T |\eta(s)|^q_{\mathscr{L}(H)}\,ds. \tag{3.46}$$

Proof. Part (i) follows from Theorem 3.6 by taking $X=H^{r,p}(\mathcal{O})$. For, firstly in view of the Reiteration Theorem, see (Triebel, 1978, Remark 4, Section 2.4.2), the real interpolation space $V:=(H^{r,p}(\mathcal{O}),H^{2+r,p}(\mathcal{O})\cap H_0^{1+r,p}(\mathcal{O}))_{1/2,2}$ equals to $B^{r+1}_{p,2}(\mathcal{O})\cap \mathring{\mathrm{B}}^{1/2+r}_{p,2}(\mathcal{O})$. Secondly, by Baxendale (1976), $||A^{-1}\eta(s)||_{M(H,V)}\leqslant C||A^{-1}||_{M(H,V)}|\eta(s)|_{\mathscr{L}(H)}$. Finally, as $D(A)=D(A_{2,2})=H^{2,2}(\mathcal{O})\cap H_0^{1,2}(\mathcal{O})$, A^{-1} belongs to $M(H,V)$ iff the imbedding $i:H^{2,2}(\mathcal{O})\cap H_0^{1,2}(\mathcal{O})\hookrightarrow V$ is γ-radonifying. The latter holds iff the imbedding i: $H^{1,2}(\mathcal{O})\hookrightarrow B^r_{p,q}(\mathcal{O})$ is γ-radonifying, which holds, see Brzeźniak (1995), if $r<\frac{1}{2}-1/p$.

To prove part (ii) we take $X=L^p(0,1)$ and $A=A_p$ with $p\geqslant 2$. Note that now $r=0$. Then $D(A^\delta)=H_0^{2\delta,p}(0,1)$. Since $r=0$ because of (3.41) we can choose a positive number σ such that both conditions (3.33) and (3.38) are satisfied. Finally, we apply Theorem 3.6.

To prove part (iii) we begin with choosing positive numbers p and δ such that $\delta>\kappa/2+1/2p$ and $\beta+\delta+1/q<\frac{1}{2}$. Then we take $X=L^p(0,1)$ and $A=A_p$. We observe that by the Sobolev imbedding theorem $H_0^{2\delta,p}(0,1)\subset\mathscr{C}_0^\kappa(0,1)$ continuously, and hence (iii) follows from (ii). □

Remark 3.15. Taking any $\beta<\frac{1}{4}$ and $\kappa=0$ one can find q such that the assumptions of Theorem 3.14(iii) are satisfied. Therefore, there exists a modification $\tilde{v}(t)$ of the process $v(t)$ which is Hölder continuous in $t \in [0,T]$ with exponent β, uniformly in $x \in [0,1]$ a.s. Analogously, taking $\beta = 0$ and any $\kappa < \frac{1}{2}$ one can find q such that the assumptions of Theorem 3.14(iii) are satisfied. Therefore, there exists a modification $\tilde{v}(t)$ of the process $v(t)$ which is Hölder continuous in $x \in [0,1]$ with exponent κ, uniformly in $t \in [0,T]$ a.s.

In particular, there exists a modification $\tilde{v}(t)$ of the process $v(t)$ which is Hölder continuous in $t \in [0,T]$ with exponent β and in $x \in [0,1]$ with exponent κ, uniformly in $t \in [0,T]$ a.s.

Remark 3.16. The results described in Theorem 3.14 hold true in the multidimensional framework as described in Remark 3.13 and with Laplace operator Δ being replaced by any uniformly elliptic second-order differential operator $\mathscr{A} = \sum_{ij} \partial/\partial x_i (a_{ij}(x) \partial/\partial x_j) + a_0(x)$ with $\mathscr{C}^2$ coefficients and boundary conditions of the form $b_0(x)u(x) + \sum_j b_j(x) \partial u/\partial x_j = 0$ for $x \in \partial\mathscr{O}$. Moreover, this holds for systems of such equations acting on vector valued functions. See also Section 6.

4. Martingale solutions

4.1. Definitions and assumptions

Consider the following stochastic equation:

$$\begin{aligned} &\mathrm{d}u(t) + Au(t)\,\mathrm{d}t = F(t,u(t))\,\mathrm{d}t + G(t,u(t))\,\mathrm{d}W(t), \\ &u(0) = u_0. \end{aligned} \tag{4.1}$$

Suppose that X is a separable Banach space and $-A$ is a generator of an analytic semigroup $\{\mathrm{e}^{-tA}\}_{t \geqslant 0}$ on X. More precise conditions on X and A are listed in assumptions (H1)–(H4). Moreover, we will need another Banach space B. Assumptions A.1–A.3 below are standing assumptions for the rest of the paper.

Assumption A.1. *B is a Banach space (with norm denoted by $|\cdot|_B$) such that*

$$D(A^\delta) \hookrightarrow B \hookrightarrow X, \tag{4.2}$$

for some $\delta \in (0, \frac{1}{2} - \sigma)$. The semigroup $\{\mathrm{e}^{-tA}\}_{t \geqslant 0}$ restricts to a strongly continuous semigroup of bounded linear operators on B. Unless we find ourselves in danger of ambiguity, the semigroup itself and its generator will be denoted without any change. Moreover, the imbedding $D(A^\delta) \hookrightarrow B$ is compact.

Assumption A.2. *The mapping $A^{-\sigma}G : [0,\infty) \times B \to M(H,X)$ is bounded, continuous with respect to the second variable and strongly measurable with respect to the first one.*

The next condition uses a notion of a subdifferential of the norm, see Da Prato (1976). Given $x, y \in B$ the map $\varphi : \mathbb{R} \ni s \mapsto |x + sy| \in B$ is convex and therefore is right and left differentiable. Define $D_\pm |x| y$ to be the right/left derivative of φ at 0.

Then the subdifferential $\partial|x|$ of $|x|$, $x \in B$, is defined by

$$\partial|x| := \{x^* \in B^*: D_-|x|y \leqslant \langle y, x^* \rangle \leqslant D_+|x|y, \ \forall y \in B\},$$

where B^* is the dual space to B. One can show that not only $\partial|x|$ is a nonempty, closed and convex set, but also

$$\partial|x| = \{x^* \in B^*: \langle x, x^* \rangle = |x| \text{ and } |x^*| \leqslant 1\}.$$

In particular, $\partial|0|$ is the unit ball in B^*.

Assumption A.3. *The mapping $F : [0,\infty) \times B \to B$ is strongly measurable with respect to the first variable and continuous with respect to the second variable. Moreover, there exist $k \in \mathbb{R}$ and an increasing function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with* $\lim_{t\to\infty} a(t) = \infty$ *such that for all $x \in D(A)$, $y \in B$ and $t \geqslant 0$*

$$\langle -Ax + F(t, x+y), z \rangle \leqslant a(|y|_B) - k|x|_B, \tag{4.3}$$

for any $z \in x^* = \partial|x|$.

Remark 4.1. The condition (H4) is the basic one from the previous sections. The last part of Assumption A.1 implies the assumptions of Theorem 2.6, i.e. that A has a compact resolvent. The only new ones (essential for our nonlinear problem (4.1)) are Assumptions A.2 and A.3. Note that the part (i) of the Assumption A.5 below coincides with the condition (2.22) from Remark 3.9.

If $a = 0$ and $F = 0$ the condition (4.3) means that the operator $A + kI$ is dissipative on B. The latter is then equivalent to the fact that the semigroup generated by $-A$ satisfies $|e^{-tA}|_{\mathscr{L}(B)} \leqslant e^{kt}$, for $t \geqslant 0$, by the Lumer Phillips Theorem, see Pazy (1983).

Later on we shall also need some of the following conditions.

Assumption A.4. *There exist nonnegative numbers $k_0 \geqslant 0$ and $N \geqslant 0$ such that Assumption* (A.3) *is satisfied with a function*

$$a(r) = k_0(1 + r^N), \quad r \geqslant 0.$$

Assumption A.5. (i) *There exists constants $M, a > 0$ such that*

$$|e^{-tA}|_{\mathscr{L}(X)} \leqslant M e^{-at}, \quad t \geqslant 0.$$

(ii) *The constant k from Assumption* A.3 *is positive, i.e. $k > 0$.*

Before we proceed any further let us now state (and prove) the following important consequence of Assumption A.3.

Lemma 4.2. *Assume that B is a Banach space, $-A$ a generator of a strongly continuous semigroup of bounded linear operators on B and a mapping $F : [0,\infty) \times B \to B$ satisfies Assumption* A.3. *Assume that for some $\tau > 0$ two continuous functions $z, v : [0, \tau) \to B$ satisfy*

$$z(t) = \int_0^t e^{-(t-s)A} F(s, z(s) + v(s))\, ds, \quad t \leqslant \tau.$$

Then

$$|z(t)|_B \leqslant \int_0^t \mathrm{e}^{-k(t-s)} a(|v(s)|_B)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau. \tag{4.4}$$

Proof of Lemma 4.2. For $\lambda > \omega_0$ where ω_0 is large enough, let $R(\lambda) = (\lambda I + A)^{-1} \in \mathscr{L}(B)$ be the resolvent of the operator A. Set $z_\lambda(t) = \lambda R(\lambda) z(t)$ and $F_\lambda = \lambda R(\lambda) \circ F$. Then

$$z_\lambda(t) = \int_0^t \mathrm{e}^{-(t-s)A} F_\lambda(s, z(s) + v(s))\,\mathrm{d}s, \quad t \geqslant 0.$$

Since the function $[0,\tau] \ni s \mapsto F_\lambda(s, z(s) + v(s)) \in D(A)$ is continuous it follows from Theorem 2.4, Section 4 in Pazy (1983) that the function $z_\lambda : [0,\tau] \to B$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} z_\lambda(t) + A z_\lambda(t) = F(t, z_\lambda(t) + v(t)) + \delta_\lambda(t), \quad t > 0,$$

where

$$\delta_\lambda(s) = F_\lambda(s, z(s) + v(s)) - F(s, z_\lambda(s) + v(s)).$$

By Gronwall Lemma and Assumption A.3 it follows that

$$|z_\lambda(t)|_B \leqslant \int_0^t \mathrm{e}^{-k(t-s)} (a(|v_\lambda(s)|_B) + |\delta_\lambda(s)|_B)\,\mathrm{d}s \quad 0 \leqslant t \leqslant \tau.$$

Since $||\lambda R(\lambda)|| \leqslant M$ for large λ and $\lambda R(\lambda) z \to z$ as $\lambda \to \infty$ for any $z \in B$, the Lebesgue-dominated convergence theorem implies that $\delta_\lambda \to 0$ in $L^1(0,\tau; B)$. Hence (4.4) follows. □

We shall define now the mild and, later on, the martingale solution to Eq. (4.1).

