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Conditions for certain ruin for the generalised Ornstein–Uhlenbeck process and the structure of the upper and lower bounds

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Abstract

For a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ the generalised Ornstein–Uhlenbeck (GOU) process is defined as

$$V_t := \mathrm{e}^{\xi_t}\left(z + \int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right), \quad t \geq 0,$$

where $z \in \mathbb{R}$. We present conditions on the characteristic triplet of (ξ, η) which ensure certain ruin for the GOU. We present a detailed analysis on the structure of the upper and lower bounds and the sets of values on which the GOU is almost surely increasing, or decreasing. This paper is the sequel to Bankovsky and Sly (2008) [2], which stated conditions for zero probability of ruin, and completes a significant aspect of the study of the GOU.

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1. Notation and theoretical background

For a review of publications for the GOU see [2]. In Section 2 of the present paper, we state results on certain ruin for the GOU. Theorem 3.1 of Paulsen [10] gives conditions for certain

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ruin for the GOU in the special case in which ξ and η are independent. In [2] it is shown that this theorem does not hold for the general case. Theorems 1 and 3 of Section 2 give the required generalisation, stated in terms of the characteristic triplet of (ξ, η). Section 3 begins with results, in particular Proposition 6 and Theorem 9, which describe the structure of the upper and lower bounds and the sets of values on which the GOU is almost surely increasing, or decreasing. Section 3 then outlines the ruin probability implications of these structural results, in particular with Theorems 13 and 14, which state conditions for certain ruin in terms of upper and lower bound structures. Section 4 contains technical propositions used to prove the major theorems. Sections 5–7 contain proofs of the results in Sections 4, 3 and 2 respectively. Section 6 also contains examples which illustrate and extend results from Section 3. We now set up some notation, which builds on that of [2], and outline some basic results.

Let (ξ, η) be a bivariate Lévy process on a filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ and define the GOU process V, and the associated stochastic integral process Z, as

$$V_t := e^{\xi_t}\left(z + \int_0^t e^{-\xi_{s-}} d\eta_s\right), \tag{1}$$

and

$$Z_t := \int_0^t e^{-\xi_{s-}} d\eta_s. \tag{2}$$

To avoid trivialities, assume that neither ξ nor η is identically zero. It was shown in [2] that

$$\Delta V_t = e^{\Delta \xi_t}(\Delta \eta_t - V_{t-}(e^{-\Delta \xi_t} - 1)). \tag{3}$$

The characteristic triplet of (ξ, η) is written $((\tilde{\gamma}_\xi, \tilde{\gamma}_\eta), \Sigma_{\xi,\eta}, \Pi_{\xi,\eta})$. The characteristic triplet of ξ as a one-dimensional Lévy process is written $(\gamma_\xi, \sigma_\xi^2, \Pi_\xi)$, where

$$\gamma_\xi = \tilde{\gamma}_\xi + \int_{\{|x|<1\}\cap\{x^2+y^2\geq 1\}} x \Pi_{\xi,\eta}(d(x, y)), \tag{4}$$

and σ_ξ^2 is the upper left entry in the matrix $\Sigma_{\xi,\eta}$. The characteristic triplet of η is similar. The random jump measure and Brownian motion components of (ξ, η) are denoted respectively by $N_{\xi,\eta,t}$ and (B_ξ, B_η). For a Lebesgue set Λ define the *hitting time* of Λ by V to be $T_{z,\Lambda} := \inf\{t > 0 : V_t \in \Lambda | V_0 = z\}$, where $T_{z,\Lambda} := \infty$ whenever $V_t \notin \Lambda$ for all $t > 0$ and $V_0 = z$. When the context makes it obvious we will simply write T_Λ. Define the *infinite horizon ruin probability* for the GOU by

$$\psi(z) := P\left(\inf_{t>0} V_t < 0 | V_0 = z\right) = P\left(\inf_{t>0} Z_t < -z\right) = P\left(T_{z,(-\infty,0)} < \infty\right).$$

For all $t > 0$, V_t is increasing as a function of the initial value z and hence, if $0 \leq z_1 \leq z_2$, then $\psi(z_1) \geq \psi(z_2)$. For further explanation of the above terms, as well as extra definitions and results for Lévy processes, see Section 1 of [2]. We now outline notation and theory needed for the present paper, which were not dealt with in Section 1 of [2].

The *total variation* of an $\mathbb{R}^d$-valued function over the interval $[a, b]$ is defined by

$$V_f([a, b]) := \sup \sum_{i=1}^n |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$. A Lévy process X on $\mathbb{R}^d$, with characteristic triplet $(\gamma_X, \Sigma_X, \Pi_X)$ and random jump measure $N_{X,t}$, is of *finite variation* if, with probability 1, its sample paths $X_t(\omega)$ are of finite total variation on $[0, t]$ for every $t > 0$. By [5, p. 86], this occurs iff $\Sigma_X = 0$ and $\int_{|z|\le 1} |z| \Pi_X(\mathrm{d}z) < \infty$. If this occurs then

$$X_t = d_X t + \int_{\mathbb{R}^d} z N_{X,t}(\cdot, \mathrm{d}z) = d_X t + \sum_{0<s\le t} \Delta X_s,$$

where

$$d_X = \gamma_X - \int_{|z|<1} z \Pi_X(\mathrm{d}z) = E\left(X_1 - \int_{\mathbb{R}^d} z N_{X,1}(\cdot, \mathrm{d}z) \right) \tag{5}$$

is called the *drift vector* of X. A one-dimensional Lévy process X is a *subordinator* if $X_t(\omega)$ is an increasing function of t, a.s. By [5, p. 88], the following are equivalent:

(1) X is a subordinator.
(2) $X_t \ge 0$ a.s. for some $t > 0$.
(3) $X_t \ge 0$ a.s. for every $t > 0$.
(4) The characteristic triplet satisfies

$$\sigma_X^2 = 0, \qquad \int_{(-\infty,0]} \Pi_X(\mathrm{d}x) = 0, \qquad \int_{(0,1)} x \Pi_X(\mathrm{d}x) < \infty, \quad \text{and} \quad d_X \ge 0.$$

That is, there is no Brownian component, no negative jumps, the positive jumps are of finite variation and the drift is non-negative.

A one-dimensional Lévy process X will drift to ∞, drift to $-\infty$ or oscillate between ∞ and $-\infty$, namely, one of the following must hold:

$$\lim_{t\to\infty} X_t = \infty \quad \text{a.s.}; \tag{6}$$

$$\lim_{t\to\infty} X_t = -\infty \quad \text{a.s.}; \tag{7}$$

$$-\infty = \liminf_{t\to\infty} X_t < \limsup_{t\to\infty} X_t = \infty \quad \text{a.s.} \tag{8}$$

Exact conditions for these cases are given in [6]. Whenever $E(X_1)$ is a well-defined member of the extended reals, cases (6)–(8) equate respectively to $E(X_1) > 0$, $E(X_1) < 0$, and $E(X_1) = 0$. When $E(X_1)$ does not exist, we need more notation. For $x > 0$, define

$$\overline{\Pi}_X^+(x) := \Pi_X((x, \infty)), \qquad \overline{\Pi}_X^-(x) := \Pi_X((-\infty, -x)),$$
$$\overline{\Pi}_X(x) := \overline{\Pi}_X^+(x) + \overline{\Pi}_X^-(x).$$

Define, for $x \ge 1$,

$$A_X^+(x) := \max\left\{\overline{\Pi}_X^+(1), 1\right\} + \int_1^x \overline{\Pi}_X^+(u)\mathrm{d}u,$$

$$A_X^-(x) := \max\left\{\overline{\Pi}_X^-(1), 1\right\} + \int_1^x \overline{\Pi}_X^-(u)\mathrm{d}u$$

and define the integrals

$$J_X^+ := \int_1^\infty \left(\frac{x}{A_X^-(x)}\right) |\overline{\Pi}_X^+(\mathrm{d}x)| \quad \text{and} \quad J_X^- := \int_1^\infty \left(\frac{x}{A_X^+(x)}\right) |\overline{\Pi}_X^-(\mathrm{d}x)|.$$

In [6] it is shown that if $E(X_1)$ is not well defined, that is, if

$$\int_1^\infty x\Pi_X(\mathrm{d}x) = \int_{-\infty}^{-1} |x|\Pi_X(\mathrm{d}x) = \infty,$$

then (6)–(8) respectively occur iff $J_X^- < \infty$, $J_X^+ < \infty$ and $J_X^- = J_X^+ = \infty$.

By [4], the GOU is a time homogeneous strong Markov process. In [7], iff conditions are stated for a.s. convergence of Z_t to a finite random variable Z_∞ as t approaches ∞. In [8], iff conditions are stated for stationarity of V. To describe these conditions, let (X, Y) be a bivariate Lévy process and define

$$I_{X,Y} := \int_{(e,\infty)} \frac{\ln(y)}{A_X^+(\ln(y))} |\overline{\Pi}_Y(\mathrm{d}y)|,$$

and the auxiliary Lévy process $K^{X,Y}$ by

$$K_t^{X,Y} := Y_t + \sum_{0<s\le t} \left(\mathrm{e}^{\Delta X_s} - 1\right) \Delta Y_s + t\,\mathrm{Cov}(B_{X,1}, B_{Y,1}),$$

where Cov denotes the covariance. Theorem 2 of [7] states that Z_t converges a.s. to a finite random variable Z_∞ as $t \to \infty$ iff $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. There is a special case in which, for some $c \in \mathbb{R}$,

$$Z_t = c\left(\mathrm{e}^{-\xi_t} - 1\right) \quad \text{and} \quad V_t = \mathrm{e}^{\xi_t}(z - c) + c, \tag{9}$$

a.s. for all $t \ge 0$. Exact conditions for this degenerate situation, given in terms of the characteristic triplet of (ξ, η), are stated in Proposition 8. In this situation, $\lim_{t\to\infty} \xi_t = \infty$ a.s. implies that Z_t converges a.s. to the constant random variable $Z_\infty = -c$ as $t \to \infty$, and in [3] it is shown that this is the only case in which Z_∞ is not continuous. Regardless of the asymptotic behaviour of ξ, if (9) holds then V is strictly stationary iff $V_0 = c$. If (9) does not hold for any $c \in \mathbb{R}$, then Theorem 2.1 of [8] states that V is strictly stationary iff $\int_0^\infty \mathrm{e}^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}$ converges a.s. or, equivalently, iff $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi,K^{\xi,\eta}} < \infty$. In this case the stationary random variable V_∞ satisfies $V_\infty =_D \int_0^\infty \mathrm{e}^{\xi_{s-}} \mathrm{d}K_s^{\xi,\eta}$.

2. Conditions for certain ruin

In Theorem 1 of [2], exact conditions were given on the characteristic triplet of (ξ, η) for the existence of $u \ge 0$ such that $\psi(u) = 0$, and a precise value was given for the value $\inf\{u \ge 0 : \psi(u) = 0\}$. For this result, and our forthcoming results, we use the convention that

$$\inf\{\emptyset \cap [0,\infty)\} = \infty, \qquad \inf\{\emptyset \cap (-\infty, 0]\} = 0,$$

whilst

$$\sup\{\emptyset \cap [0,\infty)\} = 0, \qquad \sup\{\emptyset \cap (-\infty, 0]\} = -\infty.$$

It is a consequence of Theorem 1 below, that when the relevant assumptions are satisfied, there exists $z \ge 0$ such that $\psi(z) < 1$ iff there exists $u \ge 0$ such that $\psi(u) = 0$. Thus, even though they are not stated explicitly, Theorem 1 implies exact conditions on the characteristic triplet of (ξ, η) for certain ruin. Statements (1) and (2) of Theorem 1 are generalisations to the dependent case of Paulsen's Theorem 3.1, parts (a) and (b), respectively. Statement (1) of Theorem 1 also removes Paulsen's assumption of finite mean for ξ, and replaces his moment conditions with the precise iff conditions for stationarity of V. For statement (2) of Theorem 1, a finite mean assumption and moment conditions remain necessary.

Theorem 1. *Let* $m := \inf\{u \geq 0 : \psi(u) = 0\}$.

(1) *Suppose* $\lim_{t\to\infty} \xi_t = -\infty$ *a.s. and* $I_{-\xi, K^{\xi,\eta}} < \infty$. *Then* $0 < \psi(z) < 1$ *iff* $0 \leq z < m < \infty$.

(2) *Suppose* $E(\xi_1) = 0$, $E(|\xi_1|^{2+\delta}) < \infty$ *for some* $\delta > 0$ *and there exist* $p, q > 1$ *with* $1/p + 1/q = 1$ *such that* $E(\mathrm{e}^{-p\xi_1}) < \infty$ *and* $E(|\eta_1|^q) < \infty$. *If, for all* $c \in \mathbb{R}$, *the degenerate case* (9) *does not hold, then* $0 < \psi(z) < 1$ *iff* $0 \leq z < m < \infty$. *If there exists* $c \in \mathbb{R}$ *such that Eq.* (9) *holds, then* $\psi(z) < 1$ *iff* $\psi(z) = 0$, *which occurs iff* $0 \leq c \leq z$.

Remark 2. (1) In proving [10, Theorem 3.1(b)], Paulsen discretizes the GOU at integer time points and uses the inequality $P(V_1 < 0 | V_0 = z) > 0$ for all $z \geq 0$. This holds in the independent case if either ξ or η has a Brownian component, or can have negative jumps. However, even in the independent case, this inequality can fail to hold when V_t decreases due to a deterministic drift. For example, let N and M be independent Poisson processes with parameter 1 and define $\xi_t := -t + N_t$ and $\eta_t := -t + M_t$. Let $T_z := \inf\{t > 0 : V_t < 0 | V_0 = z\}$. Then $V_t \geq (z+1)\mathrm{e}^{-t} - 1 := V'_t$ on $t \leq T_z$ and $P(V'_1 < 0 | V'_0 = z) = 0$ whenever $z > \mathrm{e}^1 - 1$. In proving statement (2) of Theorem 1 we avoid this difficulty by discretizing the GOU at random times T_i and then showing that the stated conditions result in $P(V_{T_1} < 0 | V_0 = z) > 0$ for all $z \geq 0$ in the general case.

