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Weak and strong solutions for the incompressible Navier–Stokes equations with damping

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Abstract

In this paper, we show that the Cauchy problem of the Navier–Stokes equations with damping $\alpha|u|^{\beta-1}u$ ($\alpha > 0$) has global weak solutions for any $\beta \geqslant 1$, global strong solution for any $\beta \geqslant 7/2$ and that the strong solution is unique for any $7/2 \leqslant \beta \leqslant 5$.

Keywords: Navier–Stokes equations; Damping; Weak solutions; Strong solution

1. Introduction

We consider the following incompressible Navier–Stokes equations with damping

$$\begin{cases} u_t - \mu\Delta u + u\cdot\nabla u + \alpha|u|^{\beta-1}u + \nabla p = 0, & (x,t)\in R^3\times(0,T),\\ \operatorname{div} u = 0, & (x,t)\in R^3\times[0,T),\\ u|_{t=0} = u_0, & x\in R^3,\\ |u|\to 0, & \text{as } |x|\to\infty. \end{cases} \tag{1.1}$$

The unknown functions here are $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$, which stand for the velocity field and the pressure of the flow, respectively. In damping term, $\beta \geqslant 1$ and $\alpha > 0$ are two constants. The given function $u_0 = u_0(x)$ is the initial velocity and the constant $\mu > 0$ represents the viscosity coefficient of the flow.

The existence of global weak solutions of initial value problem and initial–boundary value problem of the Navier–Stokes equations were proved by Leray [11] and Hopf [6] long before. Since then, the uniqueness and the regularity of the weak solutions and the global (in time) existence of strong solution have been extensively investigated (see [3–6, 9–18] and references therein). However, the uniqueness of weak solutions and the global existence (in time) of strong solutions remain completely open. Introducing the class $L^s(0,T;L^q)$, Serrin showed that if u is a weak solution in such a class with $2/s + 3/q < 1$ satisfying $2 < s < \infty$, $3 < q < \infty$, then u is smooth. Since Serrin's criterion, many

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efforts have been made to obtain a larger class of weak solutions in which the uniqueness and regularity hold. The obtained results show that if the weak solution $u(x,t)$ of the Navier–Stokes equations belongs to $L^s(0,T;L^q)$ with $2/s+3/q \leqslant 1$ satisfying $2 \leqslant s \leqslant \infty$, $3 \leqslant q \leqslant \infty$, then the weak solution is regular and unique (see [3–5,13–17] and references therein). The class $L^s(0,T;L^q)$ is also called Serrin's class.

The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [1,2,7,8] and references therein). The purpose of this paper is to study the well-posedness of the incompressible Navier–Stokes equations with damping. We will show that the Cauchy problem (1.1) has global weak solutions for any $\beta \geqslant 1$ and global strong solution for any $\beta \geqslant 7/2$. Moreover, we will prove that for any $7/2 \leqslant \beta \leqslant 5$, the global strong solution of (1.1) is unique.

We apply the Galerkin method to construct the approximate solutions and make more delicate a priori estimates to proceed to compactness arguments. In particular, we obtain new more a priori estimates, comparing with the Navier–Stokes equations, to guarantee that the solution u belongs to $L^\infty(0,T;W^{1,2}_{0,\sigma}(R^3)) \cap L^\infty(0,T;L^{\beta+1}(R^3)) \cap L^2(0,T;H^2(R^3))$ for $\beta \geqslant \frac{7}{2}$ and the strong solution is unique when $\frac{7}{2} \leqslant \beta \leqslant 5$. Recalling Serrin's class to the Navier–Stokes equations, we obtain that the solutions of (1.1) will belong to Serrin's class if and only if $\beta \geqslant 4$. As mentioned above, the solution of the Navier–Stokes equations lying in Serrin's class will be unique. However, for the Navier–Stokes equations with damping, when $\beta > 5$, whether the strong solution is unique or not is still open.

Before ending this section, we introduce some notations of function spaces which will be used later. The space $L^p(R^3)$, $1 \leqslant p \leqslant \infty$, represents the usual Lebesgue space of scalar functions as well as that of vector-valued functions with norm denoted by $\|\cdot\|_p$. Let $C^\infty_{0,\sigma}(R^3)$ denote the set of all C^∞ real vector-valued functions $u=(u_1,u_2,u_3)$ with compact support in R^3 such that $\operatorname{div} u = 0$. Then the function space $L^p_\sigma(R^3)$, $1<p<\infty$, is defined as the closure of $C^\infty_{0,\sigma}(R^3)$ in $L^p(R^3)$ endowed with norm $\|\cdot\|_p$. We define $W^{k,p}(R^3)$ the usual Sobolev space with the norm $\|\cdot\|_{k,p}$ and $W^{k,p}_{0,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_{k,p}$. When $p=2$, we denote $W^{k,2}(R^3)$ by $H^k(R^3)$. Given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$, the set of functions $f(t)$ defined on $(0,T)$ with values in X such that $\int_0^T \|f(t)\|_X^p\,dt < \infty$. In this paper, we use C to express an absolute constant which may change from line to line.