Definition 4.3. Assume that the conditions (H1)–(H2) and (H4) as well as the Assumptions A.1 and A.2 are satisfied. Suppose that F is a map from $[0,T) \times B$ into B. Let $x \in B$ be fixed. Let $u(t)$, $0 \leqslant t < T$, be a B-valued admissible process. Then $u(t)$ is called a mild solution to the problem (4.1) iff for $t < T$

$$u(t) = \mathrm{e}^{-tA} x + \int_0^t \mathrm{e}^{-(t-r)A} G(r, u(r))\,\mathrm{d}W(r) + \int_0^t \mathrm{e}^{-(t-r)A} F(r, u(r))\,\mathrm{d}r, \quad \text{a.s.}$$

Definition 4.4. Assume that H is a separable Hilbert space and that the conditions and (H1), (H2) and (H4) as well as Assumptions A.1 and A.2 are satisfied. Let F be a map from $[0,\infty) \times B$ into B and let $x \in B$ be fixed.

A martingale solution to Eq. (4.1) is a system

$$(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t \geqslant 0}, \{W(t)\}_{t \geqslant 0}, \{u(t)\}_{t \geqslant 0}) \tag{4.5}$$

such that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, $\{\mathscr{F}_t\}_{t \geqslant 0}$ a filtration on it, $\{W(t)\}_{t \geqslant 0}$ is an H-cylindrical Wiener process (with respect to the filtration $\mathscr{F}_t$) and $u(t)$ is a B-valued admissible process such that for any $t \in [0,T]$

$$u(t) = \mathrm{e}^{-tA} x + \int_0^t \mathrm{e}^{-(t-s)A} F(s, u(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A} G(s, u(s))\,\mathrm{d}W(s), \quad \text{a.s.} \tag{4.6}$$

We say that the martingale solution (4.5)–(4.1) is unique iff given another martingale solution to (4.1)

$$(\Omega', \mathscr{F}', \mathbb{P}', \{\mathscr{F}'_t\}_{t\geqslant 0}, \{W'(t)\}_{t\geqslant 0}, \{u'(t)\}_{t\geqslant 0}),$$

the laws of the processes $u(t)$ and $u'(t)$ on the space $C(0,T;B)$ coincide.

We shall also need (sometimes) the following:

Assumption A.6. *For any $x \in B$ the martingale solution of* (4.1) *is unique.*

Assumption A.6 results from uniqueness of strong solutions of stochastic evolution equations as in Neidhardt (1978), Bally et al. (1994), Dettweiler (1988), Brzeźniak (1995) and Brzeźniak and Elworthy (1999) by the well-known scheme of Yamada–Watanabe, see Dettweiler (1988) for details. A more sophisticated way of proving (A.6) via the Girsanov transformation is given in Gątarek and Gołdys (1997) (but with more stringent conditions on the coefficients). Let us also emphasize that uniqueness of strong solutions of stochastic evolution equations in a framework similar to the one considered in the present paper is studied in detail in Brzeźniak (1997).

In this section we show existence of martingale solution to Eq. (4.1). Let us notice at first that Assumption A.3 implies that

$$|F(t,y)|_B \leqslant a(|y|_B), \quad t\geqslant 0, \; y\in B. \tag{4.7}$$

4.2. Existence

We begin with the following.

Theorem 4.5. *Suppose that for a Banach space X, a Hilbert space H, a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an operator $A+\nu I$ (with some $\nu \geqslant 0$), Assumptions* A.1, A.2 *and* (H1)–(H4) *are satisfied. Suppose also that for some $\beta<1$ the function $A^{-\beta}F : [0,\infty)\times B \to X$ is strongly measurable with respect to the first variable and with respect to the second variable and* bounded. *Then for any $x\in B$ there exists a martingale solution of* (4.1).

Proof of Theorem 4.5. Without loss of generality we may assume that $\nu = 0$. Our proof is an infinite-dimensional modification of the original Skorohod proof, see also Gihman and Skorohod (1972) and Gątarek and Gołdys (1994). Let $T>0$ and $x\in B$ be fixed. Take $\alpha>\delta$ such that $\alpha+\sigma<\frac{1}{2}$ and next choose $q>2$ such that $1/q<\alpha-\delta$ and $1/q<1-\beta$. Finally choose γ such that $\delta<\gamma<\delta+1/q$. Consider a sequence $\{x_n\}\subset D(A^\delta)$ such that $x_n\to x$ in B as $n\to\infty$. Let $s_n=(k/2^n)T$ if $(k/2^n)T\leqslant s<((k+1)/2^n)T$. Define a sequence of admissible $D(A^\delta)$-valued processes by

$$\bar{u}_n(t) = \mathrm{e}^{-tA}x_n + \int_0^t \mathrm{e}^{-(t-s)A}F(s,\bar{u}_n(s_n))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}G(s,\bar{u}_n(s_n))\,\mathrm{d}W(s). \tag{4.8}$$

It follows from Assumption A.1 and Theorems 4.1 and A.1 from Brzeźniak (1997) that the definition of $\bar{u}_n$ is correct. We need to find some estimates on $\bar{u}_n$ independent of n. Set

$$y_n(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-(t-s)A}G(s,\bar{u}_n(s_n))\,\mathrm{d}W(s)$$

and $\bar{f}_n(s)=F(s,\bar{u}_n(s_n))$ and $\bar{g}_n(s)=G(s,\bar{u}_n(s_n))$, $s\in[0,T]$. Since $A^{-\sigma}G:[0,\infty)\times B\to M(H,X)$ is bounded, Lemma 3.7 yields that

$$\sup_{n\geqslant 1}\mathbb{E}\,|y_n|^q_{L^q(0,T;X)}<\infty. \tag{4.9}$$

Therefore the family of processes y_n, $n\in\mathbb{N}$, is uniformly bounded in probability on $L^q(0,T;X)$. The same holds for the family $A^{-\beta}\bar{f}_n$, $n\in\mathbb{N}$. Indeed, since the map $A^{-\beta}F:[0,\infty)\times B\to X$ is bounded, $|A^{-\beta}\bar{f}_n(t)|_X\leqslant C$ for some $C>0$ and all $t\geqslant 0$. Let us recall that in view of Corollary 2.8 the operators $\Lambda_T^{-\alpha}$ and $\Lambda_T^{-(1-\beta)}$ are compact from $L^q(0,T;X)$ to $C(0,T;D(A^\delta))$. Therefore, as also the map $\mathscr{A}_T\Lambda_T^{-1}$ is bounded on $L^q(0,T;X)$, the families of laws of $\Lambda_T^{-\alpha}y_n$ and $\Lambda_T^{-1}\bar{f}_n=\Lambda_T^{-(1-\beta)}(\mathscr{A}_T\Lambda_T^{-1})^\beta A^{-\beta}\bar{f}_n$ are tight on $C(0,T;D(A^\delta))$. This in conjunction with (4.2) yields tightness on $C(0,T;B)$ of the families of laws of $\Lambda_T^{-1}\bar{f}_n$ and $\Lambda_T^{-\alpha}y_n$. Define

$$u_n(t)=\mathrm{e}^{-tA}x_n+[\Lambda_T^{-1}\bar{f}_n](t)+[\Lambda_T^{-\alpha}y_n](t),\quad t\in[0,T]. \tag{4.10}$$

In view of Corollary 3.8 $u_n(t)=\bar{u}_n$, a.s., $t\in[0,T]$. Moreover, the argument preceding (4.10) yields that the family of laws of u_n is tight on $C(0,T;B)=\Omega_T$. Hence, there exists a measure μ on Ω_T and a subsequence of the sequence u_n, still denoted by u_n, such that $u_n\to\mu$ weakly. By the Skorohod imbedding theorem, see Williams (1979) or Da Prato and Zabczyk (1992a), there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ with filtration $\tilde{\mathscr{F}}_t$ and a sequence of B-valued admissible processes $\tilde{u}_n(t)$, $t\in[0,T]$, on $\tilde{\Omega}$, such that the laws of $\tilde{u}_n$ and u_n are the same, and there exists a process $\tilde{u}(t)$, $t\in[0,T]$, on $\tilde{\Omega}$ with law μ such that $\tilde{u}_n\to\tilde{u}$ in $C(0,T;B)$, a.s. on $\tilde{\Omega}$.

Set $f_n(s)=F(s,u_n(s_n))$ and $g_n(s)=G(s,u_n(s_n))$, $s\in[0,T]$. Since $1-\sigma>\frac{1}{2}$, the range of $A^{-1}G=A^{-(1-\sigma)}A^{-\sigma}G$ is contained in $M(H,D(A^{-(1-\sigma)}))\subset M(H,(X,D(A))_{1/2,2})$. Thus Lemma 3.5 implies that, for $t\leqslant T$,

$$A^{-1}u_n(t)+\int_0^t u_n(s)\,\mathrm{d}s=A^{-1}x_n+\int_0^t A^{-1}g_n(s)\,\mathrm{d}W(s)+\int_0^t A^{-1}f_n(s)\,\mathrm{d}s,\quad \text{a.s.}$$

Therefore,

$$M_n(t):=A^{-1}\bar{u}_n(t)+\int_0^t u(s)\,\mathrm{d}s-A^{-1}x_n-\int_0^t A^{-1}f(s)\,\mathrm{d}s,\quad t\in[0,T]$$

is an X-valued martingale and its cylindrical quadratic variation $[M_n]$ is of the form

$$[M_n](t)=\int_0^t Q_n(s)\,\mathrm{d}s,$$

with $Q_n(t)=A^{-1}g_n(s)\circ[A^{-1}g_n(s)]^{\mathrm{t}}$, see Dettweiler (1988) for explanation of necessary concepts (for $L\in M(H,X)\subset\mathscr{L}(H,X)$, $L^{\mathrm{t}}\in\mathscr{L}(X^*,H)$ is its transpose).

The argument from the proof of Theorem 8.1 in Da Prato and Zabczyk (1992a) yields (verbatim-verbatim) that

$$\tilde{M}_n(t):=A^{-1}\tilde{u}_n(t)-A^{-1}x_n+\int_0^t [\tilde{u}_n(s)-A^{-1}F(s,\tilde{u}_n(s_n))]\,\mathrm{d}s$$

is an X-valued square integrable martingale with respect to the filtration $\tilde{\mathscr{F}}_n(t) := \sigma\{\tilde{u}_n(s)\colon s \leqslant t\}$ and its cylindrical quadratic variation $[\tilde{M}_n]$ is of the form

$$[\tilde{M}_n](t) = \int_0^t \tilde{Q}_n(s)\,\mathrm{d}s,$$

where $\tilde{Q}_n(s) = A^{-1}\tilde{g}_n(s) \circ [A^{-1}\tilde{g}_n(s)]^{\mathrm{t}}$ and $\tilde{g}_n(s) := G(\tilde{u}_n(s))$, $s \in [0,T]$.