(2) Assume ξ and η are independent and η is not a subordinator. Whenever ξ drifts to $-\infty$ a.s. or ξ oscillates between ∞ and $-\infty$ a.s. Theorem 1 in [2] implies that $\psi(u) > 0$ for all $u \geq 0$, and hence $m = \infty$. Thus, by (1) of Theorem 1, if $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi, K^{\xi,\eta}} < \infty$, then $\psi(z) = 1$ for all $z \geq 0$. This is a slight strengthening of Paulsen's Theorem 3.1(a). Further, statement (2) simplifies exactly to Paulsen's Theorem 3.1(b). Since ξ and η are independent the conditions in (2) simplify to $E(\xi_1) = 0$, $E(|\xi_1|^{2+\delta}) < \infty$, $E\left(\mathrm{e}^{-\xi_1}\right) < \infty$ and $E(\eta_1) < \infty$. Since $m = \infty$, $\psi(z) = 1$ for all $z \geq 0$ whenever these conditions hold. This simplification occurs because Hölder's inequality is not needed in the proof, and an argument using independence suffices. When transferred onto the Lévy measure, these conditions are equivalent to those in Paulsen's Theorem 3.1(b).

We now present Theorem 3, which is the generalisation to the dependent case of Paulsen's Theorem 3.1, part (c). In addition, Paulsen's assumption of finite mean for ξ is removed, and his moment conditions are replaced with the precise iff conditions for a.s. convergence of Z_t to a finite random variable Z_∞, as $t \to \infty$. A formula for the ruin probability in this situation was given in Theorem 4 of [2], however no conditions for certain ruin were found. Theorem 3 gives exact conditions on the characteristic triplet of (ξ, η) for certain ruin. Let $A_1 := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, and similarly, let A_2, A_3 and A_4 be the quadrants in which $\{x \geq 0, y \leq 0\}$, $\{x \leq 0, y \leq 0\}$ and $\{x \leq 0, y \geq 0\}$ respectively. For each $i = 1, 2, 3, 4$ and $u \in \mathbb{R}$ let

$$B_i^u := \left\{(x, y) \in A_i : y - u(\mathrm{e}^{-x} - 1) > 0\right\}$$

and define

$$\theta_1' := \begin{cases} \inf\left\{u \le 0 : \Pi_{\xi,\eta}(B_1^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_1 \setminus A_2) = 0, \end{cases} \qquad \theta_3' := \begin{cases} \sup\left\{u \le 0 : \Pi_{\xi,\eta}(B_3^u) > 0\right\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_3 \setminus A_2) = 0, \end{cases}$$

$$\theta_2' := \begin{cases} \inf\left\{u \ge 0 : \Pi_{\xi,\eta}(B_2^u) > 0\right\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_2 \setminus A_3) = 0, \end{cases} \qquad \theta_4' := \begin{cases} \sup\left\{u \ge 0 : \Pi_{\xi,\eta}(B_4^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_4 \setminus A_3) = 0. \end{cases}$$

Theorem 3. *Suppose* $\lim_{t\to\infty} \xi_t = \infty$ *a.s. and* $I_{\xi,\eta} < \infty$. *Then* $\psi(0) = 1$ *if and only iff* $-\eta$ *is a subordinator, or there exists* $z > 0$ *such that* $\psi(z) = 1$. *The latter occurs if and only if* $\Pi_{\xi,\eta}(A_1) = 0, \theta_4' \le \theta_2'$, *and there exists* $u \in [\theta_4', \theta_2']$ *such that*

$$\Sigma_{\xi,\eta} = \begin{bmatrix} 1 & -u \\ -u & u^2 \end{bmatrix} \sigma_\xi^2, \tag{10}$$

and

$$g(u) := \tilde{\gamma}_\eta + u\tilde{\gamma}_\xi - \frac{1}{2}u\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} (ux + y)\Pi_{\xi,\eta}(\mathrm{d}(x, y)) \le 0. \tag{11}$$

If there exists $z \ge 0$ *such that* $\psi(z) = 1$ *and, for all* $c \in \mathbb{R}$, *Eq.* (9) *does not hold, then the following hold:*

(1) *If* $\sigma_\xi^2 = 0$ *then* $\psi(z) = 1$ *for all* $z \le m := \sup\{u \in [\theta_4', \theta_2'] : g(u) \le 0\}$, *and* $0 \le \psi(z) < 1$ *for all* $z > m$;
(2) *If* $\sigma_\xi^2 \ne 0$ *then* $\psi(z) = 1$ *for all* $z \le m := -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, *and* $0 < \psi(z) < 1$ *for all* $z > m$.

If there exists $z \ge 0$ *such that* $\psi(z) = 1$ *and there exists* $c \in \mathbb{R}$ *such that* (9) *holds, then* $0 < c = \theta_4' = \theta_2'$, $\psi(z) = 1$ *for all* $z < c$, *and* $\psi(z) = 0$ *for all* $z \ge c$.

Remark 4. (1) It may be that $\theta_2' = \infty$, and if so we consider $[\theta_4', \theta_2']$ to be $[\theta_4', \infty)$. To avoid unwieldy statements, we adopt similar conventions throughout the paper. Namely, if we define parameters $\epsilon, \delta \in [-\infty, \infty]$ where $\epsilon \le \delta$, then the interval $[\epsilon, \delta]$ is considered to be $(-\infty, \delta]$ when $\epsilon = -\infty$, and $[\epsilon, \infty)$ when $\delta = -\infty$. If $\epsilon = \infty$ or $\delta = -\infty$ then we consider $[\epsilon, \delta]$ to be the empty set.
(2) When $\Pi_{\xi,\eta}(A_1) = 0$, $\theta_4' \le \theta_2'$ and $u \in [\theta_4', \theta_2']$ the function $g(u)$ is a well-defined member of the extended reals. The existence and finiteness of g is fully analysed in point (1) of Remark 19.
(3) Assume ξ and η are independent, so $\sigma_{\xi,\eta} = 0$ and all jumps occur at the axes of the A_i. Theorem 3 simplifies to the following statement: Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Then $\psi(0) = 1$ iff $-\eta$ is a subordinator, or $\psi(z) = 1$ for some $z > 0$. The latter occurs iff ξ and η are each of finite variation and have no positive jumps, and $g(z) \le 0$. Note that when (ξ, η) is finite variation, g simplifies to $g(u) = d_\eta + ud_\xi$, as explained in Eq. (13). Since ξ drifts to ∞ a.s., it must be that $d_\xi > 0$. Thus, $g(z) \le 0$ for some $z > 0$ iff $d_\eta < 0$. In particular, $-\eta$ is a subordinator.
(4) Theorem 3.1(c) of Paulsen [10] states that when ξ and η are independent, $E(\xi_1) > 0$, and a set of moment conditions hold, then $\psi(z) = 1$ iff $\xi_t = \alpha t$, $\eta_t = \beta t$ and $\beta < -\alpha z$ for real constants α and β. This contradicts the independence version of Theorem 3 stated above, and is false. A counterexample is $(\xi, \eta)_t := (t, -t - N_t)$ where N is a Poisson process. Paulsen's moment conditions are satisfied trivially. However, Theorem 3 implies that $\psi(z) = 1$ for

all $z \le 1$, and this is confirmed by calculations. If we denote the jump times of N_t by $0 = T_0 < T_1 < T_2 < \cdots$ then

$$V_t = 1 + \mathrm{e}^t \left(z - 1 - \sum_{i=1}^{N_t} \mathrm{e}^{-T_i} \right).$$

Thus, if $z = 1$, then $V_{T_2} = -\mathrm{e}^{T_2 - T_1} < 0$ a.s. and so $\psi(1) = 1$.

The following proposition fully explains the ruin probability function for the degenerate situation (9). It will be used to prove that Theorems 1 and 3 correctly allow for this case.

Proposition 5. *Suppose that there exists $c \in \mathbb{R}$ such that $V_t = \mathrm{e}^{\xi_t}(z - c) + c$. If $c \ge 0$ then $\psi(z) = 0$ for all $z \ge c$, and the following statements hold for all $0 \le z < c$:*

(1) *If ξ drifts to $-\infty$ a.s. then $0 < \psi(z) < 1$;*
(2) *If ξ oscillates between ∞ and $-\infty$ a.s. then $\psi(z) = 1$;*
(3) *If ξ drifts to ∞ a.s. then $\psi(z) = 1$.*

If $c < 0$ then the following statements hold for all $z \ge 0$:

(4) *If ξ drifts to $-\infty$ a.s. then $\psi(z) = 1$;*
(5) *If ξ oscillates between ∞ and $-\infty$ a.s. then $\psi(z) = 1$;*
(6) *If ξ drifts to ∞ a.s. then $0 < \psi(z) < 1$.*

3. Structure of the upper and lower bounds, and relationship with certain ruin

Define the *lower bound function* δ and the *upper bound function* Υ by

$$\delta(z) := \inf \left\{ u \in \mathbb{R} : P\left(\inf_{t \ge 0} V_t \le u | V_0 = z \right) > 0 \right\}$$

and

$$\Upsilon(z) := \sup \left\{ u \in \mathbb{R} : P\left(\sup_{t \ge 0} V_t \ge u | V_0 = z \right) > 0 \right\}.$$

When $V_0 = z$, the probability that the sample paths V_t will ever rise above $\Upsilon(z)$, or below $\delta(z)$, is zero. In particular, the ruin probability function ψ satisfies $\psi(z) = 0$ iff $\delta(z) \ge 0$. Define the sets L and U by

$$L := \{u \in \mathbb{R} : \delta(u) = u\} \quad \text{and} \quad U := \{u \in \mathbb{R} : \Upsilon(u) = u\}.$$

It will be an important consequence of Proposition 17 that L and U must each, for some $a, b \in \mathbb{R}$, be of the form

$$\emptyset, \{a\}, [a, b], [a, \infty), \text{ or } (-\infty, b]. \tag{12}$$

This section contains an analysis of δ, Υ, U and L and their relationship with the ruin function. We examine the possible combinations of L and U and for each combination we examine the possible asymptotic behaviour of ξ. This asymptotic behaviour is closely linked with the conditions for convergence of Z_t and stationarity of V, as discussed in Section 1. As well as being of independent interest, the results contained in this section are essential for the proofs of Theorems 1 and 3.

We begin with comments on δ, and L. The analogues for Υ and U are obvious. Note that $\delta(z) \leq z$ for all $z \in \mathbb{R}$, whilst the fact that V_t is increasing in z for all $t \geq 0$ implies that $\delta(z_1) \leq \delta(z_2)$ whenever $z_1 < z_2$. The next result explains the behaviour of δ outside L, and states that L is precisely the set of starting parts $V_0 = z$ for which almost all sample paths V_t are increasing for some time period. Recall that $T_{z,\Lambda} := \inf\{t > 0 : V_t \in \Lambda\}$, and define $L^c := \mathbb{R} \setminus L$.

Proposition 6. *The following statements hold for L and δ, and the symmetric statements hold for U and Υ:*

(1) *If $z \geq \sup L$ then $\delta(z) = \sup L$;*
(2) *If $z < \inf L$ then $\delta(z) = -\infty$;*
(3) *For $z \in L$, $P(V_t$ is increasing on $0 < t \leq T_{z,L^c} \mid V_0 = z) = 1$;*
(4) *For $z \in L^c$, $P(V_t$ is increasing on $0 < t \leq T_{z,L} \mid V_0 = z) < 1$.*

In Section 1 we assumed that neither ξ nor η is identically zero in order to avoid trivialities. The following proposition explains these trivialities.

Proposition 7. (1) *$L = \mathbb{R}$ iff $\xi_t = 0$ a.s. for all $t > 0$ and η is a subordinator.*
(2) *$U = \mathbb{R}$ iff $\xi_t = 0$ a.s. for all $t > 0$ and $-\eta$ is a subordinator.*
(3) *$L = U = \mathbb{R}$ iff $\xi_t = \eta_t = 0$ a.s. for all $t > 0$.*

We again assume that neither ξ nor η is zero. The next proposition explains the degenerate situation described in Eq. (9). Note that $(\xi, \eta)_t := (\alpha, \beta)t$ for non-zero constants α and β satisfies the conditions of this proposition for $c = -\beta/\alpha$. Recall that a Borel set $\Lambda \subsetneq \mathbb{R}$ is an *absorbing* set for V, if for all $0 \leq s \leq t$, $P(V_t \in \Lambda | V_s = x) = 1$ for all $x \in \Lambda$. That is, whenever a sample path V_t hits Λ, it never leaves. The stochastic exponential is denoted by ϵ.

Proposition 8. *The following are equivalent for $c \neq 0$:*

(1) *$L \cap U \neq \emptyset$;*
(2) *$L \cap U = \{c\}$;*
(3) *$V_t = \mathrm{e}^{\xi_t}(z - c) + c$ and $Z_t = c(\mathrm{e}^{-\xi_t} - 1)$;*
(4) *$\{c\}$ is an absorbing set;*
(5) *$\Sigma_{\xi,\eta}$ satisfies (10) for $u = c$, $\Pi_{\xi,\eta} = 0$ or is supported on the curve $\{(x, y) : y - c(\mathrm{e}^{-x} - 1) = 0\}$, and $g(c) = 0$;*
(6) *$\mathrm{e}^{-\xi_t} = \epsilon(\eta/c)_t$.*

If the above conditions hold and $\Sigma_{\xi,\eta} \neq 0$ then $L = U = \{c\}$ and there exist Lévy processes (ξ, η) for this situation such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s. If the above conditions hold and $\Sigma_{\xi,\eta} = 0$ then:

(a) *$U = (-\infty, c]$ and $L = [c, \infty)$ iff ξ is a subordinator;*
(b) *$L = (-\infty, c]$ and $U = [c, \infty)$ iff $-\xi$ is a subordinator;*
(c) *$L = U = \{c\}$ iff neither ξ or $-\xi$ is a subordinator. There exist Lévy processes (ξ, η) for this situation such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s.*

We present a theorem which describes all possible combinations of L and U and the associated asymptotic behaviour of ξ, for the case in which $L \cap U = \emptyset$.