The rest of the paper is organized as follows. In Section 2, we prove the global weak solutions of (1.1) for any $\beta \geqslant 1$. In Section 3, we prove the global existence of strong solutions for any $\beta \geqslant \frac{7}{2}$ and the existence and uniqueness of strong solution for $\frac{7}{2} \leqslant \beta \leqslant 5$ for the Cauchy problem (1.1).

2. Existence of weak solutions

In this section, we prove the global existence of weak solutions for the problem (1.1). The definition of weak solutions is given as usual way.

Definition 1. The function pair $(u(x,t), p(x,t))$ is called a weak solution of the problem (1.1) if for any $T>0$, the following conditions are satisfied:

(1) $u \in L^\infty(0,T;L^2_\sigma(R^3)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(R^3)) \cap L^{\beta+1}(0,T;L^{\beta+1}(R^3))$,
(2) for any $\Phi \in C^\infty_{0,\sigma}([0,T]\times R^3)$ with $\Phi(\cdot,T)=0$, we have

$$-\int_0^T (u,\Phi_t)\,dt + \mu\int_0^T\int_{R^3} \nabla u : \nabla\Phi\,dx\,dt - \int_0^T\int_{R^3} (u\cdot\nabla)u\Phi\,dx\,dt + \alpha\int_0^T\int_{R^3} |u|^{\beta-1}u\Phi\,dx\,dt = (u_0,\Phi_0), \tag{2.1}$$

(3) $\operatorname{div} u(x,t) = 0$ for a.e. $(x,t) \in R^3 \times [0,T)$.

In (2.1), ∇u denotes matrix $(\partial_i u_j)_{3\times 3}$ and for two matrices $A=(a_{ij})$ and $B=(b_{ij})$, the matrix $A:B=\sum_{i,j=1}^3 a_{ij}b_{ij}$. Here $(\,,\,)$ means the inner product in $L^2(R^3)$.

The following lemma is a compactness result, for the proof of it one can refer to [18].

Lemma 2.1. *Let X_0, X be Hilbert spaces satisfying a compact imbedding $X_0 \hookrightarrow X$. Let $0 < \gamma \leqslant 1$ and $(v_j)_{j=1}^{\infty}$ be a sequence in $L^2(R; X_0)$ satisfying*

$$\sup_j \left(\int_{-\infty}^{\infty} \|v_j\|_{X_0}^2 \, dt \right) < \infty, \qquad \sup_j \left(\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{v}_j\|_X^2 \, d\tau \right) < \infty,$$

where

$$\hat{v}(\tau) = \int_{-\infty}^{+\infty} v(t) \exp(-2\pi i \tau t) \, dt$$

is the Fourier transformation of $v(t)$ on the time variable. Then there exists a subsequence of $\{v_j\}_{j=1}^{\infty}$ which converges strongly in $L^2(R; X)$ to some $v \in L^2(R; X)$.

Our main result of this section reads as

Theorem 1. *Suppose that $\beta \geqslant 1$ and $u_0 \in L^2_\sigma(R^3)$. Then for any given $T > 0$, there exists a weak solution $(u(x,t), p(x,t))$ to the problem* (1.1) *such that*

$$u \in L^\infty\big(0, T; L^2_\sigma\big(R^3\big)\big) \cap L^2\big(0, T; W^{1,2}_{0,\sigma}\big(R^3\big)\big) \cap L^{\beta+1}\big(0, T; L^{\beta+1}\big(R^3\big)\big). \tag{2.2}$$

Moreover,

$$\sup_{0 \leqslant t \leqslant T} \big\|u(t)\big\|_{L^2}^2 + 2\mu \int_0^T \|\nabla u\|_{L^2}^2 \, dt + 2\alpha \int_0^T \|u\|_{L^{\beta+1}}^{\beta+1} \, dt \leqslant \|u_0\|_{L^2}^2. \tag{2.3}$$

Proof. We employ the Galerkin approximations to prove the theorem. The approach is similar to that of [18] for the classical Navier–Stokes equations.

Since $W^{1,2}_{0,\sigma}$ is separable and $C^\infty_{0,\sigma}$ is dense in $W^{1,2}_{0,\sigma}$, there exists a sequence $\omega_1, \omega_2, \ldots, \omega_m$ of elements of $C^\infty_{0,\sigma}$, which is free and total in $W^{1,2}_{0,\sigma}$. For each m we define an approximate solution u_m as follows:

$$u_m = \sum_{i=1}^m g_{im}(t) \omega_i(x)$$

and

$$\begin{aligned} &\big(u_m'(t), \omega_j\big) + \mu\big(\nabla u_m(t), \nabla \omega_j\big) + \big(u_m(t) \cdot \nabla u_m(t), \omega_j\big) + \big(\alpha |u_m|^{\beta-1} u_m(t), \omega_j\big) = 0, \\ &\quad t \in [0, T],\ j = 1, 2, \ldots, m, \end{aligned} \tag{2.4}$$

and $u_{0m} \to u_0$ in L^2_σ, as $m \to \infty$.

We have a priori estimates on the approximate solutions u_m as follows.