Set also $\tilde{f}_n(s) := F(s, \tilde{u}_n(s))$, $s \in [0,T]$. Then, a.s. on $(\tilde{\Omega}, \mathbb{P})$, $\tilde{u}_n \to \tilde{u}$, $A^{-\beta}\tilde{f}_n \to A^{-\beta}\tilde{f}$ and $A^{-1}\tilde{g}_n \to A^{-1}\tilde{g}$ in respectively, $C(0,T;B)$, $C(0,T;X)$ and $\mathscr{M}^q(0,T;M(H,X))$. Here $\tilde{f}(s) := F(s,\tilde{u}(s))$ and $\tilde{g}(s) := A^{-1}G(s,\tilde{u}(s))$. Therefore, $\tilde{M}_n \to \tilde{M}$ in $C(0,T;B)$ a.s. on $(\tilde{\Omega}, \mathbb{P})$ and $\tilde{M}(t)$ is a square integrable X-valued martingale with respect to the filtration $\tilde{\mathscr{F}}(t) := \sigma\{\tilde{u}(s)\colon s \leqslant t\}$. Moreover, if $\tilde{Q}(s) = A^{-1}\tilde{g}(s) \circ [A^{-1}\tilde{g}(s)]^{\mathrm{t}}$, the cylindrical quadratic variation $[\tilde{M}]$ of $\tilde{M}(t)$ is of the following form:

$$[\tilde{M}](t) = \int_0^t \tilde{Q}(s)\,\mathrm{d}s.$$

The next step is to employ the martingale representation Theorem, see Theorem 2.4 in Dettweiler (1988). Taking into account separability of both X and its dual X^*, the just cited result of Dettweiler yields existence of an enlarged probability space $(\tilde{\tilde{\Omega}}, \tilde{\tilde{F}}, \tilde{\tilde{\mathbb{P}}})$ and existence of an H-cylindrical Wiener process $\tilde{\tilde{W}}(t)$, $\geqslant 0$, on it, such that $\tilde{M}(t) = \int_0^t A^{-1}\tilde{g}(s)\,\mathrm{d}\tilde{\tilde{W}}(s)$, a.s., $t \geqslant 0$. Therefore, for $t \in [0,T]$,

$$A^{-1}\tilde{u}(t) + \int_0^t u(s)\,\mathrm{d}s = A^{-1}x + \int_0^t A^{-1}\tilde{g}(s)\,\mathrm{d}\tilde{\tilde{W}}(s) + \int_0^t A^{-1}\tilde{f}(s)\,\mathrm{d}s, \quad \text{a.s.}$$

Since $A^{-1}G(x) = A^{-(1-\sigma)}A^{-\sigma}G(x) \in M(H, D(A^{-(1-\sigma)})) \subset M(H, (X, D(A))_{1/2,2})$ as $1 - \sigma > \frac{1}{2}$, Lemma 3.5 implies that

$$A^{-1}\tilde{u}(t) = \mathrm{e}^{-tA}A^{-1}x + \int_0^t \mathrm{e}^{-(t-s)A}A^{-1}\tilde{g}(s)\,\mathrm{d}\tilde{\tilde{W}}(s) + \int_0^t \mathrm{e}^{-(t-s)A}A^{-1}\tilde{f}(s)\,\mathrm{d}s, \quad \text{a.s.,} \quad t \in [0,T].$$

Since $\tilde{u}(t)$ is an admissible B-valued process the definitions of $\tilde{f}$ and $\tilde{g}$ imply that the system

$$(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathscr{F}}}, \tilde{\tilde{\mathbb{P}}}, \{\tilde{\tilde{\mathscr{F}}}_t\}_{t \geqslant 0}, \{\tilde{\tilde{W}}(t)\}_{t \geqslant 0}, \{\tilde{u}(t)\}_{t \geqslant 0})$$

is a martingale solution to the problem (4.6). This concludes the proof of Theorem 4.5. □

In the main result of this section we replace the boundedness assumption of F by the dissipativity of the drift $-A + F$.

Theorem 4.6. *Suppose that a Banach space X, a Hilbert space H, a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, the operator $A + \nu I$ (with some $\nu \geqslant 0$) and the function $F\colon [0,\infty) \times B \to B$ satisfy all Assumptions* A.1–A.3 *and* (H1)–(H4). *Then there exists a martingale solution of* 4.1.

Proof of Theorem 4.6. Without loss of generality we may assume that $v=0$. As in the previous proof we take $\alpha>\delta$ such that $\alpha+\sigma<\frac{1}{2}$ and then choose $q>2$ such that $\frac{1}{q}<\alpha-\delta$. We fix $x\in B$ and $T>0$. Let $F_n:[0,\infty)\times B\to B$ be defined by

$$F_n(s,x)=\begin{cases} F(s,x) & \text{if } |x|_B\leqslant n,\\ F\left(s,\dfrac{n}{|x|_B}x\right) & \text{otherwise.}\end{cases}$$

By (4.7) $|F_n(s,y)|\leqslant a(n)$, for all $s\geqslant 0$, $y\in B$. In view of Theorem 4.5 there exists a martingale solution

$$(\Omega_n,\mathscr{F}_n,\mathbb{P}_n,\{\mathscr{F}_{n,t}\}_{t\geqslant 0},\{W_n(t)\}_{t\geqslant 0},\{\bar u_n(t)\}_{t\geqslant 0}).$$

of the following equation:

$$\begin{aligned}&\mathrm{d}\bar u_n(t)=[-A\bar u_n(t)+F_n(s,\bar u_n(t))]\,dt+G(s,\bar u_n(t))\,\mathrm{d}W(t),\\ &\bar u_n(0)=x.\end{aligned}\tag{4.11}$$

Denote, for $t\in[0,T]$,

$$y_n(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}\mathrm{e}^{-(t-s)A}G(s,\bar u_n(s))\,\mathrm{d}W_n(s),\tag{4.12}$$

$$v_n=\Lambda_T^{-\alpha}y_n,\tag{4.13}$$

$$z_n(t)=\int_0^t\mathrm{e}^{-(t-s)A}F_n(s,\bar u_n(s)))\,\mathrm{d}s.\tag{4.14}$$

Notice that the process $v_n(t)$ is an admissible modification of the process given by the stochastic Itô integral $\int_0^t\mathrm{e}^{-(t-s)A}G(\bar u_n(s))\,\mathrm{d}W_n(s)$. Furthermore, $z_n(t)=\bar u_n(t)-v_n(t)-\mathrm{e}^{-tA}x$, $t\in[0,T]$. Let

$$\tau_n=\inf\{t\geqslant 0\colon |\bar u_n(t)|_B\geqslant n\}.$$

As in the proof of Theorem 4.5, because $A^{-\sigma}G:[0,\infty)\times B\to M(H,X)$ is bounded, we have

$$\sup_{n\geqslant 1}\mathbb{E}_n\int_0^T|y_n(t)|^q\,\mathrm{d}t<\infty.$$

Therefore, by Lemma 2.4 and (4.14)

$$\sup_{n\geqslant 1}\mathbb{E}_n\sup_{0\leqslant t\leqslant T}|v_n(t)|_B^q<\infty.\tag{4.15}$$

Moreover, by Corollary 2.8 and Assumption A.1, the family of laws of v_n is tight on $C(0,T;B)$. Since $\bar u_n(t)=z_n(t)+v_n(t)+\mathrm{e}^{-tA}x$ and

$$z_n(t)=\int_0^t\mathrm{e}^{-(t-s)A}F(s,\bar u_n(s))\,\mathrm{d}s,\quad t\leqslant\tau_n,\tag{4.16}$$

from Lemma 4.2 we infer that

$$|z_n(t)|\leqslant\int_0^t\mathrm{e}^{-k(t-s)}a(|v_n(s)+\mathrm{e}^{-sA}x|)\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau_n.\tag{4.17}$$

Set $C_1 := \sup_{0\leqslant t\leqslant T}|e^{-tA}x|_{D(A^\delta)} < \infty$. Then (4.17) yields

$$\sup_{0\leqslant t\leqslant \tau_n}|\bar u_n(t)|_B \leqslant C_1 + \frac{1+e^{|k|T}}{|k|}\sup_{0\leqslant t\leqslant T} a(|v_n(t)+e^{-tA}x|_B) + \sup_{0\leqslant t\leqslant T}|v_n(t)|_B. \quad (4.18)$$

Since the function a is increasing, the RHS of (4.18) is less than or equal to

$$\sup_{0\leqslant t\leqslant T}[C_1 + C_2 a(|v_n(t)|_B + C_1) + |v_n(t)|_B].$$

Let $R: m \mapsto R(m)$ be the inverse function of $r \mapsto C_1 + C_2 a(r+C_1) + r$. Since R is also increasing (and continuous). Therefore, by the Chebyshev inequality we get

$$\mathbb{P}_n\left(\sup_{0\leqslant t\leqslant \tau_n\wedge T}|\bar u_n(t)|_B \geqslant m\right) \leqslant \mathbb{P}_n\left(\sup_{0\leqslant t\leqslant T}|v_n(t)|_B \geqslant R(m)\right)$$
$$\leqslant \frac{1}{R(m)^q}\mathbb{E}_n\left(\sup_{0\leqslant t\leqslant T}|v_n(t)|_B^q\right).$$

Recall that

$$\left\{\sup_{0\leqslant t\leqslant T}|\bar u_n(t)|_B \geqslant n\right\} = \{\tau_n \leqslant T\}.$$

Thus, for any $n \geqslant m$,

$$\left\{\sup_{0\leqslant t\leqslant \tau_n\wedge T}|\bar u_n(t)|_B \geqslant m\right\} = \left\{\sup_{0\leqslant t\leqslant T}|\bar u_n(t)|_B \geqslant m\right\}.$$

Hence, by taking into account (4.15) we infer that

$$\sup_{n\geqslant m}\mathbb{P}_n\left(\sup_{0\leqslant t\leqslant T}|\bar u_n(t)|_B \geqslant m\right) \to 0 \quad \text{as } m \to 0. \quad (4.19)$$

Since $a(r)\nearrow\infty$ as $r\nearrow\infty$ by Assumption A.3, by taking into account (4.7) and(4.19) it follows that the laws of the family of processes $\bar u_n(t)$, $t\in[0,T]$, are uniformly bounded on $C(0,T;B)$. It follows, in view of (4.16) and Corollary 2.8, that the family of laws of z_n is tight on $C(0,T;B)$. On the other hand, by Corollary 3.8,

$$\bar u_n(t) = e^{-tA}x + z_n(t) + \Lambda_T^{-\alpha}y_n(t), \quad \text{a.s., } t\in[0,T].$$

Setting

$$u_n(t) = e^{-tA}x + z_n(t) + \Lambda_T^{-\alpha}y_n(t), \quad t\in[0,T],$$

we infer that the family of laws of $u_n(t)$, $t\in[0,T]$, is tight on $C(0,T;B)$. Now we can complete the proof as we have done at the end of the proof of Theorem 4.5. □

4.3. Uniqueness

In this subsection we are working within the framework of conditions (H1)–(H4) and Assumption A.1. We begin with

Theorem 4.7. *Assume that a Banach space X satisfying the conditions* (H1), *a Hilbert space H, a probability space $(\Omega,\mathscr{F},\mathbb{P})$ and an operator $A+\nu I$, for some $\nu\geqslant 0$, satisfy Assumption* A.1 *and condition* (H1)–(H4).