Theorem 9. *Suppose that $L \cap U = \emptyset$. If $\Sigma_{\xi,\eta} \neq 0$ then only the following cases can exist:*

(1) *$L = U = \emptyset$;*

(2) $L = \{a\}$ *for some* $a \in \mathbb{R}$ *and* $U = \emptyset$*;*
(3) $U = \{a\}$ *for some* $a \in \mathbb{R}$ *and* $L = \emptyset$*.*

If $\Sigma_{\xi,\eta} = 0$ *then only the following cases can exist:*

(a) *If* $L = \emptyset$ *then* U *is of the form* $\emptyset$, $\{a\}$, $[a, b]$, $[a, \infty)$*, or* $(-\infty, b]$ *for some* $a, b \in \mathbb{R}$*;*
(b) *If* $U = \emptyset$ *then* L *is of the form* $\emptyset$, $\{a\}$, $[a, b]$, $[a, \infty)$*, or* $(-\infty, b]$ *for some* $a, b \in \mathbb{R}$*;*
(c) *If* $L \neq \emptyset$ *and* $U \neq \emptyset$ *then there exist* $a < b$ *such that* $L = (-\infty, a]$ *and* $U = [b, \infty)$*, or* $U = (-\infty, a]$ *and* $L = [b, \infty)$*.*

If $U = (-\infty, a]$ *or* $L = [b, \infty)$ *(or both with* $a < b$*) then* ξ *is a subordinator. If* $L = (-\infty, a]$ *or* $U = [b, \infty)$ *(or both with* $a < b$*) then* $-\xi$ *is a subordinator. For all of the other combinations of* L *and* U *above, there exist Lévy processes* (ξ, η) *such that* ξ *drifts to* ∞ *a.s.,* ξ *drifts to* $-\infty$ *a.s. or* ξ *oscillates a.s.*

An absorbent set $\Lambda \subsetneq \mathbb{R}$ is a *maximal absorbing* set if it is not properly contained in another absorbing set. If Λ is a maximal absorbing set, then $\mathbb{R} \setminus \Lambda$ contains no absorbing sets since the union of Λ with the absorbing set is an absorbing set properly containing Λ. The following corollary is immediate. For each statement (1)–(4), the claim that the sets Λ are maximal absorbing follows from Proposition 6. The remaining statements follow from Theorem 9.

Corollary 10. *There exist Lévy processes* (ξ, η) *with* $L \cap U = \emptyset$ *such that the associated GOU has the following maximal absorbing sets* Λ*:*

(1) $\Lambda = U \cup L$*, where* $U = (-\infty, a]$ *and* $L = [b, \infty)$*;*
(2) $\Lambda = U$*, where* $U = (-\infty, a]$ *and* $L = \emptyset$*;*
(3) $\Lambda = L$*, where* $L = [b, \infty)$ *and* $U = \emptyset$*;*
(4) $\Lambda = (a, b)$ *where* $L = (-\infty, a]$ *and* $U = [b, \infty)$*.*

If (ξ, η) *has* $L \cap U = \emptyset$ *and does not have* U *and* L *satisfying one of* (1)–(4)*, then no absorbing sets exist.*

We examine two striking cases of L and U structure, and state iff conditions on the characteristic triplet of (ξ, η) for such behaviour. Similar conditions exist for the other structures in Theorem 9, however, the statements are unwieldy.

Proposition 11. *Suppose* $L \cap U = \emptyset$*. Then* $U = (-\infty, a]$ *and* $L = [b, \infty)$ *for* $-\infty < a < b < \infty$ *iff* (ξ, η) *is of finite variation and the following hold:*

- *There is no Brownian component* $(\Sigma_{\xi,\eta} = 0)$*;*
- *The drift of* ξ *is non-negative* $(d_\xi \geq 0)$*;*
- *The Lévy measure satisfies* $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_4) = 0$*,* $\theta_1' > -\infty$*, and* $\theta_2 < \infty$*.*

If these conditions hold, ξ *is a subordinator and for any finite starting random variable* V_0 *we have* $\lim_{t\to\infty} |V_t| = \infty$ *a.s.*

Similarly $L = (-\infty, a]$ *and* $U = [b, \infty)$ *for* $-\infty < a < b < \infty$ *iff* (ξ, η) *is of finite variation and the following hold:*

- *There is no Brownian component* $(\Sigma_{\xi,\eta} = 0)$*;*
- *The drift of* ξ *is non-positive* $(d_\xi \leq 0)$*;*
- *The Lévy measure satisfies* $\Pi_{\xi,\eta}(A_1) = \Pi_{\xi,\eta}(A_2) = 0$*,* $\theta_4' < \infty$ *and* $\theta_3 > -\infty$*.*

If these conditions hold, ξ *is a subordinator and there is a random variable* V_∞ *with support* (a, b) *such that* V*, starting with* $V_0 =_D V_\infty$*, is strictly stationary.*

We state a theorem describing the relationship between the sets L and U, and the upper and lower bounds of the limit random variable Z_∞ of Z_t as $t \to \infty$.

Theorem 12. *Let $a, b \in \mathbb{R}$ and suppose $Z_t \to Z_\infty$ a.s. as $t \to \infty$, where Z_∞ is a finite random variable. If, for all $c \in \mathbb{R}$, the degenerate case (9) does not hold, then $a \le \sup U$ iff $Z_\infty < -a$ a.s., whilst $b \ge \inf L$ iff $Z_\infty > -b$ a.s. Further, $-\sup U = \inf\{u \in \mathbb{R} : Z_\infty < u \text{ a.s.}\}$ and $-\inf L = \sup\{u \in \mathbb{R} : Z_\infty > u \text{ a.s.}\}$. Alternatively, if there exists $c \in \mathbb{R}$ such that Eq. (9) holds, then $Z_\infty = -c$ a.s. and $\inf L = \sup U = c$.*

The next theorem presents results on certain ruin which occur when L and U are of a particular structure.

Theorem 13. *Suppose that $L \cap U = \emptyset$. Then the following statements hold:*

(1) *If $\sup U \ge 0$ and $L \cap [0, \sup U] = \emptyset$, then $\psi(z) = 1$ for all $z \le \sup U$;*
(2) *If $\sup L \ge 0$ and $U \cap [0, \sup L] = \emptyset$, then $0 < \psi(z) < 1$ for all $0 \le z < \inf L$. If $\sup L \ge 0$ and $U \cap [0, \sup L] \ne \emptyset$, then $\psi(z) < 1$ for all $z > \sup U$.*

Note that in statement (2) above, when $\sup L \ge 0$ and $L \cap U \ne \emptyset$, Theorem 9 ensures that $\sup U < \inf L$, and statement (1) above ensures that $\psi(z) = 1$ for all $z \le \sup U$. Also, by definition of L, $\psi(z) = 0$ whenever $z \ge \inf L$.

We state a major theorem which uses Theorems 9, 12 and 13, and is used to prove Theorems 1 and 3. For the non-degenerate case, and for (ξ, η) which satisfies various asymptotic and stability criteria, this theorem presents iff conditions for certain ruin, stated in terms of L and U structure. In particular, it completely describes the L and U structures for which certain ruin occurs.

Theorem 14. *Suppose $L \cap U = \emptyset$.*

(1) *Suppose $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi, K^{\xi,\eta}} < \infty$. There exists $z \ge 0$ such that $\psi(z) < 1$ iff $L \cap [0, \infty) \ne \emptyset$. If this occurs then $0 < \psi(z) < 1$ for all $0 \le z < \inf L$, $\psi(z) = 0$ for all $z \ge \inf L$, and one of the following must hold:*
   (a) *$L = [a, b]$ and $U = \emptyset$, where $-\infty \le a \le b < \infty$, and $b \ge 0$;*
   (b) *$L = (-\infty, a]$ and $U = [b, \infty)$ where $0 \le a < b < \infty$.*
(2) *Suppose $E(\xi_1) = 0$, $E(|\xi_1|^{2+\delta}) < \infty$ for some $\delta > 0$ and there exist $p, q > 1$ with $1/p + 1/q = 1$ such that $E(\mathrm{e}^{-p\xi_1}) < \infty$ and $E(|\eta_1|^q) < \infty$. There exists $z \ge 0$ such that $\psi(z) < 1$ iff $L \cap [0, \infty) \ne \emptyset$. If this occurs then $L = [a, b]$ and $U = \emptyset$, where $-\infty < a \le b < \infty$ and $b \ge 0$, in which case $0 < \psi(z) < 1$ for all $0 \le z < a$ and $\psi(z) = 0$ for all $z \ge a$;*
(3) *Suppose $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. There exists $z \ge 0$ such that $\psi(z) = 1$ iff $U \cap [0, \infty) \ne \emptyset$. If this occurs then one of the following must hold:*
   (c) *$U = [a, b]$ and $L = \emptyset$, where $-\infty \le a \le b < \infty$ and $b \ge 0$, in which case $\psi(z) = 1$ for all $z \le b$ and $0 < \psi(z) < 1$ for all $z > b$;*
   (d) *$U = (-\infty, a]$ and $L = [b, \infty)$ where $0 \le a < b < \infty$, in which case $\psi(z) = 1$ for all $z \le a$, $0 < \psi(z) < 1$ for all $a < z < b$ and $\psi(z) = 0$ for all $z \ge b$.*

Remark 15. The characteristic triplet conditions which equate to the iff result in statement (3) above, are given in Theorem 3, and are obtained using the forthcoming Proposition 20. Exact characteristic triplet conditions for $U = (-\infty, a]$ and $L = [b, \infty)$ in case (d) above, are given in Proposition 11.

4. Technical results on the upper and lower bounds

We state some propositions on δ, L, Υ and U which are essential in proving the previous theorems. The first proposition combines and restates Proposition 6, Theorem 7 and Theorem 9 of [2]. No proof is given. This proposition completely describes the relationship between $\Pi_{\xi,\eta}$ and δ. For A_i as in Section 2, define $A_i^u := \{(x, y) \in A_i : y - u(e^{-x} - 1) < 0\}$. For $u \le 0$ define

$$\theta_1 := \begin{cases} \sup\left\{u \le 0 : \Pi_{\xi,\eta}(A_1^u) > 0\right\} \\ -\infty \quad \text{if } \Pi_{\xi,\eta}(A_1 \setminus A_4) = 0, \end{cases} \qquad \theta_3 := \begin{cases} \inf\left\{u \le 0 : \Pi_{\xi,\eta}(A_3^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_3 \setminus A_4) = 0, \end{cases}$$

and for $u \ge 0$ define

$$\theta_2 := \begin{cases} \sup\left\{u \ge 0 : \Pi_{\xi,\eta}(A_2^u) > 0\right\} \\ 0 \quad \text{if } \Pi_{\xi,\eta}(A_2 \setminus A_1) = 0, \end{cases} \qquad \theta_4 := \begin{cases} \inf\left\{u \ge 0 : \Pi_{\xi,\eta}(A_4^u) > 0\right\} \\ \infty \quad \text{if } \Pi_{\xi,\eta}(A_4 \setminus A_1) = 0. \end{cases}$$

Throughout, let W be the Lévy process such that $e^{-\xi_t} = \epsilon(W)_t$.

Proposition 16 (*Lower Bound*). *The following statements are equivalent:*

(1) *The lower bound function satisfies $\delta(z) > -\infty$ for some $z \in \mathbb{R}$;*
(2) *There exists $u \in \mathbb{R}$ such that $\delta(u) = u$;*
(3) *There exists $u \in \mathbb{R}$ such that the Lévy process $\eta - uW$ is a subordinator.*

Statement (2) *holds for a particular value $u \in \mathbb{R}$ iff* (3) *holds for the same u, and vice versa. Statements* (2) *and*(3) *hold for a particular value $u \neq 0$ iff the following three conditions are satisfied:* (i) *the Gaussian covariance matrix satisfies Eq. (10);* (ii) *one of the following is true:*

(a) $\Pi_{\xi,\eta}(A_3) = 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$, $\theta_2 \le \theta_4$ *and* $u \in [\theta_2, \theta_4]$*;*
(b) $\Pi_{\xi,\eta}(A_2) = 0$, $\Pi_{\xi,\eta}(A_3) \neq 0$, $\theta_1 \le \theta_3$ *and* $u \in [\theta_1, \theta_3]$*;*
(c) $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ *and* $u \in [\theta_1, \theta_4]$*;*

and, (iii)*, in addition, u satisfies $g(u) \ge 0$ for the function g in Eq. (11).*

From the definition of L it is an immediate corollary, firstly, that $L = \emptyset$ iff none of conditions (1)–(3) of Proposition 16 hold, and secondly, that η is a subordinator iff $0 \in L$. The next proposition adds further information concerning L. Most importantly, it shows that the set L is always connected, and gives concrete values for the endpoints.

Proposition 17. *If $\sigma_\xi^2 \neq 0$ and any of conditions* (1)–(3) *of Proposition 16 hold, then $L = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$. If $\sigma_\xi^2 = 0$ and any of* (1)–(3) *hold, then $\sigma_\eta^2 = 0$ and one of the following holds:*

- *η is a subordinator and condition* (ii) *of Proposition 16 does not hold for any $u \neq 0$, in which case $L = \{0\}$;*
- *Condition* (ii) *is satisfied for some $u \neq 0$, in which case there exists $-\infty \le a \le b \le \infty$ such that $L = [a, b]$.*

In the latter case, if condition (a) *of Proposition 16 holds then $0 \le a = \max\{\theta_2, m_1\}$ and $b = \min\{\theta_4, m_2\}$ for $m_1 := \inf\{u \in \mathbb{R} : g(u) \ge 0\}$ and $m_2 := \sup\{u \in \mathbb{R} : g(u) \ge 0\}$. If* (b) *holds then $a = \max\{\theta_1, m_1\}$ and $b = \min\{\theta_3, m_2\} \le 0$. If* (c) *holds then $a = \max\{\theta_1, m_1\}$ and $b = \min\{\theta_4, m_2\}$.*

Define L^* to be the set of starting values on which the GOU has no negative jumps, namely

$$L^* := \{u \in \mathbb{R} : \forall t > 0 P\left(\Delta V_t < 0 | V_{t-} = u\right) = 0\}.$$

It is a consequence of Proposition 6 that $L \subseteq L^*$. The next proposition describes L^*. In particular, it shows that the set L^* is always connected, and gives concrete values for the endpoints. It also shows that whenever $V_{t-} > \sup L^*$ and a negative jump ΔV_t occurs, then the jump cannot be so negative as to cause $V_t \le \sup L^*$. Thus, L^* acts as a barrier for negative jumps of V.

Proposition 18. (1) *If* $L^* \neq \emptyset$ *then, for any* $t \ge 0$, $V_{t-} > \sup L^*$ *implies* $V_t > \sup L^*$ *a.s.;*
(2) $L^* = \{u \in \mathbb{R} : \eta - uW$ *has no negative jumps*$\}$*;*
(3) $L^* \neq \emptyset$ *iff condition* (ii) *of Proposition* 16 *is satisfied for some* $u \neq 0$, *or* η *has no negative jumps;*
(4) $L^* = \{0\}$ *iff* η *has no negative jumps and condition* (ii) *does not hold for any* $u \neq 0$*;*
(5) *If condition* (ii) *of Proposition* 16 *holds for some* $u \neq 0$ *then* $L^* = [\theta_2, \theta_4], [\theta_1, \theta_3]$ *or* $[\theta_1, \theta_4]$, *corresponding to conditions* (a), (b) *or* (c) *of Proposition* 16.