Lemma 2.2. *Suppose that $u_0 \in L^2_\sigma$. Then for any given $T > 0$ and any $\beta \geqslant 1$, we have*

$$\sup_{0 \leqslant t \leqslant T} \big\|u_m(t)\big\|_{L^2_\sigma} + 2\mu \|u_m\|_{L^2(0,T; W^{1,2}_{0,\sigma})} + 2\alpha \|u_m\|_{L^{\beta+1}(0,T; L^{\beta+1})} \leqslant \|u_0\|_{L^2}^2.$$

Proof. Multiplying on both sides of (2.4) by $g_{jm}(t)$ and summing over $j = 1, \ldots, m$, we have

$$\frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2}^2 + \mu \|\nabla u_m\|_{L^2}^2 + \alpha \|u_m\|_{L^{\beta+1}}^{\beta+1} \leqslant 0,$$

where we have used the fact that $((u \cdot \nabla) v, v) = 0$ for $u \in W^{1,2}_{0,\sigma}$ and $v \in W^{1,2}$.

Integrating over $(0,T)$ we obtain

$$\sup_{0\leqslant t\leqslant T}\left\|u_m(t)\right\|_{L^2}^2+2\mu\int_0^T\|\nabla u_m\|_{L^2}^2\,dt+2\alpha\int_0^T\|u_m\|_{L^{\beta+1}}^{\beta+1}\,dt\leqslant\|u_0\|_{L^2}^2. \tag{2.5}$$

The proof of Lemma 2.2 is finished. □

By a standard procedure, applying Lemma 2.2, we obtain the global existence of the approximate solutions $u_m\in L^\infty(0,T;L^2_\sigma(R^3))\cap L^2(0,T;W^{1,2}_{0,\sigma}(R^3))\cap L^{\beta+1}(0,T;L^{\beta+1}(R^3))$. Next, we will use Lemma 2.1 to prove the strong convergence of u_m (or its subsequence) in $L^2\cap L^\beta([0,T]\times\Omega)$ for any $\Omega\subset R^3$. To this end, we denote by $\tilde u_m$ the function from R into $W^{1,2}_{0,\sigma}$, which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. Similarly, we prolong $g_{im}(t)$ to R by defining $\tilde g_{im}(t)=0$ for $t\in R\backslash[0,T]$. The Fourier transforms on time variable of $\tilde u_m$ and $\tilde g_{im}$ are denoted by $\hat{\tilde u}_m$ and $\hat{\tilde g}_{im}$ respectively.

Note that the approximate solutions $\tilde u_m$ satisfy

$$\begin{aligned}\frac{d}{dt}(\tilde u_m,\omega_j)&=\mu\big(\nabla\tilde u_m(t),\nabla\omega_j\big)+\big(\tilde u_m(t)\cdot\nabla\tilde u_m(t),\omega_j\big)\\&\quad+\big(\alpha|\tilde u_m|^{\beta-1}\tilde u_m(t),\omega_j\big)+(u_{0m},\omega_j)\delta_0-\big(u_m(T),\omega_j\big)\delta_T\\&\equiv(\tilde f,\omega_j)+\big(\alpha|\tilde u_m|^{\beta-1}\tilde u_m(t),\omega_j\big)+(u_{0m},\omega_j)\delta_0-\big(u_m(T),\omega_j\big)\delta_T,\quad j=1,2,\dots,m,\end{aligned} \tag{2.6}$$

where δ_0, δ_T are Dirac distributions at 0 and T and

$$(\tilde f_m,\omega_j)=\mu\big(\nabla\tilde u_m(t),\nabla\omega_j\big)+\big(\tilde u_m(t)\cdot\nabla\tilde u_m(t),\omega_j\big).$$

Taking the Fourier transform about the time variable, (2.6) gives

$$2\pi i\tau(\hat{\tilde u}_m,\omega_j)=(\hat{\tilde f}_m,\omega_j)+\alpha\big(\widehat{|\tilde u_m|^{\beta-1}\tilde u_m}(t),\omega_j\big)+(u_{0m},\omega_j)-\big(u_m(T),\omega_j\big)\exp(-2\pi iT\tau), \tag{2.7}$$

where $\hat{\tilde f}_m$ denotes the Fourier transform of $\tilde f_m$.

Multiply (2.7) by $\hat{\tilde g}_{jm}(\tau)$ and add the resulting equations for $j=1,\dots,m$ to get:

$$2\pi i\tau\left\|\hat{\tilde u}_m(\tau)\right\|_2^2=\big(\hat{\tilde f}_m(\tau),\hat{\tilde u}_m\big)+\alpha\big(\widehat{|\tilde u_m|^{\beta-1}\tilde u_m}(\tau),\hat{\tilde u}_m\big)+(u_{0m},\hat{\tilde u}_m)-\big(u_m(T),\hat{\tilde u}_m\big)\exp(-2\pi iT\tau). \tag{2.8}$$

For any $v\in L^2(0,T;H^1_0)\cap L^{\beta+1}(0,T;L^{\beta+1})$, we have

$$\big(f_m(t),v\big)=(\nabla u_m,\nabla v)+(u_m\cdot\nabla u_m,v)\leqslant C\big(\|u_m\|_2^2+\|\nabla u_m\|_2^2+\|\nabla u_m\|_2\big)\|v\|_{H^1}.$$