Assume that $A^{-\sigma}G(t,\cdot)$ is a locally Lipschitz map from $D(A^\delta)$ to $M(H,X)$ uniformly in $t\geqslant 0$ and that F is a locally Lipschitz map from $D(A^\delta)$ to X uniformly in $t\geqslant 0$. This means that for each $R>0$ there exists a constant K_R such that for all $t\geqslant 0$ and all $u,v\in D(A^\delta)$ satisfying $|u|_{D(A^\delta)},|v|_{D(A^\delta)}\leqslant R$

$$||A^{-\sigma}G(t,u)-A^{-\sigma}G(t,v)||_{M(H,X)}\leqslant K_R|u-v|_{D(A^\delta)}, \tag{4.20}$$

$$|F(t,u)-F(t,v)|_X\leqslant K_R|u-v|_{D(A^\delta)}. \tag{4.21}$$

Then, for any $T>0$, there exists at most one mild solution $u(t)$, $t<T$, to the problem (4.1).

Remark. The above result is similar to the uniqueness part of Theorem 4.9 from Brzeźniak (1997). However some technical details make them incomparable and we have to provide a complete proof of our Theorem 4.7. Nevertheless, our proof follows the lines of the corresponding proof of Brzeźniak (1997).

Proof of Theorem 4.7. It is sufficient to prove the lemma below.

Lemma 4.8. *Under the assumptions of Theorem* 4.7, *let U be an open subset of $D(A^\delta)$ on which functions F and $A^{-\sigma}G$ are uniformly Lipschitz continuous in the sense of* (4.20) *and* (4.21), *respectively. Let $\Omega_0\in\mathscr{F}_0$ be such $\mathbb{P}(\Omega_0)>0$ and*

$$u_1(0)|_{\Omega_0}=u_2(0)|_{\Omega_0}\in U \quad a.s.$$

Let $u_1(t)$, $t\geqslant 0$ and $u_2(t)$, $t\geqslant 0$ be two mild solutions to problem (4.1). *Let τ_i be the first exit time of $u_i(t)$ from U. Then*

$$\tau_1=\tau_2 \quad a.s.$$

and the processes

$$u_1|_{[0,\tau_1)\times\Omega_0}, \quad u_2|_{[0,\tau_2)\times\Omega_0} \tag{4.22}$$

are equivalent.

Proof of Lemma 4.8. We may assume that $\Omega_0=\Omega$ since we can normalize $\mathbb{P}$ on Ω_0 such that $\mathbb{P}(\Omega_0)=1$. In the same way we can assume that $u_i(0)\in U$, $i=1$, 2, a.s. Let $\tau=\tau_1\wedge\tau_2$. Denote $u(t)=u_1(t)-u_2(t)$, $t\geqslant 0$. Then, with $f(t)=F(t,u_1(t))-F(t,u_2(t))$ and $g(t)=G(t,u_1(t))-G(t,u_2(t))$, $u(t)$, $t\geqslant 0$ is a mild solution to the problem

$$\begin{aligned} &\mathrm{d}u(t)+Au(t)=f(t)\,\mathrm{d}t+g(t)\,\mathrm{d}W(t),\\ &u(0)=0. \end{aligned} \tag{4.23}$$

Since $u_1(t),u_2(t)\in U$ for $t<\tau$, we have

$$||A^{-\sigma}g(t)||_{M(H,X)}\leqslant K|u(t)|_{D(A^\delta)}, \quad t<\tau \text{ a.s.}, \tag{4.24}$$

$$||f(t)||_X\leqslant K|u(t)|_{D(A^\delta)}, \quad t<\tau \text{ a.s.} \tag{4.25}$$

Now we encounter a delicate (but a simple) problem. Take first δ,σ as in Assumption A.1. Then, in particular, $\delta+\sigma<\frac{1}{2}$ and $A^{-\sigma}\in M(H,X)$. Then there exist $q\geqslant 2$ and

$\gamma > 0$ such that $0 \leqslant \delta < \gamma < \delta + 1/q < \frac{1}{2} - \sigma$. With this choice applying a stopping time modification of Theorem 4.1 from Brzeźniak (1997) yields existence of a constant $C > 0$ such that for any $t \in [0, T]$ and any stopping time ξ such that $\xi < \tau$ a.s., the following holds:

$$\mathbb{E} \int_0^{t\wedge\xi} |u(s)|^q_{D(A^\gamma)}\, \mathrm{d}s + \mathbb{E} \sup_{0\leqslant s\leqslant t\wedge\xi} |u(s)|^p_{D(A^\delta)} \leqslant C\mathbb{E} \int_0^{t\wedge\xi} ||A^{-\sigma} g(s)||^q_{M(H,X)}\, \mathrm{d}s + C\mathbb{E} \int_0^{t\wedge\xi} |f(s)|^q_X\, \mathrm{d}s.$$

Thus, by taking into account (4.2), (4.25) and (4.24) we infer that

$$\mathbb{E} \sup_{0\leqslant s\leqslant t\wedge\xi} |u(s)|^q_{D(A^\delta)} \leqslant CK\mathbb{E} \int_0^{t\wedge\xi} |u(s)|^q_{D(A^\delta)}\, \mathrm{d}s.$$

Set

$$\alpha = \begin{cases} 1 & \text{if } t < \xi, \\ 0 & \text{if } t \geqslant \xi. \end{cases}$$

Then the following sequence of inequalities holds a.s.:

$$\sup_{0\leqslant s\leqslant t\wedge\xi} |u(s)|^q_{D(A^\delta)} = \sup_{0\leqslant s\leqslant t} |\alpha(s) u(s)|^q_{D(A^\delta)},$$

$$\int_0^{t\wedge\xi} |u(s)|^q_{D(A^\delta)}\, \mathrm{d}s \leqslant \int_0^t ||\alpha(s) u(s)||^q_{D(A^\delta)}\, \mathrm{d}s,$$

Therefore

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} |\alpha(s) u(s)|^q_{D(A^\delta)} \leqslant C \int_0^t \mathbb{E}|\alpha(s) u(s)|^q_{D(A^\delta)}\, \mathrm{d}s,$$

and hence the Gronwall Lemma yields that

$$\mathbb{E}|\alpha(t) u(t)|^q_{D(A^\delta)} = 0, \quad t \geqslant 0,$$

which implies that $u(t) = 0$ a.s. on $\{t < \xi\}$. Taking a sequence ξ_n of stopping times such that $\xi_n \nearrow \tau$ a.s. we infer that $u(t) = 0$ a.s. on $\{t < \tau\} = \Omega_t$ which proves (4.22).

The proof that $\tau_1 = \tau_2$ can be then performed in the same way as the corresponding part of the proof of Theorem 5 from Section VI of Elworthy (1982). □

Remark 4.9. For a different approach to the uniqueness question, more analogous to the finite-dimensional one as described in Kunita (1990); see Carroll's (1999) thesis.

Following the scheme of Yamada–Watanabe uniqueness theorem, see the proof of Theorem 1.1 in Chapter 4 of Ikeda and Watanabe (1981), and using the just proven uniqueness of mild solutions, we get the following result on uniqueness of martingale solutions.

Theorem 4.10. *Assume that a Banach space X satisfying the conditions* (H1), *a Hilbert space H, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an operator $A + \nu I$, for some $\nu \geqslant 0$, satisfy Assumptions* A.1–A.3 *and conditions* (H1)–(H4). *Assume that the maps $A^{-\sigma}G$ and F are locally Lipschitz maps uniformly in $t \geqslant 0$ in the sense of* (4.20) and (4.21). *Then for any $x \in B$ there exists a unique martingale solution of* (4.1).

4.4. Feller property

If the martingale solution is unique, i.e. when Assumption A.6 is satisfied, $u(t)$ is a Markov process, see Da Prato and Zabczyk (1992a, Theorem 9.14). For $x \in B$ let $(\Omega, \mathscr{F}, \mathbb{P}_x, \{\mathscr{F}_t\}, u(t), \{W(t)\})$ be the unique martingale solution to (4.1) with $u_0 = x$. Note that although all the objects depend on x we use the subscript x only in denoting the probability measure $\mathbb{P}_x$ and the expectation $\mathbb{E}_x$.

Define the transition operator P_t by a standard formula: let $\varphi \in \mathscr{C}_b(B)$, then

$$P_t\varphi(x) = \mathbb{E}_x[\varphi(u(t))]. \tag{4.26}$$

Theorem 4.11. *Assume that a Banach space X satisfying the conditions* (H1), *a Hilbert space H, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an operator $A + \nu I$, for some $\nu \geqslant 0$, satisfy Assumptions* A.1–A.3 *and* A.6 *and* conditions (H1)–(H4). *Let for $t \geqslant 0$, P_t be the transition operator of the process $u(t)$. Then P_t is a family of Feller operators, i.e. $P_t : C_b(B) \to C_b(B)$ and, for any $\varphi \in \mathscr{C}_b(B)$ and $x \in B$,*

$$P_t\varphi(x) \to \varphi(x) \quad \text{as } t \to 0. \tag{4.27}$$

Proof of Theorem 4.11. Let $x \in B$ and $\varphi \in C_b(B)$. Let $(\Omega, \mathscr{F}, \mathbb{P}_x, \{\mathscr{F}_t\}, u(t), \{W(t)\})$ be the martingale solution to (4.1). Since $u(t)$ is a B-valued admissible process, $u(t) \to x$ as $t \searrow 0$ $\mathbb{P}_x$-a.s. Thus, since φ is bounded and continuous, (4.27) follows by applying Lebesgue-dominated convergence theorem.