Remark 19. (1) If (ξ, η) is an infinite variation Lévy process then, as noted in Section 1, $\int_{\{x^2+y^2<1\}} |(x, y)| \Pi_{\xi,\eta}(\mathrm{d}(x, y)) = \infty$. Thus, for some $u \in \mathbb{R}$ the integral $\int_{\{x^2+y^2<1\}}(ux + y)\Pi_{\xi,\eta}(\mathrm{d}(x, y))$, and hence the function $g(u)$ in (11), may not exist as a well-defined member of the extended real numbers. However, it is a consequence of the proof of Theorem 9 in [2], that if $u \in L^*$ then $g(u)$ is a well-defined member of the extended reals, and $g(u) \in [-\infty, \infty)$. Under such conditions, it is also shown that

$$\Pi_{\xi,\eta}\left(\{y - u(\mathrm{e}^{-x} - 1) < 0\}\right) = 0$$

and so the domain of integration for the integral component of g can be decreased to $\{x^2 + y^2 < 1\} \cap \{y - u(\mathrm{e}^{-x} - 1) \ge 0\}$.

(2) Note that g is a linear function on $\mathbb{R}$ iff the Lévy measure of (ξ, η) is of finite variation, namely

$$\int_{\{x^2+y^2<1\}} |(x, y)| \Pi_{\xi,\eta}(\mathrm{d}(x, y)) < \infty.$$

In this case the drift vector (d_ξ, d_η) is finite, and we can write

$$\begin{aligned} g(u) &= \gamma_\eta - \int_{(-1,1)} y \Pi_\eta(\mathrm{d}y) + u\left(\gamma_\xi - \frac{1}{2}\sigma_\xi^2 - \int_{(-1,1)} x \Pi_\xi(\mathrm{d}x)\right) \\ &= d_\eta + u\left(d_\xi - \frac{1}{2}\sigma_\xi^2\right), \end{aligned} \tag{13}$$

where the first equality follows by converting $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)$ to $(\gamma_\xi, \gamma_\eta)$ using Eq. (4) and the symmetric version for η, and the second equality follows by converting $(\gamma_\xi, \gamma_\eta)$ to (d_ξ, d_η) using Eq. (5). It will be a consequence of the proof of Proposition 17, that if $a, b \in L$ and $a \neq b$ then g is a linear function on $\mathbb{R}$.

(3) Section 1 stated exact conditions for a Lévy process to be a subordinator. When $u \neq 0$ the conditions in Proposition 16 are exactly the requirements for $\eta - uW$ to be a subordinator. Eq. (10) is equivalent to $\sigma_{\eta-uW} = 0$. The requirement that one of the conditions (a), (b) and (c) holds is equivalent to the requirement that there exists $u \neq 0$ such that $\Pi_{\eta-uW}((-\infty, 0)) = 0$. This implies that $L^* \setminus \{0\}$ is the set of all $u \neq 0$ such that $\eta - uW$ has no negative jumps. Finally, if $u \in L^*$ then $g(u) = d_{\eta-uW}$, and hence condition (11) is

equivalent to the requirement that $\eta - uW$ has positive drift. The fact that $\eta - uW$ is of finite variation actually follows from the two conditions $\Pi_{\eta-uW}((-\infty, 0)) = 0$ and $d_{\eta-uW} \geq 0$. To see this, note that when $\Pi_{\eta-uW}((-\infty, 0)) = 0$, Eq. (5) simplifies to

$$d_{\eta-uW} = \gamma_{\eta-uW} - \int_{(0,1)} x \Pi_{\eta-uW}(\mathrm{d}x),$$

and hence $d_{\eta-uW}$ is a member of the extended reals regardless of whether $\eta - uW$ is finite variation. In particular, $d_{\eta-uW} \in [-\infty, \infty)$, and $d_{\eta-uW} = -\infty$ iff $\int_{(0,1)} x \Pi_{\eta-uW}(\mathrm{d}x) = \infty$ which occurs iff $\eta - uW$ is infinite variation.

We state, without proof, the parallel version for U and Υ, to Proposition 16. This statement is needed to prove Theorem 3 and will be combined with Proposition 16 to prove Theorems 9, 13 and 14. If we define

$$U^* := \{u \in \mathbb{R} : \forall t > 0 \; P(\Delta V_t > 0 | V_{t-} = u) = 0\},$$

then the parallel versions of Propositions 17 and 18 and Remark 19 hold. We use these results, however the parallels are obvious so we do not state explicitly.

Proposition 20 (*Upper Bound*). *The following are equivalent:*

(1) *The upper bound function satisfies $\Upsilon(z) < \infty$ for some $z \in \mathbb{R}$;*
(2) *There exists $u \in \mathbb{R}$ such that $\Upsilon(u) = u$;*
(3) *There exists $u \in \mathbb{R}$ such that the Lévy process $-(\eta - uW)$ is a subordinator.*

Statement (2) *holds for a particular value $u \in \mathbb{R}$ iff* (3) *holds for the same u, and vice versa. Statements* (2) *and* (3) *hold for a particular value $u \neq 0$ iff the following three conditions are satisfied:* (i) *the Gaussian covariance matrix satisfies Eq. (10);* (ii) *one of the following is true:*

(a) *$\Pi_{\xi,\eta}(A_1) = 0$, $\Pi_{\xi,\eta}(A_4) \neq 0$, $\theta'_4 \leq \theta'_2$ and $u \in [\theta'_4, \theta'_2]$;*
(b) *$\Pi_{\xi,\eta}(A_4) = 0$, $\Pi_{\xi,\eta}(A_1) \neq 0$, $\theta'_3 \leq \theta'_1$ and $u \in [\theta'_3, \theta'_1]$;*
(c) *$\Pi_{\xi,\eta}(A_1) = \Pi_{\xi,\eta}(A_4) = 0$ and $u \in [\theta'_3, \theta'_2]$;*

and, (iii)*, in addition, u satisfies $g(u) \leq 0$ for the function g in Eq. (11).*

Remark 21. Symmetric statements to those for L and L^* in Remark 19, hold for U and U^*. The following remarks relate to the combination of L and U, and L^* and U^*.

(1) Parallel to 1 and 2 of Remark 19, whenever $u \in U^*$, $g(u)$ from (11) is a well-defined member of the extended reals, $g(u) \in (-\infty, \infty]$, and $-g(u) = d_{-(\eta-uW)}$. Since $d_{-(\eta-uW)} = -d_{\eta-uW}$, we know that if $u \in U^* \cup L^*$ then $g(u)$ is a well-defined member of the extended reals and $g(u) = d_{\eta-uW}$.
(2) If $a \in L$, $b \in U$ and $a \neq b$ then g is linear and (ξ, η) is finite variation. This statement is proved easily using similar arguments to those in the proof of Proposition 17.

We state a proposition, describing the possible combinations of L^* and U^*, which will be essential for proving Theorem 9.

Proposition 22. *The following statements hold for L^*, and the symmetric statements hold for U^*:*

(1) *If $L^* = \mathbb{R}$ then $U^* = \emptyset$ or $U^* = \mathbb{R}$;*
(2) *If $L^* = [a, b]$ for some $-\infty < a \leq b < \infty$, then $U^* = \emptyset$ or $U^* = L^* = \{a\} = \{b\}$;*

(3) *If* $L^* = [b, \infty)$ *for some* $b \in \mathbb{R}$*, then* $U^* = \emptyset$ *or* $U^* = (-\infty, a]$ *for some* $-\infty < a \le b < \infty$*;*
(4) *If* $L^* = (-\infty, a]$ *for some* $a \in \mathbb{R}$*, then* $U^* = \emptyset$ *or* $U^* = [b, \infty)$ *for some* $-\infty < a \le b < \infty$.

We end the section with two lemmas. No proof will be given. The first follows by considering the definitions of θ_i and θ_i'. It will be used several times as a calculation tool. The second gives conditions on the Lévy measure of ξ and η which ensure that $\sup_{0\le t\le 1} |Z_t|$ has finite mean. It will be needed to prove statement (2) of Theorem 1. The proof is similar to that of Lemma 11 in [2] and uses the Burkholder–Davis–Gundy inequalities, and various Doob's inequalities.

Lemma 23. (1) *If* $\Pi_{\xi,\eta}(A_1) \ne 0$ *then* $\theta_1' \le \theta_1 \le 0$*;*
(2) *If* $\Pi_{\xi,\eta}(A_2) \ne 0$ *then* $0 \le \theta_2' \le \theta_2$*;*
(3) *If* $\Pi_{\xi,\eta}(A_3) \ne 0$ *then* $\theta_3 \le \theta_3' \le 0$*;*
(4) *If* $\Pi_{\xi,\eta}(A_4) \ne 0$ *then* $0 \le \theta_4 \le \theta_4'$.

Further:

(a) $\Pi_{\xi,\eta}(A_1) = 0$ *iff* $\theta_1 = -\infty$ *and* $\theta_1' = 0$*;*
(b) $\Pi_{\xi,\eta}(A_2) = 0$ *iff* $\theta_2 = 0$ *and* $\theta_2' = \infty$*;*
(c) $\Pi_{\xi,\eta}(A_3) = 0$ *iff* $\theta_3 = 0$ *and* $\theta_3' = -\infty$*;*
(d) $\Pi_{\xi,\eta}(A_4) = 0$ *iff* $\theta_4 = \infty$ *and* $\theta_4' = 0$.

Lemma 24. *Suppose there exist* $r > 0$ *and* $p, q > 1$ *with* $1/p + 1/q = 1$ *such that* $E(\mathrm{e}^{-\max\{1,r\}p\xi_1}) < \infty$ *and* $E(|\eta_1|^{\max\{1,r\}q}) < \infty$*. Then*

$$E\left(\sup_{0\le t\le 1}\left|\int_0^t \mathrm{e}^{-\xi_{s-}}\mathrm{d}\eta_s\right|^{\max\{1,r\}}\right) < \infty. \tag{14}$$

5. Proofs for Section 4

Throughout the remaining sections, with the exception of the proof of Proposition 7, we retain the assumption that neither ξ nor η is identically zero. We also note that Proposition 16 will not be proved, but will be used repeatedly.

Proof (Proposition 18). We prove statements (2), (3) then (1). The proof of (4) and (5) follows trivially from the proof of (2) and (3).

(2) Proposition 6 in [2] implies that $\Delta(\eta_t - uW_t) = \Delta\eta_t - u(\mathrm{e}^{-\Delta\xi_t} - 1)$. Thus, Eq. (3) implies that whenever $V_{t-} = u$, a jump $(\Delta\xi_t, \Delta\eta_t)$ causes a negative jump ΔV_t iff $\Delta(\eta_t - uW_t)$ is negative. Hence L^* is precisely the set of all u such that $\eta_t - uW_t$ has no negative jumps.

(3) By (2), $L^* \ne \emptyset$ iff $\eta - uW$ has no negative jumps for some $u \in \mathbb{R}$. If $u = 0$, this occurs iff η has no negative jumps. If $u \ne 0$, it is noted in point (3) of Remark 19, that this occurs iff $u \ne 0$ satisfies condition (ii) of Proposition 16.

(1) Suppose $L^* \ne \emptyset$. If $0 \in L^*$ then (1) is trivial. If $0 \notin L^*$ then (ii) of Proposition 16 holds for some $u \ne 0$. We assume (a) of Proposition 16 holds. If (b) or (c) of Proposition 16 holds the proof is similar. Since (a) holds, property (5) implies that $L^* = [\theta_2, \theta_4]$. Suppose $V_{t-} > \theta_4$ and recall that Eq. (3) states

$$\Delta V_t = (\mathrm{e}^{\Delta\xi_t} - 1)V_{t-} + \mathrm{e}^{\Delta\xi_t}\Delta\eta_t.$$

By the definitions of θ_4 and A_4^u, and Eq. (3), there exists $(x, y) \in A_4^{V_{t-}}$ such that $(e^x - 1)\theta_4 + e^x y \geq 0$ and $(e^x - 1)V_{t-} + e^x y < 0$. Thus,

$$\begin{aligned} V_t &= V_{t-} + (e^x - 1)V_{t-} + e^x y \\ &= V_{t-} + (e^x - 1)(V_{t-} - \theta_4) + (e^x - 1)\theta_4 + e^x y \\ &\geq V_{t-} + (e^x - 1)(V_{t-} - \theta_4) \\ &> \theta_4. \quad \square \end{aligned}$$

Proof (Proposition 17). Assume $\sigma_\xi^2 \neq 0$ and (2) and (3) of Proposition 16 hold for some $u \neq 0$. Then Eq. (10) holds for u, which implies that $u = -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$, and hence is the unique non-zero number satisfying (2) and (3) of Proposition 16. Since $-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$ satisfies condition (2), $L = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$ by definition.

Assume $\sigma_\xi^2 \neq 0$ and (2) and (3) of Proposition 16 hold for $u = 0$. By (2), $0 \in L$. By (3), η is a subordinator, and hence $\sigma_\eta^2 = \sigma_{\xi,\eta} = 0$. Thus, by the above, no non-zero number can satisfy statements (2), (3), and so $L = \{0\} = \{-\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}\}$.

Assume $\sigma_\xi^2 = 0$. If (2) and (3) of Proposition 16 hold for $u = 0$ then η is a subordinator by (3) and hence $\sigma_\eta^2 = 0$. Alternatively, if (1)–(3) of Proposition 16 hold for some $u \neq 0$ then Eq. (10) holds for u, which implies that $\sigma_\eta^2 = u^2\sigma_\xi^2$, and so $\sigma_\eta^2 = 0$.

Assume $\sigma_\xi^2 = 0$ and condition (ii) of Proposition 16 does not hold for any $u \neq 0$. This implies that $L \cap (\mathbb{R} \setminus \{0\}) = \emptyset$. If, further, η is a subordinator, then $0 \in L$, and hence $L = \{0\}$.