It follows that for any given $T>0$

$$\int_0^T\left\|f_m(t)\right\|_{H^{-1}}dt\leqslant\int_0^T C\big(\|u_m\|_2^2+\|\nabla u_m\|_2^2+\|\nabla u_m\|_2\big)\,dt\leqslant C,$$

and hence

$$\sup_{\tau\in R}\left\|\hat{\tilde f}_m(\tau)\right\|_{H^{-1}}\leqslant\int_0^T\left\|f_m(t)\right\|_{H^{-1}}dt\leqslant C. \tag{2.9}$$

Moreover, it follows from Lemma 2.2 that

$$\int_0^T\left\||u_m|^{\beta-1}u_m\right\|_{\frac{\beta+1}{\beta}}dt\leqslant\int_0^T\|u_m\|_{\beta+1}^\beta\,dt\leqslant C,$$

which implies that

$$\sup_{\tau\in R}\left\|\widehat{|u_m|^{\beta-1}u}(\tau)\right\|_{\frac{\beta+1}{\beta}}\leqslant C. \tag{2.10}$$

From Lemma 2.2, we have

$$\|u_m(0)\|_2 \leqslant C, \qquad \|u_m(T)\|_2 \leqslant C. \tag{2.11}$$

We deduce from (2.8)–(2.11) that

$$|\tau|\|\hat{\tilde{u}}_m(\tau)\|_2^2 \leqslant C\big(\|\hat{\tilde{u}}_m(\tau)\|_{H^1} + \|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}\big).$$

For any γ fixed, $0 < \gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \leqslant C\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \quad \forall \tau \in R.$$

Thus

$$\begin{aligned}\int_{-\infty}^{+\infty} |\tau|^{2\gamma}\|\hat{\tilde{u}}_m(\tau)\|_2^2\,d\tau &\leqslant C\int_{-\infty}^{+\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}\|\hat{\tilde{u}}_m(\tau)\|_2^2\,d\tau \\ &\leqslant C\int_{-\infty}^{+\infty}\|\hat{\tilde{u}}_m(\tau)\|_2^2\,d\tau + C\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{H^1}}{1+|\tau|^{1-2\gamma}}\,d\tau + C\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}}{1+|\tau|^{1-2\gamma}}\,d\tau. \end{aligned}\tag{2.12}$$

Thanks to the Parseval equality and Lemma 2.2, the first integral on the right-hand side of (2.12) is bounded uniformly on m.

By the Schwartz inequality, the Parseval equality and Lemma 2.2, we have

$$\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}_m(\tau)\|_{H^1}}{1+|\tau|^{1-2\gamma}}\,d\tau \leqslant \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^2}\right)^{\frac{1}{2}}\left(\int_0^T \|u_m(\tau)\|_{H^1}^2\,d\tau\right)^{\frac{1}{2}} \leqslant C \tag{2.13}$$

for $0 < \gamma < \frac{1}{4}$,

Similarly, when $0 < \gamma < \frac{1}{2(\beta+1)}$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\|\hat{\tilde{u}}(\tau)\|_{\beta+1}}{1+|\tau|^{1-2\gamma}}\,d\tau &\leqslant \left(\int_{-\infty}^{+\infty}\frac{d\tau}{(1+|\tau|^{1-2\gamma})^{\frac{\beta+1}{\beta}}}\right)^{\frac{\beta}{\beta+1}}\left(\int_{-\infty}^{+\infty}\|\hat{\tilde{u}}_m(\tau)\|_{\beta+1}^{\beta+1}\,d\tau\right)^{\frac{1}{\beta+1}} \\ &\leqslant C\left(\int_{-\infty}^{+\infty}\|\tilde{u}_m(\tau)\|_{\beta+1}^{\frac{\beta+1}{\beta}}\,d\tau\right)^{\frac{\beta}{\beta+1}} \\ &\leqslant CT^{\frac{\beta-1}{\beta+1}}\left(\int_0^T\|u_m(\tau)\|_{\beta+1}^{\beta+1}\,d\tau\right)^{\frac{1}{\beta}}. \end{aligned}\tag{2.14}$$

It follows from (2.12) that

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}\|\hat{\tilde{u}}_m(\tau)\|_2^2\,d\tau \leqslant C. \tag{2.15}$$

Thanks to Lemma 2.2, there exists a function $u(x,t)$ such that

$$u \in L^\infty\big(0,T;L^2_\sigma(R^3)\big) \cap L^2\big(0,T;W^{1,2}_{0,\sigma}(R^3)\big) \cap L^{\beta+1}\big(0,T;L^{\beta+1}(R^3)\big), \tag{2.16}$$

and there exists a subsequence of $\{u_m\}_{m=1}^\infty$, still denoted by itself, such that $u_m \rightharpoonup u$ weakly-* in $L^\infty(0,T;L^2_\sigma(R^3))$ and weakly in $L^2(0,T;W^{1,2}_{0,\sigma}(R^3))$ and $u_m \rightharpoonup u$ in $L^{\beta+1}(0,T;L^{\beta+1}(R^3))$. Moreover, we choose $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots$ with smooth boundary, satisfying $\bigcup_{i=1}^\infty \Omega_i = R^3$. For any fixed $i = 1,2,\ldots$, we take $X_0 = W_0^{1,2}(\Omega_i)$, $X = L^2(\Omega_i)$ in