Let $x_n \to x \in B$ and $t > 0$. Fix $T > t$. Let $(\Omega_n, \mathscr{F}^n, \mathbb{P}_{x_n}, \{\mathscr{F}_t^n\}, u(t, x_n), \{W_n(t)\})$ be the martingale solution to (4.1) with the initial condition x replaced by x_n. Let μ_{x_n} be the law of $u(\cdot, x_n)$ on $C(0, T; B)$.

We will show that the family of measures μ_{x_n} is tight (on $C(0, T; B)$ of course). Set (see also (4.14), for $t \in [0, T]$,

$$y(t, x_n) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \mathrm{e}^{-(t-s)A} G(s, u(s, x_n))\, \mathrm{d}W_n(s), \tag{4.28}$$

$$v(\cdot, x_n) = \Lambda_T^{-\alpha} y(\cdot, x_n), \tag{4.29}$$

$$z(t, x_n) = \int_0^t \mathrm{e}^{-(t-s)A} F(u(s, x_n)))\, \mathrm{d}s. \tag{4.30}$$

Note that it follows from Corollary 3.8 that $\mathbb{P}_{x_n}$ a.s.

$$u(\cdot, x_n) = \mathrm{e}^{-\cdot A} x + \Lambda_T^{-1} F(s, u(\cdot, x_n)) + \Lambda_T^{-\alpha} y(\cdot, x_n). \tag{4.31}$$

As in the proof of Theorem 4.5 there is an $M > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{E}|y(\cdot, x_n)|^q_{L^q(0,T;X)} \leqslant M,$$

$$\mathbb{E}|F(u(\cdot, x_n))|^q_{L^q(0,T;X)} \leqslant M.$$

Thus, as $\mathrm{e}^{-\cdot A} x_n \to \mathrm{e}^{-\cdot A} x$ in $C(0, T; B)$, and $\Lambda_T^{-\alpha}$ and Λ_T^{-1} are compact operators from $L^q(0, T; X)$ to $C(0, T; B)$, our claim follows.

Let μ be any cluster point of μ_{x_n}. By using the Skorohod imbedding theorem, compare with the end of the proof of Theorem 4.5, we may show that there is a martingale solution $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}_x, \{\tilde{\mathscr{F}}_t\}, \tilde{u}(t), \{\tilde{W}_t\})$ to the problem (4.1) with initial condition x such that the law on $C(0,T;B)$ of this solution is equal to μ. By uniqueness of martingale solutions (i.e. by Assumption A.6), $\mu = \mu_x$. Hence μ_x is a weak limit of some subsequence of the measures μ_{x_n}. A standard subsequence–subsubsequence argument shows that in fact $\langle \mu_{x_n}, \varphi \rangle \to \langle \mu_x, \varphi \rangle$. In other words, $P_t\varphi(x_n) \to P_t\varphi(x)$. Hence P_t is a Feller semigroup. □

5. Invariant measures

Let us point out that it is exactly this section where we make use of Assumption A.4. Furthermore, in this section, contrary to the previous one, we assume that the operator A itself, not simply $A + \nu I$ for some $\nu \geqslant 0$, satisfies conditions (H2) and (H3) and Assumptions A.1–A.3.

Definition 5.1. Suppose that $(P_t)_{t\geqslant 0}$ is a Feller semigroup on a Polish space B. A Borel probability measure μ on B is called an invariant measure for $(P_t)_{t\geqslant 0}$ iff

$$P_t^* \mu = \mu, \quad t \geqslant 0,$$

where $(P_t^*\mu)(\Gamma) = \int_B P_t(x,\Gamma)\mu(\mathrm{d}x)$ for $\Gamma \in \mathscr{B}(B)$ and the $P_t(x,\cdot)$ is the transition probability, $P_t(x,\Gamma) = P_t(1_\Gamma)(x)$, $x \in B$.

Theorem 5.2. *Assume that a Banach space X satisfying the conditions* (H1), *a Hilbert space H, a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an operator A satisfy Assumptions* A.1–A.6 *and conditions* (H1)–(H4). *Then there exists an invariant measure μ for the semigroup P_t, i.e. a probability measure μ on B such that $\mu P_t = \mu$.*

Proof of Theorem 5.2. Let $x \in B$ be a fixed point. Let $(\Omega, \mathscr{F}, \mathbb{P}_x, \{\mathscr{F}_t\}, u(t), \{W(t)\})$ be the martingale solution to (4.1). We need

Proposition 5.3. *There exists $x \in B$ such that the family of laws of $u(t)$, $t \geqslant 1$ is tight on B.*

Since the process is Feller on B, invoking the above Proposition 5.3 by standard Krylov–Bogoliubov technique there exists an invariant measure μ on B for the semigroup P_t. For details we refer to Da Prato and Zabczyk (1992a), Proposition 11.2.

Proof of Proposition 5.3. Take $x = 0$ and define

$$v(t) = \int_0^t \mathrm{e}^{-(t-s)A} G(s, u(s))\,\mathrm{d}W(s), \tag{5.1}$$

$$y(t) = \int_0^t (t-s)^{-\alpha} \mathrm{e}^{-(t-s)A} G(s, u(s))\,\mathrm{d}W(s), \tag{5.2}$$

$$z(t) = u(t) - v(t) = \int_0^t \mathrm{e}^{-(t-s)A} F(s, u(s))\,\mathrm{d}s. \tag{5.3}$$

Our first tool will be the following obvious lemma.

Lemma 5.4. *Assume that B is a complete topological vector space and $(\Omega,\mathscr{F},\mathbb{P})$ a probability space. Let I be any nonempty index set. Given are two families η_i, $i\in I$ and ζ_i, $i\in I$ of B-valued random variables such that the laws of each family are tight on B (separately), i.e. for any $\varepsilon >$ there exist compact sets K and L in B such that*

$$\mathbb{P}\{\eta_i \notin K\} < \varepsilon, \quad i\in I, \tag{5.4}$$

$$\mathbb{P}\{\zeta_i \notin L\} < \varepsilon, \quad i\in I, \tag{5.5}$$

Set $\xi_i := \eta_i + \zeta_i$, $i \in I$. Then the family of laws of $\xi_i, i\in I$, is tight on B.

In order to prove Proposition 5.3 it suffices to show the following two lemmata.

Lemma 5.5. *The family of laws of $v(t)$, $t\geqslant 1$ is tight on B.*

Lemma 5.6. *The family of laws of $z(t)$, $t\geqslant 1$ is tight on B.*

Proof of Lemma 5.5. Let us first put together some basic facts about tightness of laws of families of random variables. Assume that B and Y are complete topological vector spaces and $(\Omega,\mathscr{F},\mathbb{P})$ is a probability space. Let I be any nonempty index set. Given is a compact (not necessarily linear) map $\Phi : B\to Y$, i.e. for any bounded set $G\subset B$ the image $\Phi(G)$ is relatively compact in Y. Given is a uniformly bounded in probability family ξ_i, $i\in I$, of B-valued random variables, i.e. for any $\varepsilon>0$ there exist a bounded set $G\subset B$ such that

$$\mathbb{P}\{\xi_i \notin G\} < \varepsilon, \quad i\in I. \tag{5.6}$$

Then the family of laws of $\eta_i := \Phi(\xi_i)$, $i\in I$ is tight on Y.

Secondly, if B is a Banach space and for some $q\geqslant 1$ $\sup_{i\in I}\mathbb{E}|\xi_i|^q<\infty$, then the family ξ_i, $i\in I$, is uniformly bounded in probability. This follows from the Chebyshev inequality $\mathbb{P}(|\xi_i|>R)\leqslant R^{-q}\mathbb{E}|\xi_i|^q$.

Let $\delta\in(0,\frac{1}{2})$ be as in Assumption A.1. Take $\alpha>\delta$ such that $\alpha+\sigma<\frac{1}{2}$ and finally choose q such that $1/q<\alpha-\delta$. Note that necessarily $q>2$. Set, for $t\geqslant 0$,

$$u_t(s) := u(t+s), \quad s\in[0,1], \tag{5.7}$$

$$y_t(s) := \frac{1}{\Gamma(1-\alpha)}\int_0^s (s-r)^{-\alpha}\mathrm{e}^{-(s-r)A}G(r,u_t(r))\,\mathrm{d}W(r), \quad s\in[0,1]. \tag{5.8}$$

Then, for $t\geqslant 0$, a.s.

$$v(t+1) = \mathrm{e}^{-A}v(t) + K_\alpha(y_t), \tag{5.9}$$

where

$$K_\alpha(f) := (\Lambda_1^{-\alpha}f)(1), \quad f\in L^q(0,1;X). \tag{5.10}$$

Since, due to Assumption A.1, e^{-A} is a compact operator from X into B and, due to Corollary 2.8 and the fact that $\delta+1/q<\alpha$, K_α is a compact operator from $L^q(0,1;X)$ into B, by virtue of Lemma 5.4 and the discussion above it suffices to show that

$$\sup_{t\geqslant 0}\mathbb{E}|v(t)|_X^q < \infty, \tag{5.11}$$

$$\sup_{t\geqslant 0}\mathbb{E}\int_0^1 |y_t(s)|_X^q\,\mathrm{d}s<\infty. \tag{5.12}$$

Proof of (5.11). This proof is similar to the proof of Lemma 3.7. Fix $t>0$. From Burkholder inequality

$$\mathbb{E}|v(t)|_X^q\leqslant C_q\mathbb{E}\left(\int_0^t ||\mathrm{e}^{-(t-s)A}G(s,u(s))||_{M(H,X)}^2\,\mathrm{d}s\right)^{q/2}.$$

Using condition (3.36) in view of Assumptions A.5 and A.2 we get

$$\begin{aligned}|\mathrm{e}^{-\tau A}G(s,u(s))|_{M(H,X)}&\leqslant ||A^\sigma \mathrm{e}^{-\tau A}||_{\mathscr{L}(X)}||A^{-\sigma}G(s,u(s))||_{M(H,X)}\\&\leqslant C\tau^{-\sigma}\mathrm{e}^{-a\tau}\end{aligned}$$

for some generic (independent of $\tau>0$) constant $C>0$. Thus

$$\begin{aligned}\mathbb{E}|v(t)|_X^q&\leqslant C\left(\int_0^t (t-s)^{-2\sigma}\mathrm{e}^{-2a(t-s)}\,\mathrm{d}s\right)^{q/2}\\&\leqslant C\left(\int_0^\infty s^{-2\sigma}\mathrm{e}^{-2as}\,\mathrm{d}s\right)^{q/2}. \qquad\square\end{aligned}$$

Proof of (5.12). This inequality follows from Lemma 3.7 by noting that due to Assumption A.2

$$\sup_{t\geqslant 0}\mathbb{E}\int_0^1 ||A^{-\sigma}G(t,u_t)||_{M(H,X)}^q<\infty.$$