Assume $\sigma_\xi^2 = 0$ and (ii) of Proposition 16 holds for some $u \neq 0$. This occurs precisely when one of conditions (a), (b) or (c) of Proposition 16 holds, and Eq. (11) holds. Thus, $\inf L = a$ and $\sup L = b$ for the values of a and b given in the proposition statement. Since L^* is connected, L is connected iff $\{u \in \mathbb{R} : g(u) \geq 0\}$ is connected, which follows from the analysis below.

As noted in point (1) of Remark 19, whenever $u \in L^*$ we know $g(u) \in [-\infty, \infty)$. There are three possibilities for behaviour of g on L^*. Firstly, it may be that $g(u) = -\infty$ for all $u \in L^*$. Secondly there may exist $v \in L^*$ such that $g(v)$ is finite and $g(u) = -\infty$ for all $u \in L^*$ with $u \neq v$. We show that the only other possibility is that g is linear on $\mathbb{R}$. Suppose there exists $u_1, u_2 \in L^*$ with $u_1 \neq u_2$, such that $g(u_1)$ and $g(u_2)$ are both finite. Then

$$g(u_1) - g(u_2) = \left(\tilde{\gamma}_\xi - \frac{1}{2}\sigma_\xi^2 - \int_{\{x^2+y^2<1\}} x\Pi_{\xi,\eta}(d(x, y))\right)(u_1 - u_2)$$

is finite, which implies that $\int_{\{x^2+y^2<1\}} x\Pi_{\xi,\eta}(d(x, y))$ exists, and is finite. Since $g(u_1)$ is finite, this implies that $\int_{\{x^2+y^2<1\}} y\Pi_{\xi,\eta}(d(x, y))$ exists and is finite. Thus, g is a linear function on $\mathbb{R}$. $\square$

Proof (Proposition 22). We prove statements (1), (2), and (3). The proof of (4) is similar to the proof of (3).

(1) Assume $L^* = \mathbb{R}$. Then condition (c) of Proposition 16 must hold, and so $\Pi_{\xi,\eta}(A_2) = \Pi_{\xi,\eta}(A_3) = 0$, and $L^* = [\theta_1, \theta_4]$. Since $\theta_1 = -\infty$ and $\theta_4 = \infty$, it must be that $\Pi_{\xi,\eta}(A_1 \setminus A_4) = 0$ and $\Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$, respectively. Thus, if $\Pi_{\xi,\eta}(A_1 \cap A_4) = 0$ then $\Pi_{\xi,\eta}(\mathbb{R}^2) = 0$, in which case condition (c) of Proposition 20 holds, and $U^* = \mathbb{R}$. Alternatively, if $\Pi_{\xi,\eta}(A_1 \cap A_4) \neq 0$ then η has positive jumps and so $0 \notin U^*$, and (ii) of Proposition 20 cannot hold. Hence $U^* = \emptyset$.

(2) Assume $L^* = [a, b]$ for some $-\infty < a \le b < \infty$. There are four ways in which this is possible, namely, when (a), (b) or (c) of Proposition 16 hold, or when $L^* = \{0\}$. For each case we show $U^* = \emptyset$ or $U^* = L^* = \{a\} = \{b\}$.

Suppose (a) of Proposition 16 holds, and $U^* \neq \emptyset$. The case in which (b) holds and $U^* \neq \emptyset$, is similar. Propositions 16 and 18 imply that $\Pi_{\xi,\eta}(A_3) = 0, \Pi_{\xi,\eta}(A_2) \neq 0, \theta_2 \le \theta_4$ and $L^* = [\theta_2, \theta_4]$. Since $\theta_4 < \infty$, it must be that $\Pi_{\xi,\eta}(A_4 \setminus A_1) \neq 0$. Since $\Pi_{\xi,\eta}(A_3) = 0$, this implies that $-\eta$ is not a subordinator, and so $0 \notin U^*$. Thus, since $U^* \neq \emptyset$, condition (a) of Proposition 20 holds, and so $\Pi_{\xi,\eta}(A_1) = 0, \theta_4' \le \theta_2'$, and $U^* = [\theta_4', \theta_2']$. However, statements (2), (4) of Lemma 23 state that $\theta_2' \le \theta_2$ and $\theta_4 \le \theta_4'$. Hence $\theta_2' = \theta_2 = \theta_4 = \theta_4'$.

Suppose (c) of Proposition 16 holds. Then $\Pi_{\xi,\eta}(A_2) = \Pi_{\xi,\eta}(A_3) = 0$, and $L^* = [\theta_1, \theta_4]$. Since $\theta_4 < \infty$ and $\theta_1 > -\infty$ it must be that $\Pi_{\xi,\eta}(A_4 \setminus A_1) \neq 0$ and $\Pi_{\xi,\eta}(A_1 \setminus A_4) \neq 0$, respectively. Hence, (ii) of Proposition 20 cannot hold, and so $U^* \setminus \{0\} = \emptyset$. Further, $-\eta$ is not a subordinator, and so $U^* = \emptyset$.

Suppose $L^* = \{0\}$, and $U^* \neq \emptyset$. By (4) of Proposition 18, $L^* = \{0\}$ iff η has no negative jumps and at the same time $\Pi_{\xi,\eta}(A_3 \cap A_4) \neq 0$ and $\Pi_{\xi,\eta}(A_2 \cap A_1) \neq 0$. Hence, (ii) of Proposition 20 fails to hold, which implies $U^* \setminus \{0\} = \emptyset$. Thus, since $U^* \neq \emptyset$, it must be that $U^* = L^* = \{0\}$.

(3) Assume $L^* = [b, \infty)$ for some $b \in \mathbb{R}$ and $U^* = \emptyset$. We prove $U^* = (-\infty, a]$ for some $-\infty < a \le b < \infty$. By the symmetric version of (2) of Proposition 22, we have $U^* \neq \{0\}$. Since $L^* = [b, \infty)$, condition (a) or (c) of Proposition 16 holds, with $\theta_4 = \infty$. Thus, $\Pi_{\xi,\eta}(A_3) = 0$, which implies $\theta_3' = -\infty$. Also, since $\theta_4 = \infty$, we have $\Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$. Since $U^* \neq \emptyset$, we have $\Pi_{\xi,\eta}(A_1 \cap A_4) = 0$, and so $\Pi_{\xi,\eta}(A_4) = 0$. This implies that (b) or (c) of Proposition 20 holds, and so $U^* = (-\infty, \theta_1']$ or $U^* = (-\infty, \theta_2']$ respectively. Now, if condition (a) of Proposition 16 holds, then $L^* = [\theta_2, \infty)$. Note that Lemma 23 states that $\theta_1' \le 0 \le \theta_2' \le \theta_2$, and hence the result is proved for either form of U^*. Alternatively, if (c) of Proposition 16 holds, then $L^* = [\theta_1, \infty)$ where $\theta_1 > -\infty$, which implies $\Pi_{\xi,\eta}(A_1 \setminus A_4) \neq 0$. Hence, (b) of Proposition 20 holds and $U^* = (-\infty, \theta_1']$. Now, Lemma 23 states that $\theta_1' \le \theta_1$. □

6. Proofs for Section 3 and associated examples

Proof (Proposition 6). Proposition 16 implies that $\delta(\delta(z)) = \delta(z)$ and

$$\delta(z) = \sup\{u \le z : \delta(u) = u\}. \tag{15}$$

Statement (1) of Proposition 6 follows from (15). To prove (2), assume $z < \inf L$. Suppose $-\infty < m := \delta(z)$. Since $\delta(z) \le z$, we have $-\infty < m \le z < \inf L$. However, (15) implies that $m \in L$. Hence $\delta(z) = -\infty$. Statements (3) and (4) follow from the definitions of δ and L. □

Proof (Proposition 7). Assume $L = \mathbb{R}$. This implies, using Proposition 16 and point (2) of Remark 19, that $\Sigma_{\xi,\eta} = 0$ and g is linear. Further, $\Pi_{\xi,\eta}(A_3) = \Pi_{\xi,\eta}(A_2) = 0$ and $L^* = [\theta_1, \theta_4] = (-\infty, \infty)$. Now $\theta_1 = -\infty$ iff $\Pi_{\xi,\eta}((0, \infty) \times [0, \infty)) = 0$, whilst $\theta_4 = -\infty$ iff $\Pi_{\xi,\eta}((-\infty, 0) \times [0, \infty)) = 0$. Hence ξ has no jumps and η has no negative jumps. By Proposition 16, $g(u) \ge 0$ on $\mathbb{R}$. Since $g(u) = d_\eta + u d_\xi$, this implies that $d_\xi = 0$ and $d_\eta \ge 0$, thus proving one direction of (1). The converse is trivial since V simplifies to $V_t = z + \eta_t$. The proof of (2) is similar and (3) follows from (1) and (2). □

Proof (Proposition 8).

(1) ⇔ (2) Assume $L \cap U \neq \emptyset$ and let $z_1, z_2 \in L \cap U$. We show $z_1 = z_2 \neq 0$. By Proposition 16, $z \in L$ iff $\eta - zW$ is increasing and by Proposition 20, $z \in U$ iff $\eta - zW$ is decreasing. Thus, $\eta - z_1 W = \eta - z_2 W = 0$, which implies $z_1 W = z_2 W$. Since ξ is not zero, W is not zero, and thus $z_1 = z_2$. Further, if $z_1 = z_2 = 0$, then η must be both increasing and decreasing, which requires that η be identically zero. Thus, $z_1 = z_2 \neq 0$.

(2) ⇔ (3) Suppose $L \cap U = \{c\}$. Then $V_t = c$ for all $t \geq 0$ whenever $V_0 = c$, which implies $\mathrm{e}^{\xi_t}(c + Z_t) = c$, which implies $V_t = \mathrm{e}^{\xi_t}(z - c) + c$, as required. Conversely, suppose $V_t = \mathrm{e}^{\xi_t}(z - c) + c$. Clearly, $c \in L \cap U$ and so $L \cap U \neq \emptyset$, which implies $L \cap U = \{c\}$ by the above.

(2) ⇔ (4) By definition of δ and Υ, c is an absorbing point iff $\delta(c) = \Upsilon(c) = c$. The definitions of L and U imply that this occurs iff $c \in L \cap U$.

(2) ⇒ (5) Assume $L \cap U = \{c\}$ where $c \neq 0$. Propositions 16 and 20 imply Eq. (10) is satisfied for $u = c$, and imply respectively that $g(c) \geq 0$ and $g(c) \leq 0$, thus giving $g(c) = 0$. Finally, since (2) ⇒ (3), the equation $Z_t := \int_0^t \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s = c\left(\mathrm{e}^{-\xi_t} - 1\right)$ holds, which implies that $\mathrm{e}^{-\xi_{t-}} \Delta\eta_t = c\left(\mathrm{e}^{-\xi_t} - 1\right) - c\left(\mathrm{e}^{-\xi_{t-}} - 1\right)$ and so $\Delta\eta_t = c(\mathrm{e}^{-\Delta\xi_t} - 1)$.

(5) ⇒ (2) Assume (5) holds for $c \neq 0$. We prove $c \in L$, and a symmetric argument proves $c \in U$. Since (10) is satisfied for $u = c$, and $g(c) = 0$ holds, conditions (i) and (iii) of Proposition 16 are respectively satisfied for $u = c$. Thus it suffices to prove condition (ii) of Proposition 16 is satisfied for $u = c$, or equivalently, show $c \in L^*$. If $\Pi_{\xi,\eta} = 0$ then this is trivial since $L^* = \mathbb{R}$. Suppose that $\Pi_{\xi,\eta}$ is supported on the curve $\{(x, y) : y - c(\mathrm{e}^{-x} - 1) = 0\}$ for $c \in \mathbb{R}$. If $c > 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$ and $\Pi_{\xi,\eta}(A_4) \neq 0$, then $\theta_2 = \theta_4 = c$ and so $L^* = \{c\}$. If $c \geq 0$, $\Pi_{\xi,\eta}(A_2) = 0$ and $\Pi_{\xi,\eta}(A_4) \neq 0$, then $\theta_2 = 0$ and $\theta_4 = c$, and so $L^* = [0, c]$. If $c \geq 0$, $\Pi_{\xi,\eta}(A_2) \neq 0$ and $\Pi_{\xi,\eta}(A_4) = 0$, then $\theta_2 = c$ and $\theta_4 = \infty$, and so $L^* = [c, \infty)$. In each of these three cases, $c \in L^*$. The proof for $c < 0$ is similar. Hence, $c \in L \cap U$ which, by the equivalence of statements (1), (2), implies that $L \cap U = \{c\}$, as required.

(2) ⇔ (6) $L \cap U = \{c\}$ iff $\eta - cW = 0$ where $\mathrm{e}^{-\xi_t} = \epsilon(W)_t$ which occurs iff $\mathrm{e}^{-\xi_t} = \epsilon(\eta/c)_t$.

Now assume statements (1)–(6) hold. If $\Sigma_{\xi,\eta} \neq 0$ and both L and U are non-empty, then Propositions 16 and 20 imply that $L = U = \{c\}$ where $c = -\frac{\sigma_{\xi,\eta}}{\sigma_\xi^2}$. For examples of Lévy processes (ξ, η) satisfying statements (1)–(6) and such that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s., see Example 26.

If $\Sigma_{\xi,\eta} = 0$ then the statements (a), (b) and (c) follow from the equation for V in statement (3) above. For examples of (ξ, η) satisfying statement (c) and satisfying each of the three asymptotic behaviours, see Example 27. □

Proof (Theorem 9). Assume $L \cap U = \emptyset$. Suppose, firstly, that $\Sigma_{\xi,\eta} \neq 0$. We prove (ξ, η) exists such that (1), (2) or (3) occurs, and for each case, we show that ξ can satisfy each of the three asymptotic behaviours. For (1), this is obvious. Choosing (ξ, η) such that $\Sigma_{\xi,\eta}$ does not satisfy Eq. (10) implies that (ξ, η) fails both propositions, and so $L = U = \emptyset$, regardless of the choice of $(\tilde{\gamma}_\xi, \tilde{\gamma}_\eta)$ and $\Pi_{\xi,\eta}$. Clearly, we can make suitable choices for these objects to obtain the desired asymptotic behaviour of ξ. For (2), our existence claims are proven by Example 25, and (3) is symmetric. It follows from Proposition 17, and the symmetric version for U, that whenever L and U are non-zero, they equal $\{-\sigma_{\xi,\eta}/\sigma_\xi^2\}$. Hence, no cases, other than (1), (2), and (3) can exist.