Lemma 2.1. Then in view of Lemmas 2.1 and 2.2, and (2.15), we obtain that there exists a subsequence of $\{u_m\}_{m=1}^{\infty}$, still denoted by itself, such that $u_m \to u$ strongly in $L^2(0,T;L^2(\Omega_i))$. By the diagonal principle, there exists a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ of $\{u_m\}_{m=1}^{\infty}$, such that $u_{m_j} \to u$ strongly in $L^2(0,T;L^2(\Omega_i))$ for any $i = 1, 2, \ldots$ and hence in $L^2(0,T;L^2_{\mathrm{loc}}(R^3))$. Noting that $\int_0^T \int_{R^3} |u_m|^{\beta+1}\,dx\,dt \leqslant C$, we obtain that $u_{m_j} \to u$ strongly in $L^p(0,T;L^p_{\mathrm{loc}}(R^3))$ for $2 \leqslant p < \beta + 1$ if $\beta > 1$. These convergence guarantee that $u(x,t)$ is a weak solution of (1.1). Furthermore, (2.2) is a direct consequence of (2.16) and (2.3) is clearly satisfied due to Lemma 2.2. The proof of Theorem 1 is finished. □

3. Existence and uniqueness of strong solution

We call the function pair $(u(x,t), p(x,t))$ the strong solution of the problem (1.1) if it is a weak solution of (1.1) satisfying

$$u \in L^\infty\big(0,T;W^{1,2}_{0,\sigma}\big(R^3\big)\big) \cap L^2\big(0,T;H^2\big(R^3\big)\big) \cap L^\infty\big(0,T;L^{\beta+1}\big(R^3\big)\big).$$

It should be remarked that just as the case of the classical Navier–Stokes equations, if $(u(x,t), p(x,t))$ is a strong solution of (1.1), then the pressure function $p(x,t)$ can be determined uniquely (up to a constant) by the velocity field $u(x,t)$.

As a preliminary, we recall the known Gagliardo–Nirenberg inequality as follows.

Lemma 3.1 *(Gagliardo–Nirenberg inequality). Assume that q and r satisfy $1 \leqslant q, r \leqslant \infty$, and j, m are arbitrary integers satisfying $0 \leqslant j < m$. Assume $u \in C_0^\infty(R^n)$, then*

$$\big\|D^j u\big\|_{L^p} \leqslant C\big\|D^m u\big\|_{L^r}^a \|u\|_{L^q}^{1-a}, \tag{3.1}$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$, $\frac{j}{m} \leqslant a \leqslant 1$, and the constant C only depends on n, m, j, q, r, a. If $m - j - \frac{n}{r}$ is a nonnegative integer, the above inequality holds for $\frac{j}{m} \leqslant a < 1$.

Our main result of this section is stated as

Theorem 2. *Suppose that $\beta \geqslant \frac{7}{2}$ and $u_0 \in W^{1,2}_{0,\sigma} \cap L^{\beta+1}$. Then there exists a strong solution $(u(x,t), p(x,t))$ to the problem (1.1) satisfying*

$$u \in L^\infty\big(0,T;W^{1,2}_{0,\sigma}\big(R^3\big)\big) \cap L^\infty\big(0,T;L^{\beta+1}\big(R^3\big)\big) \cap L^2\big(0,T;H^2\big(R^3\big)\big),$$
$$\nabla u|u|^{\frac{\beta-1}{2}} \in L^2\big(0,T;L^2\big(R^3\big)\big); \qquad u_t \in L^2\big(0,T;L^2\big(R^3\big)\big).$$

Moreover when $\frac{7}{2} \leqslant \beta \leqslant 5$, the strong solution is unique.

Proof. The existence of strong solution is based on the following a priori estimates.

Lemma 3.2. *Suppose that $(u(x,t), p(x,t))$ is a smooth solution of the problem (1.1). Then for any $\beta \geqslant \frac{7}{2}$, we have*

$$\sup_{0\leqslant t\leqslant T}\big(\big\|\nabla u(t)\big\|_2^2 + \big\|u(t)\big\|_{\beta+1}^{\beta+1}\big) + \|u_t\|_{2,2;T}^2 + \|\Delta u\|_{2,2;T}^2 + \big\||\nabla u||u|^{\frac{\beta-1}{2}}\big\|_{2,2;T}^2$$
$$+ \frac{\alpha(\beta-1)}{2}\int_0^T\int_{R^3} |u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\,dt \leqslant C. \tag{3.2}$$

Proof. Multiply the first equation of (1.1) by u_t, $-\Delta u$ and integrate the resulting equations on R^3, respectively, to obtain

$$\frac{\mu}{2}\frac{d}{dt}\int_{R^3} |\nabla u|^2\,dx + \frac{\alpha}{\beta+1}\frac{d}{dt}\int_{R^3} |u|^{\beta+1}\,dx + \int_{R^3} |u_t|^2\,dx = -\int_{R^3} u_t u\cdot\nabla u\,dx, \tag{3.3}$$

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|\nabla u|^2\,dx+\mu\int_{R^3}|\Delta u|^2\,dx+\alpha\int_{R^3}|u|^{\beta-1}|\nabla u|^2\,dx$$
$$+\frac{\alpha(\beta-1)}{4}\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx=\int_{R^3}(u\cdot\nabla u)\Delta u\,dx. \tag{3.4}$$