The proof of Lemma 5.5 is therefore complete. $\square$

Proof of Lemma 5.6. From Lemma 4.2

$$|z(t)|_B\leqslant\int_0^t \mathrm{e}^{-k(t-s)}a(|v(s)|_B)\,ds,\quad t\geqslant 0.$$

Hence

$$\mathbb{E}|z(t)|_B\leqslant\int_0^t \mathrm{e}^{-k(t-s)}\mathbb{E}a(|v(s)|_B)\,\mathrm{d}s,\quad t\geqslant 0, \tag{5.13}$$

and so

$$\mathbb{E}|z(t)|_B\leqslant\frac{1}{k}\sup_{s\geqslant 0}\mathbb{E}a(|v(s)|_B),\quad t\geqslant 0. \tag{5.14}$$

On the other hand, from (5.11), in view of Assumptions A.4 and A.5, we infer that

$$\sup_{t\geqslant 0}\mathbb{E}a(|v(t)|_B)<\infty, \tag{5.15}$$

from which and (5.14) it follows that

$$\sup_{t\geqslant 0}\mathbb{E}|z(t)|_B<\infty. \tag{5.16}$$

Analogously as in Da Prato and Zabczyk (1992) we represent

$$z(t+1)=\mathrm{e}^{-A}z(t)+K_\alpha(F(u_t)),$$

where K_α and y_t were defined in (5.10) and (5.7), respectively. Since e^{-A} is a compact operator in B it follows from (5.16) that the family of laws of $\{\mathrm{e}^{-A}z(t)\}$, $t\geqslant 1$ is tight on B. In virtue of Lemma 5.4, Assumption A.1 and Corollary 2.8, the proof of Lemma 5.6 will be completed as soon as we show the following.

Lemma 5.7. *The family of laws of* $F(u_t)$, $t \geqslant 0$ *is uniformly bounded in probability on* $L^q(0,1;B)$.

Proof of Lemma 5.7. Since $F(x) \leqslant a(|x|_B)$ by (4.7) it is sufficient to show that the family of laws of $a(u_t(\cdot))$, $t \geqslant 0$ is uniformly bounded in probability on $L^q(0,1;B)$. Since, in view of Assumption A.4, $a(|u_t(s)|_B) \leqslant k_0(1 + |u_t(s)|_B^N)$ for some $k_0, N > 0$, it suffices to show that laws of u_t, $t \geqslant 0$, are uniformly bounded in probability on $L^{qN}(0,1;B)$. The last statement follows from (5.11) and (5.16) by means of Chebyshev inequality. Indeed, if $t \geqslant 0$ then for some C> 0 and all $t \geqslant 0$

$$
\begin{aligned}
\mathbb{E}|u_t|^{qN}_{L^{qN}(0,1;B)} &= \mathbb{E} \int_t^{t+1} |u(s+t)|_B^{qN}\, \mathrm{d}s \\
&= \mathbb{E} \int_t^{t+1} |v(t+s) + z(t+s)|_B^{qN}\, \mathrm{d}s \\
&\leqslant 2^{qN-1} \left(\sup_{t \leqslant r \leqslant +1} \mathbb{E}|v(r)|_B^{qN} + \sup_{t \leqslant r \leqslant t+1} \mathbb{E}|z(r)|_B^{qN} \right) \\
&\leqslant C.
\end{aligned}
$$

This concludes the proof of Lemma 5.7 which completes the proof of Proposition 5.3 and the same the proof of Theorem 5.2 is complete. □

6. Examples and applications

6.1. Reaction–diffusion equation

Let $\mathcal{O}$ be a bounded open interval in $\mathbb{R}^d$, $d \geqslant 1$. Let $H = H^{\theta,2}(\mathcal{O})$ for fixed $\theta > d/2 - 1$. Let

$$
\mathscr{A} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + a_0(x)
$$

be a second-order differential uniformly elliptic operator, i.e. such that for some $C > 0$ and all $x \in \mathcal{O}$, $\lambda \in \mathbb{R}^d$

$$
\sum_{i,j=1}^{d} a_{ij}(x)\lambda_i\lambda_j \geqslant C|\lambda|^2.
$$

If $\theta = 0$, $d = 1$ and $\mathscr{A} = \Delta$ we are in the case studied in Da Prato and Pardoux (1995). Assume that the functions a_{ij} and a_0 are of $\mathscr{C}^2$ class on $\bar{\mathcal{O}}$. Assume that f and g are separately continuous real valued functions defined on $[0,\infty) \times \mathcal{O} \times \mathbb{R}$. Assume that g is a bounded function. Consider the following condition

$$
-K(1 + |u|^q 1_{\{u \geqslant 0\}}) \leqslant f(t,x,u) \leqslant K(1 + |u|^q 1_{\{u \leqslant 0\}}), \quad t \geqslant 0,\, x \in \mathcal{O},\, u \in \mathbb{R}, \tag{6.1}
$$

where $K > 0$. It is easy to prove that if f satisfies the condition (6.1) then $f(t,x, v+z)\,\mathrm{sgn}\, v \leqslant K(1 + |z|^q)$ for all $v, z \in \mathbb{R}$ and $t \geqslant 0$, $x \in \mathcal{O}$.

Let $\xi(t,x)$ be a space-time white noise on D with Cameron–Martin space equal to H. We are interested in solutions to the following initial value problem:

$$\begin{aligned}
&\mathrm{d}u(t,x)+\mathscr{A}u(t,x)\,\mathrm{d}t = f(t,x,u(t,x))\,\mathrm{d}t + g(t,x,u(t,x))\,\xi(t,x),\\
&u(t,\cdot)=0 \quad \text{on } \partial\mathcal{O},\\
&u(0,x)=u_0(x), \quad x\in\mathcal{O}.
\end{aligned} \tag{6.2}$$

which will be interpreted as a solution to the problem (4.1) with $W(t)$ being an H-cylindrical Wiener process on some complete probability space in the following framework.

Define $B=C_0(\mathcal{O})$, $X=L^p(\mathcal{O})$, $D(A)=H_0^{2,p}(\mathcal{O})$ with $Au=\mathscr{A}u$ for $u\in D(A)$. Obviously, X satisfies the conditions (H1) and the conditions (H2) and (H3) hold for λ_0+A for some $\lambda_0\geqslant 0$, see Seeley (1971). Moreover, Assumptions A.1 and A.3 are satisfied as well. Indeed, by Lemma 3.12 and Remark 3.13 there exists $\sigma\in(0,\frac{1}{2})$ such that $A^{-\sigma}\in M(H,X)$. Moreover, $D(A^s)=H_0^{2,p}(\mathcal{O})$ for $d/2p<s<\frac{1}{2}$, and thus by choosing p large enough (to be precise satisfying $d/2p+\sigma<\frac{1}{2}$) we see that $D(A^s)\hookrightarrow B$ and so the first part of Assumption A.1 is satisfied. The second part of the latter is satisfied due to Theorem 5 in Stewart (1974).

It follows from the maximum principle and Stewart (1974) that the operator $-A$ on B is m-dissipative, if $k:=\inf a_0(x)\geqslant 0$. Therefore, Assumption A.3 is satisfied. Moreover, if $\inf a_0(x)>0$ then also Assumptions A.4 and A.5 are satisfied.

In the framework described above the results from Sections 4 and 5 hold true. In particular we have

Theorem 6.1. *In the framework described above and under the assumptions listed therein there exists a martingale solution to the problem* (6.2). *Moreover, if f and g are locally Lipschitz continuous functions with respect to the third variable* (*i.e.* u) *then the martingale solution is unique. Finally, if also* $\inf a_0(x)>0$, *then there exists an invariant measure for the problem* (6.2).

Proof. First of all we need to verify Assumption A.2. Let G be the Nemytski map associated with function g. Since $\theta>d/2-1$ one can find $\sigma<1/2$ such that $A^{-\sigma}$ is a γ-radonifying operator from H into X. Hence by involving a result of Neidhardt (see Brzeźniak, 1997, Theorem 2.1), it is sufficient to show that G is bounded, continuous with respect to u and strongly measurable with respect to t as a map from $[0,\infty)\times B$ into $\mathscr{L}(H)$, what is obviously satisfied. □

Assumption A.3 is a consequence of the above and following preparatory result a proof of which in the case when f depends only on u, the third variable can be found in Da Prato and Zabczyk (1992b, pp. 193–194).

Proposition 6.2. *In the framework described above, assume that f satisfies the condition* (6.1) *and let $F(t,\cdot)$ be the Nemytski operators associated with f, i.e.*

$$F(t,u)(x)=f(t,x,u(x)), \quad u\in B,\ x\in\mathcal{O},\ t\geqslant 0.$$

Then for all $t\geqslant 0$ and any $u,v\in B$ and any $z\in u^*=\partial|u|$ the following holds:

$$\langle F(t,u+v),z\rangle\leqslant K(1+|v|_B^q). \tag{6.3}$$

Let us also note that obviously F is a continuous map from B into itself.

Remarks. (i) The question of existence of invariant measures in the case A symmetric and $g(x,u)=1$ with some different assumptions on f was studied in Manthey and Maslowski (1992).

(ii) Any function f of the form $f(t,u)=-u^{2n+1}+\sum_{i=0}^{2n} a_j(t)u^j$ with a_j being continuous real-valued functions satisfies the condition (6.1) and the appropriate assumptions of Theorem 6.1. As a function g we take the following (see Tribe, 1995):

$$g(u)=\begin{cases}\sqrt{u(1-u)} & \text{if } u\in[0,1],\\ 0 & \text{otherwise.}\end{cases} \tag{6.4}$$

6.2. *Equations of QFT type*

Assume that $\mathcal{O}\subset\mathbb{R}^d$ is a bounded domain with smooth boundary and let $X=L^p(\mathcal{O})$ with $p\geqslant 2$ and $A=\sqrt{-\Delta_D+m^2}$, where $m\geqslant 0$ and $-\Delta_D$ is the Laplace operator with Dirichlet boundary conditions. Also let $B=C_0(D)$. The Neumann boundary conditions can be treated without any significant difference. We have $D(A)=H_0^{1,p}(\mathcal{O})$. It follows from Seeley (1971) that the operator A satisfies the conditions (H2) and (H3). As in many examples earlier in this paper the condition (H1) is satisfied as well. Let $H=H^{\theta,2}(\mathcal{O})$ with $\theta>\frac{1}{2}(d-1)$ and let $W(t)$ be the H-cylindrical Wiener process on some complete probability space. It follows from Remark 3.13 that $A^{-\sigma}$ is a γ-radonifying operator from H into X if $\sigma+\theta>d/2$. Suppose that p is chosen large enough so that

$$\frac{d}{2}-\theta<\frac{1}{2}-\frac{d}{p}.$$

Note that under these assumptions we can find $\delta,\sigma\geqslant 0$ such that $\sigma+\delta<\frac{1}{2}$, $A^{-\sigma}\in M(H;X)$ and $D(A^{\delta})\hookrightarrow B$. Hence Assumption A.1 is satisfied.