Now suppose $\Sigma_{\xi,\eta} = 0$. We prove (ξ, η) exists such that (a), (b) or (c) occurs, and for case, we show that ξ can satisfy the specified asymptotic behaviours. Examples 28 and 29 present (ξ, η)

such that $L = \emptyset$, whilst U may be of form $\emptyset$, $\{a\}$ or $[a, b]$ for $-\infty < a < b < \infty$, and for each of these combinations, it is shown that ξ can satisfy the three asymptotic behaviours. In Example 30, $L = \emptyset, U$ is of form $[b, \infty)$ for $b \in \mathbb{R}$, and ξ drifts to $-\infty$ a.s. In Example 32, $L = \emptyset, U$ is of form $(-\infty, a]$ for $a \in \mathbb{R}$, and ξ drifts to ∞ a.s. These four examples prove the existence claims for (a), and the case (b) is symmetric. In Example 31, $L = (-\infty, a], U = [b, \infty)$ for $-\infty < a < b < \infty$ and ξ drifts to $-\infty$ a.s. In Example 33, $U = (-\infty, a], L = [b, \infty)$ for $-\infty < a < b < \infty$, and ξ drifts to ∞ a.s. These two examples prove the existence claims for (c).

Now assume $\Sigma_{\xi,\eta} = 0, L \neq \emptyset, U \neq \emptyset$ and $L \cap U = \emptyset$. We prove that no cases, other than those listed in (c), can exist. As noted in point (2) of Remark 21, it follows from our assumptions that (ξ, η) is finite variation and g is linear.

Suppose $L = [a, b]$ for some $-\infty < a \leq b < \infty$. We obtain a contradiction. If $L^* = [c, d]$ for some $-\infty < c \leq a \leq b \leq d < \infty$, then (2) of Proposition 22 states that $U^* = \emptyset$ or $U^* = L^* = \{c\} = \{d\}$. Thus, $U = \emptyset$ or $U = L = \{a\} = \{b\}$, both of which contradict our assumptions. Hence, $L^* = [c, \infty)$ for some $-\infty < c \leq a$, or $L^* = (-\infty, d]$ for some $b \leq d < \infty$.

Thus, suppose $L = [a, b]$ and $L^* = [c, \infty)$ for some $-\infty < c \leq a \leq b < \infty$. The case $L^* = (-\infty, d]$ for some $b \leq d < \infty$ is symmetric. We know $g(u) = d_\eta + ud_\xi$. If $d_\xi \geq 0$ then $b = \infty$, which we have rejected. Hence $d_\xi < 0$, and so $b = -\frac{d_\eta}{d_\xi} \geq a$. Thus, since U is non-empty, $L \cap U = \emptyset$, and $g(u) \leq 0$ on U, we must have $U \subset [b, \infty)$. However, (3) of Proposition 22 implies that $U^* \cap [b, \infty) = \emptyset$. Hence U is empty, which is a contradiction. This completes the proof that $L \neq [a, b]$ for some $-\infty < a \leq b < \infty$.

Now assume $L = [b, \infty)$ for $b \in \mathbb{R}$. We prove ξ is a subordinator. Proposition 17 and point (2) of Remark 19, imply respectively, that (ξ, η) has no Brownian component, and (ξ, η) is of finite variation. Thus, $g(u) = d_\eta + ud_\xi$. Proposition 16 implies that $g(u) \geq 0$ on $[b, \infty)$ and hence $d_\xi \geq 0$. Finally, $L^* = [c, \infty)$ for some $-\infty \leq c \leq b$. It is a consequence of the proofs of statements (1), (3) of Proposition 22, that ξ has no negative jumps. Thus ξ is a subordinator.

Now, assume $L = [b, \infty)$ for $b \in \mathbb{R}$ and $U = \emptyset$. We prove $U = (-\infty, a]$ for some $-\infty < a < b < \infty$. Note that $L^* = [c, \infty)$ for some $-\infty \leq c \leq b$, so statement (3) of Proposition 22 implies that $U^* = (-\infty, d]$ for some $-\infty < d \leq c$. Since $g(u) = d_\eta + ud_\xi$ and $d_\xi \geq 0$, we have $U = (-\infty, a]$ for some $-\infty < a \leq d$. Since we have assumed $L \cap U = \emptyset$, we have $a < b$.

If we assume that $U = (-\infty, a]$ for $a \in \mathbb{R}$, it can be shown, using a method of proof similar to the one above, that ξ is a subordinator, and $L = \emptyset$ or $L = [b, \infty)$ for some $-\infty < a < b < \infty$. We omit the details.

If $L = (-\infty, a]$ for $a \in \mathbb{R}$, then symmetric proofs to the above, show that $-\xi$ is a subordinator, and $U = \emptyset$ or $U = [b, \infty)$ for $-\infty < a < b < \infty$. Similarly, if $U = [b, \infty)$ for $b \in \mathbb{R}$, then symmetric proofs show that $-\xi$ is a subordinator, and $L = \emptyset$ or $L = (-\infty, a]$ for $-\infty < a < b < \infty$. □

Proof (Proposition 11). Assume $L \cap U = \emptyset$. In the proof of Theorem 9, it was shown that if $L = [b, \infty)$ for $b \in \mathbb{R}$ then (ξ, η) is of finite variation, $\Sigma_{\xi,\eta} = 0, d_\xi \geq 0, \Pi_{\xi,\eta}(A_3) = 0, \Pi_{\xi,\eta}(A_4 \setminus A_1) = 0$, and $\theta_2 < \infty$. By Propositions 16 and 17 the converse also holds. A similar proof shows that $U = (-\infty, a]$ for $a \in \mathbb{R}$ iff (ξ, η) is of finite variation, $\Sigma_{\xi,\eta} = 0, d_\xi \geq 0, \Pi_{\xi,\eta}(A_4) = 0, \Pi_{\xi,\eta}(A_3 \setminus A_2) = 0$, and $\theta_1' > -\infty$. Combining these two sets of iff conditions gives iff conditions for the case $U = (-\infty, a]$ and $L = [b, \infty)$ with $-\infty < a < b < \infty$. Since V is increasing on L and decreasing on U, and V is a strong Markov process, it is clear that $\lim_{t\to\infty} |V_t| = \infty$ a.s. for any finite starting random variable V_0.

It follows by symmetric methods that $L = (-\infty, a]$ and $U = [b, \infty)$ for $-\infty < a < b < \infty$ iff the conditions in Proposition 11 hold. We must show that in this situation V can be strictly stationary. In [8] it is shown that

$$V_t =_D e^{\xi_t} z + \int_0^t e^{\xi_{s-}} dK_s^{\xi,\eta}.$$

Theorem 2 in [7] states that if $\lim_{t\to\infty} \xi_t = -\infty$ and $I_{-\xi,K^{\xi,\eta}} = \infty$ then $\left|\int_0^t e^{\xi_{s-}} dK_s^{\xi,\eta}\right| \to_P \infty$ as $t \to \infty$. As noted, if $L = (-\infty, a]$ and $U = [b, \infty)$ with $-\infty < a < b < \infty$ then $-\xi$ is a subordinator, so $\lim_{t\to\infty} \xi_t = -\infty$ a.s. Now if $I_{-\xi,K^{\xi,\eta}} = \infty$ then by the above, and since $\lim_{t\to\infty} e^{\xi_t} = -\infty$ a.s, we have $|V_t| \to_D \infty$. However this is impossible since V is increasing on L and decreasing on U. Thus, $I_{-\xi,K^{\xi,\eta}} < \infty$. Hence, by Theorem 2.1 in [8], there is a finite random variable $V_\infty := \int_0^\infty e^{\xi_{s-}} dK_s^{\xi,\eta}$ such that V, starting with $V_0 = V_\infty$, is strictly stationary. Since V is increasing on L and decreasing on U, and V is a strong Markov process, it is clear that V_∞ has support (a, b). □

Proof (Theorem 12). Assume $Z_t \to Z_\infty$ a.s. as $t \to \infty$, where Z_∞ is finite. Suppose that for all $c \in \mathbb{R}$, Eq. (9) does not hold. This implies that Z_∞ is continuous. As noted in Section 1, a necessary condition for convergence of Z_t, is $\lim_{t\to\infty} \xi_t = \infty$ a.s., which implies that $e^{\xi_t} \to \infty$ a.s. Since Z_∞ is finite a.s., and $e^{\xi_t} \to \infty$ a.s., it is clear from the definition $V_t := e^{\xi_t}(z + Z_t)$, that

$$P\left(\lim_{t\to\infty} V_t = \infty | V_0 = z\right) = P(Z_\infty > -z). \tag{16}$$

Now let $a \le \sup U$. By definition of U, $P(\lim_{t\to\infty} V_t = \infty | V_0 = a) = 0$ which implies, by Eq. (16), that $Z_\infty < -a$ a.s., as required.

Conversely, let $a > \sup U$. We prove $P(Z_\infty > -a) > 0$. Since we have assumed that $|Z_\infty| < \infty$ a.s., we can choose $x > a$ such that $P(Z_\infty > -x) > 0$. Note that $\Upsilon(a) = \infty$ and so there exists a fixed time $T > 0$ such that $P(V_T \ge x | V_0 = a) > 0$.

Hence, using (16), the law of conditional probability and the Markov property,

$$\begin{aligned} P(Z_\infty > -a) &= P\left(\lim_{t\to\infty} V_t = \infty | V_0 = a\right) \\ &\ge P\left(\lim_{t\to\infty} V_t = \infty | V_T \ge x\right) P(V_T \ge x | V_0 = a) \\ &\ge P\left(\lim_{t\to\infty} V_t = \infty | V_0 = x\right) P(V_T \ge x | V_0 = a) \end{aligned}$$

which is greater than zero by (16) and the choice of x and T. Thus,

$$a \le \sup U \quad \text{iff } Z_\infty < -a \text{ a.s.} \tag{17}$$

Now we prove $-\sup U = m$ where $m := \inf\{u \in \mathbb{R} : Z_\infty < u \text{ a.s.}\}$. By (17), $Z_\infty < -\sup U$, so $-\sup U \ge m$. By assumption, Z_∞ has no atoms and so $Z_\infty < m$ a.s. Thus, (17) implies $-m \le \sup U$. The proofs for L are similar.

Now assume there exists $c \in \mathbb{R}$ such that Eq. (9) holds, and assume $Z_t \to Z_\infty$ a.s. as $t \to \infty$. By (9), $Z_\infty = -c$ a.s. Further, since ξ drifts to ∞ a.s., Proposition 8 implies that $L = U = \{c\}$, or $U = (-\infty, c]$ and $L = [c, \infty)$. In both these cases, $\inf L = \sup U = c$. □

Proof (Theorem 13). (1) Assume $L \cap U = \emptyset$, $\sup U \ge 0$, $L \cap [0, \sup U] = \emptyset$, and let $0 \le u \le \sup U$. We prove $\psi(u) = 1$. There exists $z \ge u$ such that $z \in U$, and so $\Upsilon(z) = z$. Since $\psi(u) \ge \psi(z)$, it suffices to prove that $\psi(z) = 1$. Since $L \cap [0, \sup U] = \emptyset$, we know $\delta(z) < 0$, which implies that $P_z(\inf_{t>0} V_t < 0) > 0$. Thus, there exists a fixed time $T \in \mathbb{R}$ such that $P_z(\inf_{0<t\le T} V_t < 0) := m > 0$. Let $n \in \mathbb{N}$ and let A be the distribution of V_{nT} conditional on both $V_0 = z$ and $\inf_{0<t\le nT} V_t \ge 0$. Since $\Upsilon(z) = z$ we know $A \le z$ a.s. Now

$$P_z\left(\inf_{nT<t\le(n+1)T} V_t < 0 \middle| \inf_{0<t\le nT} V_t \ge 0\right) = P_A\left(\inf_{0<t\le T} V_t < 0\right) \ge m,$$

where the equality follows from the Markov property and the inequality follows from the fact that $A \le z$ and V_t is increasing in z. Define $P^n := P_z\left(\inf_{0<t\le nT} V_t < 0\right)$ for all $n \in \mathbb{N}$. By the law of total probability

$$P^{n+1} = P^n + P_z\left(\inf_{nT<t\le(n+1)T} V_t < 0 \middle| \inf_{0<t\le nT} V_t \ge 0\right)\left(1 - P^n\right)$$

and so $P^{n+1} \ge P^n + (1 - P^n)m$ where $P^1 = m \in (0, 1)$. This implies that $P^n \ge 1 - (1 - m)^n$ which implies that $\lim_{n\to\infty} P^n = 1$, and hence $P_z\left(\inf_{0<t} V_t < 0\right) = 1$ by the continuity property of measures.

(2) Assume $L \cap U = \emptyset$, $\sup L \ge 0$, and $U \cap [0, \sup L] = \emptyset$. We let $z \ge 0$ and prove that $\psi(z) < 1$. If $z \ge \inf L$ then $\psi(z) = 0$ by definition. Thus, it suffices to assume $0 \le z < \inf L$. Suppose $\psi(z) = 1$. By assumption, $\Upsilon(z) > \inf L$ and so, by definition, $P(C) > 0$ where $C := \{\sup_{t\ge0} V_t \ge \inf L\}$. By definition of L, $\lim_{t\to\infty} V_t \ge \inf L$ a.s. for all $\omega \in C$. Let $T_1 := \inf\{t > 0 : V_t < 0\}$ and $T_n := \inf\{t > T_{n-1} : V_t < V_{T_{n-1}}\}$ for integers $n > 1$. By assumption, $\psi(z) = 1$ and so T_1 is finite a.s. Further, the strong Markov property of V implies that $\{T_n\}$ is a sequence of stopping times increasing towards infinity as $n \to \infty$, and each T_i is a.s. finite. In particular, each T_i is a.s. finite on C. However $V_{T_n} < 0$ a.s. which contradicts the fact that $\lim_{t\to\infty} V_t > \inf L$ a.s. on C. Hence $\psi(z) < 1$. The proof of the case $U \cap [0, \sup L] \ne \emptyset$ is similar. □

Proof (Theorem 14). (1): Assume $L \cap U = \emptyset$, $\lim_{t\to\infty} \xi_t = -\infty$ a.s. and $I_{-\xi,K^{\xi,\eta}} < \infty$. Suppose that $L \cap [0, \infty) \ne \emptyset$. Since ξ drifts to $-\infty$ a.s., Proposition 8 and Theorem 9 imply that condition (a) or (b) holds. Further, by statement (2) of Theorem 13 and the definition of L, we have $0 < \psi(z) < 1$ for all $0 \le z < \inf L$, and $\psi(z) = 0$ for all $z \ge \inf L$.