Adding (3.3), (3.4) and using Hölder inequality, Young inequality yield

$$\frac{\mu+1}{2}\frac{d}{dt}\int_{R^3}|\nabla u|^2\,dx+\frac{\alpha}{\beta+1}\frac{d}{dt}\int_{R^3}|u|^{\beta+1}\,dx+\frac{3\mu}{4}\int_{R^3}|\Delta u|^2\,dx$$
$$+\frac{1}{2}\int_{R^3}|u_t|^2\,dx+\alpha\int_{R^3}|u|^{\beta-1}|\nabla u|^2\,dx+\frac{\alpha(\beta-1)}{4}\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\leqslant C\int_{R^3}|u\cdot\nabla u|^2\,dx\equiv J. \tag{3.5}$$

The estimates of J are divided into the following two cases.

Case I. Using Gagliardo–Nirenberg inequality (3.1), we have

$$\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}\leqslant C\|\Delta u\|_2^a\|u\|_{\beta+1}^{1-a} \tag{3.6}$$

where β satisfies

$$\frac{1}{2}\leqslant a=\frac{11-\beta}{\beta+7}\leqslant 1, \tag{3.7}$$

that is,

$$2\leqslant\beta\leqslant 5. \tag{3.8}$$

Using Hölder inequality, (3.6) and Young inequality, we have

$$\begin{aligned}J&\leqslant C\|u\|_{\beta+1}^2\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}^2\\&\leqslant C\|u\|_{\beta+1}^2\|\Delta u\|_2^{\frac{2(11-\beta)}{\beta+7}}\|u\|_{\beta+1}^{\frac{4(\beta-2)}{\beta+7}}\\&\leqslant C\|\Delta u\|_2^{\frac{2(11-\beta)}{\beta+7}}\|u\|_{\beta+1}^{\frac{6(\beta+1)}{\beta+7}}\\&\leqslant\frac{\mu}{4}\|\Delta u\|_2^2+C\|u\|_{\beta+1}^{\frac{3(\beta+1)}{\beta-2}}.\end{aligned} \tag{3.9}$$

If $\frac{3(\beta+1)}{\beta-2}\geqslant\beta+1$, that is, $2<\beta\leqslant 5$, it directly follows that

$$J\leqslant\frac{\mu}{4}\|\Delta u\|_2^2+C\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\frac{4\beta-\beta^2+5}{\beta-2}}. \tag{3.10}$$

In (3.10) we demand that

$$\begin{cases}4\beta-\beta^2+5\geqslant 0 & \Rightarrow\quad -1\leqslant\beta\leqslant 5,\\ 4\beta-\beta^2+5\leqslant(\beta-2)(\beta+1) & \Rightarrow\quad \beta\geqslant\frac{7}{2}.\end{cases} \tag{3.11}$$

Combining (3.8) with (3.11), we obtain the restrictions on β:

$$\frac{7}{2}\leqslant\beta\leqslant 5. \tag{3.12}$$

Substituting (3.10) into (3.5), we have

$$(\mu+1)\sup_{0\leqslant t\leqslant T}\|\nabla u(t)\|_2^2+\frac{2\alpha}{\beta+1}\sup_{0\leqslant t\leqslant T}\|u(t)\|_{\beta+1}^{\beta+1}+\mu\|\Delta u\|_{2,2;T}^2$$

$$+2\|u_t\|_{2,2;T}^2+2\alpha\big\||u|^{\frac{\beta-1}{2}}|\nabla u|\big\|_{2,2;T}^2+\frac{\alpha(\beta-1)}{2}\int_0^T\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\,dt$$

$$\leqslant C\exp\big(\|u\|_{\beta+1,\beta+1;T}^{\frac{4\beta-\beta^2+5}{\beta-2}}T^{\frac{2\beta-7}{\beta-2}}\big)\times\big(\|\nabla u_0\|_2^2+\|u_0\|_{\beta+1}^{\beta+1}\big), \tag{3.13}$$

where β satisfies (3.12).

Case II. Using Gagliardo–Nirenberg inequality, we have

$$\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}\leqslant C\|\Delta u\|_2^a\|u\|_2^{1-a}, \tag{3.14}$$

$$\frac{1}{2}\leqslant a=\frac{\beta+4}{2(\beta+1)}\leqslant 1, \tag{3.15}$$

that is,

$$\beta\geqslant 2. \tag{3.16}$$

Using Hölder inequality, (3.14) and Young inequality, we obtain

$$J\leqslant C\|u\|_{\beta+1}^2\|\nabla u\|_{\frac{2(\beta+1)}{\beta-1}}^2$$

$$\leqslant C\|u\|_{\beta+1}^2\|\Delta u\|_2^{\frac{\beta+4}{\beta+1}}\|u\|_2^{\frac{\beta-2}{\beta+1}}$$

$$\leqslant\frac{\mu}{4}\|\Delta u\|_2^2+C\|u\|_{\beta+1}^{\frac{4(\beta+1)}{\beta-2}}\|u\|_2^2. \tag{3.17}$$

Now we divide the index $\frac{4(\beta+1)}{\beta-2}$ into two cases and then we discuss them.