Assume that f and g are separately continuous real functions defined on $[0,\infty)\times\mathcal{O}\times\mathbb{R}$ which are locally bounded in time, i.e. for each $T>0$ there is $C>0$ such that

$$|f(t,x,u)|,|g(t,x,u)|\leqslant C;\quad \text{if } x\in\mathcal{O},\ u\in\mathbb{R},\ 0\leqslant t\leqslant T.$$

We consider the following SPDE:

$$\begin{aligned}&\mathrm{d}u+\sqrt{-\Delta_D+m^2}u\,\mathrm{d}t=f(t,x,u)\,\mathrm{d}t+g(t,x,u)\,\mathrm{d}W(t),\quad t>0,\\ &u(0)=u_0,\end{aligned} \tag{6.5}$$

where $u_0\in B$. This equation is similar to the equation of free field found by Hida and Streit (1977); see also Rozovskij (1983) and Brzeźniak (1995).

Theorem 4.5 is applicable and so we have

Theorem 6.3. *For any $u_0\in C_0(D)$ there exists a martingale solution of* (6.5). *Moreover, if the functions f and g are locally Lipschitz continuous with respect to u then the solution is unique.*

6.3. Higher-order equations

This section is devoted to presentation of general applications of the results from Sections 4 and 5. These results are generalizations of the two examples from the previous subsections to more systems of elliptic operators with more general boundary conditions. For the sake of completeness of the exposition, we present the precise results below.

Let $\mathcal{O}$ be a bounded open domain in $\mathbb{R}^d$ with a boundary of class $\mathscr{C}^\infty$. We make the following assumptions

(i) The differential operator $-A$

$$-A = \sum_{|\alpha|\leqslant 2k} a_\alpha(x) D^\alpha \tag{6.6}$$

is properly elliptic, see Triebel (1978, 4.9.1). The coefficients a_α are $\mathscr{C}^\infty$ functions on the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$.

(ii) A system $\{C_j\}_{j=1}^k$ of differential operators on $\partial\mathcal{O}$ is given,

$$C_j = \sum_{|\alpha|\leqslant m_j} c_{j,\alpha}(x) D^\alpha, \tag{6.7}$$

with the coefficients $c_{j,\alpha}$ being $\mathscr{C}^\infty$ functions on $\partial\mathcal{O}$. The orders m_j of the operators C_j are ordered in the following way:

$$0 \leqslant m_1 < m_2 < \cdots < m_k.$$

The system $\{C_j\}$ is normal, i.e. $m_k < 2k$ and

$$\sum_{|\alpha|=m_j} c_{j,\alpha}(x) \nu_x^\alpha \neq 0, \quad x \in D,\ j=1,\dots,k, \tag{6.8}$$

where ν_x is the unit outer normal vector to $\partial\mathcal{O}$ at $x \in \partial\mathcal{O}$.

(iii)

$$(-1)^k \frac{a(x,\xi)}{|a(x,\xi)|} \neq -1, \quad x \in \bar{\mathcal{O}},\ \xi \in \mathbb{R}^n \setminus \{0\}, \tag{6.9}$$

where $a(x,\xi) = \sum_{|\alpha|=2k} a_\alpha(x)\xi^\alpha$.

(iv) If $c_j(x,\xi) = \sum_{|\alpha|=m_j} c_{j,\alpha}(x)\xi^\alpha$ then for all $x \in \partial\mathcal{O}, \xi \in T_x(\partial\mathcal{O}), t \in (-\infty, 0]$ the polynomials

$$\{\tau \to c_j(x, \xi + \tau\nu_x)\}, \quad j=1,\dots,k$$

are linearly independent modulo polynomial $\{\tau \to \prod_{j=1}^k (\tau - \tau_j^+(t)\}$. Here $\tau_j^+(t)$ are the roots with positive imaginary part of the polynomial $\mathbb{C} \ni \tau \to a(x, \xi + \tau\nu_x) - t$.

The differential operator A gives rise to a linear unbounded operator A_p in a Banach space $X = L^p(\mathcal{O})$ with a domain $D(A_p)$ defined by

$$D(A_p) = H^{2k,p}_{\{C_j\}}(\mathcal{O}) = \left\{ u \in H^{2k,p}(\mathcal{O}) \colon C_j u|_{\partial\mathcal{O}} = 0 \text{ for } m_j < 2k - \frac{1}{p} \right\}. \tag{6.10}$$

It has been shown by Seeley in Seeley (1971), see also Triebel (1978, 4.9.1), that for any $\gamma > 0$ there is $C = C_\gamma > 0$ such that

$$||A_p^{it}|| \leqslant C e^{\gamma|t|}, \quad t \in \mathbb{R},$$

and therefore the operator A_p satisfies the condition (H3) from Section 2, see also Dore and Venni (1987).

When there is no danger of ambiguity, the operator A_p will be denoted simply as A. In order to be able to apply our results from Sections 4 and 5 we need to determine the spaces of the fractional powers of the operator A. From Triebel (1978, Theorem 4.3.3) we have

$$D(A^\alpha) = H^{2k\alpha}_{p,2;\{C_j\}}(\mathcal{O}), \tag{6.11}$$

where $H^{2k\alpha}_{p,2;\{C_j\}}(\mathcal{O}) = \{u \in H^{2k\alpha}_{p,2}(D)\colon C_j u|_{\partial D} = 0 \text{ for } m_j < 2k\alpha - 1/p\}$.

Example 6.1. In this example $\mathcal{O}$ is as before but we take $k = 1$, i.e. the operator A is of second order. We assume that it is given in the following divergence form:

$$(Au)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) + \sum_{i=1}^{n} d_i(x)\frac{\partial u(x)}{\partial x_i} + e(x)u(x), \quad x \in \mathcal{O}, \tag{6.12}$$

with all coefficients of $\mathscr{C}^\infty$ class on the closure $\bar{\mathcal{O}}$ of a bounded domain $\mathscr{C}^\infty$ domain $\mathcal{O}$ and the matrix $[a_{ij}(x)]$ not necessarily symmetric. The boundary operator C is given by

$$(Cu)(x) = \varepsilon\frac{\partial u}{\partial \nu_A}(x) + \beta(x)u(x), \quad x \in \partial\mathcal{O}, \tag{6.13}$$

again with $\mathscr{C}^\infty$ coefficients, where the first term is the "co-normal" derivative with respect to A,

$$\frac{\partial u}{\partial \nu_A}(x) = \sum_{i,j} a_{ij}(x)\nu_x^j \frac{\partial u}{\partial x_i}(x),$$

where $\nu_x = (\nu_x^1, \ldots, \nu_x^n)$ is the unit outer normal vector to ∂D at point $x \in \partial\mathcal{O}$.

We take $X = L^p(\mathcal{O}), D(A) = H^{2,p}_{\{C\}}(\mathcal{O})$ and the operator A with domain $D(A)$ acting in X via formula (6.12).

Now we present the main result of this section.

Theorem 6.4. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with boundary $\partial\mathcal{O}$ of $\mathscr{C}^\infty$ class. Let A be a differential operator satisfying the properties* (i)–(iv) *above. Also let $A = A_p$ denote a linear operator in $X = L^p(\mathcal{O})$ with domain as in* (6.10), *where* $p \geqslant 2$. *Let $H = H^{\theta,2}(\mathcal{O})$ with $d/2 > \theta > d/2 - k$. Assume that $W(t)$, $t \geqslant 0$, is an H-cylindrical Wiener process and $w_1(t), \ldots, w_n(t)$, $t \geqslant 0$, be an independent n-dimensional Wiener process on the same complete probability space. Suppose that δ is a positive number such that $d/2k - \theta/2 < 1/2 - \delta$. Suppose that D_0 is a differential operator on $\mathcal{O}$ of order $d_0 < 2k$. Finally let us assume that $B_1, \ldots, B_n$ are linear differential operators of orders $< k$,*

$$B_j = \sum_{|\alpha| < k} b_{j,\alpha}(x) D^\alpha, \quad j = 1, \ldots, n, \tag{6.14}$$

with the coefficients $b_{j,\alpha}$ of $\mathscr{C}^\infty$ class.

Assume that f and g are separately continuous real functions defined on $[0,\infty)\times\mathcal{O}\times\mathbb{R}$ which are locally bounded in time, i.e. for each $T>0$ there is $C>0$ such that

$$|f(t,x,u)|,|g(t,x,u)|\leqslant C \quad \text{if } x\in\mathcal{O},\ u\in\mathbb{R},\ 0\leqslant t\leqslant T.$$

Suppose finally that $u_0\in B=D(A^\delta)$. Then the problem (6.15) *below has a martingale solution*

$$\begin{aligned} &\mathrm{d}u(t,x)+Au(t,x)\,\mathrm{d}t+\sum_{j=1}^{d}B_ju(t,x)\,\mathrm{d}w_j(t)\\ &\quad=D_0f(t,x,u)\,dt+g(t,x,u)\,\mathrm{d}W(t),\quad t>0,x\in\mathcal{O}, \qquad (6.15)\\ &u(0,x)=u_0(x)\quad \text{for } x\in\mathcal{O},\\ &C_ju(t,x)=0\quad \text{for } x\in\partial\mathcal{O},\ t>0. \end{aligned}$$

If the functions f and g are locally Lipschitz continuous with respect to the variable u, the solution is unique.

Proof. By the assumption we can find a positive number σ such that $1/2-\delta>\sigma>(1/k)(d/4-\theta/2)$. With $B=D(A^\delta)$ the first of the last two inequalities implies the first part of Assumption A.1. The second one implies that $k\sigma+\theta/2>d/4$ which in view of Remark 3.13 implies that $A^{-\sigma}\in M(H,X)$. The result follows from Theorem 4.5 by standard procedure. We only have to choose $\beta=d_0/2k$. □

Remark 6.5. Note that the condition $d/2>\theta>d/2-k$ implies that one can find a positive number δ such that $d/2k-\theta/2<1/2-\delta$.

We close this paper with a generalization of result from Funaki (1989) (but in a bounded domain) on a stochastic Ginzburg–Landau equation.