Suppose $L \cap [0, \infty) = \emptyset$. We let $z \ge 0$ and prove $\psi(z) = 1$. Let N be a Poisson process with parameter λ, let D_i be an iid sequence of one-dimensional exponential random variables and let $C_i = 1$ for all i. Suppose N, D_i and (ξ, η) are mutually independent and define the compound Poisson process $(X_t, Y_t) := \sum_{i=1}^{N_t}(C_i, D_i)$. Define a Lévy process $(\xi_t^\diamond, \eta_t^\diamond) := (\xi_t, \eta_t) + (X_t, Y_t)$, and denote the associated GOU by $V^\diamond$. For $V^\diamond$, denote the upper and lower bound functions, the sets of upper and lower bounds, and the ruin probability function by $\Upsilon^\diamond$, $\delta^\diamond$, $U^\diamond$, $L^\diamond$ and $\psi^\diamond$ respectively.

Recall $T_z := \inf\{t > 0 : V_t < 0 | V_0 = z\}$. Since $\sup L < 0$, we know $\delta(z) < 0$ and hence T_z is finite a.s. Note that $V_0 = V_0^\diamond = z$. Also, whenever $V_{t-} \ge 0$, every jump $\Delta(X, Y)_t$ causes a non-negative ΔV_t. Hence $V_t \le V_t^\diamond$ a.s. on $t \le T_z$. This implies that $\psi(z) \ge \psi^\diamond(z)$. Thus it suffices to show $\psi^\diamond(z) = 1$. We first prove that $V^\diamond$ can be strictly stationary.

We show that $\lambda > 0$ can be chosen small enough such that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$. Since $\lim_{t\to\infty} \xi_t = -\infty$, either $E(\xi_1) \in [-\infty, 0)$ or $E(\xi_1)$ does not exist. If $E(\xi_1) \in [-\infty, 0)$ then $E(\xi_1^\diamond) = E(\xi_1) + \lambda$ and so we can choose λ small enough that $E(\xi_1^\diamond) < 0$, which implies $\lim_{t\to\infty} \xi_t^\diamond = -\infty$. If $E(\xi_1)$ does not exist then $E(\xi_1^\diamond)$ does not exist. We show that $\lim_{t\to\infty} \xi_t^\diamond = -\infty$ holds for any $\lambda > 0$. Note that $\xi^\diamond = \xi + N$ and, by Section 1, $J_\xi^+ < \infty$ since $E(\xi_1)$ does not exist and $\lim_{t\to\infty} \xi_t = -\infty$. Also, $\overline{\Pi}_{\xi^\diamond}^- = \overline{\Pi}_\xi^-$ and so $A_{\xi^\diamond}^- = A_\xi^-$. Since ξ and N

are independent we have $\overline{\Pi}^+_{\xi^\diamond} = \overline{\Pi}^+_{\xi} + \overline{\Pi}^+_{N}$. Further $\overline{\Pi}^+_{N}(x) = 0$ for all $x \geq 1$. Hence $J^+_{\xi^\diamond} = J^+_{\xi}$ and so is finite. By Section 1, this implies that $\lim_{t\to\infty} \xi^\diamond_t = -\infty$.

We now show that $(\xi^\diamond, \eta^\diamond)$ satisfies $I_{-\xi^\diamond, K^{\xi^\diamond,\eta^\diamond}} < \infty$. Since (ξ, η) and (X, Y) are independent, the definitions in Section 1 imply that $K^{\xi^\diamond,\eta^\diamond}_t = K^{\xi,\eta}_t + K^{X,Y}_t$ and $\overline{\Pi}_{K^{\xi^\diamond,\eta^\diamond}}(y) = \overline{\Pi}_{K^{\xi,\eta}}(y) + \overline{\Pi}_{K^{X,Y}}(y)$. And, as above, $A^+_{-\xi^\diamond} = A^+_{-\xi}$. Hence

$$I_{-\xi', K^{\xi^\diamond,\eta^\diamond}} = I_{-\xi, K^{\xi,\eta}} + \int_{(e,\infty)} \left(\frac{\ln(y)}{A^+_{-\xi}(\ln(y))}\right) |\overline{\Pi}_{K^{X,Y}}(\mathrm{d}y)|.$$

By the choice of (X, Y) it is clear that $K^{X,Y}_1$ has a finite expected value which implies that $\int_{(e,\infty)} y|\overline{\Pi}_{K^{X,Y}}(\mathrm{d}y)| < \infty$. Hence $I_{-\xi', K^{\xi^\diamond,\eta^\diamond}} < \infty$. Thus $V^\diamond$ can be assumed to be strictly stationary.

For a Lebesgue set Λ define $T^\diamond_\Lambda := \inf\{t > 0 : V^\diamond_t \in \Lambda\}$. Since $\theta'^\diamond_1 = -\infty$, Proposition 20 implies that $\Upsilon^\diamond(u) = \infty$ for all $u \in \mathbb{R}$, or equivalently, $U^\diamond = \emptyset$. Also, $\theta^\diamond_1 = 0$, and so Proposition 16 implies that $L^\diamond \cap (-\infty, 0) = \emptyset$, whilst the fact that $L \cap (0, \infty) = \emptyset$ clearly implies that $L' \cap (0, \infty) = \emptyset$.

These facts imply that, for all a and u in $\mathbb{R}$, $P(T^\diamond_{(-\infty,a]} < \infty | V^\diamond_0 = u) > 0$ and $P(T^\diamond_{[a,\infty]} < \infty | V^\diamond_0 = u) > 0$. Since D is an exponential random variable, $V^\diamond_t$ has a continuous density with respect to Lebesgue measure. Hence $P(T^\diamond_\Lambda < \infty) > 0$ for any set Λ with positive Lebesgue measure. This result, and the fact that $V^\diamond$ is strictly stationary, allows us to mimic the argument of Theorem 3.1(a) in Paulsen [10]. Let S be an independent standard exponential variable and define the resolvent kernel

$$K(z, \Lambda) := \int_0^\infty P_z(V^\diamond_t \in \Lambda)\mathrm{e}^{-t}\mathrm{d}t = P_z(V^\diamond_S \in \Lambda).$$

Proposition 2.1 of [9] implies that $V^\diamond$ is ϕ-irreducible for the measure $\phi = \lambda K$. Using the language of [9, p. 495 and 496], it is clear that K has a continuous nontrivial component for all z and hence is a T-process. Since $V^\diamond$ is strictly stationary it is clear that $V^\diamond$ is non-evanescent, as defined in [9, p. 494]. Thus Theorem 3.2 of [9, p. 494] implies that $V^\diamond$ is Harris recurrent, as defined in [9, p. 490], which implies that $\psi^\diamond(z) = 1$.

(2) Assume that $L \cap U = \emptyset$, $E(\xi_1) = 0$, $E(\mathrm{e}^{|\xi_1|}) < \infty$ and there exist $p, q > 1$ with $1/p + 1/q = 1$ such that $E(\mathrm{e}^{-p\xi_1}) < \infty$ and $E(|\eta_1|^q) < \infty$.

Suppose that $L \cap [0, \infty) \neq \emptyset$. Since ξ oscillates a.s., Theorem 9 implies that $L = [a, b]$ and $U = \emptyset$ where $-\infty < a \leq b < \infty$ and $b \geq 0$. Hence, it follows from statement (2) of Theorem 13 and the definition of L, that $0 < \psi(z) < 1$ for all $0 < z < a$ and $\psi(z) = 0$ for all $z \geq a$.

Now suppose that $L \cap [0, \infty) = \emptyset$. We let $z \geq 0$ and prove that $\psi(z) = 1$. We know that $P(\inf_{t>0} V_t < 0 | V_0 = z) > 0$. However, it is possible that for some $z > 0$, $P(V_1 < 0 | V_0 = z) = 0$. For example, this would happen if (ξ, η) has no Brownian component and $\sup L^* > 0$. Let $0 = T_0 < T_1 < T_2 < \cdots$ be random times such that $T_i - T_{i-1}$ are iid with exponential distribution and parameter λ. Since T_1 has infinite support it is clear that $\sup L < 0$ implies $P(V_{T_1} < 0 | V_0 = z) > 0$ for all $z \geq 0$. Eq. (1) implies that a.s.

$$V_{T_n} = \mathrm{e}^{\xi_{T_n} - \xi_{T_{n-1}}} \left(\mathrm{e}^{\xi_{T_{n-1}}} \left(z + \int_0^{T_{n-1}} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s\right)\right) + \mathrm{e}^{\xi_{T_n}} \int_{T_{n-1}+}^{T_n} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s.$$

Thus, if we define $A_n := \mathrm{e}^{\xi_{T_n}-\xi_{T_{n-1}}}$, $B_n := \mathrm{e}^{\xi_{T_n}} \int_{T_{n-1}+}^{T_n} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s$ and the stochastic difference equation $Y_n := A_n Y_{n-1} + B_n$ with $Y_0 := V_0 = z$ then $Y_n = V_{T_n}$ a.s. for all $n \in \mathbb{N}$. The term $\mathrm{e}^{\xi_{T_n}}$ in B_n cannot be brought under the integral sign because it is not predictable. Since a Lévy process has independent increments it is clear that (A_n, B_n) is an independent sequence. Now,

$$\begin{aligned} (A_2, B_2) = &\left(\mathrm{e}^{\xi_{T_2}-\xi_{T_1}}, \mathrm{e}^{\xi_{T_2}-\xi_{T_1}} \mathrm{e}^{\xi_{T_1}} \int_{T_1+}^{T_2} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s\right) \\ = &\left(\mathrm{e}^{\xi_{T_2}-\xi_{T_1}}, \mathrm{e}^{\xi_{T_2}-\xi_{T_1}} \int_{T_1+}^{T_2} \mathrm{e}^{-\left(\xi_{s-}-\xi_{T_1}\right)} \mathrm{d}\eta_s\right) \\ = &\left(\mathrm{e}^{\xi_{T_2}-\xi_{T_1}}, \mathrm{e}^{\xi_{T_2}-\xi_{T_1}} \int_{T_1+}^{T_2} \mathrm{e}^{-\left(\xi_{s-}-\xi_{T_1}\right)} \mathrm{d}(\eta_s - \eta_{T_1})\right) \\ =_D &\left(\mathrm{e}^{\xi_{T_1}}, \mathrm{e}^{\xi_{T_1}} \int_0^{T_1} \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s\right) = (A_1, B_1), \end{aligned}$$

where the second equality holds because $\mathrm{e}^{\xi_{T_1}}$ is predictable and the fourth equality holds because a Lévy process has identically distributed increments. The argument for general n is identical, and thus (A_n, B_n) is an iid sequence.

Now Proposition 1.1 and Corollary 4.2 of [1] state that if $P(A_1 z + B_1 = z) < 1$ for all $z \in \mathbb{R}$, $E(\ln A_1) = 0$, $A_1 \not\equiv 1$ and there exists $\delta > 0$ such that

$$E\left(\left(|\ln A_1| + \ln^+ |B_1|\right)^{2+\delta}\right) < \infty \tag{18}$$

then W has an invariant unbounded Radon measure μ unique up to a constant factor such that the sample paths W_n, with $W_0 = z$, visit every open set of positive μ-measure infinitely often with probability 1, for every $z \in \mathbb{R}$. The first of these conditions follows from the assumption $L \cap U = \emptyset$, using Proposition 8. The second and third conditions follow respectively from our assumption that $E(\xi_1) = 0$, and ξ_1 is not identically zero. We show later that our moment conditions on ξ and η ensure Eq. (18) holds. Note that the Babillot result implies that $\psi(z) = 1$ if $\mu((-\infty, 0)) > 0$. However by invariance,

$$\mu\left((-\infty, 0)\right) = \int_{z\in\mathbb{R}} P(A_1 z + B_1 < 0)\mu(\mathrm{d}z) \geq \int_{z\in\mathbb{R}} P(V_{T_1} < 0 | V_0 = z)\mu(\mathrm{d}z).$$

Thus if $\mu([0, \infty)) > 0$ then $\mu((-\infty, 0)) > 0$ since $P(V_{T_1} < 0 | V_0 = z) > 0$ for all $z \geq 0$. And if $\mu([0, \infty)) = 0$ then $\mu((-\infty, 0)) > 0$ since $\mu(\mathbb{R}) > 0$. Thus we are done if we can prove Eq. (18).

To do this, it suffices to assume $T_1 = 1$ and $(A_1, B_1) := (\mathrm{e}^{\xi_1}, \mathrm{e}^{\xi_1} \int_0^1 \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s)$ since we can choose the parameter λ of the increments to be arbitrarily small. Note that if $x, y > 0$ and $\alpha > 0$ then there exists $c_1 > 0$ such that

$$(x + y)^\alpha \leq c_1 \left(x^\alpha + y^\alpha\right). \tag{19}$$

Hence, to prove (18), it suffices to prove that $E(|\xi_1|^{2+\delta}) < \infty$ and

$$E\left(\left(\ln^+ \left|\mathrm{e}^{\xi_1} \int_0^1 \mathrm{e}^{-\xi_{s-}} \mathrm{d}\eta_s\right|\right)^{2+\delta}\right) < \infty. \tag{20}$$

Note that the former inequality is assumed as a condition. If $x, y > 0$ then $\ln^+(xy) \leq \ln^+(x) + \ln^+(y)$, and hence, using (19), Eq. (20) holds if

$$E\left(\left(\ln^+\left|\int_0^1 e^{-\xi_{s-}} d\eta_s\right|\right)^{2+\delta}\right) < \infty. \tag{21}$$

Whenever $0 < \delta \leq 1$ and $x > 0$, there exists $c_2 > 0$ such that $(\ln^+ x)^{2+\delta} \leq c_2 x^\delta$. It suffices to assume $0 < \delta \leq 1$, and hence (21) holds if $E(|\int_0^1 e^{-\xi_{s-}} d\eta_s|^\delta) < \infty$. However, with our assumptions on p and q, this follows from Lemma 24.

(3) Assume $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Suppose $-\infty \leq \sup U < z$. Assume, for contradiction, that $\psi(z) = 1$. Theorem 12 implies that $P(C) > 0$ where $C := \{Z_\infty > -z\}$. Since $\lim_{t\to\infty} \xi_t = \infty$, we have $\lim_{t\to\infty} V_t = \infty$ a.s. on C. The same strong Markov property argument used in the proof of statement (2) of Theorem 13, gives a contradiction. Hence $\psi(z) < 1$.