If $\frac{4(\beta+1)}{\beta-2}\leqslant\beta+1$, that is,

$$\beta\geqslant 6, \tag{3.18}$$

substituting (3.17) into (3.5), we have

$$(\mu+1)\sup_{0\leqslant t\leqslant T}\|\nabla u(t)\|_2^2+\frac{2\alpha}{\beta+1}\sup_{0\leqslant t\leqslant T}\|u(t)\|_{\beta+1}^{\beta+1}+\mu\|\Delta u\|_{2,2;T}^2$$

$$+\|u_t\|_{2,2;T}^2+2\alpha\big\||u|^{\frac{\beta-1}{2}}|\nabla u|\big\|_{2,2;T}^2+\frac{\alpha(\beta-1)}{2}\int_0^T\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\,dt$$

$$\leqslant C\big(\|u\|_{2,\infty;T}^2\|u\|_{\beta+1,\beta+1;T}^{\frac{4(\beta+1)}{\beta-2}}T^{\frac{\beta-6}{\beta-2}}\big)+\big(\|\nabla u_0\|_2^2+\|u_0\|_{\beta+1}^{\beta+1}\big), \tag{3.19}$$

where β satisfies (3.18).

If $\frac{4(\beta+1)}{\beta-2}\geqslant\beta+1$, that is,

$$\beta\leqslant 6, \tag{3.20}$$

it directly follows that

$$J\leqslant\frac{\mu}{4}\|\Delta u\|_2^2+C\|u\|_{\beta+1}^{\beta+1}\|u\|_{\beta+1}^{\frac{5\beta-\beta^2+6}{\beta-2}}\|u\|_2^2. \tag{3.21}$$

In (3.21) we demand that

$$\begin{cases} 5\beta-\beta^2+6\geqslant 0 \quad\Rightarrow\quad -1\leqslant\beta\leqslant 6,\\ 5\beta-\beta^2+6\leqslant(\beta-2)(\beta+1) \quad\Rightarrow\quad \beta\geqslant 4. \end{cases} \tag{3.22}$$

Combining (3.20) and (3.22), we obtain

$$4\leqslant\beta\leqslant 6. \tag{3.23}$$

Substituting (3.17) into (3.5), we have

$$\begin{aligned}
&(\mu+1)\sup_{0\leqslant t\leqslant T}\big\|\nabla u(t)\big\|_2^2+\frac{2\alpha}{\beta+1}\sup_{0\leqslant t\leqslant T}\big\|u(t)\big\|_{\beta+1}^{\beta+1}+\mu\|\Delta u\|_{2,2;T}^2\\
&\quad+2\|u_t\|_{2,2;T}^2+2\alpha\big\||u|^{\frac{\beta-1}{2}}|\nabla u|\big\|_{2,2;T}^2+\frac{\alpha(\beta-1)}{2}\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\\
&\leqslant C\exp\big(\|u\|_{2,\infty;T}^2\|u\|_{\beta+1,\beta+1;T}^{\frac{(\beta+1)(6-\beta)}{\beta-2}}T^{\frac{2\beta-8}{\beta-2}}\big)\times\big(\|\nabla u_0\|_2^2+\|u_0\|_{\beta+1}^{\beta+1}\big)
\end{aligned} \tag{3.24}$$

where β satisfies (3.23).

Combining (3.5), (3.13), (3.19) and (3.24), we obtain that, for any $\beta\geqslant\frac{7}{2}$,

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\big(\big\|\nabla u(t)\big\|_2^2+\big\|u(t)\big\|_{\beta+1}^{\beta+1}\big)+\|u_t\|_{2,2;T}^2+\|\Delta u\|_{2,2;T}^2+\big\||\nabla u||u|^{\frac{\beta-1}{2}}\big\|_{2,2;T}^2\\
&\quad+\frac{\alpha(\beta-1)}{2}\int_0^2\int_{R^3}|u|^{\beta-3}\big|\nabla|u|^2\big|^2\,dx\,dt\leqslant C.
\end{aligned} \tag{3.25}$$

The proof of Lemma 3.2 is finished. □

Now we proceed to proving the uniqueness of the strong solution of Theorem 2. Assume that under the same initial data, there exist two strong solutions $(u,p),(\bar u,\bar p)$ of (1.1) satisfying

$$(u_t,\Phi)+\mu\int_{R^3}\nabla u:\nabla\Phi\,dx-\int_{R^3}(u\cdot\nabla)u\Phi\,dx+\alpha\int_{R^3}|u|^{\beta-1}u\Phi\,dx=0, \tag{3.26}$$

$$(\bar u_t,\Phi)+\mu\int_{R^3}\nabla\bar u:\nabla\Phi\,dx-\int_{R^3}(\bar u\cdot\nabla)\bar u\Phi\,dx+\alpha\int_{R^3}|\bar u|^{\beta-1}\bar u\Phi\,dx=0 \tag{3.27}$$

for $\Phi\in C_{0,\sigma}^\infty([0,T]\times R^3)$ and by the density argument (3.26) and (3.27) hold actually for $\Phi\in L^2(0,T;H^1)$.