Example 6.2. Let $\mathcal{O}=(0,1)$ and $A=A_p=\Delta^2-\gamma\Delta$ for some $\gamma>0$ with $X=L^p(\mathcal{O})$ and $D(A)=\{u\in H^{4,p}(\mathcal{O}): u(0)=u(1), u'(0)=u'(1)\}$. Let $H=H^{1,2}(\mathcal{O})$. The norm on H is chosen in such a way the the derivative $\nabla: L^2(\mathcal{O})\to H$ is a unitary isomorphism. Suppose that $V:\mathbb{R}\to\mathbb{R}$ is a $\mathscr{C}^1$ function with a bounded derivative and that $g:\mathbb{R}\to\mathbb{R}$ is a $\mathscr{C}^1$ function. Consider the following problem, see Eq. (5.1) in Funaki (1989):

$$\mathrm{d}u(t,x)+(\Delta^2-\gamma\Delta)u(t,x)\,\mathrm{d}t=\nabla(V'(u(t,x)))\,\mathrm{d}t+g(u(t,x))\,\mathrm{d}W(t), \qquad (6.16)$$

$$u(t,0)=u(t,1),\quad u'(t,0)=u'(t,1),\ t>0, \qquad (6.17)$$

$$u(0)=u_0. \qquad (6.18)$$

It is assumed in Funaki (1989) that $V\in\mathscr{C}_b^3(\mathbb{R})$ and $g=\sqrt{2}$ which implied that his problem had all coefficients globally Lipschitz. Obviously, this is not our case. Existence of martingale solutions to problem (6.16)–(6.18) follows from Theorem 6.4. Indeed, $\theta=-1$ is larger than $d/2-k=1/2-2$ and the order of the operator $D_0=\Delta$ is less that $2k=4$. If the functions g and V' are also locally Lipschitz, then the martingale solution is unique.

7. For further reading

The following references are also of interest to the reader: Agmon et al., 1959; Agmon et al., 1964; Bergh and Löfström, 1976; Henry, 1981; Pisier, 1976; Stewart, 1980; Walsh, 1986

Acknowledgements

We would like to thank various people for their interest and stimulating discussions. In particular we thank Andrew Carroll, Nigel Cutland, Giuseppe Da Prato, David Elworthy, Franco Flandoli, Ben Gołdys and Szymon Peszat. This paper is a modified and improved version of a preprint, Brzeźniak and Gątarek (1996.)

References

Agmon, S., Douglis, A., Nirenberg, L., 1959. Estimates near boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. Comm. Pure Appl. Math. 12, 623–727.

Agmon, S., Douglis, A., Nirenberg, L., 1964. Estimates near boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Comm. Pure Appl. Math. 17, 35–92.

Albeverio, S., Röckner, M., 1991. Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Probab. Theory Related Fields 89, 347–386.

Bally, V., Gyöngy, I., Pardoux, E., 1994. White noise driven parabolic SPDE's with measurable drift. J. Funct. Anal. 120, 484–510.

Baxendale, P., 1976. Gaussian measures on Function Spaces. Amer. J. Math. 98, 891–952.

Bergh, J., Löfström, J., 1976. Interpolation Spaces. Springer, Berlin.

Brzeźniak, Z., 1995. Stochastic partial differential equations in M-type 2 Banach spaces. Potential Anal. 4, 1–45.

Brzeźniak, Z., 1996. On Sobolev and Besov spaces regularity of Brownian paths. Stoch. Stoch. Rep. 56, 1–15.

Brzeźniak, Z., 1997. On stochastic convolutions in Banach spaces and applications. Stoch. Stoch. Rep. 61, 245–295.

Brzeźniak, Z., Elworthy, K.D., 1999. Stochastic differential equations on Banach manifolds; applications to diffusions on loop spaces, in preparation.

Brzeźniak, Z., Gątarek, D., 1996. Martingale solutions and invariant measures for stochastic evolution equations in Banach spaces. Hull preprint.

Brzeźniak, Z., Peszat, Sz., 1999a. Space-time continuous solutions to SPDEs driven by a homogeneous Wiener process. Stud. Math., to appear.

Brzeźniak, Z., Peszat, Sz., 1999b. Euler Equations in random environment, submitted.

Burkholder, D.L., 1986. Martingales and Fourier Analysis in Banach Spaces. In Probability and Analysis, Varenna, 1985, Lecture Notes in Mathematics, vol. 1206. Springer, Berlin, pp. 61–108.

Capiński, M., Cutland, N., 1991. Stochastic Navier–Stokes equations. Acta Appl. Math. 25 (1), 59–85.

Capiński, M., Gątarek, D., 1994. Stochastic equations in Hilbert space with application to Navier-Stokes equations in any dimension. J. Funct. Anal. 126, 26–35.

Carroll, A., 1999. Ph.D. Thesis, University of Hull, 1999.

Cwikel, M., Kalton, N.J., 1993. Interpolation of compact operators by the method of Calderon and Gustavsson–Petre. Proc. Edinburgh Math. Soc. 38, 261–276.

Da Prato, G., 1976. Applications Croissantes Et Équations D'Évolution Dans Les Espaces De Banach, Institutiones Mathematicae, Vol. II. Instituto Nazionale di Alta Matematica, Roma; distributed throughout the world by Academic Press Inc., London.

Da Prato, G., Kwapień, S., Zabczyk, J., 1987. Regularity of Solutions of linear Stochastic Equations in Hilbert spaces. Stochastics 23, 1–23.

Da Prato, G., Pardoux, E., 1995. Invariant measures for white noise driven stochastic partial differential equations. Stochastic Anal. Appl. 13, 295–305.

Da Prato, G., Zabczyk, J., 1992a. Stochastic Equations In Infinite Dimensions, Encyclopedia of Mathematics and Its Applications, Vol. 44. Cambridge University Press, Cambridge.

Da Prato, G., Zabczyk, J., 1992bb. Non explosion, boundedness and ergodicity for stochastic semilinear equations. J. Differential Equations 98, 181–195.

Dettweiler, E., 1988. Representation of Banach space valued martingales as stochastic integrals. In Probability in Banach spaces, Progr. Probab., Vol. 21, 7 (Oberwolfach), pp. 43–62.

Dettweiler, E., 1991. Stochastic integration relative to Brownian motion on a general Banach space. Boga – Tr. J. Math. 15, 6–44.

Dettweiler, E., 1992. The martingale problem on Banach spaces. Exposition.-Math. 10 (1), 3–59.

Dore, G., Venni, A., 1987. On the closedness of the sum of two closed operators. Math. Z. 196, 189–201.

Elworthy, K.D., 1982. Stochastic Differential Equations On Manifolds. London Mathematical Society. Lecture Notes Series, Vol. 70. Cambridge University Press, Cambridge.

Flandoli, F., Gątarek, D., 1995. Martingale and stationary solutions for stochastic Navier–Stokes equations. Probab. Theory Related Fields 102, 367–391.

Funaki, T., 1989. Derivation of the hydrodynamical equation for one-dimensional Ginzburg–Landau model. Probab. Theory Related Fields 82 (1), 39–93.

Gątarek, D., Gołdys, B., 1994. On weak solutions of stochastic equations in Hilbert spaces. Stoch. Stoch. Rep. 46, 41–51.

Gątarek, D., Gołdys, B., 1997. On invariant measures for diffusions on Banach spaces. Potential Anal. 7, 539–553.

Gątarek, D., Sobczyk, J., 1994. On the existence of optimal controls of Hilbert-space valued diffusions. SIAM Control Optim. 32, 170–175.

Giga, Y., Sohr, H., 1991. Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains. J. Funct. Anal. 102, 72–94.

Gihman, I.I., Skorohod, A.V., 1972. Stochastic Differential Equations, Ergebnisse der Mathematik und ihrer Grenzbegiete. Springer, Berlin.

Henry, D., 1981. Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Vol. 840. Springer, Berlin.

Hida, T., Streit, L., 1977. On quantum theory in terms of white noise. Nagoya Math. J. 68, 21–34.

Kunita, H., 1990. Stochastic Flows And Stochastic Differential Equations. Cambridge University Press, Cambridge.

Kuo, H.H., 1975. Gaussian Measures In Banach Spaces. Lecture Notes in Mathematics, Vol. 463. Springer, Berlin.

Ikeda, N., Watanabe, S., 1981. Stochastic Differential Equations And Diffusion Processes. North-Holland, Amsterdam.

Manthey, R., Maslowski, B., 1992. Qualitative behaviour of solutions of stochastic-reaction diffusion equations. Stochastic Process. Appl. 43, 265–289.

Metivier, M., 1988. Stochastic Partial Differential Equations In Infinite Dimensional Spaces. Scuola Normale Superiore di Pisa, Pisa.

Mueller, C., 1993. Coupling and invariant measures for the heat equation with noise. Ann. Probab. 21, 2189–2199.

Neidhardt, A.L., 1978. Stochastic Integrals in 2-uniformly smooth Banach Spaces. University of Wisconsin.

Pisier, G., 1976. Martingales with values in uniformly convex spaces. Israel J. Math. 20 (3–4), 326–350.

Pazy, A., 1983. Semigroups Of Linear Operators and Applications To Partial Differential Equations, Applied Mathematical Sciences. Springer, Berlin.

Peszat, S., 1995. Existence and uniqueness of the solution for stochastic equations on Banach spaces. Stochastics 55, 167–193.

Prüss, J., Sohr, H., 1990. On operators with bounded imaginary powers. Math. Z. 203, 429–452.

Rozovskij, B.L., 1983. Stochastic Evolution Systems. Moscow, 1983 (in Russian).

Seeley, R., 1971. Norms and domains of complex powers A_B^z. Amer. J. Math. 93, 299–309.

Sowers, R., 1992. Large deviations for the invariant measure of the reaction-diffusion equation with non-Gaussian perturbations. Probab. Theory Related Fields 92, 393–421.

Stewart, H.B., 1974. Generation of analytic semigroups by strongly elliptic operators. Trans. Amer. Math. Soc. 199, 141–162.

Stewart, H.B., 1980. Generation of analytic semigroups by strongly elliptic operators under general boundary conditions. Trans. Amer. Math. Soc. 259, 299–310.

Temam, R., 1977. Navier–Stokes Equations; Theory And Numerical Analysis. North-Holland, Amsterdam.

Tribe, R., 1995. Large time behavior of interface solutions to the heat equation with Fisher-Wright white noise. Probab. Theory Related Fields 102 (3), 289–311.

Triebel, H., 1978. Interpolation Theory, Function Spaces, Differential Operators. North-Holland, Amsterdam.
Viot, M., 1976. Solutions Faibles D'Equations Aux Derivees Partielles Non Lineaires. Thesis, Universite Paris VI.
Walsh, J.B., 1986. An introduction to stochastic partial differential equations. In: Hennequin, P.L. (Ed.), Ecole d'été de Probabilités de Saint Flour, Lecture Notes in Mathematics, Vol. 1180. Springer, Berlin, pp. 265–437.
Williams, D., 1979. Markov Processes And Martingales. Wiley, New York.