Now suppose $U \cap [0, \infty) \neq \emptyset$. Since ξ drifts to ∞ a.s., Theorem 9 implies that either $U = [a, b]$ and $L = \emptyset$ where $-\infty \leq z \leq b < \infty$ and $b \geq 0$, or $U = (-\infty, a]$ and $L = [b, \infty)$ for some $0 \leq a < b < \infty$. In both cases, statement (1) of Theorem 13 implies that $\psi(z) = 1$ for all $z \leq \sup U$. By the definition of L and the above result, $0 < \psi(z) < 1$ for all $\sup U < z < \inf L$ and $\psi(z) = 0$ for all $z \geq \sup L$. $\square$

Propositions 8 and 11 and Theorem 9 claim that Lévy processes (ξ, η) exist which satisfy particular combinations of L and U, and particular asymptotic behaviour for ξ. We present examples to prove these claims. The Lévy measures will always be finite activity, namely $\Pi_{\xi,\eta}(\mathbb{R}^2) < \infty$. Hence, $(\xi, \eta)_t = (d_\xi, d_\eta)t + (B_{\xi,t}, B_{\eta,t}) + \sum_{i=1}^{N_t} Y_i$ where $(B_{\xi,t}, B_{\eta,t})$ is Brownian motion with covariance matrix $\Sigma_{\xi,\eta}$, N is a Poisson process with parameter Λ and $\{Y_i\}_{i=1}^\infty$ is an iid sequence of two-dimensional random variables with distribution Y.

Examples with Brownian component. The first example has $L = \{a\}, U = \emptyset$ and the second example has $L = U = \{a\}$. For each, we choose parameters so that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. or ξ oscillates a.s.

Example 25. Let $(\xi, \eta)_t := (d_\xi, 2)t + (B_t, B_t) + \sum_{i=1}^{N_t} Y_i$ where B is a one-dimensional Brownian motion with variance 1, $P(Y = (10, 10)) = 1/2$ and $P(Y = (-10, 10)) = 1/2$. The covariance matrix Eq. (10) holds for $u = -1$. Condition (ii) of Proposition 16 holds for $u = -1$, whilst condition (ii) of Proposition 20 fails to hold. By Eq. (13), $g(-1) = 3/2 - d_\xi$. Thus, if $d_\xi \leq 3/2$ then $L = \{-1\}$ and $U = \emptyset$. However $E(\xi_1) = d_\xi$ so if $0 < d_\xi < 3/2$ then ξ drifts to ∞ a.s., if $d_\xi < 0$ then ξ drifts to $-\infty$ a.s., and if $d_\xi = 0$ then ξ oscillates a.s.

Example 26. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + (B_t, -B_t)$. Eq. (10) holds for $u = 1$, whilst condition (ii) of Proposition 16 and condition (ii) of Proposition 20 hold trivially. Eq. (13) implies $g(1) = d_\eta + d_\xi - 1/2$. Thus, if $d_\xi = 1/2 - d_\eta$, then $L = U = \{1\}$. Note $E(\xi_1) = d_\xi$, so if $d_\eta < 1/2$ then ξ drifts to ∞ a.s., if $d_\eta > 1/2$ then ξ drifts to $-\infty$ a.s., and if $d_\eta = 1/2$ then ξ oscillates a.s.

Examples with no Brownian component. We present seven examples of Lévy processes (ξ, η) with no Brownian component. In Example 27, $L = U = \{a\}$ and we indicate how the parameters can be changed in order to obtain each of the three asymptotic behaviours for ξ. In Examples 28 and 29, $L = \emptyset$, whilst U may be of form $\emptyset$, $\{a\}$ or $[a, b]$ for $-\infty < a < b < \infty$. We indicate how parameters can be changed in order to obtain these different sets, and for each set, to obtain the three possible asymptotic behaviours for ξ. In Example 30, $L = \emptyset$ whilst U is of form $[b, \infty)$ for $b \in \mathbb{R}$. In Example 31, $L = (-\infty, a]$ and $U = [b, \infty)$ for $-\infty < a < b < \infty$. For both these examples we show that ξ drifts to $-\infty$ a.s. In Example 32, $L = \emptyset$ whilst U is of form $(-\infty, a]$ for $a \in \mathbb{R}$. In Example 33, $U = (-\infty, a]$ and $L = [b, \infty)$ for $-\infty < a < b < \infty$. For both these examples we show that ξ drifts to ∞ a.s.

Example 27. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (3, 2e^{-3} - 2)) = 1/2$ and $P(Y = (-3, 2e^3 - 2)) = 1/2$. Then $\theta_2 = \theta_2' = \theta_4 = \theta_4' = 2, L^* = U^* = \{2\}$ and $g(u) = d_\eta + ud_\xi$. If $d_\eta = -2d_\xi$ then $g(2) = 0$ and hence $L = U = \{2\}$. Since $E(\xi_1) = d_\xi$, choosing $d_\xi > 0$, $d_\xi < 0$, and $d_\xi = 0$, implies respectively that ξ drifts to ∞ a.s., ξ drifts to $-\infty$ a.s. and ξ oscillates a.s.

Example 28. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (4, -2)) = 1/3, P(Y = (-2, -3)) = 1/3$ and $P(Y = (-2, 1)) = 1/3$. Then $L = \emptyset$ since $\Pi_{\xi,\eta}(A_2)$ and $\Pi_{\xi,\eta}(A_3)$ are both non-zero, whilst $U^* = [\theta_4', \theta_2'] = \left[\frac{1}{e^2-1}, \frac{-2}{e^{-4}-1}\right] \cong [0.2, 2]$. Now $U = \{u \in U^* : g(u) \leq 0\}$ and g simplifies to $g(u) = d_\eta + ud_\xi$. Note that $E(\xi_1) = d_\xi$.

Choosing $d_\xi = 0$ and $d_\eta > 0$ implies that $U = \emptyset$ and ξ oscillates a.s. Choosing $d_\xi > 0$ and $d_\eta > -\theta_4' d_\xi$ implies that $U = \emptyset$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta > -\theta_2' d_\xi$ implies that $U = \emptyset$ and ξ drifts to $-\infty$ a.s.

Choosing $d_\xi = 0$ and $d_\eta < 0$ implies that $U = U^* \cong [0.2, 2]$ and ξ oscillates a.s. Choosing $d_\xi > 0$ and $d_\eta < -\theta_2' d_\xi$ implies that $U = U^* \cong [0.2, 2]$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta < -\theta_4' d_\xi$ implies that $U = U^* \cong [0.2, 2]$ and ξ drifts to $-\infty$ a.s.

Choosing $d_\xi > 0$ and $d_\eta = -\theta_4' d_\xi$ implies that $U = \{\theta_4'\} \cong \{0.2\}$ and ξ drifts to ∞ a.s. Choosing $d_\xi < 0$ and $d_\eta = -\theta_2' d_\xi$ implies that $U = \{\theta_2'\} \cong \{2\}$ and ξ drifts to $-\infty$ a.s.

Note that for Example 32, no adjustment of d_ξ and d_η can result in $U = \{a\}$ with ξ oscillating a.s. We now present a different example with this behaviour.

Example 29. Let $(\xi, \eta)_t := (0, -2)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (2, e^{-2} - 1)) = 1/3$ and $P(Y = (-1, e - 1)) = 1/3$ and $P(Y = (-1, -2)) = 1/3$. Then $L = \emptyset, \theta_2 = \theta_2' = \theta_4 = \theta_4' = 1$, and $U^* = \{1\}$. Since g simplifies to $g(u) = -2$ for all $u \in \mathbb{R}$ we obtain $U = \{1\}$. Since $E(\xi_1) = 0$, ξ oscillates a.s.

Example 30. Let $(\xi, \eta)_t := (0, -2)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (-1, 2)) = 1/3$ and $P(Y = (-2, -3)) = 1/3$ and $P(Y = (0, -5)) = 1/3$. Then $L^* = \emptyset$ whilst $U^* = [\theta_4', \theta_2'] = \left[\frac{2}{e-1}, \infty\right) \cong [1.2, \infty)$. Since $g(u) = -2$ for all $u \in \mathbb{R}$ we obtain $L = \emptyset$ and $U = U^*$ Since $E(\xi_1) = -1.5$, ξ drifts to $-\infty$ a.s.

Example 31. Let $(\xi, \eta)_t := (d_\xi, d_\eta)t + \sum_{i=1}^{N_t} Y_i$ where $P(Y = (-1, 2)) = 1/2$ and $P(Y = (-2, -3)) = 1/2$. Then $L^* = [\theta_1, \theta_3] = \left(-\infty, \frac{-3}{e^2-1}\right] \cong (-\infty, -0.5]$ and $U^* = [\theta_4', \theta_2'] = \left[\frac{2}{e-1}, \infty\right) \cong [1.2, \infty)$. Note that g simplifies to $g(u) = d_\eta + ud_\xi$ and hence choosing $d_\xi \leq 0$ and $d_\eta = 0$ gives $L = L^*$ and $U = U^*$. Since $E(\xi_1) = -1.5 + d_\xi$, ξ drifts to $-\infty$ a.s.

Example 32. Let $(\xi, \eta)_t := \sum_{i=1}^{N_t} Y_i$ where $P(Y = (1, 2)) = 1/3$ and $P(Y = (1, 8)) = 1/3$ and $P(Y = (0, -5)) = 1/3$. Then $L^* = \emptyset$ whilst $U^* = [\theta_3', \theta_1'] = \left(-\infty, \frac{8}{e^{-1}-1}\right] \cong (-\infty, -12.6]$. Note that $g(u) = 0$ for all $u \in \mathbb{R}$ so $L = L^*$ and $U = U^*$. Since $E(\xi_1) = 1$, ξ drifts to ∞ a.s.

Example 33. Let $(\xi, \eta)_t := \sum_{i=1}^{N_t} Y_i$ where $P(Y = (1, 2)) = 1/2$ and $P(Y = (1, 8)) = 1/2$. Then $L^* = [\theta_1, \theta_4] = \left[\frac{2}{e^{-1}-1}, \infty\right) \cong [-3.2, \infty)$ and $U^* = [\theta_3', \theta_1'] = \left(-\infty, \frac{8}{e^{-1}-1}\right] \cong (-\infty, -12.6]$. Note that $g(u) = 0$ for all $u \in \mathbb{R}$ so $L = L^*$ and $U = U^*$. Since $E(\xi_1) = 1$, ξ drifts to ∞ a.s.

7. Proofs for Section 2

Proof (Proposition 5). Assume that $V_t = e^{\xi_t}(z - c) + c$. By definition of L, if $c \geq 0$ then $\psi(z) = 0$ for all $z \geq c$.

Let $0 \leq z < c$. If ξ drifts to $-\infty$ a.s. then $\lim_{t\to\infty} V_t = c$ a.s. Thus, the strong Markov property of V implies that $\psi(z) < 1$, using a proof similar to that used for statement (2) of Theorem 13. If ξ oscillates a.s. then $-\infty = \liminf_{t\to\infty} V_t < \limsup_{t\to\infty} V_t = c$, and so $\psi(z) = 1$. If ξ drifts to ∞ a.s. then $\lim_{t\to\infty} V_t = -\infty$ a.s. which implies $\psi(z) = 1$.

Let $c < 0 \leq z$. If ξ drifts to $-\infty$ a.s. then $\lim_{t\to\infty} V_t = c$ a.s. and so $\psi(z) = 1$. If ξ oscillates a.s. then $c = \liminf_{t\to\infty} V_t < \limsup_{t\to\infty} V_t = \infty$, and so $\psi(z) = 1$. If ξ drifts to ∞ a.s. then $\lim_{t\to\infty} V_t = \infty$ a.s. which implies $\psi(z) < 1$, using a strong Markov property argument. □

Proof (Theorem 1). Suppose that for all $c \in \mathbb{R}$ the degenerate case (9) does not hold. Then, by Proposition 8, $L \cap U = \emptyset$. It follows immediately from Theorem 14 that $0 < \psi(z) < 1$ iff $0 \leq z < m < \infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 1 are satisfied. Now suppose that there exists $c \in \mathbb{R}$ such that Eq. (9) holds. Then it follows immediately from Proposition 5 that $0 < \psi(z) < 1$ iff $0 \leq z < m < \infty$ whenever the assumptions for statement (1), or statement (2), of Theorem 1 are satisfied. In both these situations, $m = c$. □

Proof (Theorem 3). Assume $\lim_{t\to\infty} \xi_t = \infty$ a.s. and $I_{\xi,\eta} < \infty$. Assume that for all $c \in \mathbb{R}$ Eq. (9) does not hold, or equivalently, $L \cap U = \emptyset$. Theorem 3 claims that $\psi(0) = 1$ iff $-\eta$ is a subordinator, or there exists $z > 0$ such that $\psi(z) = 1$. This claim follows from two results: firstly, $\psi(z) = 1$ iff $\sup U \geq 0$ and $z < \sup U$, which is implied by statement (3) of Theorem 13; secondly, $0 \in U$ iff $-\eta$ is a subordinator, which is stated in Proposition 20.

Theorem 3 states conditions on the characteristic triplet of (ξ, η) and claims these are equivalent to the fact that there exists $z > 0$ such that $\psi(z) = 1$. By statement (3) of Theorem 13, there exists $z > 0$ such that $\psi(z) = 1$ iff $\sup U > 0$. And Proposition 20 gives iff conditions on the characteristic triplet of (ξ, η) for the case $\sup U > 0$. These conditions are precisely the conditions stated in Theorem 3.

Finally, statements (1) and (2) of Theorem 3 contain values for $\sup\{z \geq 0 : \psi(z) = 1\}$. These follow from the unstated parallel version of Proposition 17 which gives exact values for the endpoints of U.

Now, assume that there exists $c \in \mathbb{R}$ such that the degenerate Eq. (9) holds, and $L = U = \{c\}$. Since ξ drifts to ∞ a.s., Proposition 8 implies that $\sup U = c$. Thus, Proposition 5 implies that $\psi(z) = 1$ iff $\sup U \geq 0$ and $z < \sup U$. Theorem 3 is proved for the degenerate case by combining this statement with Proposition 20 and the parallel version of Proposition 17, in an identical manner to the above. The only difference is that the set $\{z \geq 0 : \psi(z) = 1\}$ does not contain its supremum in the degenerate case, since $\sup\{z \geq 0 : \psi(z) = 1\} = U = L$, and is an absorbing point. □

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