Subtracting (3.26) from (3.27) and taking $\Phi=u-\bar u$ in the resulting equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u-\bar u\|_2^2+\mu\big\|\nabla(u-\bar u)\big\|_2^2+\alpha\big\||u|^{\frac{\beta-1}{2}}|u-\bar u|\big\|_2^2\\
&\quad\leqslant\int_{R^3}|u-\bar u|^2|\nabla\bar u|\,dx+\alpha\int_{R^3}|u-\bar u||\bar u|\big||u|^{\beta-1}-|\bar u|^{\beta-1}\big|\,dx\\
&\quad\equiv I_1+I_2,
\end{aligned} \tag{3.28}$$

where we have used the fact that $((u\cdot\nabla)v,v)=0$, $u\in W_{0,\sigma}^{1,2}$, $v\in W^{1,2}$.

Apply Hölder and Sobolev inequalities to yield

$$\begin{aligned}
I_1&\leqslant\|u-\bar u\|_4^2\|\nabla\bar u\|_2\\
&\leqslant C\big(\big\|\nabla(u-\bar u)\big\|_2^{\frac{3}{4}}\|u-\bar u\|_2^{\frac{1}{4}}\big)^2\|\nabla u\|_2\\
&\leqslant C\big\|\nabla(u-\bar u)\big\|_2^{\frac{3}{2}}\|u-\bar u\|_2^{\frac{1}{2}}\|\nabla\bar u\|_2\\
&\leqslant\varepsilon\big\|\nabla(u-\bar u)\big\|_2^2+C\|u-\bar u\|_2^2\|\nabla\bar u\|_2^4
\end{aligned} \tag{3.29}$$

and

$$
\begin{aligned}
I_2 &\leqslant \alpha \int_{R^3} |u-\bar{u}||\bar{u}|\big||u|^{\beta-1}-|\bar{u}|^{\beta-1}\big|\,dx \\
&\leqslant C(\beta-1)\int_{R^3} |u-\bar{u}|\big||u|^{\beta-2}+|\bar{u}|^{\beta-2}\big||u-\bar{u}||\bar{u}|\,dx \\
&\leqslant C\|u-\bar{u}\|_4^2\|\bar{u}\|_6\big\||u|^{\beta-2}+|\bar{u}|^{\beta-2}\big\|_3 \\
&\leqslant C\big(\big\|\nabla(u-\bar{u})\big\|_2^{\frac{3}{4}}\|u-\bar{u}\|_2^{\frac{1}{4}}\big)^2\|\bar{u}\|_6\big\||u|+|\bar{u}|\big\|_{3(\beta-2)}^{\beta-2} \\
&\leqslant C\big\|\nabla(u-\bar{u})\big\|_2^{\frac{3}{2}}\|u-\bar{u}\|_2^{\frac{1}{2}}\|\bar{u}\|_6\big\||u|+|\bar{u}|\big\|_{3(\beta-2)}^{\beta-2} \\
&\leqslant \varepsilon\big\|\nabla(u-\bar{u})\big\|_2^2+C\|u-\bar{u}\|_2^2\|\bar{u}\|_6^4\big\||u|+|\bar{u}|\big\|_{3(\beta-2)}^{4(\beta-2)}.
\end{aligned}
\tag{3.30}
$$

In the second inequality of I_2, we used the fact that

$$\big|x^p-y^p\big|\leqslant Cp\big(|x|^{p-1}+|y|^{p-1}\big)|x-y|$$

for any $x, y \geqslant 0$, where C is an absolute constant.

Substituting the estimates of I_1, I_2 into inequality (3.28), choosing $\varepsilon=\frac{\mu}{4}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|u-\bar{u}\|_{L^2}^2+\mu\big\|\nabla(u-\bar{u})\big\|_{L^2}^2+2\alpha\big\||u|^{\frac{\beta-1}{2}}|u-\bar{u}|\big\|_2^2 \\
&\quad\leqslant C\|u-\bar{u}\|_{L^2}^2\big(\|\nabla\bar{u}\|_2^4+\|\bar{u}\|_6^4\big[\|u\|_{3(\beta-2)}^{4(\beta-2)}+\|\bar{u}\|_{3(\beta-2)}^{4(\beta-2)}\big]\big).
\end{aligned}
\tag{3.31}
$$

Note that

$$\int_0^T\big\||u|\big\|_{3(\beta-2)}^{4(\beta-2)}\leqslant\int_0^T\|u\|_{\beta+1}^{\frac{4(\beta^2+\beta)}{\beta+7}}\|\Delta u\|_2^{\frac{8(2\beta-7)}{\beta+7}}\leqslant\sup_{0\leqslant t\leqslant T}\|u\|_{\beta+1}^{\frac{4(\beta^2+\beta)}{\beta+7}}\|\Delta u\|_{2,2;T}^{\frac{8(2\beta-7)}{\beta+7}}T^{\frac{35-7\beta}{\beta+7}}\tag{3.32}$$

and similar estimate holds true for $\bar{u}$ instead of u in (3.32). In (3.32), we have a restriction of β such that $0\leqslant\frac{8(2\beta-7)}{\beta+7}\leqslant 2$, that is,

$$\frac{7}{2}\leqslant\beta\leqslant 5.\tag{3.33}$$

Substituting (3.32) into (3.31) and applying the Gronwall inequality, we obtain that $u=\bar{u}$ for a.e. $(x,t)\in R^3\times[0,T]$ under (3.33). This completes the proof of Theorem 2. □

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